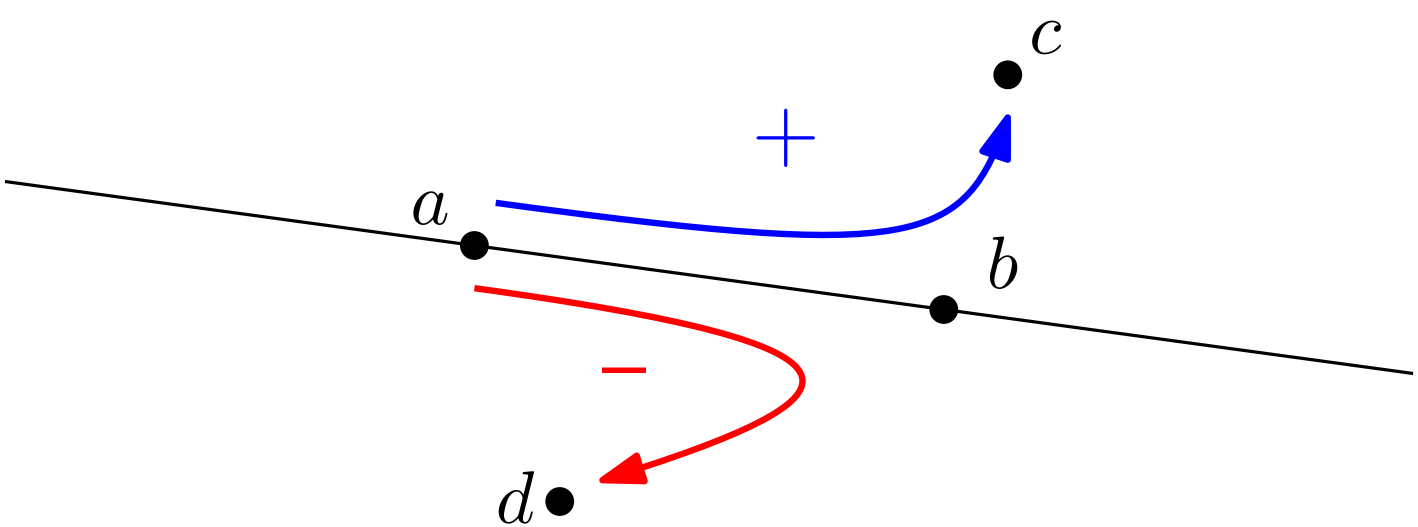
Many Order Types on Integer Grids of Polynomial Size

Manfred Scheucher

Order Types

- point set $\{p_1, \ldots, p_n\}$ induces triple-orientations $\chi : [n]^3 \to \{+, 0, -\}$



$$\chi(a, b, c) = \operatorname{sgn} \det \begin{pmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{pmatrix}$$

Order Types

- point set $\{p_1, \ldots, p_n\}$ induces triple-orientations $\chi : [n]^3 \to \{+, 0, -\}$
- equiv. classes are called *order types*

Order Types

- point set $\{p_1, \ldots, p_n\}$ induces triple-orientations $\chi : [n]^3 \to \{+, 0, -\}$
- equiv. classes are called *order types*
- *non-degenerated* if no three points on a line, i.e., $\chi(a, b, c) \neq 0$ for a, b, c distinct

Order Types

- point set $\{p_1, \ldots, p_n\}$ induces triple-orientations $\chi : [n]^3 \to \{+, 0, -\}$
- equiv. classes are called *order types*
- *non-degenerated* if no three points on a line, i.e., $\chi(a, b, c) \neq 0$ for a, b, c distinct
- mapping $\chi : [n]^3 \to \{+, 0, -\}$ fulfills *chirotope axioms*:
  - *alternating*: $\chi(i_{\pi(1)}, i_{\pi(2)}, i_{\pi(3)}) = \chi(i_1, i_2, i_3) \cdot \mathrm{sgn}(\pi)$
  - *exchange axioms*:
    if $\chi(y_i, x_2, \ldots, x_r) \cdot \chi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r) \geq 0$
    holds for all i, then $\chi(x_1, \ldots, x_r) \cdot \chi(y_1, \ldots, y_r) \geq 0$

