Orthogeodesic Point Set Embeddings of Outerplanar Graphs

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Graz, June 18, 2015

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Motivation

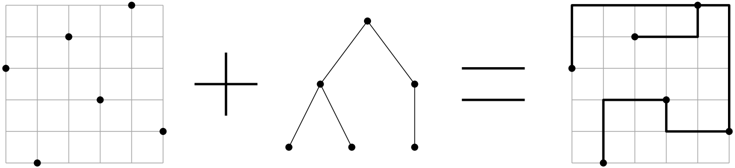

Figure : Point Set Embedding of a Tree

Motivation

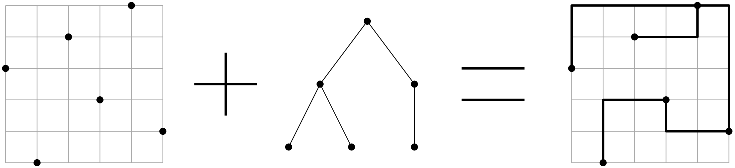

Figure : Point Set Embedding of a Tree

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n	n	n
3-Tree	$n^2 - 2n + 2$	n	n
4-Cat.	$3n - 2$	$n + 1$	$\lfloor 1.5n \rfloor$
4-Tree	$n^2 - 2n + 2$	$4n - 3$	$4n$

Table : Upper bounds given by Giacomo et al.

Graph

- A **graph** is a tuple $G = (V, E)$ with

$$E \subseteq \{\{u, v\} \mid u \neq v \in V\}.$$

- V is said to be the set of **vertices** and E the set of **edges**.
- An example:

$$K_3 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\})$$

Embedding of a Graph

- An **embedding** of a graph G is a tuple (ν, μ) where
  - ν is an injective mapping of the vertices into the plane
  - and μ maps every edge $e = \{u, v\}$ to a polygonal arc with endpoints $\nu(u)$ and $\nu(v)$

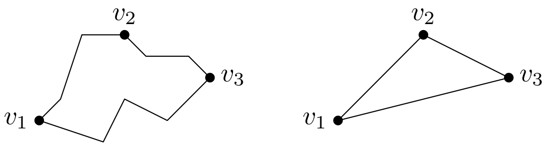

Figure : Two embeddings of K_3.

Embedding of a Graph

- An embedding is said to be **planar** if the interior of every edge neither intersects other edges nor contains vertices.

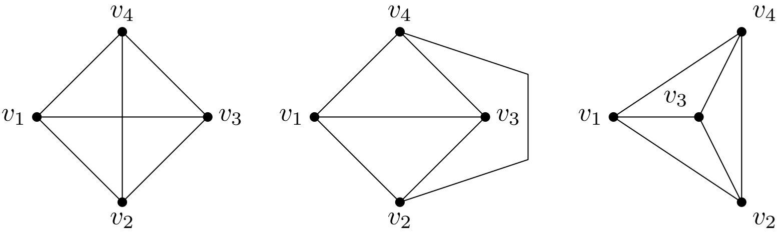

Figure : A nonplanar, a planar, and a straight-line planar embedding of K_4.

Point Set Embedding

- Let $P \subset \mathbb{R}^2$ be a set of points. We call an embedding with $\nu(V) \subseteq P$ an **embedding in** P (PSE).
- Planar PSE in every point set of size n with at most two bends per edge [Kaufmann and Wiese, 1999]
- Deciding whether a planar PSE with at most one bend per edge exists is NP-complete [Kaufmann and Wiese, 1999]

Orthogeodesic PSE

- A PSE is said to be **orthogeodesic** if
  - every edge (polygonal arc) has minimal L^1-length
  - edges are drawn on the grid of horizontal and vertical lines induced by the points in P
  - all edges incident to a vertex enter from distinct directions

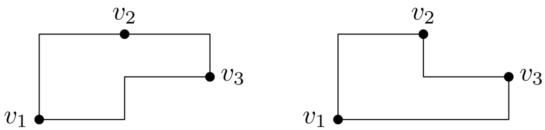

Figure : Orthogeodesic embedding of K_3.

Orthogeodesic PSE

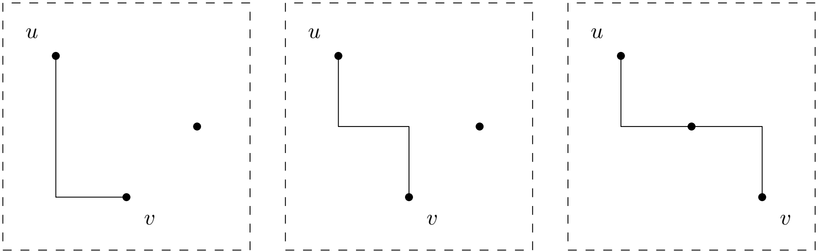

Figure : Edges in planar orthogeodesic embeddings.

Orthogeodesic PSE

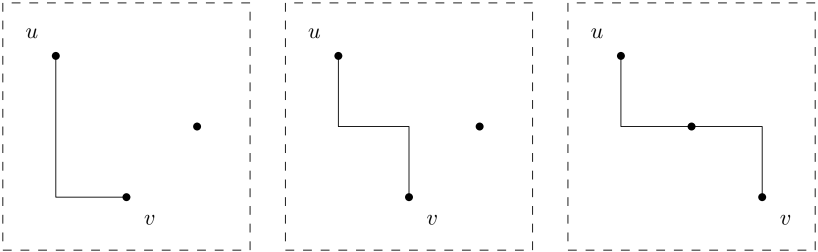

Figure : Edges in planar orthogeodesic embeddings.

- An orthogeodesic PSE is said to be **L-shaped** if every edge has at most one bend

Orthogeodesic PSE

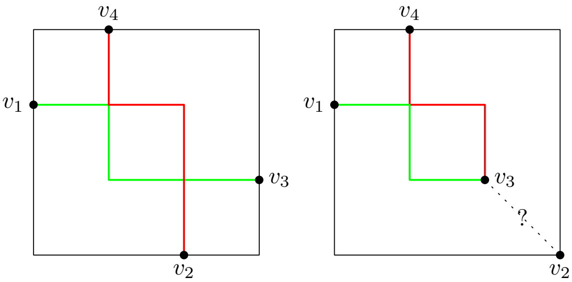

Figure : Orthogeodesic PSE of K_4.

