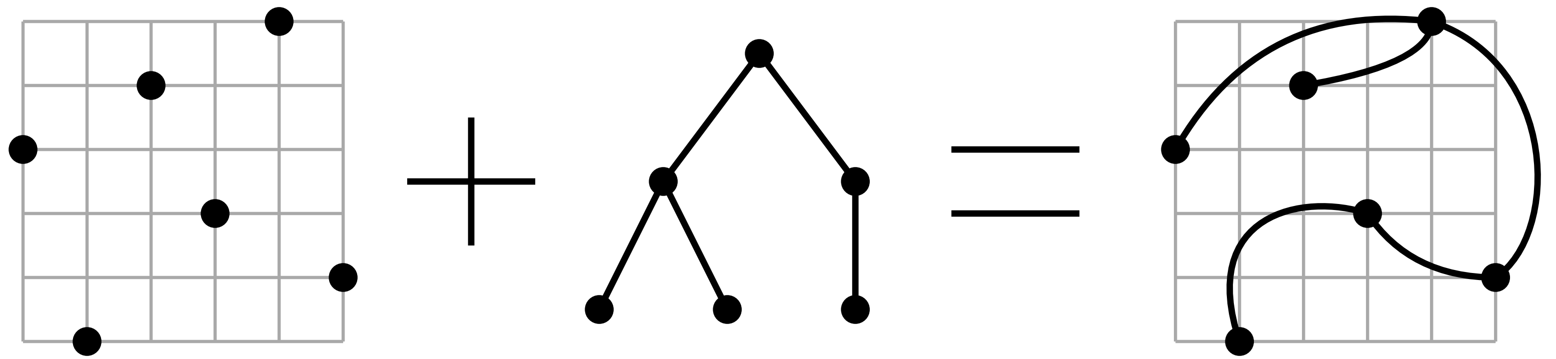
L-shaped Point Set Embeddings of Trees

Manfred Scheucher

Point Set Embeddings

T . . . tree on n vertices
P . . . set of m points

point set embedding . . . drawing of T, vertices drawn as points of P



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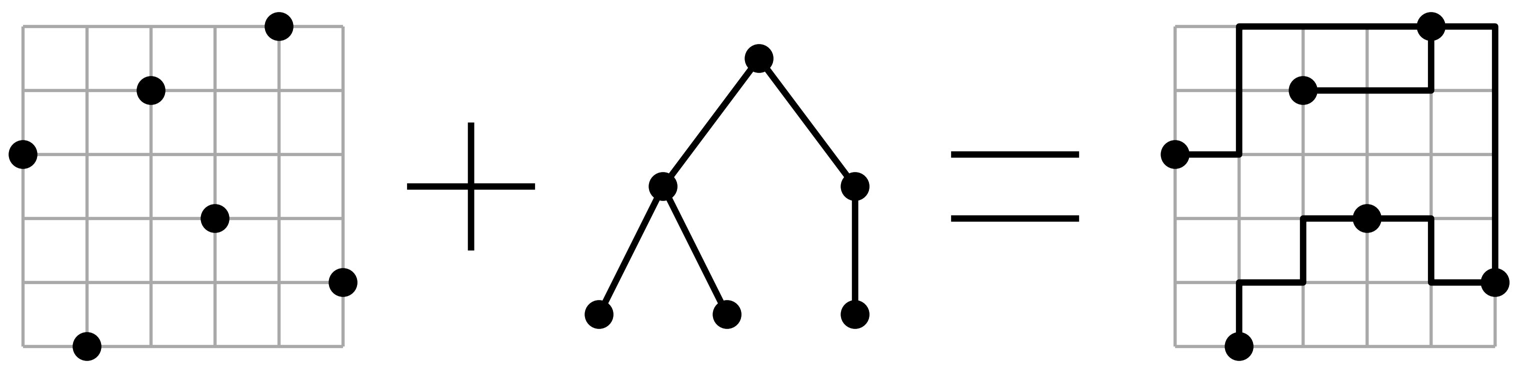

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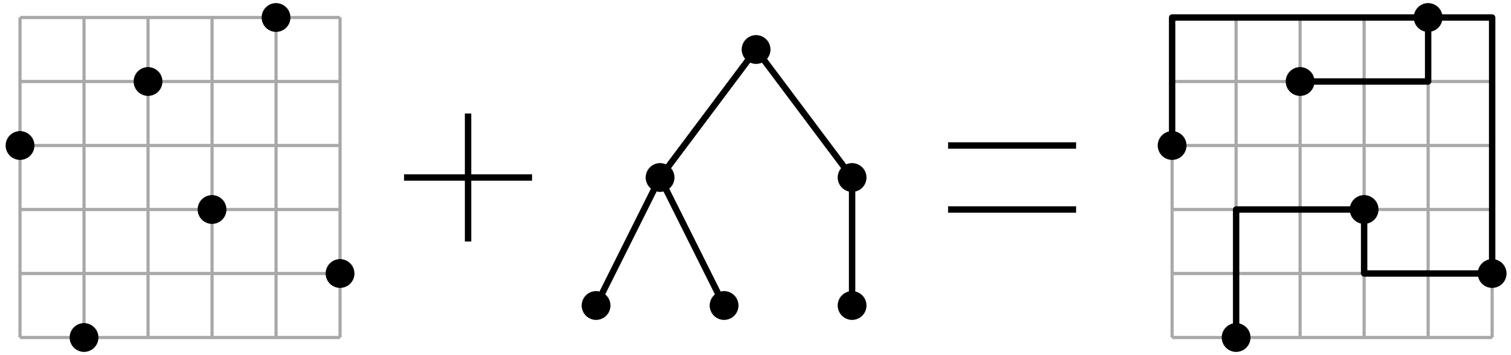

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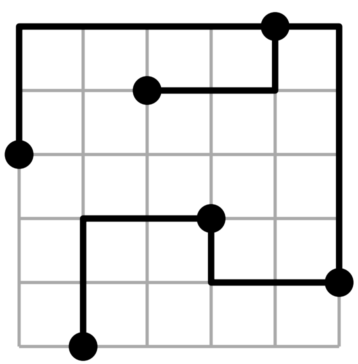
Assumptions:

- distinct x- and y-coordinates
- $P = \{(1, \pi_1), \ldots, (m, \pi_m)\}$



Point Set Embeddings

$f(T)$ ... minimum number m s.t. tree T admits a planar L-shaped embedding in any set of m points

$$f_d(n) := \max_{\substack{T\,:\,\text{tree on } n \text{ vertices} \\ \text{max. deg. } \Delta(T) \leq d}} f(T)$$

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- $f_4(n) \leq n^2$ [Di Giacomo et al.'13]
- $f_4(n) \leq O(n^{1.58})$ [S.'15, Aichholzer-Hackl-S.'16]
- $f_3(n) \leq O(n^{1.22})$, $f_4(n) \leq O(n^{1.55})$ [Biedl et al.'17]
- non-trivial lower bound [Mütze-S. '18]

