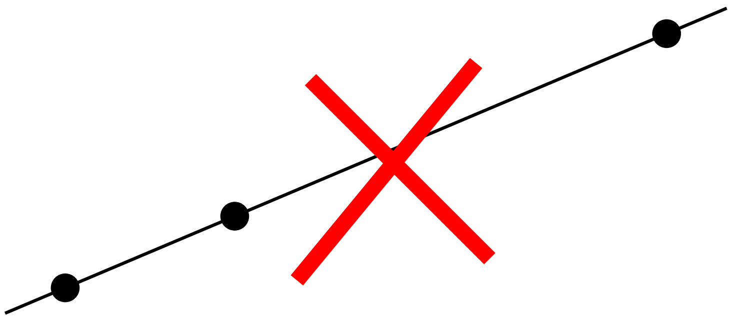
A SAT attack on higher dimensional Erdős–Szekeres numbers

Manfred Scheucher

Planar k-Gons

a finite point set P in the plane is
in *general position* if $\nexists$ collinear points in P



throughout this presentation, every set is in general position

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a finite point set P in the plane is
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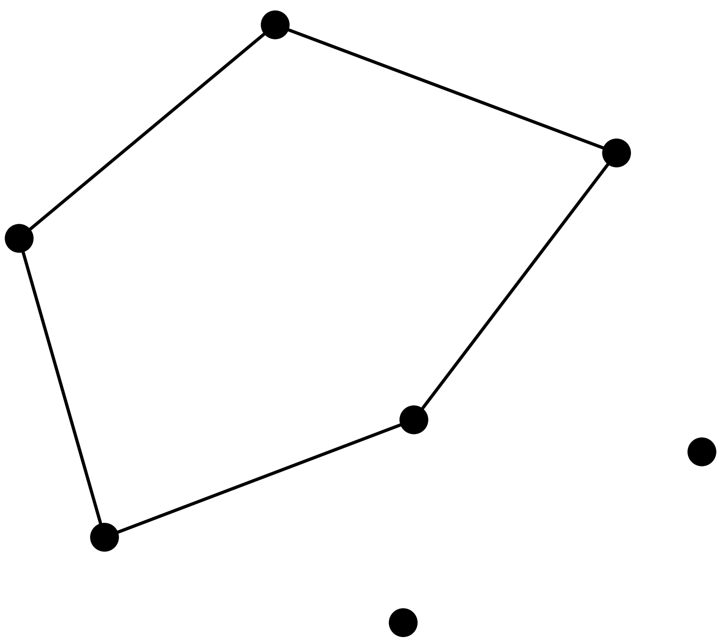
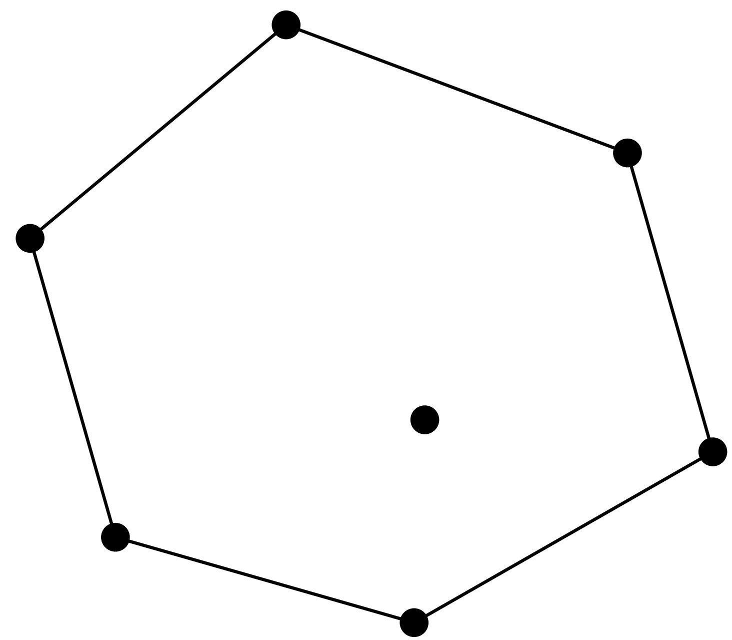
a *k-gon (in P)* is the vertex set of a convex k-gon