- Deciding whether an orthogeodesic PSE exists is NP-complete. [Katz et al., 2010]

Orthogeodesic PSE

- Restriction to general point sets
- Restriction to certain classes of graphs

General Point Set

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and

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  hold for a permutation σ.
- W.l.o.g.,

$$P = \{(1, \pi_1), (2, \pi_2), \ldots, (n, \pi_n)\}$$

  holds for a permutation π. (Actually, $\pi = \sigma^{-1}$)

General Point Set

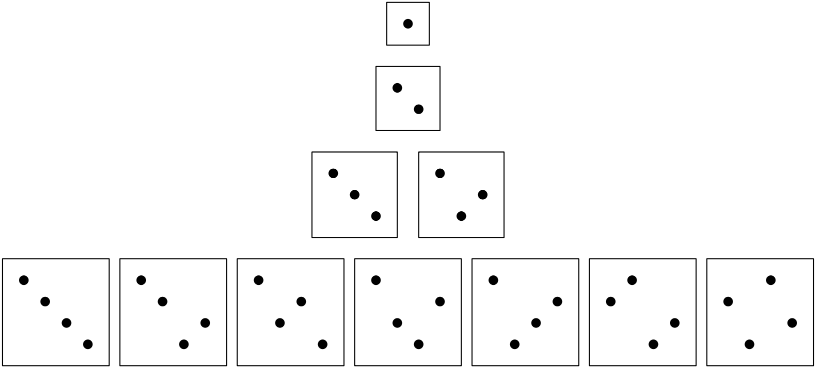

Figure : General point sets up to size 4 (+symmetry).

Diagonal Point Sets

- Every point set of size $n^2 + 1$ admits a diagonal point set of size $n + 1$ [Erdős and Szekeres, 1935]

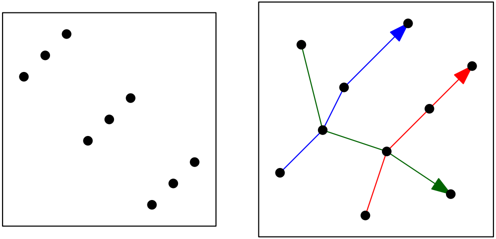

Figure : An illustration.

Diagonal Point Sets

- If G admits an embedding in a diagonal point set of size n, then it admits an embedding in every point set of size $(n-1)^2+1$
- Otherwise, it can not be embedded in certain point sets (e.g., in diagonal point sets)

Classes of Graphs

- Planar graphs
- Outerplanar graphs
- Trees
- Caterpillars

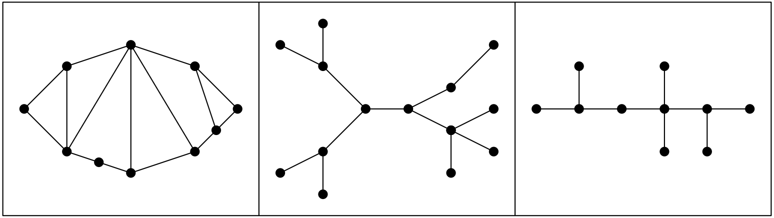

Figure : An outerplanar graph, a tree, and a caterpillar.

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- Outerplanar with $\Delta = 3$:
  - Planar L-shaped: NO

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- Planar with $\Delta \in \{3, 4\}$: NO
- Outerplanar with $\Delta = 4$: NO
- Outerplanar with $\Delta = 3$:
  - Planar L-shaped: NO
  - L-shaped: YES
  - Planar orthogeodesic: YES
- Trees and caterpillars: YES [Giacomo et al., 2013]

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- For caterpillars: $O(n)$ [Giacomo et al.]

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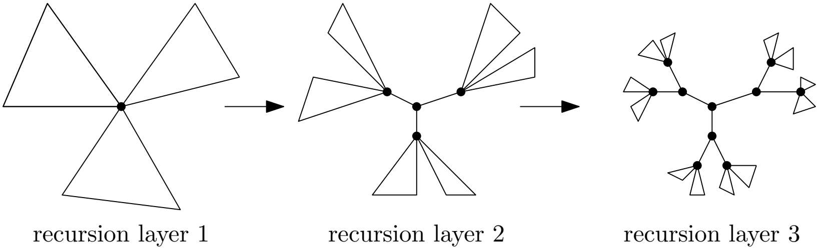
- For caterpillars: $O(n)$ [Giacomo et al.]
- For trees: Recursive Embedding



Embedding 3-Trees

- Start with root (e.g., a leaf) and continue recursively

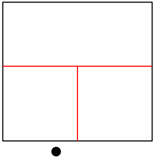


Figure : Recursive embedding of a 3-tree.

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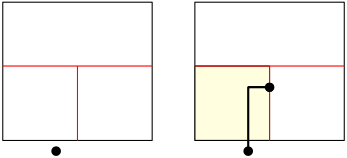

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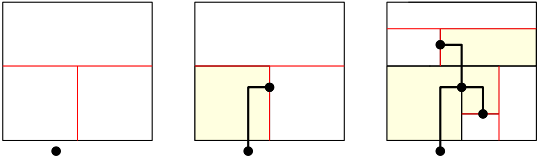

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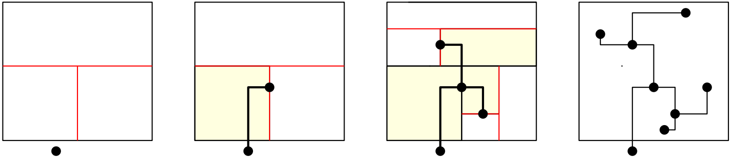

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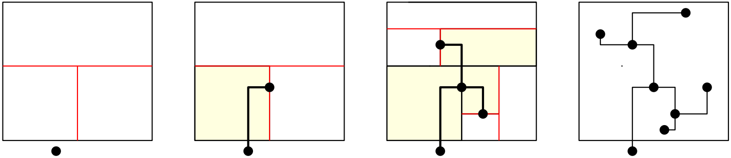

Figure : Recursive embedding of a 3-tree.