Order Types

- point set $\{p_1, \ldots, p_n\}$ induces triple-orientations $\chi : [n]^3 \to \{+, 0, -\}$
- equiv. classes are called *order types*
- *non-degenerated* if no three points on a line, i.e., $\chi(a, b, c) \neq 0$ for a, b, c distinct
- mapping $\chi : [n]^3 \to \{+, 0, -\}$ fulfills *chirotope axioms*:
  - *alternating*: $\chi(i_{\pi(1)}, i_{\pi(2)}, i_{\pi(3)}) = \chi(i_1, i_2, i_3) \cdot \mathrm{sgn}(\pi)$
  - *exchange axioms*:
    if $\chi(y_i, x_2, \ldots, x_r) \cdot \chi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r) \geq 0$
    holds for all i, then $\chi(x_1, \ldots, x_r) \cdot \chi(y_1, \ldots, y_r) \geq 0$
- equiv. classes are called *abstract order types*

Number of Order Types

realizable order types

line arrangement
great-circle arr.

abstract order types

*pseudo*line arr.
great-*pseudo*circle arr.

Number of Order Types

realizable order types $\exp(4n \log n + O(n)) = n^{4n+o(n)}$

↑

abstract order types $\exp(\Theta(n^2))$

Number of Order Types

realizable order types $\exp(4n \log n + O(n)) = n^{4n+o(n)}$

bounds tight, for general setting
and for non-degenerate setting!

abstract order types $\exp(\Theta(n^2))$

Number of Order Types

realizable order types $\exp(4n \log n + O(n)) = n^{4n+o(n)}$

↑

bounds tight, for general setting
and for non-degenerate setting!

abstract order types $\exp(\Theta(n^2))$

- realizability ETR-complete

Number of Order Types

realizable order types $\exp(4n \log n + O(n)) = n^{4n+o(n)}$

bounds tight, for general setting
and for non-degenerate setting!

abstract order types $\exp(\Theta(n^2))$

- realizability ETR-complete
- Generalization #1: In higher dimensions
  $\exp(\Theta(n \log n))$ realizable, $\exp(\Theta(n^d))$ abstract OT

Number of Order Types

realizable order types $\exp(4n \log n + O(n)) = n^{4n+o(n)}$

↑

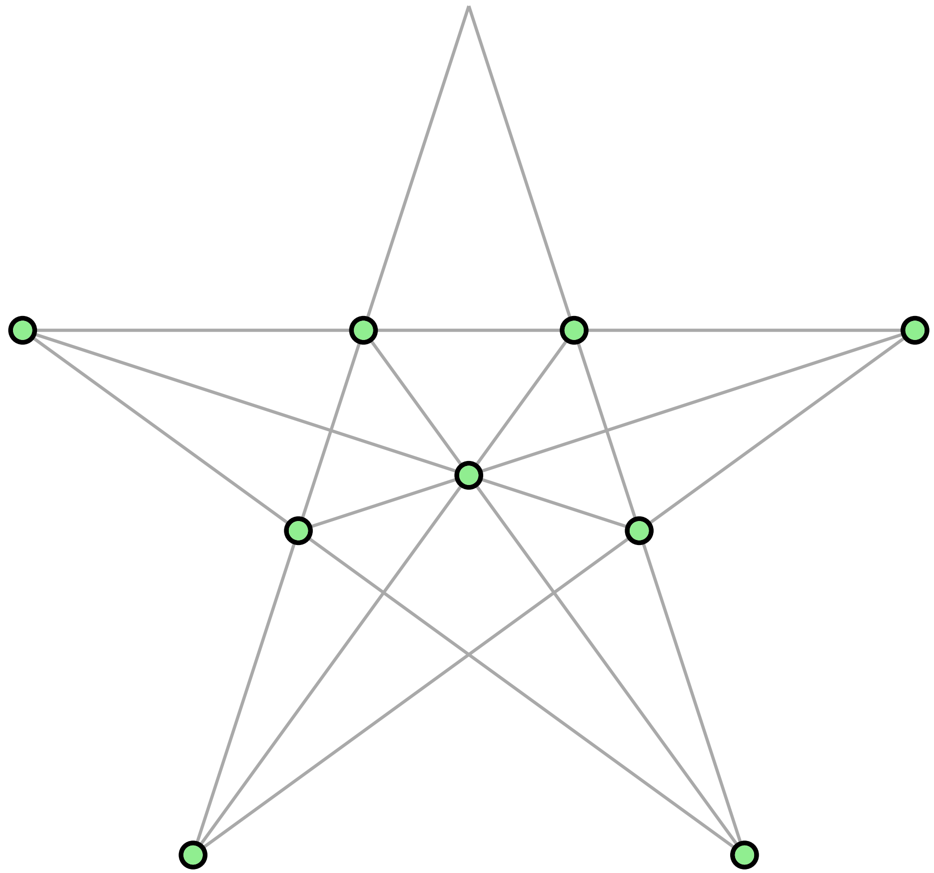
bounds tight, for general setting
and for non-degenerate setting!

abstract order types $\exp(\Theta(n^2))$

- realizability ETR-complete
- Generalization #1: In higher dimensions $\exp(\Theta(n \log n))$ realizable, $\exp(\Theta(n^d))$ abstract OT
- Generalization #2: $\exp(\Theta(n \log n))$ circle arr., $\exp(\Theta(n^2))$ pseudocircle arr.