Recursive Embedding Approach

T . . . tree on n vertices ($\Delta \leq 4$), rooted at deg. 1 vertex r

P . . . set of u_n points

suitable sequence

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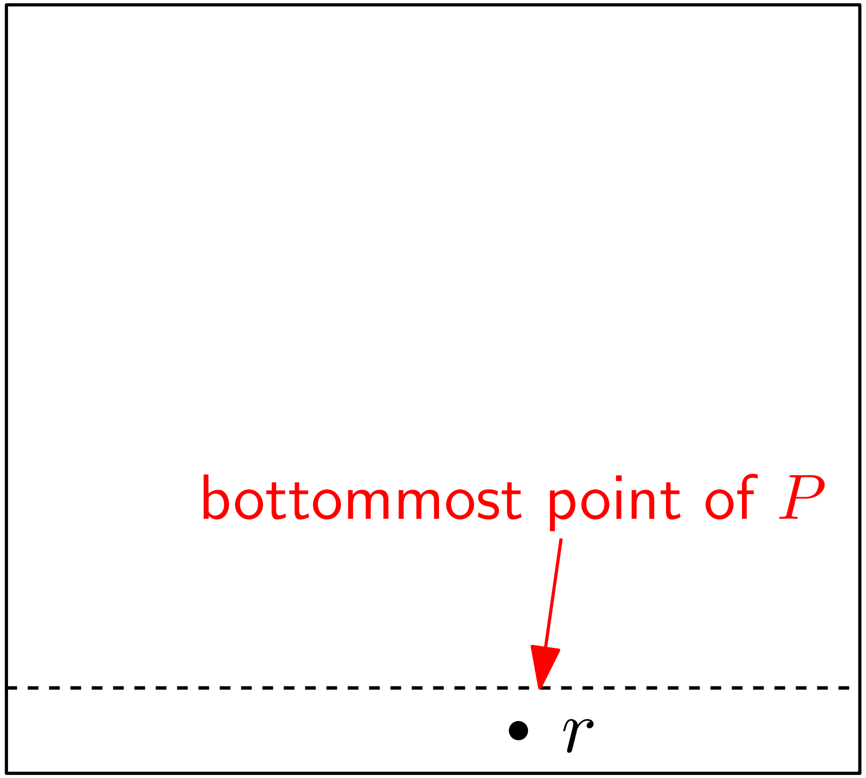

Recursive Embedding Approach

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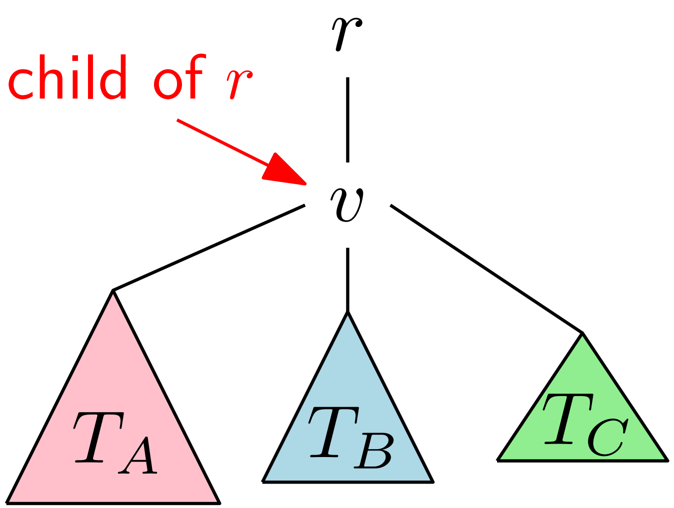
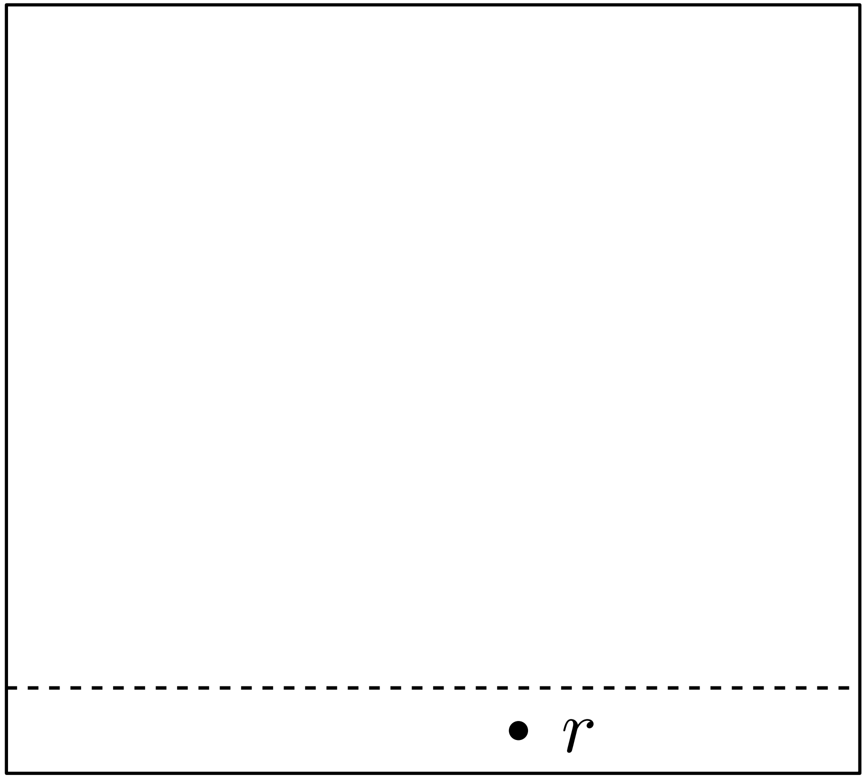
subtrees T_A, T_B, T_C
of sizes $a \geq b \geq c$
$(n - 1 = 1 + a + b + c)$





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$$1 + u_a + 2u_b + 2u_c \leq u_{n-1} \leq u_n - 1$$

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$$P = A \cup (B_1 \cup C \cup B_2) \cup \{r\}$$

- $|A| = u_a$
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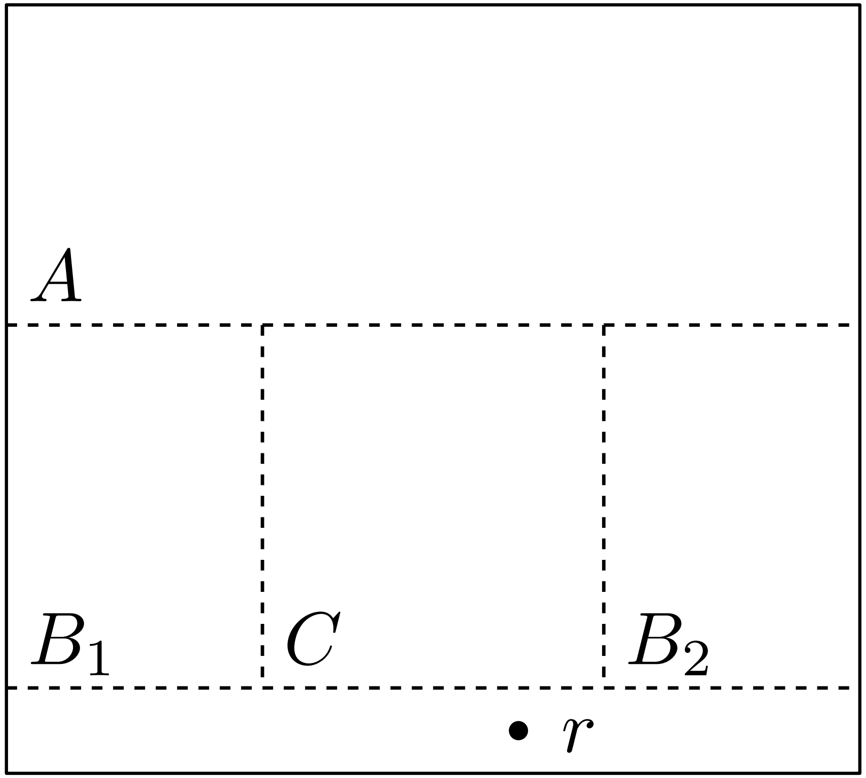