- $f(n) \overset{!}{\geq} 1 + f(a) + 2f(b)$ for $a \geq b$ and $a + b = n - 1$

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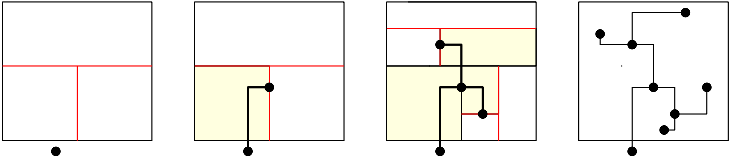

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- $f(n) \overset{!}{\geq} 1 + f(a) + 2f(b)$ for $a \geq b$ and $a + b = n - 1$
- Trivial solution: $f(n) = n^2$

Embedding 3-Trees

- Consider f convex with $f(0) = 0$ and

$$f(n) \geq \max_{0 \leq b \leq \frac{n-1}{2}} \underbrace{1 + f(n-1-b) + 2f(b)}_{=:\phi_n(b)}$$

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- sum of convex functions is convex
- convex function on convex set (Maximum Principle)

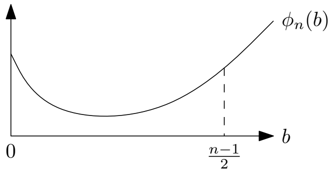

Figure : An illustration of the ϕ_n function.

Embedding 3-Trees

- $f(n) \geq \max\{\phi_n(0), \phi_n(\frac{n-1}{2})\} = \phi_n(\frac{n-1}{2}) = 3f(\frac{n-1}{2}) + 1$, since

$$\phi_n(0) = f(n-1) + \underbrace{1}_{\leq f'(\xi)} \leq f(n)$$

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- A solution: $f(n) = n^{\log_2 3}$, where $\log_2 3 = 1.5849\cdots$
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Theorem

$$f(n) = O(n^{\log_2 3})$$

Embedding 4-Trees

- Analogous

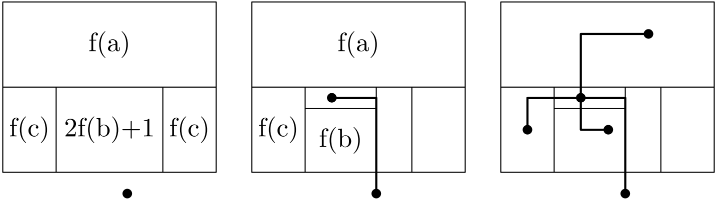


Figure : Recursive embedding of a 4-tree.

- $f(n) \geq 1 + f(a) + 2f(b) + 2f(c)$ with $a \geq b \geq c \ldots$
- $f(n) = O(n^{\log_2 3})$

Embedding 4-Trees

- Consider f convex with $f(0) = 0$ and

$$f(n) \geq \max_{\substack{0 \leq c \leq b \\ b \leq n-1-b-c}} \underbrace{1 + f(n-1-b-c) + 2f(b) + 2f(c)}_{=:\phi_n(b,c)}$$

- Maximum Principle: analyze the corners of the convex set

$$C = \{(b, c) \mid 0 \overset{(1)}{\leq} c \overset{(2)}{\leq} b \overset{(3)}{\leq} n-1-b-c\}$$

Embedding 4-Trees

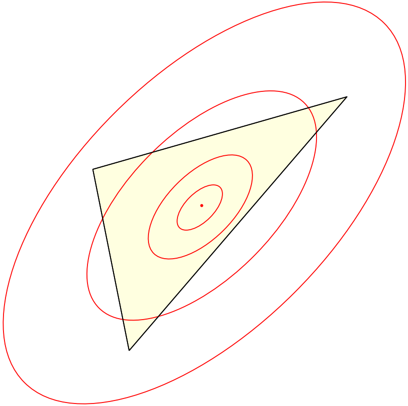


Figure : Maximum Principle illustration.

Embedding 4-Trees

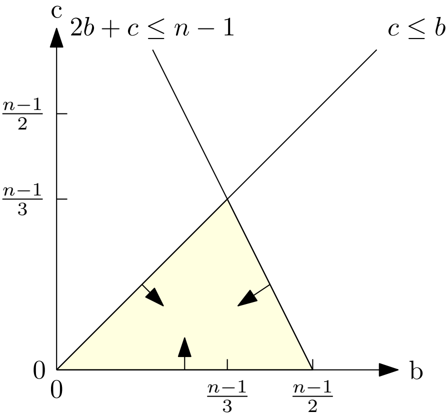


Figure : Corners of the convex set C.

Results for the General Case

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n	n	n
3-Tree	$n^2 - 2n + 2$	n	n
4-Cat.	$3n - 2$	$n + 1$	$\lfloor 1.5n \rfloor$
4-Tree	$n^2 - 2n + 2$	$4n - 3$	$4n$

Table : Upper bounds given by Giacomo et al.

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n [Giacomo et al.]	n [Giacomo et al.]	n [Giacomo et al.]
3-Tree	$0.334n^{1.585} + \mathcal{O}(n)$	n [Giacomo et al.]	n [Giacomo et al.]
4-Cat.	$1.334n + \mathcal{O}(1)$	n	$1.334n + \mathcal{O}(1)$
4-Tree	$0.339n^{1.585} + \mathcal{O}(n)$	$2.334n + \mathcal{O}(1)$	$1.5n + \mathcal{O}(1)$

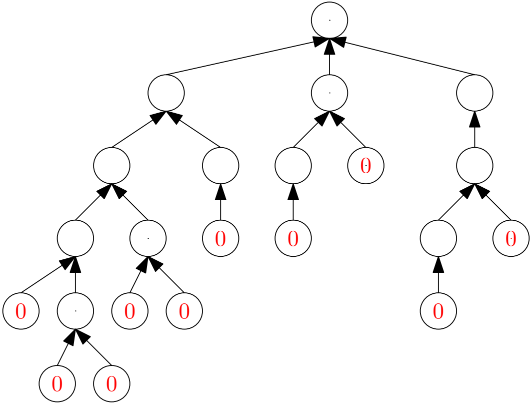
Table : Best upper bounds currently known

What else can we do?