Coordinate Sizes

- some deg. OT are only realizable with irrational coordinates [Grünbaum and Perles '03]



Coordinate Sizes

- some deg. OT are only realizable with irrational coordinates [Grünbaum and Perles '03]

[Goodman, Pollack, Sturmfels '89]

- every non-deg. OT on n points is realizable with integer coords of size $\leq \exp(\exp(c_1 \cdot n))$
- some non-deg. OTs on n points require integer coords of size $\geq \exp(\exp(c_2 \cdot n))$

Coordinate Sizes

- some deg. OT are only realizable with irrational coordinates [Grünbaum and Perles '03]

[Goodman, Pollack, Sturmfels '89]

- every non-deg. OT on n points is realizable with integer coords of size $\leq \exp(\exp(c_1 \cdot n))$
- some non-deg. OTs on n points require integer coords of size $\geq \exp(\exp(c_2 \cdot n))$

[Caraballo, Díaz-Báñez, Fabila-Monroy, Hidalgo-Toscano, Leaños, Montejano '18]

- $\geq n^{2n+o(n)}$ non-deg. OTs on $n^2 \times n^2$ grid
- $\geq n^{3n+o(n)}$ non-deg. OTs on $n^{2.5} \times n^{2.5}$ grid

Coordinate Sizes

Theorem (S.'21):
$n^{4n+o(n)}$ non-deg. OTs on $\Theta(n^4) \times \Theta(n^4)$ grid

- exponent is essentially best possible up to a lower-order error term
- a significant proportion of all n-point order types can be stored as point sets with $\Theta(\log n)$ bits per point
- what about $\Theta(n^c) \times \Theta(n^c)$ grid for $c < 4$?