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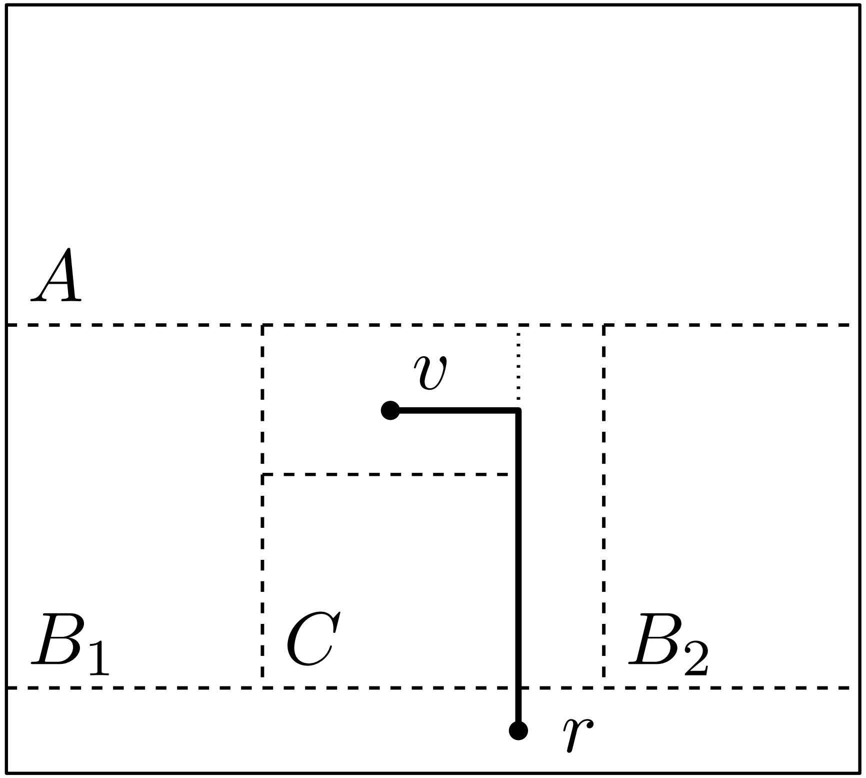

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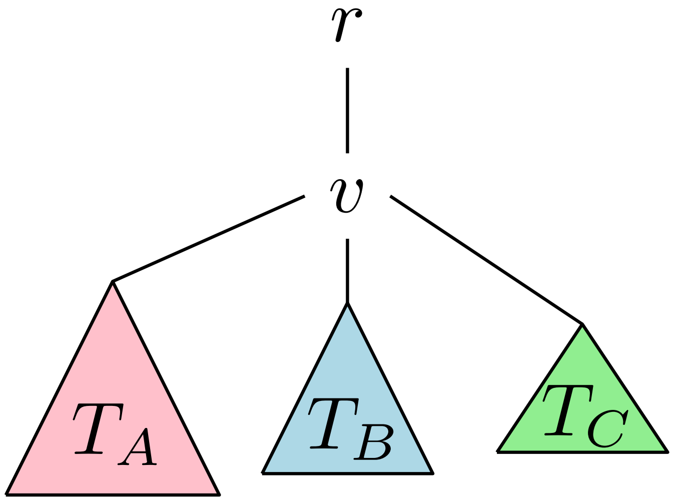

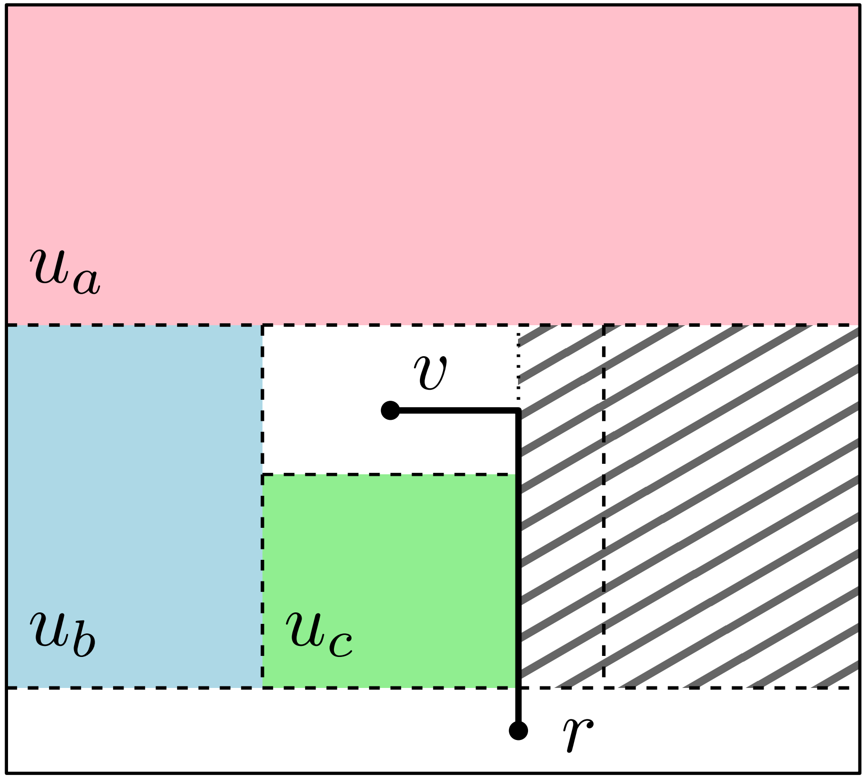

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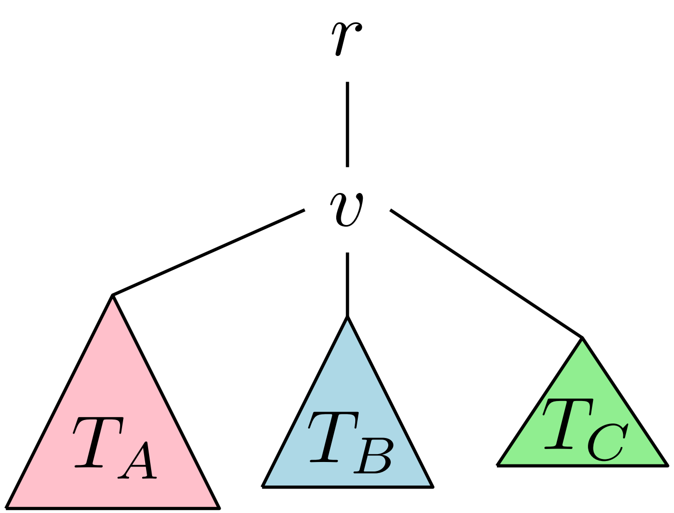