- Analyze Trees
- Analyze Point Sets
- Probabilistic Analysis

Saturation Property

$$\sigma_{T,r}(v) := \max\{0, \sigma_{T,r}(u_1), \sigma_{T,r}(u_2) + 1, \ldots, \sigma_{T,r}(u_k) + 1\}$$



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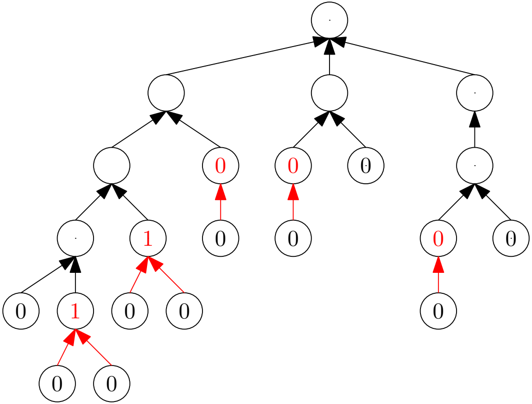

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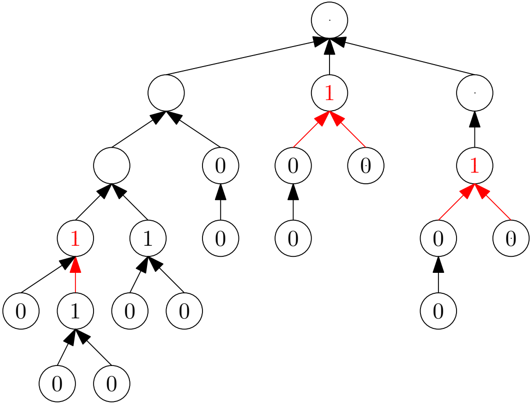

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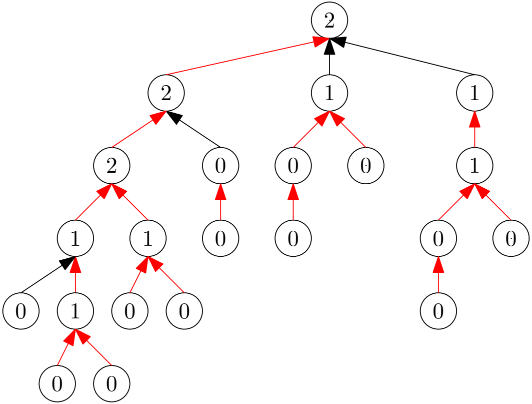

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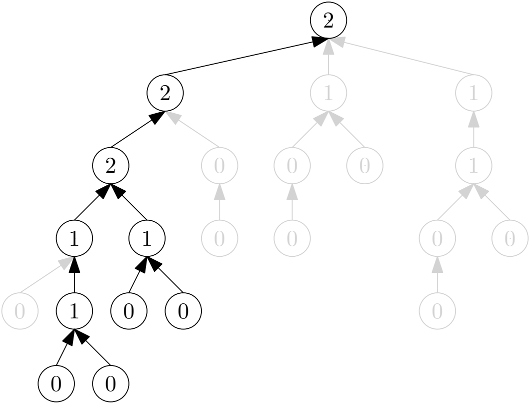


Saturation Property

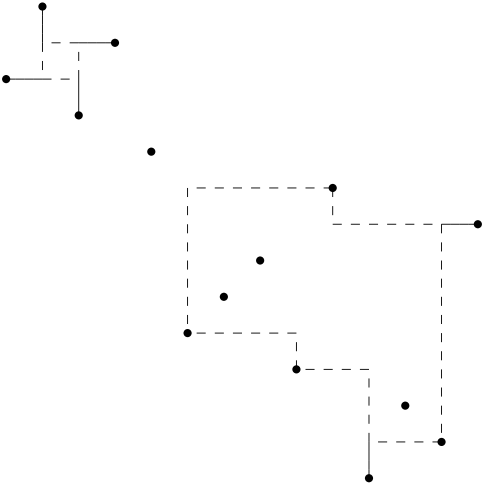
- $\sigma(T) := \min_{r \in V} \sigma_{T,r}(r)$
- $f(T) = O(n \cdot 2^{\sigma(T)})$
- For caterpillars:

$$\sigma(T) = O(1) \Rightarrow f(T) = O(n)$$

- For trees:

$$\sigma(T) \leq \log_2(n+1) \Rightarrow f(T) = O(n^2)$$

Orthogonal Convex Hull



Orthogonal Convex Hull

- l ... number of layers in onion peeling
- k_i ... number of points in layer i
- P contains diag. PS of size

$$n := \max \left\{ 2l - 1, \left\lceil \frac{k_1}{4} \right\rceil, \dots, \left\lceil \frac{k_l}{4} \right\rceil \right\}$$

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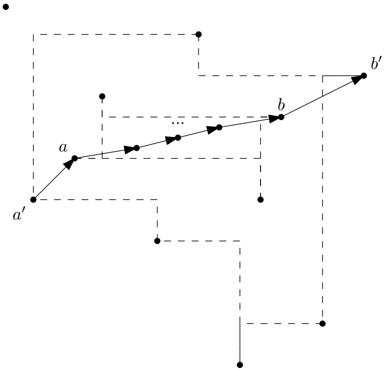
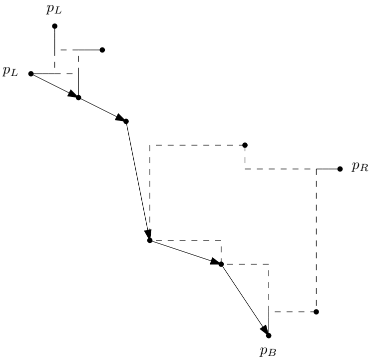
- Yet another proof of the $m = O(n^2)$ bound, because $n = \Omega(\sqrt{m})$