5-gon


6-gon

Planar k-Gons

a finite point set P in the plane is
in *general position* if $\nexists$ collinear points in P

a *k-gon (in P)* is the vertex set of a convex k-gon

Theorem (Erdős and Szekeres '35).
$\forall\ k \geq 3$, $\exists$ a smallest integer $g(k)$ such that
every set of $g(k)$ points contains a k-gon.

Planar k-Gons

Theorem. $2^{k-2}+1 \le g(k) \le \binom{2k-4}{k-2}$. [Erdős–Szekeres '35]

↑

equality conjectured by Szekeres, Erdős offered 500$ for a proof

Planar k-Gons

Theorem. $2^{k-2} + 1 \leq g(k) \leq \binom{2k-4}{k-2}$. [Erdős–Szekeres '35]

$\vdots$ several improvements of order $4^{k-o(k)}$

Theorem. $g(k) \leq 2^{k+o(k)}$. [Suk '17]

Planar k-Gons

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Known: $g(4) = 5$, $g(5) = 9$, $g(6) = 17$

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computer assisted proof, 1500 CPU hours [Szekeres–Peters '06]

Planar k-Gons

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< 1 hour using SAT solvers [S.'18, Marić '19]

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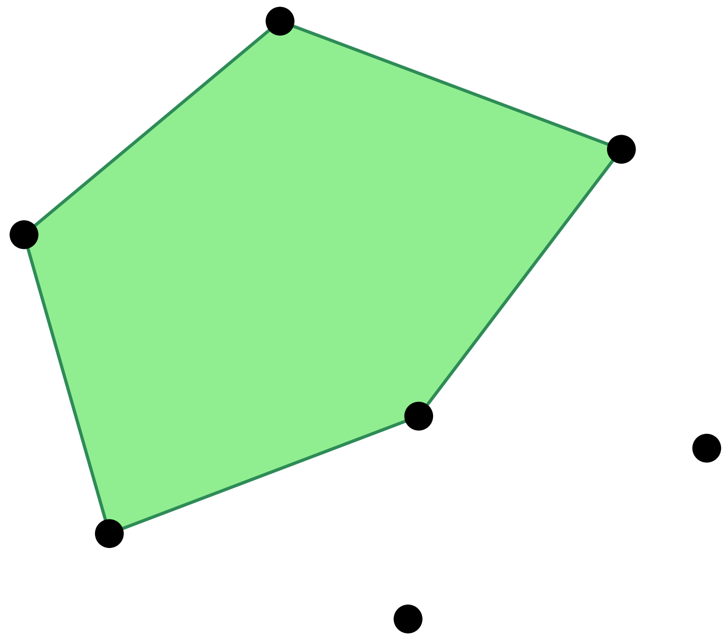
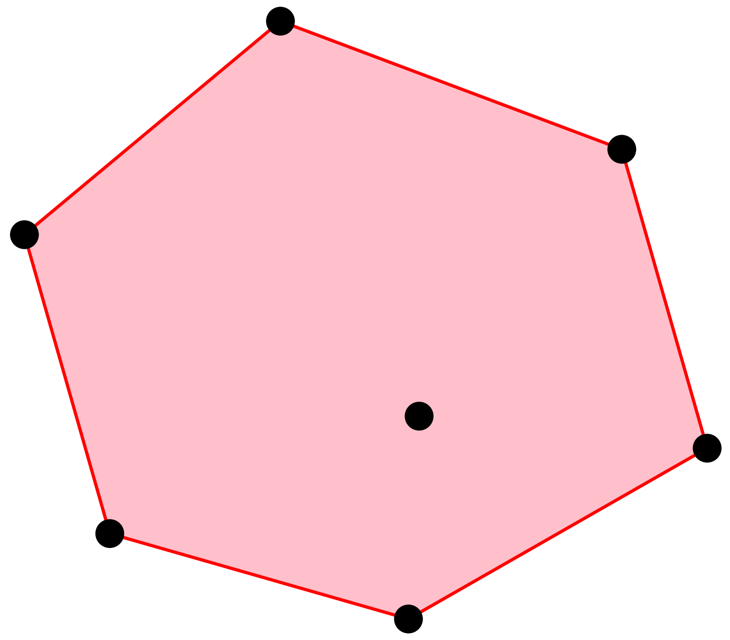
Planar k-Holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P