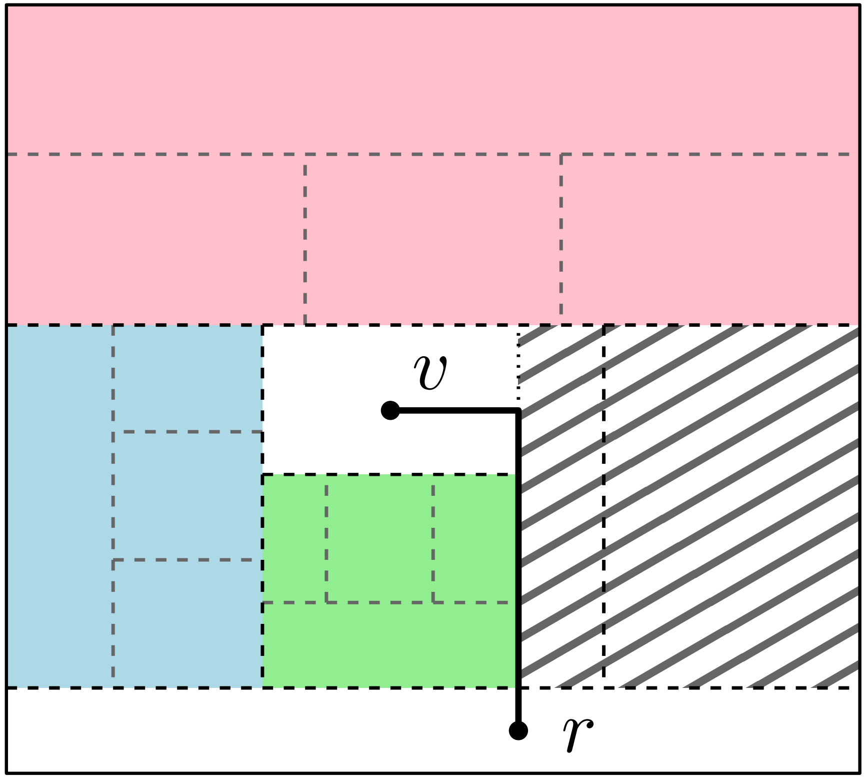

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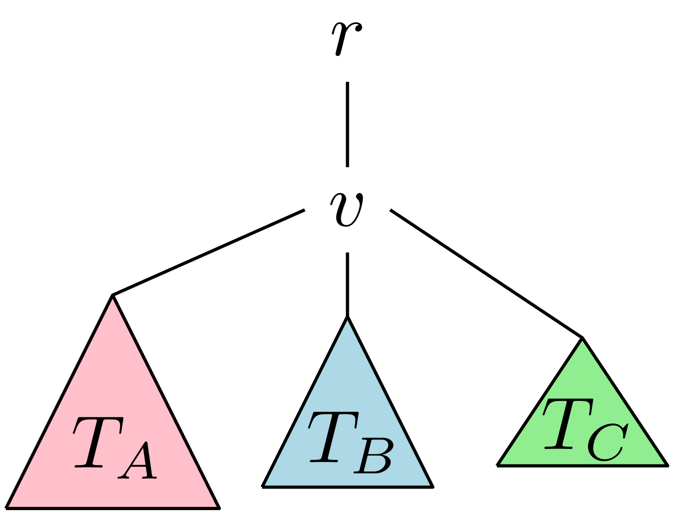

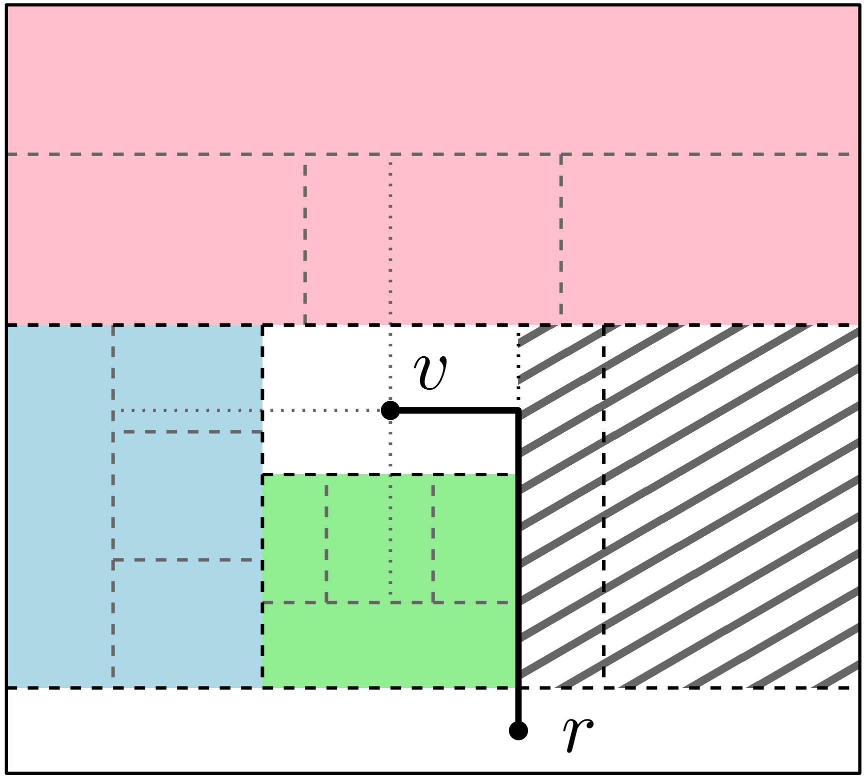

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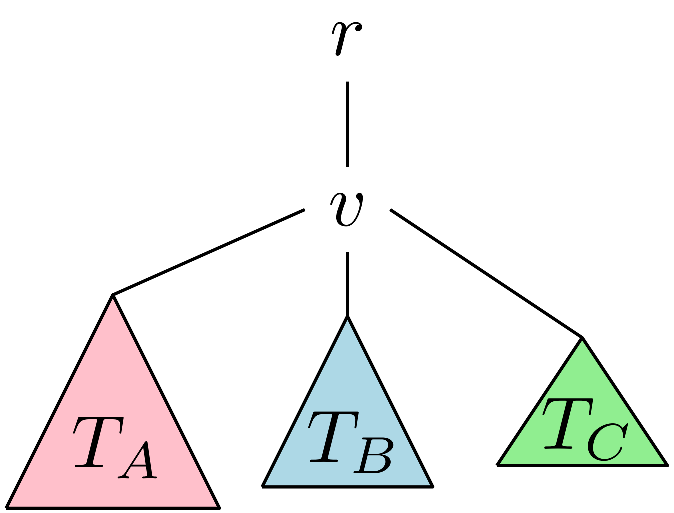

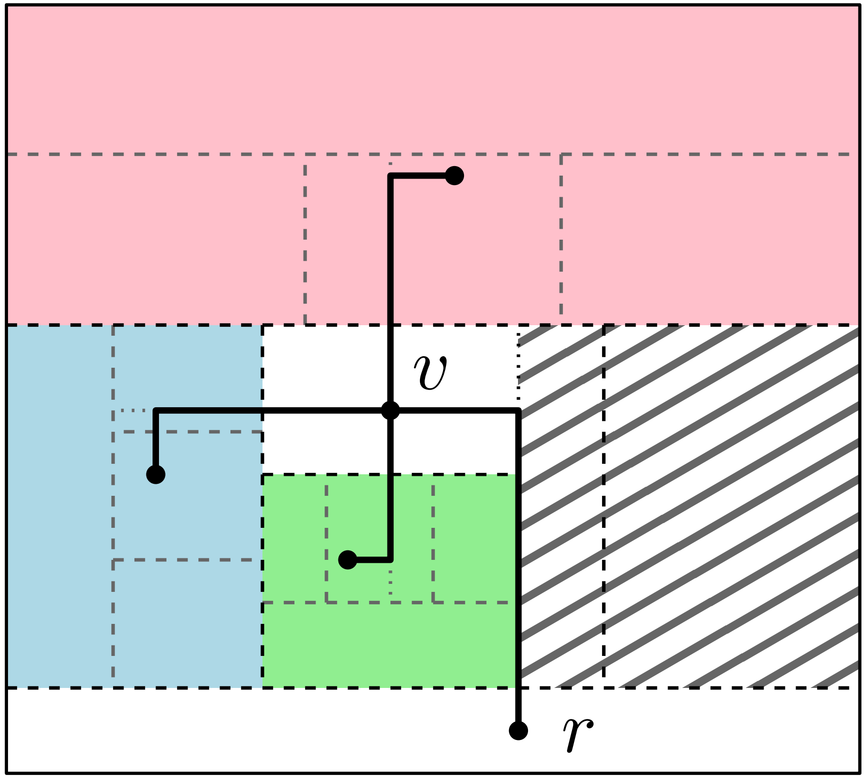

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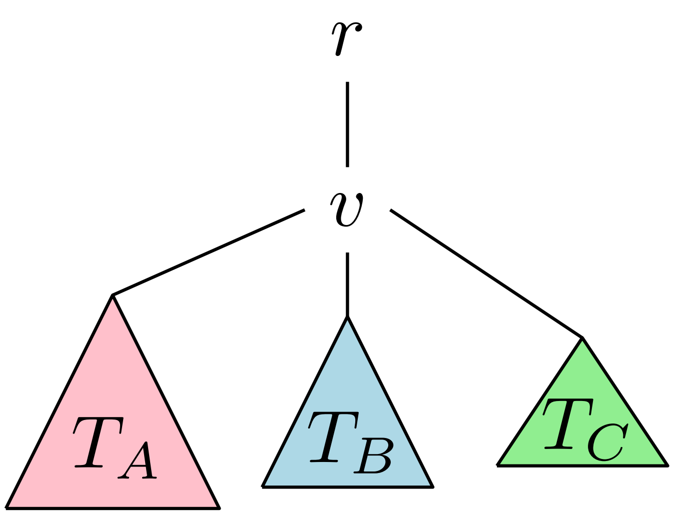