Orthogonal Convex Hull

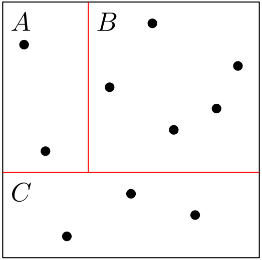


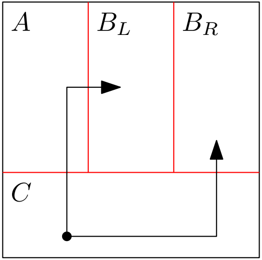
(a) l layers in onion peeling



(b) Layer i of size k_i

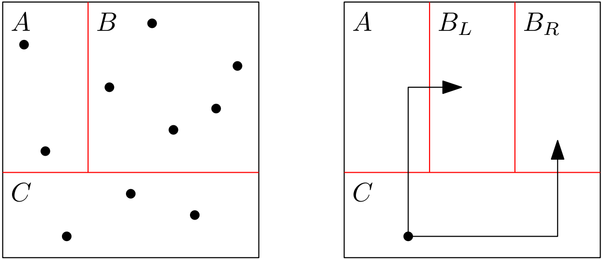
Probabilistic Results





- Point exists with probability at least $1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}$

Probabilistic Results



- Point exists with probability at least $1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}$
- For 3-Trees: $O(n \log n (\log \log n)^2)$
- For 4-Trees: $O(n^{\gamma_0 + \varepsilon})$ where $\gamma_0 = 1.3319\cdots$

Summary

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
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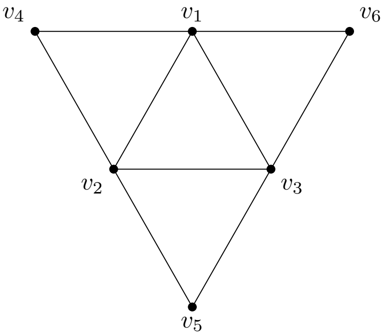
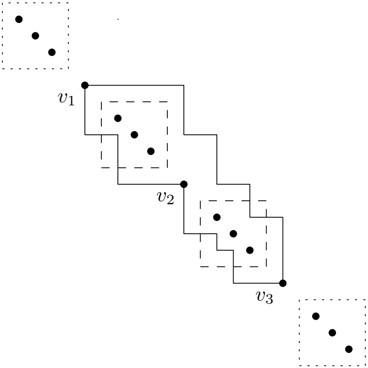
- $f_{LT4}(T) = O(n \cdot 2^{\sigma(T)})$
- $n \geq \max\left\{2l - 1, \left\lceil \frac{k_1}{4} \right\rceil, \ldots, \left\lceil \frac{k_l}{4} \right\rceil\right\}$
- $f_{LT3}^{1/2}(n) = O(n \log n (\log \log n)^2)$
- $f_{LT4}^{1/2}(n) = O(n^{\gamma_0 + \varepsilon})$ where $\gamma_0 = 1.3319\cdots$

Thank you for your attention!

Classes of Graphs

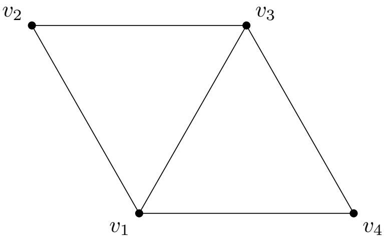


(a) An outerplanar graph with $\Delta = 4$ that does not admit an embedding

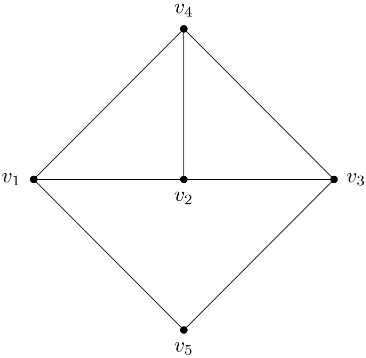


(b) Sketch of the proof.

Classes of Graphs



(a) An outerplanar graph with $\Delta = 3$ that does not admit a planar L-shaped embedding



(b) A planar graph with $\Delta = 3$ that does not admit an embedding

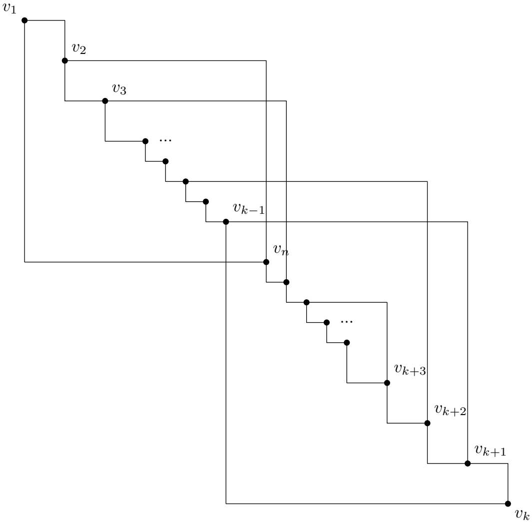


Figure : Outerplanar Graphs $\Delta = 3$ (L-Shaped)

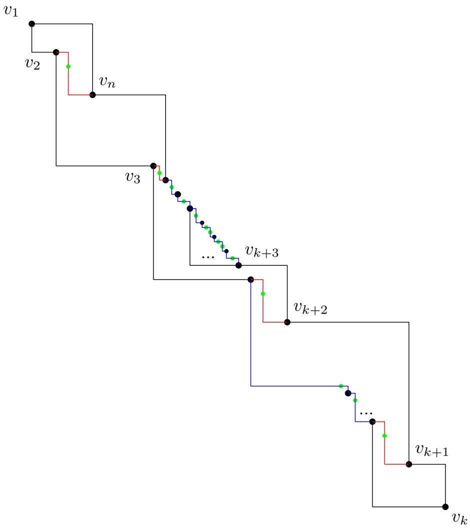
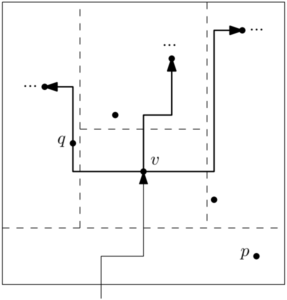


Figure : Outerplanar Graphs $\Delta = 3$ (Planar Orthog.)