5-hole



not a 6-hole

Planar k-Holes

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Erdős, 1970': Is there a smallest integer $h(k)$ s.t. every set of $h(k)$ points contains k-hole?

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- 3 points $\Rightarrow \exists$ 3-hole
- 5 points $\Rightarrow \exists$ 4-hole

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- 10 points $\Rightarrow \exists$ 5-hole [Harborth '78]

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- $\exists$ arbitrarily large point sets with no 7-hole [Horton '83]

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- 10 points $\Rightarrow \exists$ 5-hole [Harborth '78]
- $\exists$ arbitrarily large point sets with no 7-hole [Horton '83]
- Sufficiently large point sets $\Rightarrow \exists$ 6-hole [Gerken '08 and Nicolás '07, independently]

Planar k-Holes

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq 463$, $h(7) = \infty$

Harborth '78

Horton'83

Overmars '02

Gerken '07, Nicolas '07, Koshelev '09

Planar k-Holes

exact value still unknown

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Higher Dimensions

a finite point set P in $\mathbb{R}^d$ is in *general position*
if no d points lie in a common hyperplane

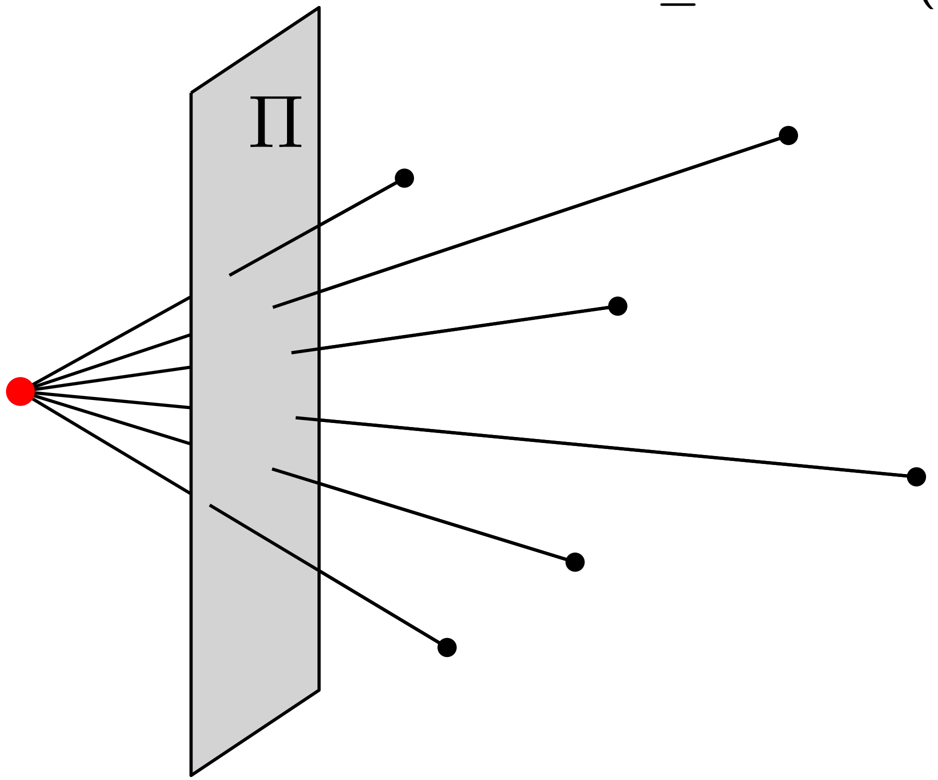
k-*gon* $= k$ points in convex position

k-*hole* $= k$-gon with no other points of P in its convex hull

Higher Dimensional k-Gons

dimension reduction (Károlyi '01):