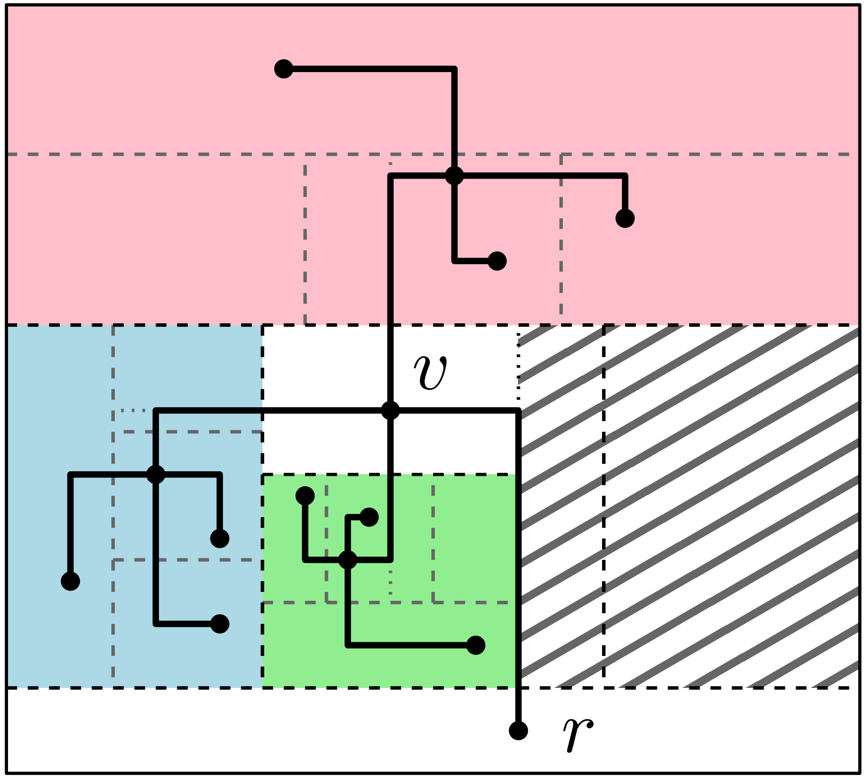

Subquadratic Upper Bound

Theorem: $f_4(n) \le n^{\log_2 3}$ $\quad (\log_2 3 \approx 1.585)$

Proof:

- Use recursive approach with

$$u_0 := 0,\ u_{n+1} := \max_{\substack{a \ge b \ge c \\ a+b+c=n}} 1 + u_a + 2u_b + 2u_c.$$

- It remains to show $u_n \le n^{\log_2 3}$.

Subquadratic Upper Bound

Induction Base:

$u_0 = 0 = g(0)$ and $u_1 = 1 = g(1)$, with $g(x) = x^{\log_2 3}$

convex

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Step:

Since $u_{n+1} = 1 + u_a + 2u_b + 2u_c$ for some $a, b, c \in \mathbb{N}$ with $a + b + c = n$,

$$\begin{aligned} u_{n+1} &\overset{IA}{\leq} 1 + g(a) + 2g(b) + 2g(c) \\ &\leq \max_{(x,y,z)\in S} \underbrace{1 + g(x) + 2g(y) + 2g(z)}_{=: G(x,y,z) \text{ (convex)}} \end{aligned}$$

convex

Subquadratic Upper Bound

$$S := \{(x,y,z) \in \mathbb{R}^3 \colon x \geq y \geq z \geq 0,\ x+y+z=n\}$$

spanned by $s_1 = (n,0,0)$, $s_2 = (\frac{n}{2}, \frac{n}{2}, 0)$, $s_3 = (\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$

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$$\Longrightarrow u_{n+1} \leq 1 + g(n) \leq g(n+1) \qquad \square$$

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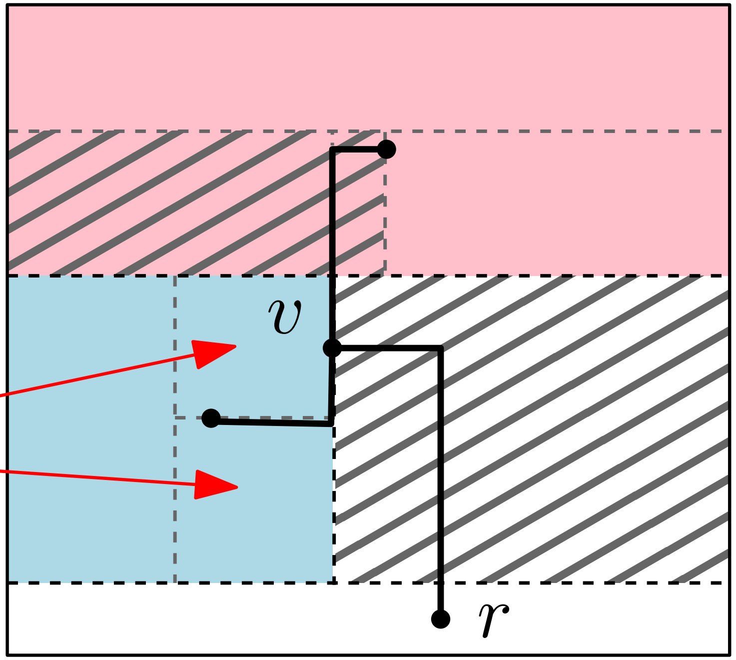
more precise analysis