Recursive Embeddings



- $f_{OT4}(n) \leq 2n$
- Actually, $f_{OT4}(n) \leq \frac{3}{2}n - 1$, because

$$|\{v \in V\colon deg(v) \geq 3\}| \leq \frac{n-2}{2}$$

Recursive Embeddings

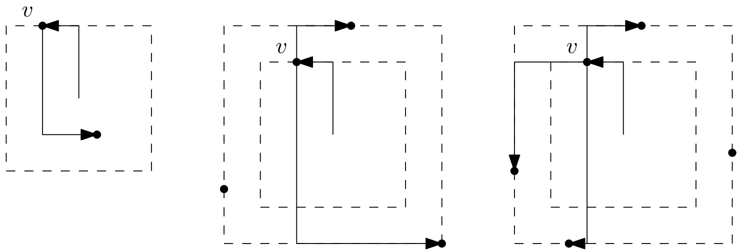
- With $U := \{v \in V \colon deg(v) \geq k\}$

$$2n - 2 = 2|E| = \sum_{v \in V} \deg(v) \geq n + (k-1)|U|,$$

- or equally,

$$|U| \leq \frac{n-2}{k-1}$$

Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$



- Ring-partition
- Case 1: recycle points, $O(1)$ wasted points
- Case 2: At most 2 wasted point per vertex
- Case 3: At most 4 wasted points per vertex

Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

$$\text{maximize } 2x_3 + 4x_4$$

$$\text{subject to } \sum_{i=1}^{4} x_i = n$$

$$\sum_{i=1}^{4} (i-2)x_i = -2 \quad \text{// holds for every tree}$$

$$x_i \in \mathbb{N}_0 \qquad , 1 \leq i \leq 4$$

- $x_2^* = 0$ must hold . . .

Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

$$
\begin{aligned}
\text{maximize } & 2x_3 + 4x_4 \\
\text{subject to } & x_1 + x_3 + x_4 = n \\
& x_1 = 2 + x_3 + 2x_4 \\
& x_1, x_3, x_4 \geq 0
\end{aligned}
$$

Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

$$
\begin{aligned}
\text{maximize } & 2x_3 + 4x_4 \\
\text{subject to } & 2x_3 + 3x_4 = n - 2 \\
& 2 + x_3 + 2x_4 \geq 0 \\
& x_3, x_4 \geq 0
\end{aligned}
$$

Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

$$\begin{aligned} \text{maximize } & n - 2 + x_4 \\ \text{subject to } & 3x_4 \leq n - 2 \\ & x_4 \geq 0 \end{aligned}$$

- number of wasted points at most $\frac{4}{3}n + O(1)$
- $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

Results for the General Case

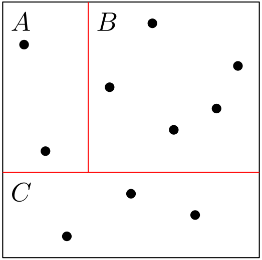
	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n	n	n
3-Tree	$n^2 - 2n + 2$	n	n
4-Cat.	$3n - 2$	$n + 1$	$\lfloor 1.5n \rfloor$
4-Tree	$n^2 - 2n + 2$	$4n - 3$	$4n$

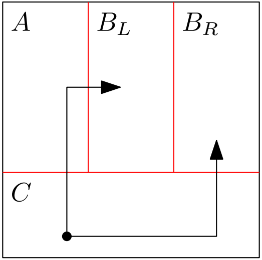
Table : Upper bounds given by Giacomo et al.

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n [Giacomo et al.]	n [Giacomo et al.]	n [Giacomo et al.]
3-Tree	$0.334n^{1.585} + \mathcal{O}(n)$	n [Giacomo et al.]	n [Giacomo et al.]
4-Cat.	$1.334n + \mathcal{O}(1)$	n	$1.334n + \mathcal{O}(1)$
4-Tree	$0.339n^{1.585} + \mathcal{O}(n)$	$2.334n + \mathcal{O}(1)$	$1.5n + \mathcal{O}(1)$

Table : Best upper bounds currently known

Probabilistic Results





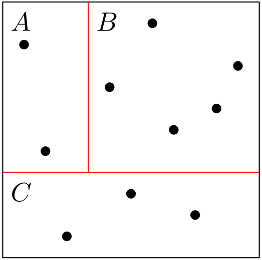
- Point exists with probability at least

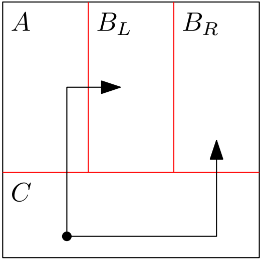
$$1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}$$

Probabilistic Results

- $\mathbb{P}(\cup_{i=1}^{h} E_i) = 1 - \mathbb{P}(\cap_{i=1}^{h} \overline{E_i}) = 1 - \prod_{i=1}^{h} \mathbb{P}(\overline{E_i} | \cap_{j=1}^{i-1} \overline{E_j})$
- $$\mathbb{P}(\overline{E_1}) = 1 - \frac{|A| + 1}{|A| + |B| + 1}$$
- $$\mathbb{P}(\overline{E_i} | \cap_{j=1}^{i-1} \overline{E_j}) = \frac{|B| + (i-1)}{|A| + |B| + (i-1) + 1} = 1 - \frac{|A| + 1}{|A| + |B| + i}$$
- $$1 - \prod_{i=1}^{h} \left(1 - \frac{|A| + 1}{|A| + |B| + i}\right) \geq 1 - \left(1 - \frac{|A|}{|A| + |B| + |C|}\right)^{|C|}$$

Probabilistic Results





- Point exists with probability at least

$$1-\left(1-\frac{|A|}{|P|}\right)^{|C|}$$

Probabilistic Results

- $|P| = \alpha n \log_2 n$
- $|A| := |C| := \alpha \frac{n}{2}$ and $|B| := 2\alpha \frac{n}{2} \log_2 \frac{n}{2}$
- Point exists with probability at least

$$1 - \left(1 - \frac{1}{2 \log_2 n}\right)^{\alpha \frac{n}{2}} \geq 1 - \left(\frac{1}{e}\right)^{\alpha \frac{n}{4 \log_2 n}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha \ln 2}{2}}$$

since $(1 - \frac{1}{x})^x \leq \frac{1}{e}$ on $[1, \infty)$ and $\frac{x}{\log x} \geq e$ on $(1, \infty)$

- $\ldots$
- $O(n \log^2 n)$, success with probability at least $\frac{1}{2}$
- Actually, $O(n \log n (\log \log n)^2)$