$$g^{(d)}(k) \leq g^{(d-1)}(k-1)+1 \leq \ldots \leq \underbrace{g^{(2)}(k-d+1)+d-2}_{\leq 2^{k+o(k)} \text{ (Suk'17)}}$$



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asymptotic behavior remains unknown for $d \geq 3$

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central problem: determine the largest value $k = H(d)$ such that every sufficiently large set in d-space contains a k-hole

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- $H(2) = 6$ because $h^{(2)}(6) < \infty$ and $h^{(2)}(7) = \infty$
  [Gerken '07, Nicolás '07] [Horton '87]

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  $H(d) < d^{d+o(d)}$
- in particular, $7 \leq H(3) \leq 22$

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- Bukh, Chao, and Holzman '20: $H(d) < 2^{7d}$

Precise Values for Small Gons and Holes

Bisztriczky, Harborth, Soltan, Morris '90s:

- $g^{(d)}(k) = h^{(d)}(k) = 2k - d - 1$ for $d + 2 \leq k \leq \frac{3d}{2} + 1$
  in particular values for
  $(k, d) = (3, 5), (4, 6), (4, 7), (5, 7), (5, 8)$

- and $g^{(3)}(6) = h^{(3)}(6) = 9$

Szekeres and Peters '06

NEW!

	$k=4$	5	6	7	8	9	10
$d=2$	5	9	17	?			
3	4	6	9	≤ 13			
4	4	5	7	9	≤ 13		
5	4	5	6	8	10	≤ 13	
6	4	5	6	7	9	11	13

Known values and bounds for $g^{(d)}(k)$.

NEW!

	$k=4$	5	6	7	8	9	10
$d=2$	5	10	30..463	∞	∞	∞	∞
3	4	6	9	≤ 14	?	?	?
4	4	5	7	9	≤ 13	?	?
5	4	5	6	8	10	≤ 13	?
6	4	5	6	7	9	11	13

Known values and bounds for $h^{(d)}(k)$.

Our Results

Theorem: $g^{(3)}(7) \leq 13$, that is,
every set of 13 points from $\mathbb{R}^3$ contains a 7-gon.

Theorem: $h^{(3)}(7) \leq 14$, that is,
every set of 14 points from $\mathbb{R}^3$ contains a 7-hole.

Our Results

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Theorem: $g^{(4)}(8) \leq h^{(4)}(8) \leq 13$, that is,
every set of 13 points from $\mathbb{R}^4$ contains a 8-gon/hole.

Theorem: $g^{(5)}(9) \leq h^{(5)}(9) \leq 13$, that is,
every set of 13 points from $\mathbb{R}^5$ contains a 9-gon/hole.

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Theorem: $g^{(5)}(9) \leq h^{(5)}(9) \leq 13$, that is,
every set of 13 points from $\mathbb{R}^5$ contains a 9-gon/hole.