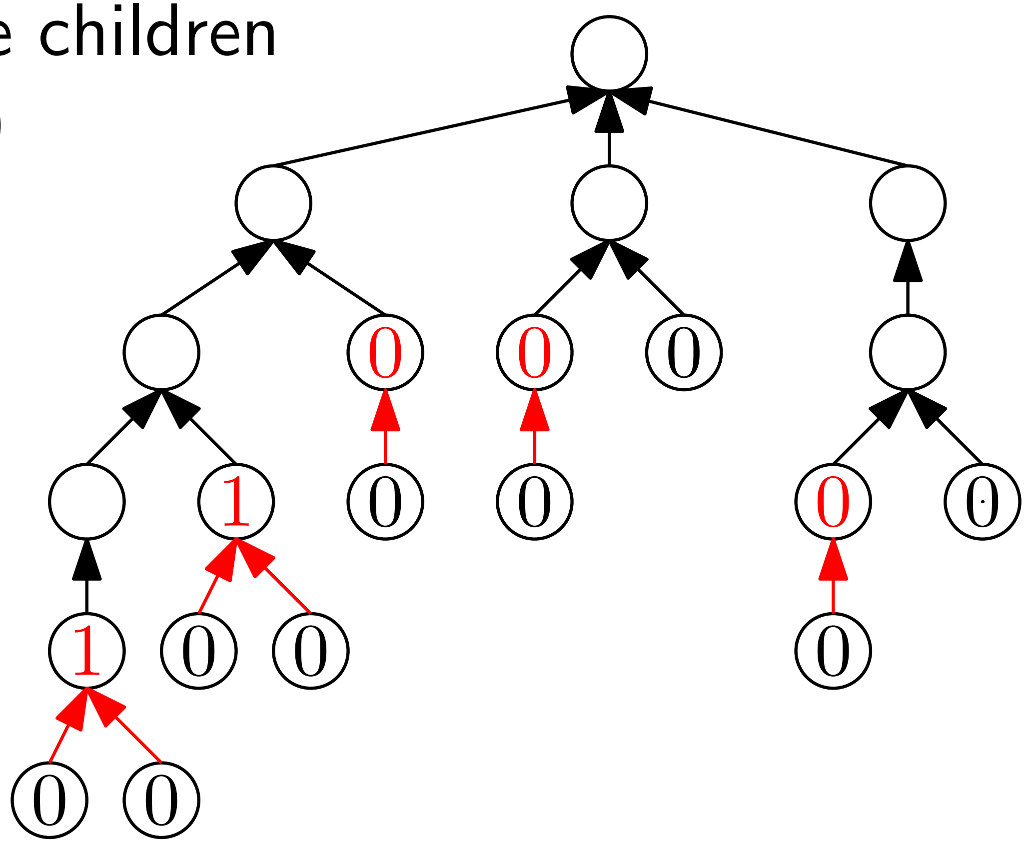
e.g.: everything usable
(deg. 3 trees)



Special Cases

$T = (V, E)$ ... tree

For a fixed "root" $r \in V$, we define
$\sigma_r(v) := \max\{0, \sigma_r(u_1), \sigma_r(u_2) + 1, \ldots, \sigma_r(u_k) + 1\}$,
where $u_1, \ldots, u_k$ denote the children of v, $\sigma_r(u_1) \geq \ldots \geq \sigma_r(u_k)$



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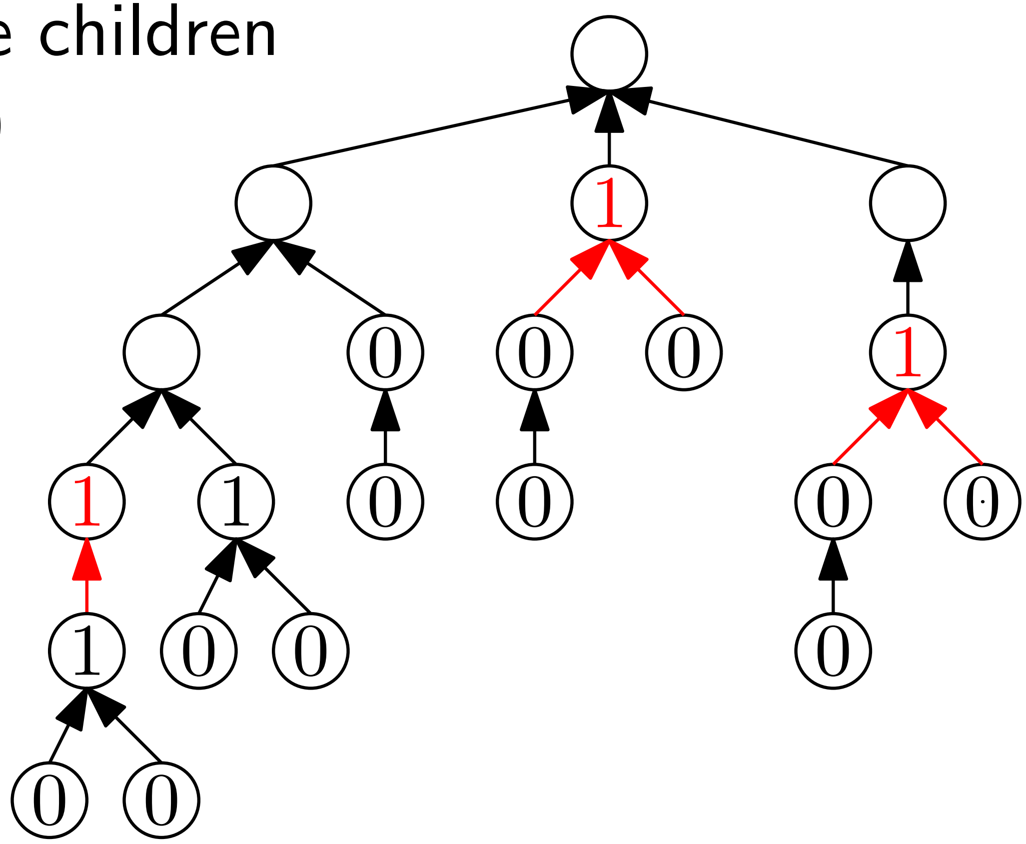

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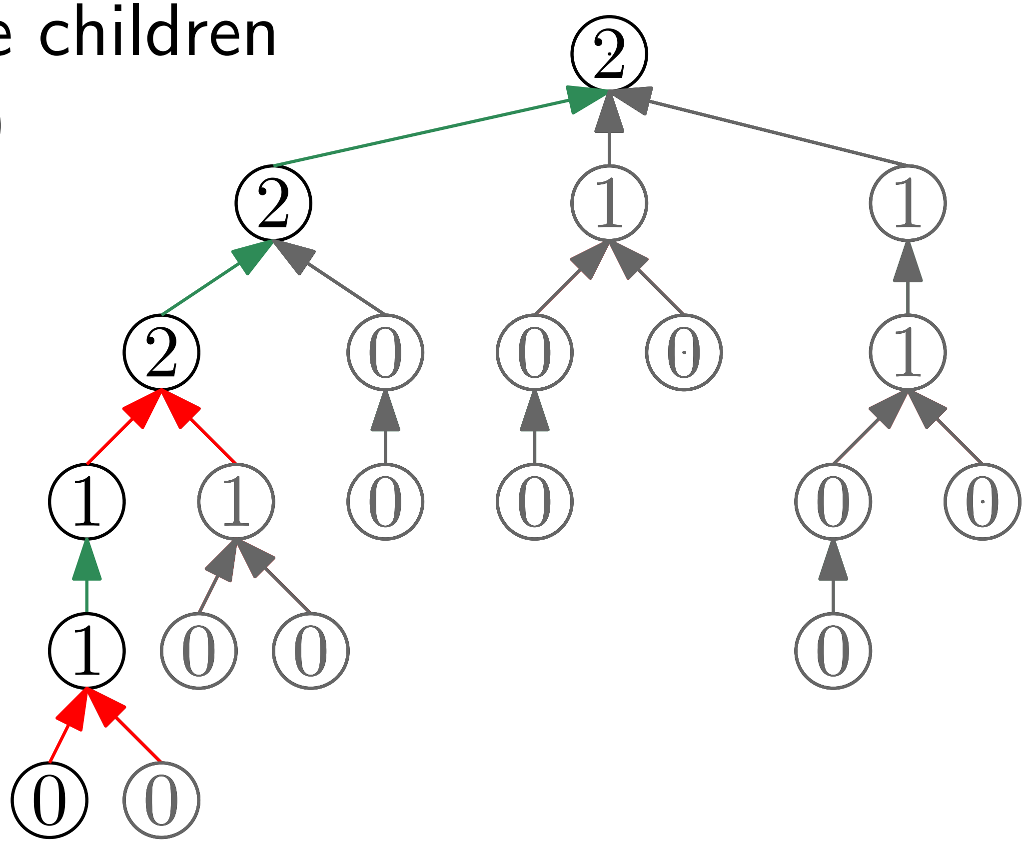