Probabilistic Results

$$(x+1)^{\varepsilon} - x^{\varepsilon} = \varepsilon\xi^{\varepsilon-1} \geq \varepsilon\xi^{-1} \geq \varepsilon(x+1)^{-1} \geq \varepsilon(2x)^{-1}$$

$$\begin{aligned}
\lceil \alpha(2m)\log_2{}^{\varepsilon}(2m) + 8m - 4 \rceil &\geq \alpha 2m\log_2{}^{\varepsilon}(2m) + 8m - 4 \\
&= \alpha 2m(\log_2 m + 1)^{\varepsilon} + 8m - 4 \\
&\geq \alpha 2m\left(\log_2{}^{\varepsilon} m + \frac{\frac{\varepsilon}{2}}{\log_2 m}\right) + 8m - 4 \\
&= \alpha 2m\log_2{}^{\varepsilon} m + \alpha 2\frac{\frac{\varepsilon}{2}m}{\log_2 m} + 8m - 4 \\
&= 2\left(\alpha m\log_2{}^{\varepsilon} m + 4m - 4\right) + 2\left(\frac{\alpha\frac{\varepsilon}{2}m}{\log_2 m}\right) + 4 \\
&\geq 2\left\lceil \alpha m\log_2{}^{\varepsilon} m + 4m - 4\right\rceil + 2\left\lfloor \frac{\alpha\frac{\varepsilon}{2}m}{\log_2 m}\right\rfloor
\end{aligned}$$

Probabilistic Results

- $\alpha := \frac{C \log_2 n}{\varepsilon^2}$ leads to

$$\frac{C}{\varepsilon^2} n \log_2^{1+\varepsilon} n + O(n)$$

- $\varepsilon := \frac{1}{\log_2 \log_2 n}$ leads to

$$C n \log_2 n (\log_2 \log_2 n)^2 + O(n)$$

- where $C := \frac{80}{e^3 (\ln 2)^2} = 8.290 \cdots$

Probabilistic Results

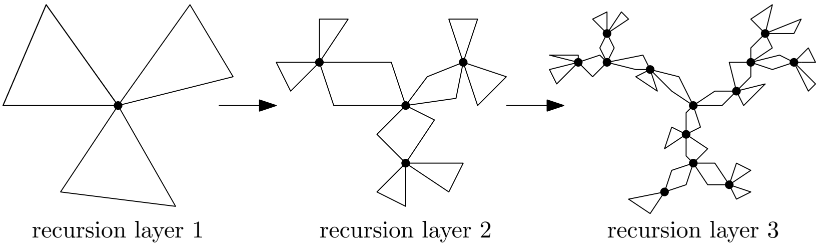


Figure : Illustration of the algorithm

- Many cases . . .

Probabilistic Results

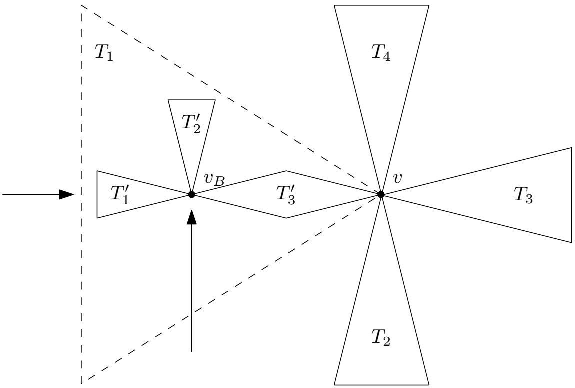


Figure : Tree in Case 3a

Probabilistic Results

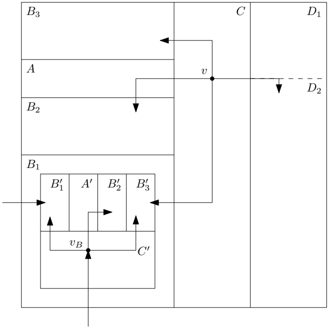


Figure : Embedding in Case 3a

$$f(n'_1) + f(n'_2) + f(n'_3) + f(n_2) + f(n_3) + 2f(n_4) + 4\alpha n.$$

Probabilistic Results

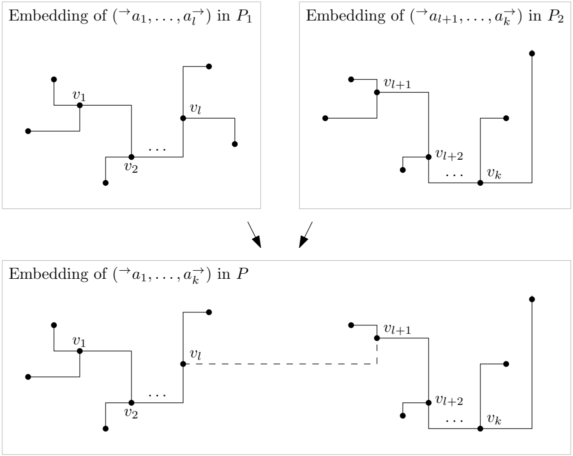
- For 4-Trees: $O(n^{\gamma_0+\varepsilon})$
- $\gamma_0 := 1.3319\cdots$ unique solution of the equation

$$\left(\frac{1}{2}\right)^\gamma + \left(\frac{1}{24}\right)^\gamma + 2\left(\frac{1}{3}\right)^\gamma + 2\left(\frac{1}{8}\right)^\gamma = 1$$

- Let $\gamma > \gamma_0$ and let $\delta_\gamma = \frac{1}{24^{\gamma_0}} - \frac{1}{24^\gamma}$. Then $f_\gamma(x) = x^\gamma$ fulfills

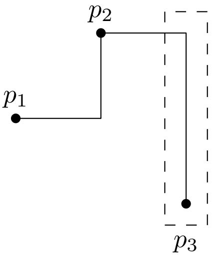
  1. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{1}{2}x) + \delta_\gamma x$,
  2. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{1}{6}x) + 2f_\gamma(\frac{1}{3}x) + \delta_\gamma x$,
  3. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{3}{8}x) + 2f_\gamma(\frac{1}{8}x) + \delta_\gamma x$, and
  4. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{1}{24}x) + 2f_\gamma(\frac{1}{3}x) + 2f_\gamma(\frac{1}{8}x) + \delta_\gamma x$