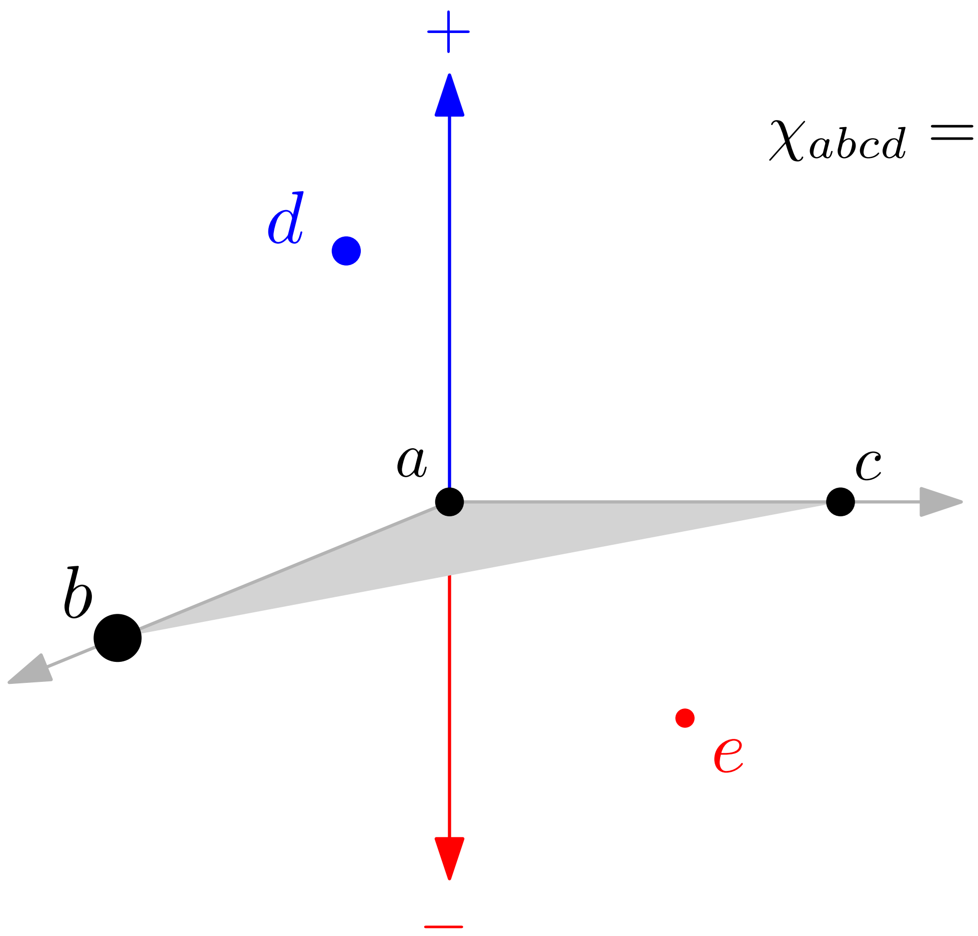
All statements hold for *chirotopes* of rank 4, 5, and 6, respectively, and the bounds are *tight for chirotopes*.

SAT Model for $\mathbb{R}^3$

- use SAT solver to test $g^{(3)}(k) \stackrel{?}{>} n$:
  does there exist $\{p_1, \ldots, p_n\}$ from $\mathbb{R}^3$ without k-gon?

SAT Model for $\mathbb{R}^3$

- variables for *quadruple-orientations*: $\chi_{abcd} \in \{+, -\}$



$$\chi_{abcd} = \operatorname{sgn} \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_a & x_b & x_c & x_d \\ y_a & y_b & y_c & y_d \\ z_a & z_b & z_c & z_d \end{pmatrix}$$

SAT Model for $\mathbb{R}^3$

- variables for *quadruple-orientations*: $\chi_{abcd} \in \{+, -\}$
- *chirotope* axioms

Grassmann-Plücker relations for r-dim. vectors (we have rank $r = 4$):

$$\det(a_1, \ldots, a_r) \cdot \det(b_1, \ldots, b_r) = \sum_{i=1}^{r} \det(b_i, a_2, \ldots, a_r) \cdot \det(b_1, \ldots, b_{i-1}, a_1, b_{i+1}, \ldots, b_r)$$

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If $\chi_{b_i,a_2,\ldots,a_r}\cdot\chi_{b_1,\ldots,b_{i-1},a_1,b_{i+1},\ldots,b_r} \geq 0$ for every i,
then $\chi_{a_1,\ldots,a_r}\cdot\chi_{b_1,\ldots,b_r} \geq 0$

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necessary conditions but not sufficient (realizability!)