Sketch of the $n^{2n-o(n)}$ bound by Caraballo et al.'18

- Let us first construct $n^{2n-o(n)}$ non-deg. order types on n points

Sketch of the $n^{2n-o(n)}$ bound by Caraballo et al.'18

- Let us first construct $n^{2n-o(n)}$ non-deg. order types on n points

- Bertrand's postulate: $\exists$ prime $p \approx \frac{n}{\log n}$

Sketch of the $n^{2n-o(n)}$ bound by Caraballo et al.'18

- Let us first construct $n^{2n-o(n)}$ non-deg. order types on n points

- Bertrand's postulate: $\exists$ prime $p \approx \frac{n}{\log n}$

- For p prime, let $Q_p = \{(i, i^2 \mod p) : i = 1, \ldots, p\}$

Sketch of the $n^{2n-o(n)}$ bound by Caraballo et al.'18

- Let us first construct $n^{2n-o(n)}$ non-deg. order types on n points

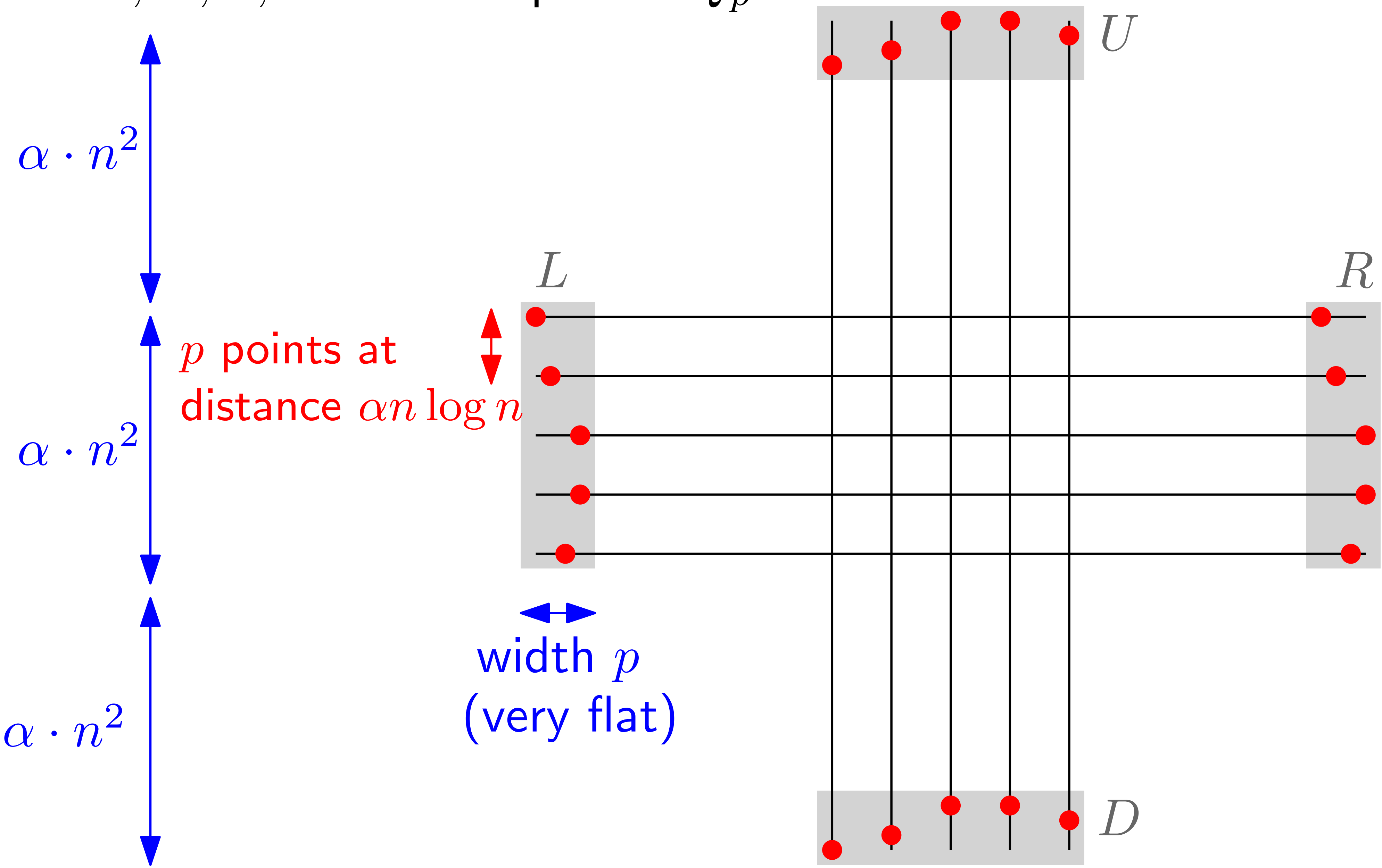
- Bertrand's postulate: $\exists$ prime $p \approx \frac{n}{\log n}$

- For p prime, let $Q_p = \{(i, i^2 \mod p) : i = 1, \ldots, p\}$

- Q_p is non-degenerate, via Vandermonde determinant:

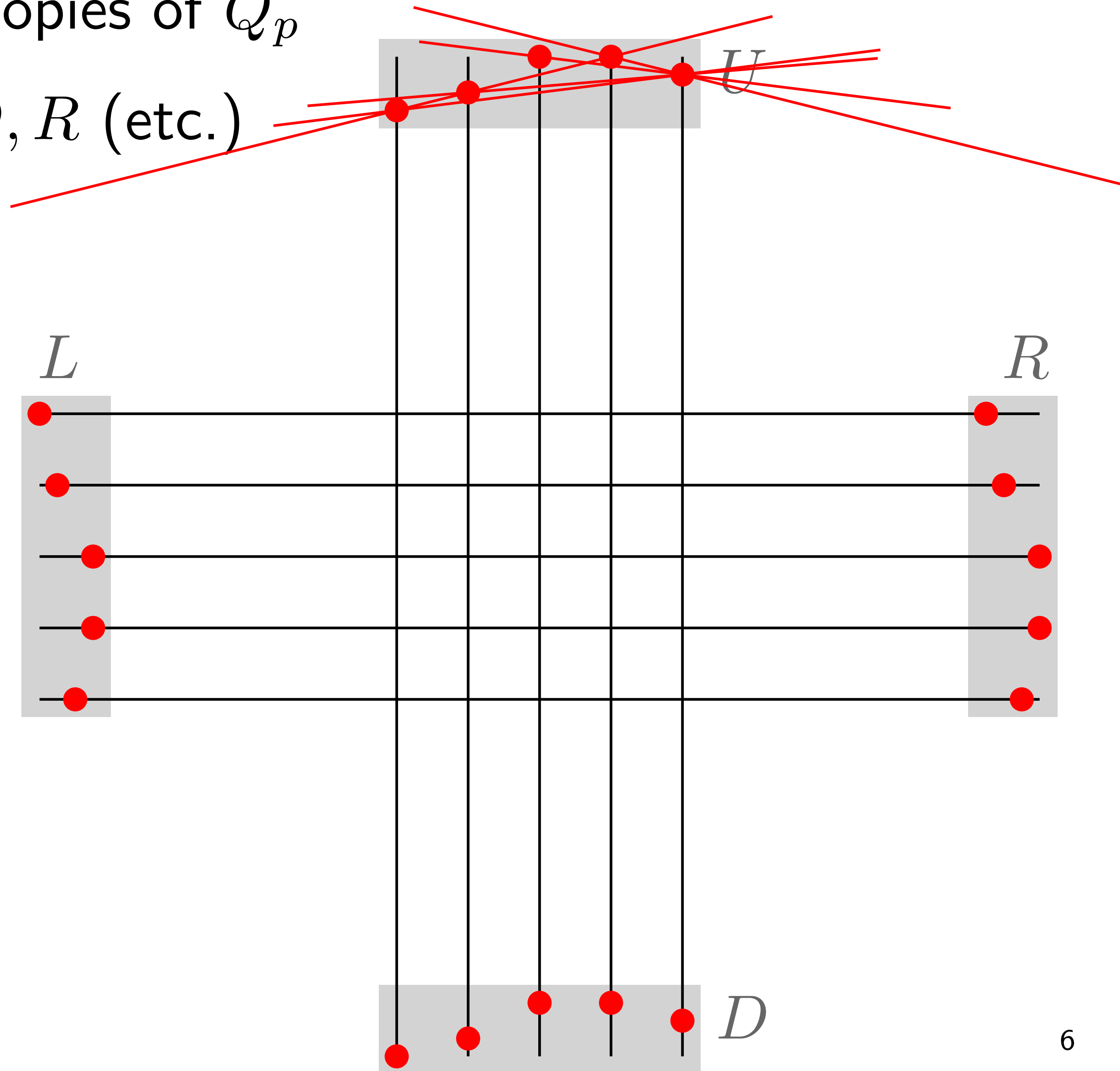
$$\chi(a,b,c) = \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = (b-a)(c-a)(c-b) \neq 0$$

- $p \approx \frac{n}{\log n}$ prime , and let α be a suitable constant
- U, D, L, R scaled copies of Q_p