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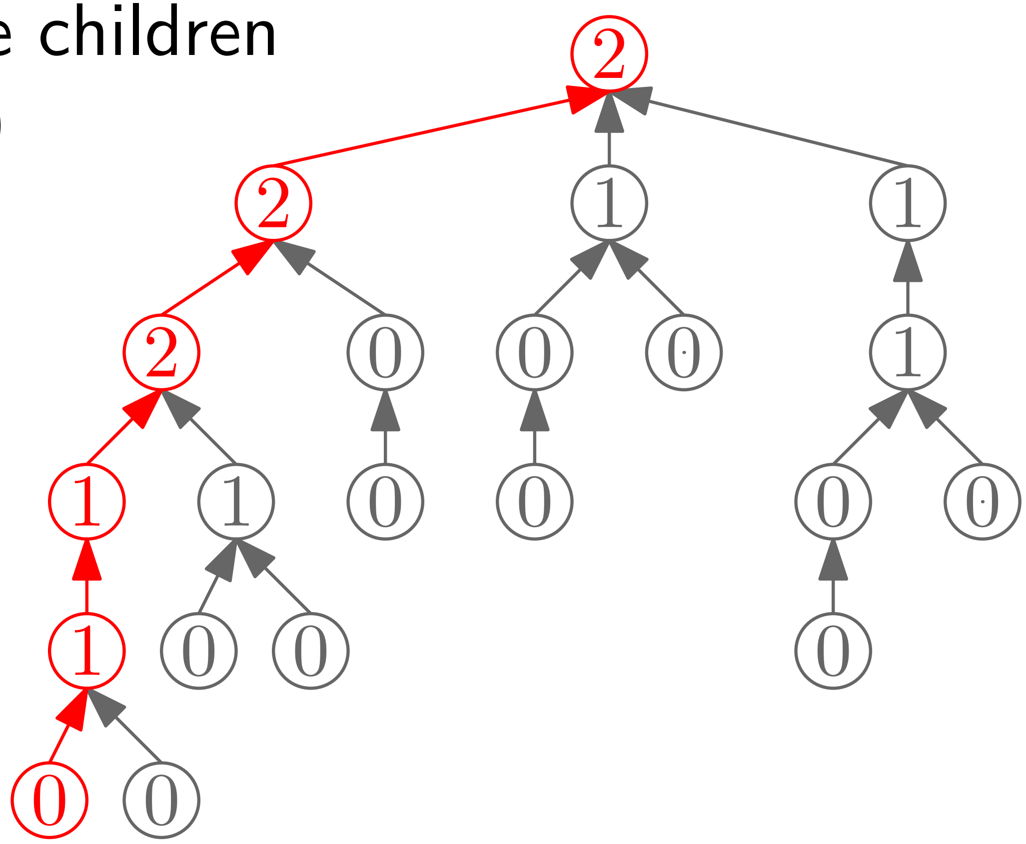

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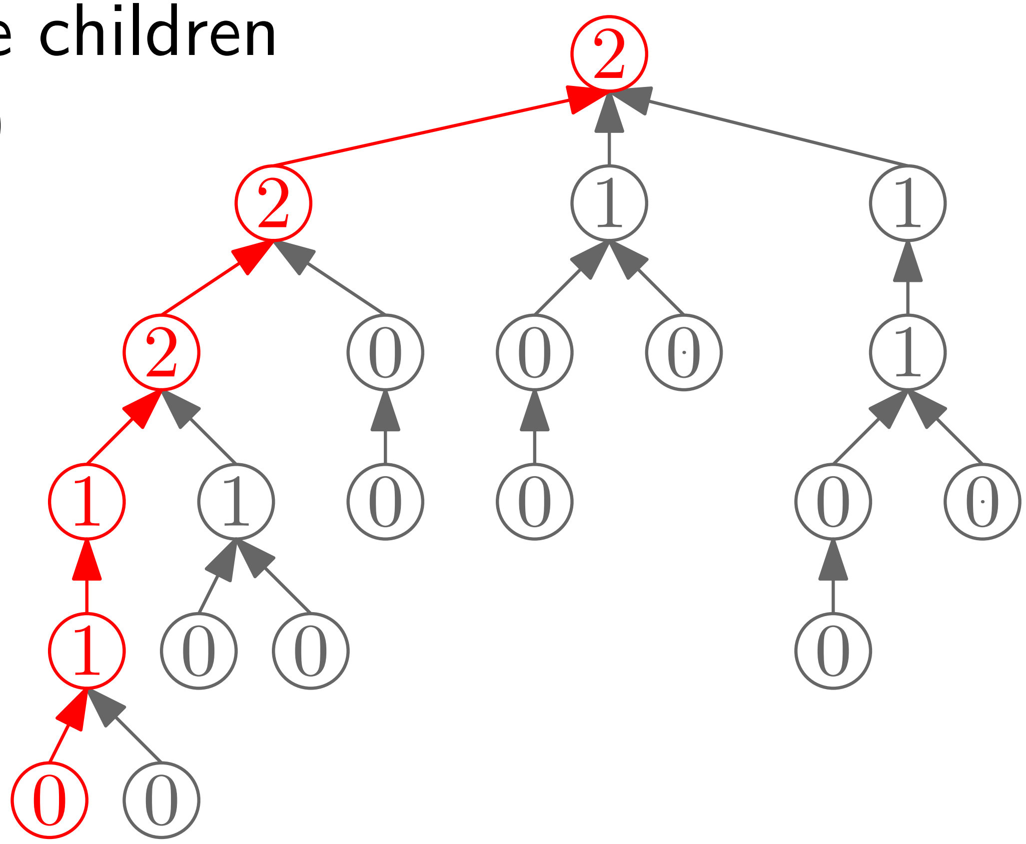
Definition (Saturation):
$\sigma(T) := \min_{r \in V} \sigma_r(r)$.



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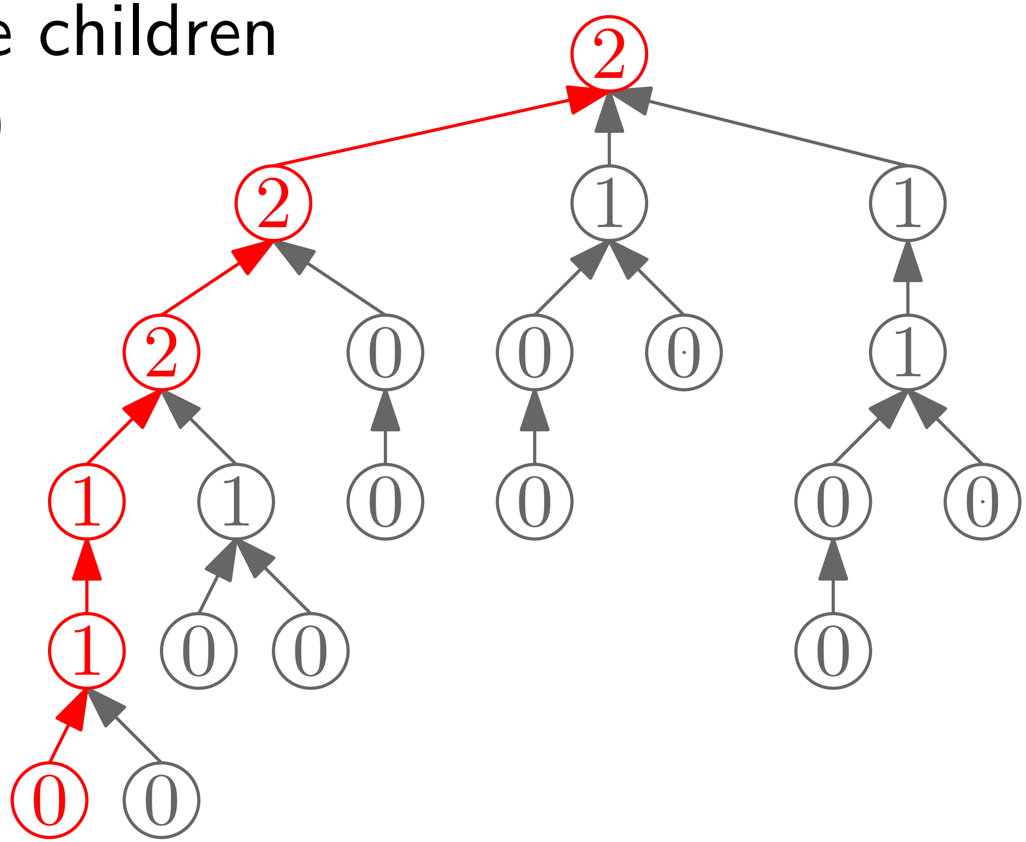

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Theorem:
$f(T) \leq 2^{\sigma(T)} n$.