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- variables for *quadruple-orientations*: $\chi_{abcd} \in \{+,-\}$
- *chirotope* axioms
  - Alternating axioms: $\Theta(n^r)$ constraints
    $\chi_{i_{\pi(1)},i_{\pi(2)},i_{\pi(3)},i_{\pi(4)}} = \mathrm{sgn}(\pi) \cdot \chi_{i_1,i_2,i_3,i_4}$
  - Exchange axioms: $\Theta(n^{2r})$ constraints
    For any $a_1,\ldots,a_r,b_1,\ldots,b_r$:
    If $\chi_{b_i,a_2,\ldots,a_r} \cdot \chi_{b_1,\ldots,b_{i-1},a_1,b_{i+1},\ldots,b_r} \geq 0$ for every i,
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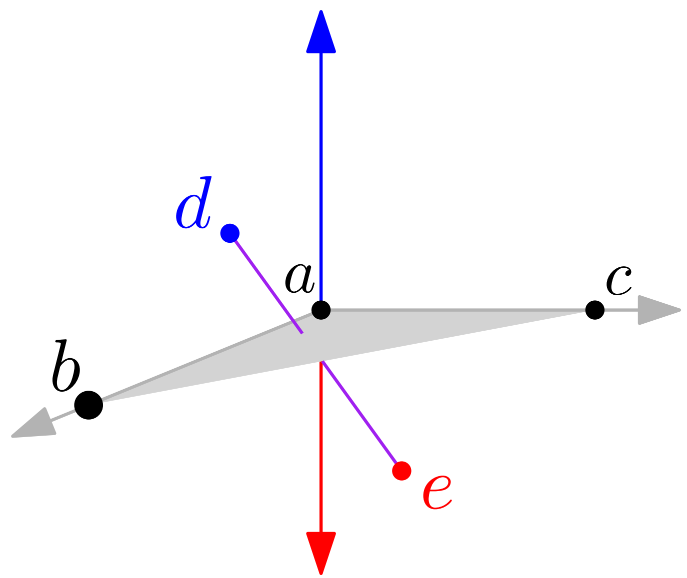
    then $\chi_{a_1,\ldots,a_r} \cdot \chi_{b_1,\ldots,b_r} \geq 0$
  - *3-term Grassmann Plücker relations* $\rightarrow \Theta(n^{r+2})$
    $(a_3 = b_3, \ldots, a_r = b_r)$

SAT Model for $\mathbb{R}^3$

- variables for *quadruple-orientations*: $\chi_{abcd} \in \{+, -\}$
- *chirotope* axioms
- auxiliary *separation* variables

$$S_{abc;de} = \text{plane } \operatorname{aff}\{a, b, c\} \text{ separates } d \text{ and } e$$



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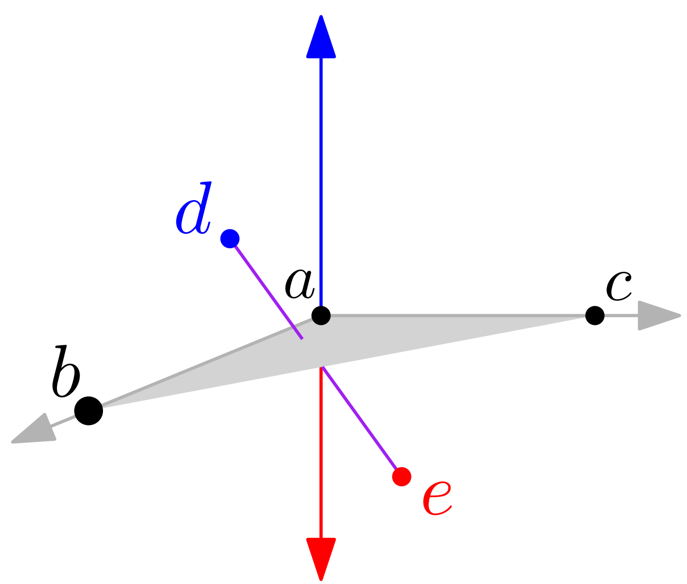
$$S_{abc;de} := \chi_{abcd} \neq \chi_{abce}$$