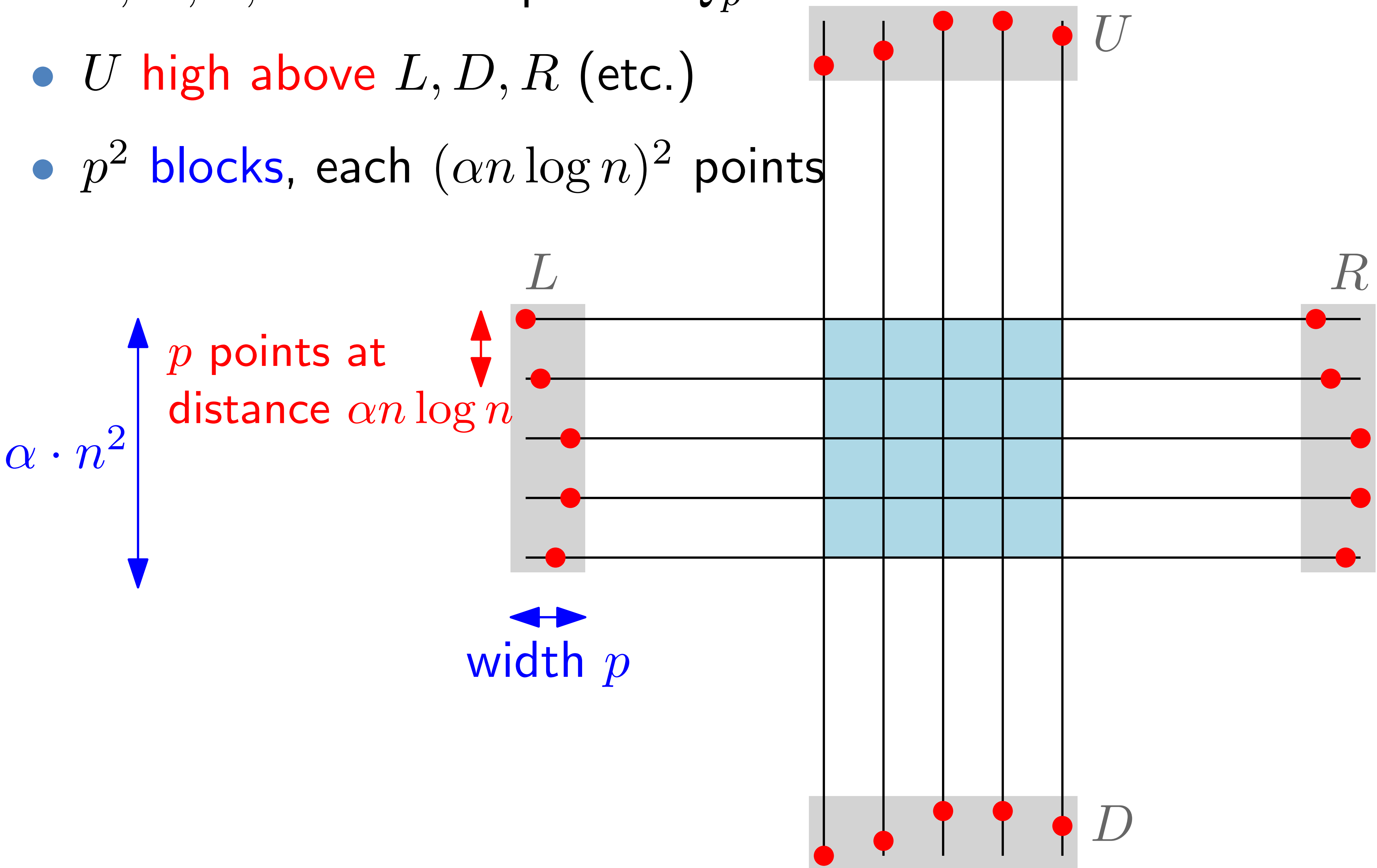


- $p \approx \frac{n}{\log n}$ prime , and let α be a suitable constant
- U, D, L, R scaled copies of Q_p
- U high above L, D, R (etc.)

  $\Rightarrow$ non-degenerate



- $p \approx \frac{n}{\log n}$ prime , and let α be a suitable constant
- U, D, L, R scaled copies of Q_p
- U high above L, D, R (etc.)
- p^2 blocks, each $(\alpha n \log n)^2$ points



- idea: iteratively place remaining $n-4p$ points in blocks, each assignment gives different OT

(relative position to L, U, D, R)

- idea: iteratively place remaining $n - 4p$ points in blocks, each assignment gives different OT
- a block is dead if no further points can be placed (we only want non-degenerate OTs)

- idea: iteratively place remaining $n - 4p$ points in blocks, each assignment gives different OT
- a block is dead if no further points can be placed (we only want non-degenerate OTs)
- each of the $\leq \binom{n}{2}$ lines between two already-placed points is either vertical (and kills at most αn^2) or kills at most 1 point per x-coordinate (kills at most αn^2). Hence at most αn^4 points are killed in total