Embedding Caterpillars - Basic Idea



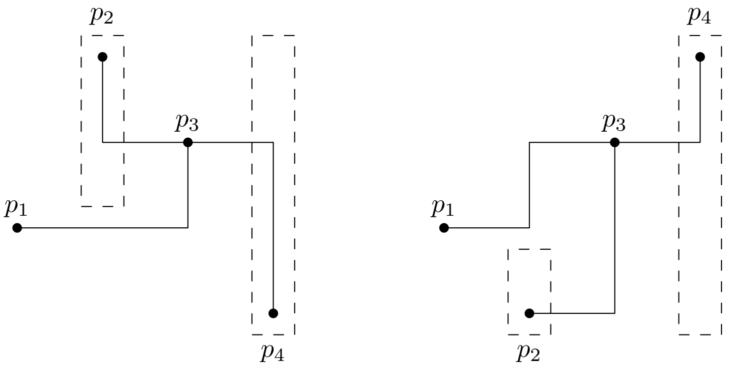
Proof of $f_{OC4}(n) \leq \frac{4}{3}n + O(1)$

- $(^{\rightarrow}2^{\rightarrow})$ admits a planar orthogeodesic embedding in any point set P of size 3



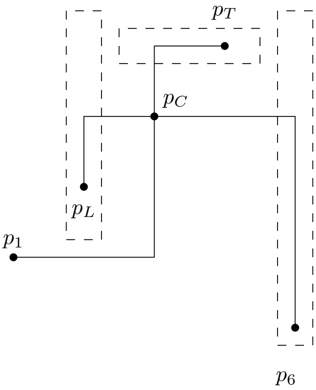
Proof of $f_{OC4}(n) \leq \frac{4}{3}n + O(1)$

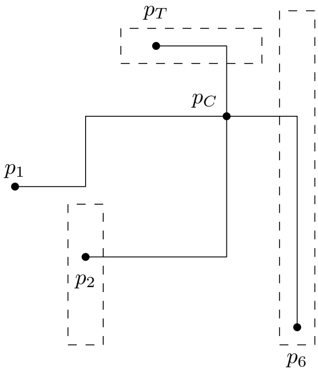
- $(^{\rightarrow}3^{\rightarrow})$ admits a planar orthogeodesic embedding in any point set P of size 4



Proof of $f_{OC4}(n) \leq \frac{4}{3}n + O(1)$

- $(^{\rightarrow}4^{\rightarrow})$ admits a planar orthogeodesic embedding in any point set P of size 6





Results for the General Case

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3-Cat.	n	n	n
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Table : Upper bounds given by Giacomo et al.

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n [Giacomo et al.]	n [Giacomo et al.]	n [Giacomo et al.]
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4-Cat.	$1.334n + \mathcal{O}(1)$	n	$1.334n + \mathcal{O}(1)$
4-Tree	$0.339n^{1.585} + \mathcal{O}(n)$	$2.334n + \mathcal{O}(1)$	$1.5n + \mathcal{O}(1)$

Table : Best upper bounds currently known

Burnside's Lemma

Notation:

- orbit $\overline{x} := \{gx \mid g \in G\}$
- fixed points $X_g := \{x \mid gx = x\}$
- stabilisators $G_x := \{g \mid gx = x\}$

Burnside's Lemma

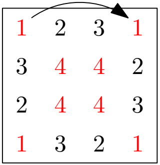
- $$\sum_{g\in G} |X_g| = \sum_{\substack{g\in G, x\in X\\ gx=x}} 1 = \sum_{x\in X} |G_x|$$
- $\overline{x} \simeq G/G_x$, because $\phi_g \colon x \mapsto gx$ has image $\phi_g(\overline{x}) = \overline{x}$, $\phi_g = \phi_{gh}$ only for $h \in G_x \ldots$
- $|\overline{x}| = \frac{|G|}{|G_x|}$ (Lagrange's Theorem)
- $$\#orbits = \sum_{x\in X} \frac{1}{|\overline{x}|} = \sum_{x\in X} \frac{|G_x|}{|G|} = \frac{1}{|G|} \sum_{g\in G} |X_g|$$

Burnside's Lemma - An Example ($n = 4$)

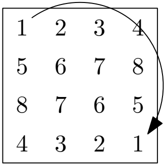
- Group actions: Rotate + Mirror

$$G = \{r_0, r_{90}, r_{180}, r_{270}, m_0, m_{90}, m_{180}, m_{270}\}$$

- $r_0 = id$: all $4! = 24$ elements are left unchanged
- r_{90} and r_{270}: 2 each
- r_{180}: 8



fixed points:
2-2-2-2,
3-3-3-3



fixed points:
1-6-6-1, 1-7-7-1,
2-5-5-2, 2-8-8-2,
3-5-5-3, 3-8-8-3,
4-6-6-4, 4-7-7-4

Figure : Fixed points for r_{90} (left) and r_{180} (right).

Burnside's Lemma - An Example ($n = 4$)

- m_0: 10
- m_{90} and m_{270}: 0 (not possible)
- m_{180}: 10 (analogous to m_0)

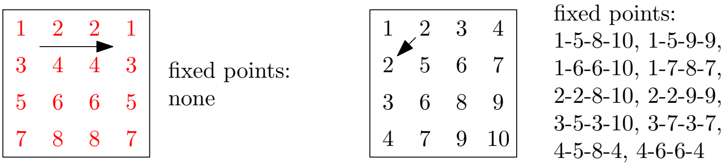


Figure : Fixed points for m_0 (left) and m_{90} (right).

Burnside's Lemma - An Example ($n = 4$)

- $\#orbits(4) = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{24+2+8+2+10+0+10+0}{8} = 7$
- $\#orbits(n) \geq \frac{n!}{8}$
- 1, 1, 2, 7, 23, 115, 694, 5 282, 46 066, 456 454, . . .
  `https://oeis.org/A000903`

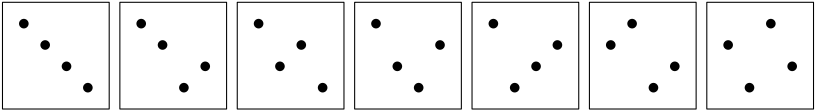


Figure : General point sets of size 4 (+symmetry).

Thank you for your attention!