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$$C_{abcd;e} := \text{"}\mathrm{conv}\{a,b,c,d\} \text{ contains point } e\text{"}$$

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$C_{abcd;e} :=$ "$\mathrm{conv}\{a,b,c,d\}$ contains point e"

$=$ "no hyperplane (abc, abd, acd, or bcd) separates e from the remaining point"

$$C_{abcd;e} \Leftrightarrow \neg S_{abc;de} \wedge \neg S_{abd;ce} \wedge \neg S_{acd;be} \wedge \neg S_{bcd;ae}$$

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- $I \subset S$ is *k-gon* $\Leftrightarrow$ no 4-tuple contains a point of I

  $\Leftrightarrow$ every 5-tuple in convex position

  (Carathéodory's theorem)

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- $I \subset S$ is *k-hole* $\Leftrightarrow$ no 4-tuple contains a point of S

Performance of SAT Solver

- $g^{(3)}(7) \leq 13$: *CaDiCaL* found chirotope $n = 12$ without 7-gons, and disproved existence for $n = 13$ (2 cpu days)
- 39GB unsat-certificate checked via *DRAT-trim* (1 additional cpu day)

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- $h^{(4)}(8) \leq 13$ (7+6 cpu days, 297GB certificate)
- $h^{(5)}(9) \leq 13$ (3+3 cpu days, 117GB certificate)

Performance of SAT Solver

- bounds tight for chirotopes
- problem: realizability as point set?

Further Results and Projects

- $\exists$ rank 4 chirotope on 18 elements with no *8-gon* (100 cpu days)
- $\exists$ rank 4 chirotope on 19 elements with no *8-hole* (24 cpu days)

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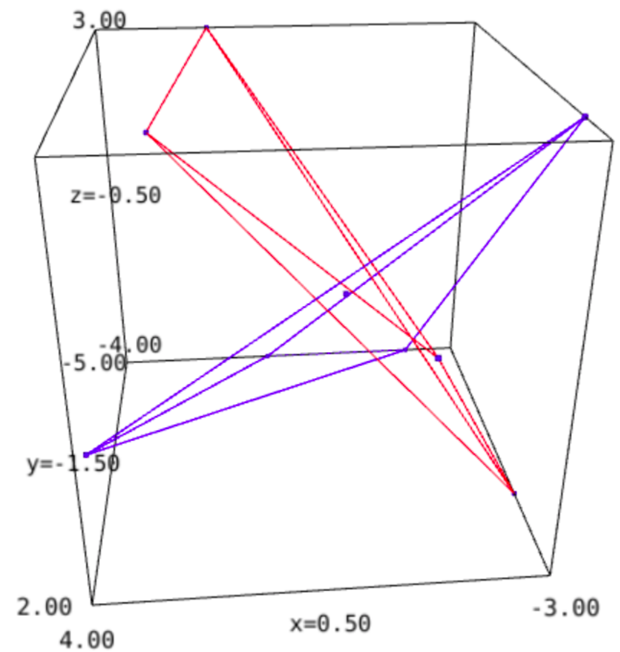
- given two intersecting simplices in $\mathbb{R}^d$,
  $\exists$ reassignment of the vertices such that the two new tetrahedra are linked?

  [Fulek, Gärtner, Kupavskii, Valtr, Wagner '18, "The Crossing Tverberg Theorem"]

  true in the plane, false for $d \geq 3$!



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- non-crossing triangle-representation of 3-uniform hypergraphs:
  $\forall\ S(2,3,n)$ with $n \geq 13$ $\exists$ non-crossing drawing using triangles?
  [Evans, Rzazewski, Saeedi, Shin, Wolff '19]

  $\rightarrow$ chirotope representation for $S(2,3,13)$

	$k=4$	5	6	7	8	9	10
$d=2$	5	9	17	?			
3	4	6	9	≤ 13			
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THANK YOU!

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