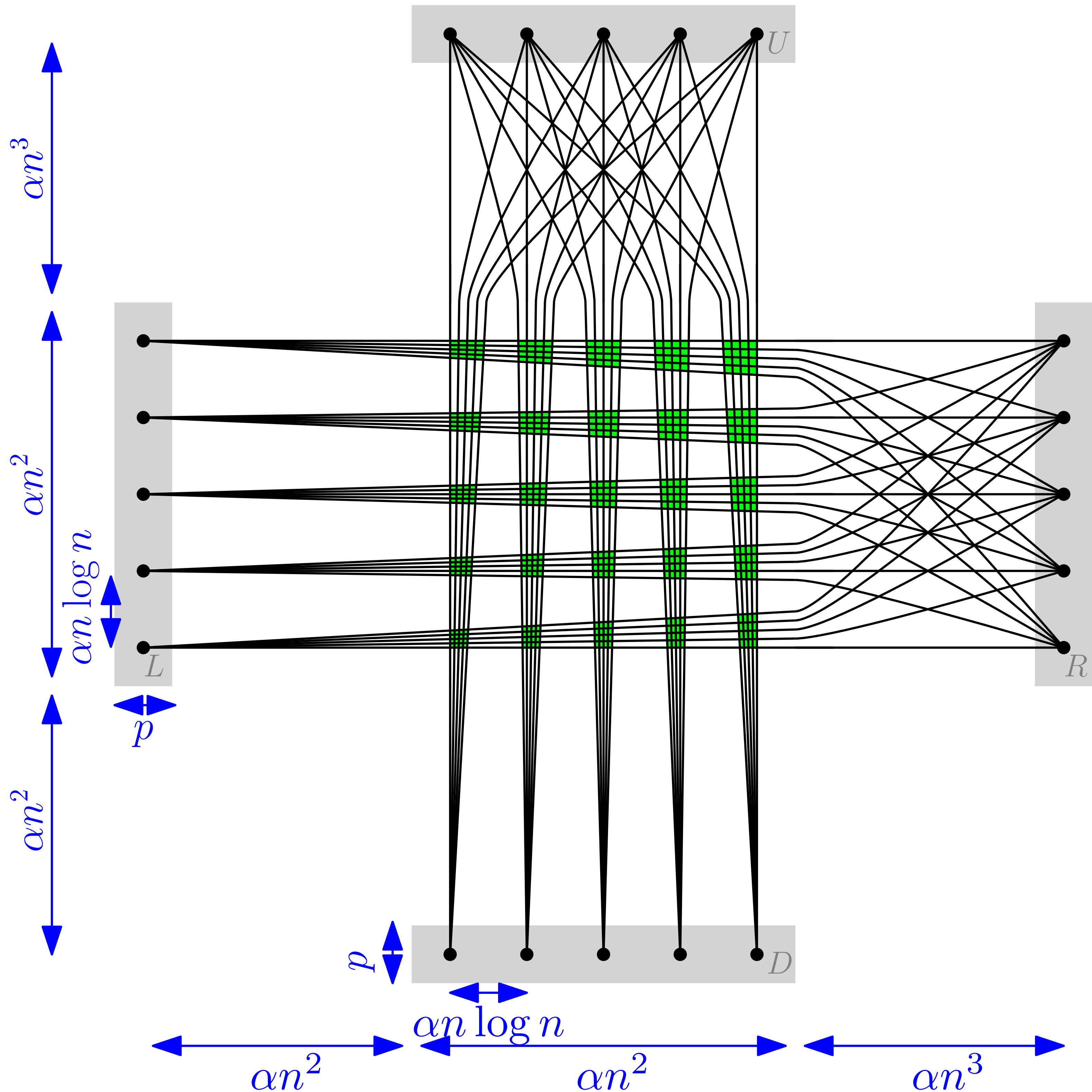
- idea: iteratively place remaining $n - 4p$ points in blocks, each assignment gives different OT
- a block is dead if no further points can be placed (we only want non-degenerate OTs)
- each of the $\leq \binom{n}{2}$ lines between two already-placed points is either vertical (and kills at most αn^2) or kills at most 1 point per x-coordinate (kills at most αn^2). Hence at most αn^4 points are killed in total
- $\geq p^2 - \frac{\alpha n^4}{(\alpha n \log n)^2} \geq c \cdot \frac{n^2}{(\log n)^2}$ blocks alive at any time

$p^2 = \frac{n^2}{(\log n)^2}$; $\frac{\alpha n^4}{(\alpha n \log n)^2} = \frac{n^2}{\alpha(\log n)^2}$

- idea: iteratively place remaining $n - 4p$ points in blocks, each assignment gives different OT
- a block is dead if no further points can be placed (we only want non-degenerate OTs)
- each of the $\leq \binom{n}{2}$ lines between two already-placed points is either vertical (and kills at most αn^2) or kills at most 1 point per x-coordinate (kills at most αn^2). Hence at most αn^4 points are killed in total
- $\geq p^2 - \frac{\alpha n^4}{(\alpha n \log n)^2} \geq c \cdot \frac{n^2}{(\log n)^2}$ blocks alive at any time
- $\Rightarrow$ # possibilities $\geq \left(c \cdot \frac{n^2}{(\log n)^2}\right)^{n-4p} = n^{2n-o(n)}$ $\square$

Construction for $n^{4n-o(n)}$

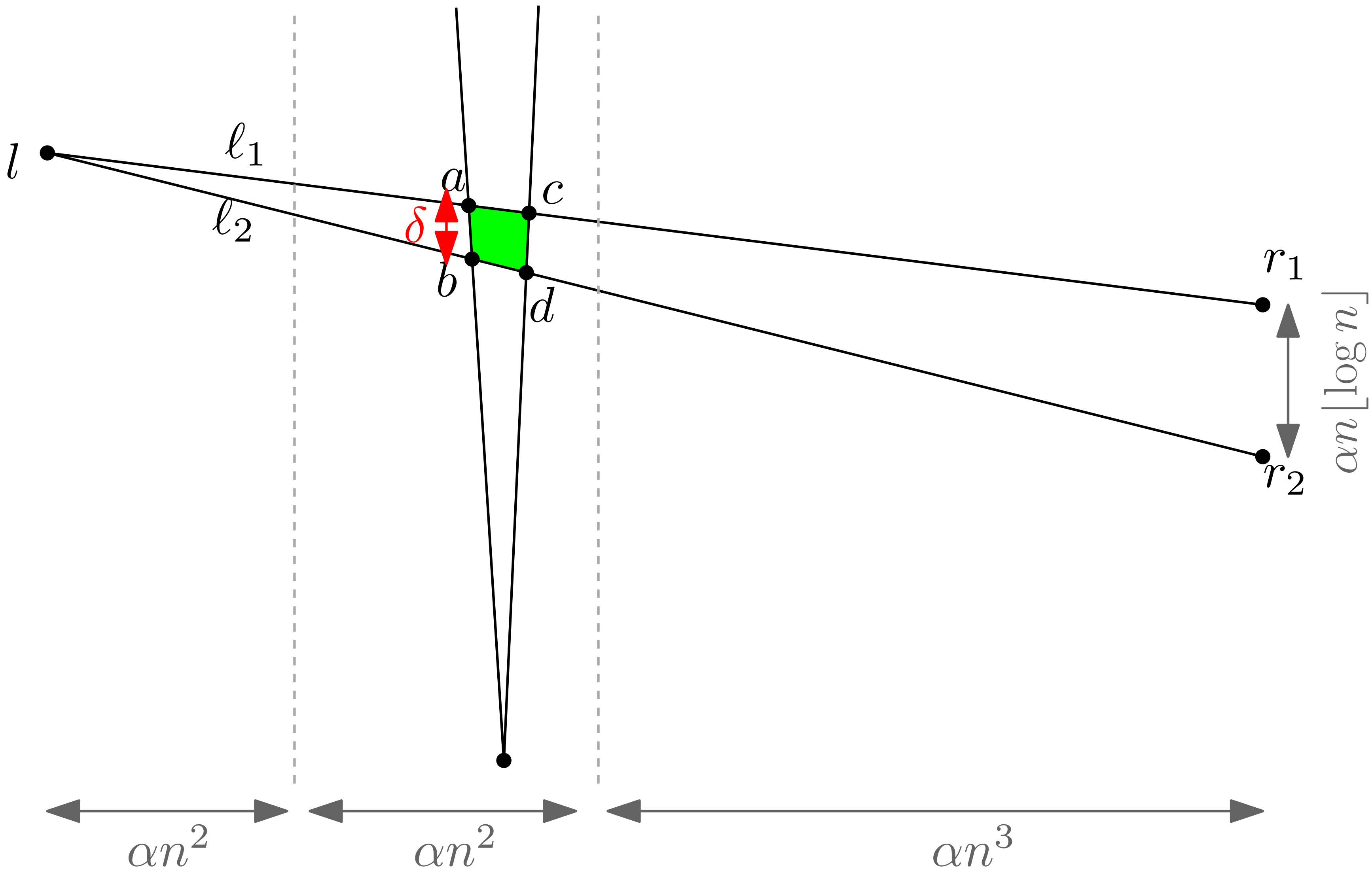
- similar idea, but different placement of U, D, L, R and more technical



U
L
R
D
αn^3
αn^2
αn^2
$\alpha n \log n$
p
p
$\alpha n \log n$
αn^2
αn^2
αn^3

- $(p^2 - p)^2 \approx p^4 \approx \frac{n^4}{(\log n)^4}$ blocks

- some technical details (blocks behave "nicely") ...



- $(p^2 - p)^2 \approx p^4 \approx \frac{n^4}{(\log n)^4}$ blocks
- some technical details (blocks behave "nicely") ...
- with $\alpha \approx n$,
  $$\# \text{ possibilities} \approx \left(c \cdot \frac{n^4}{(\log n)^4}\right)^{n-4p} = n^{4n-o(n)}$$

Theorem (S.'21):
$n^{4n+o(n)}$ non-deg. OTs on $\Theta(n^4) \times \Theta(n^4)$ grid

Thank you very much for your attention!