Caterpillars

caterpillar . . . tree s.t. removal of leaves results in a path

image source:
http://scottsofstow.co.uk/sos/plush-alphabet-caterpillar-soft-toy (price: £29.95)

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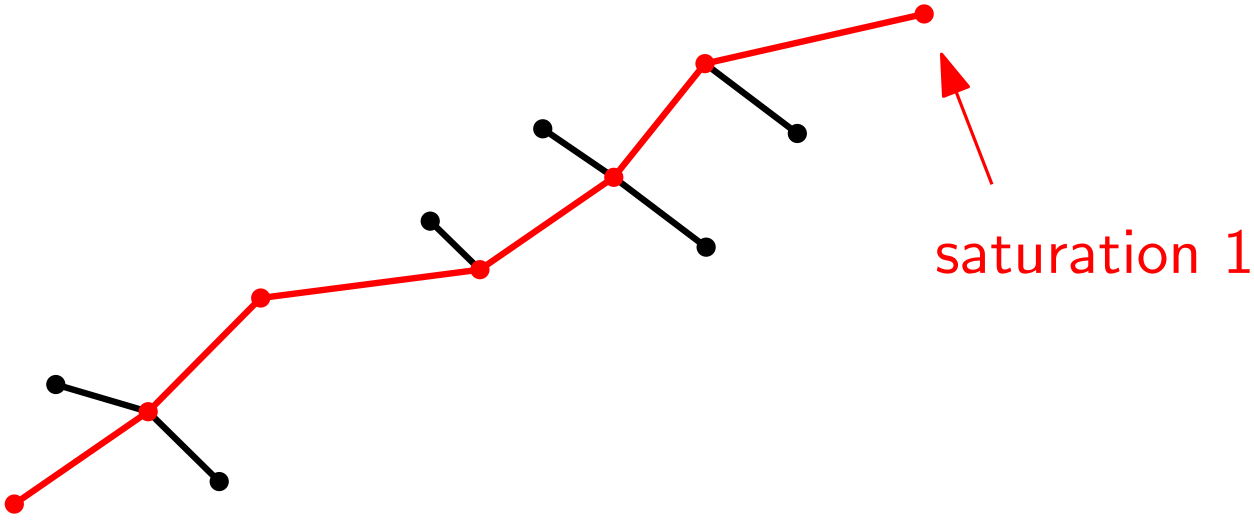

Corollary: $f(C) \leq 2n$.

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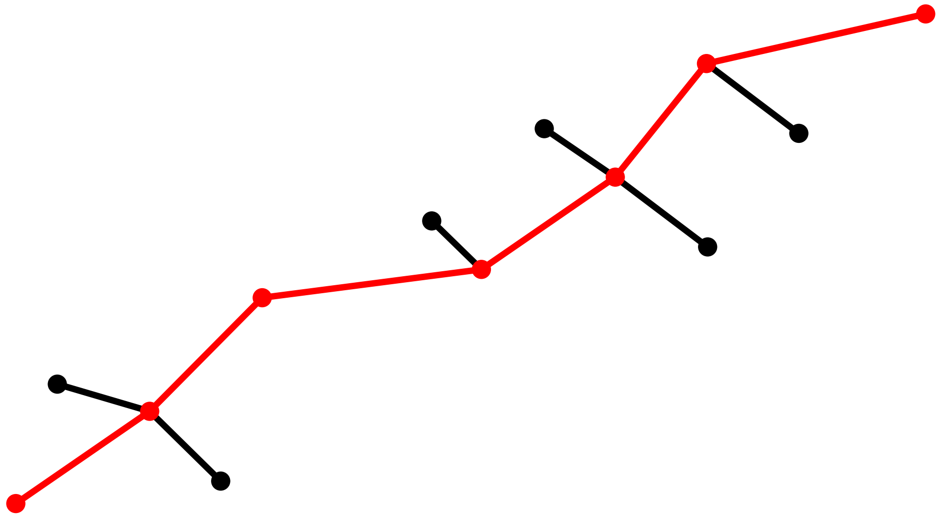

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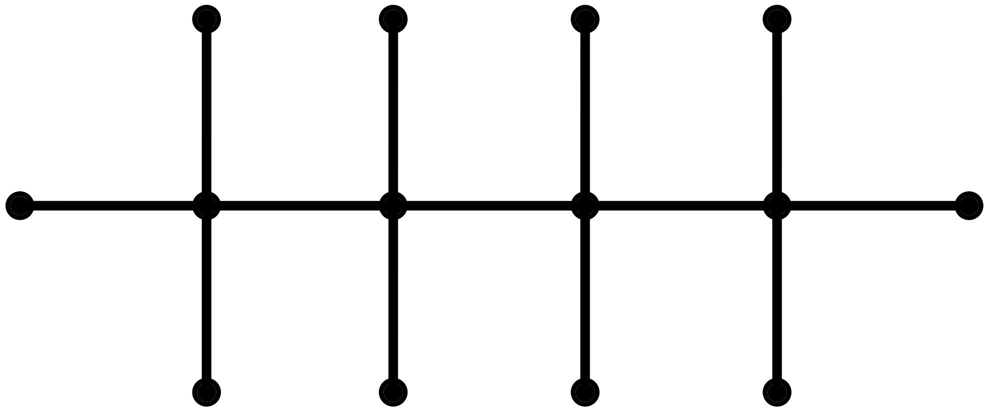
piecewise, computer assisted

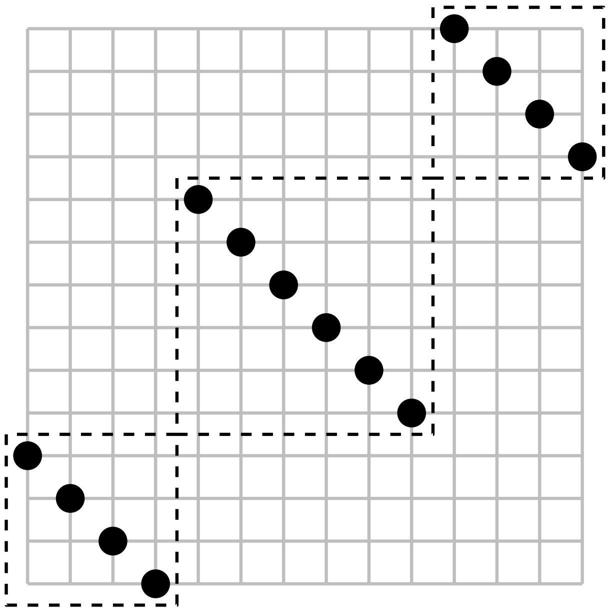
Further Improvements: $f(C) \leq (4/3 + \varepsilon)n$.

Lower Bound

- only in more restrictive setting:
  ∃ example which does not always admit an L-shaped embedding if **cyclic order** around each vertex is **fixed** (Biedl et al. 2017):



ordered tree on 14 vertices

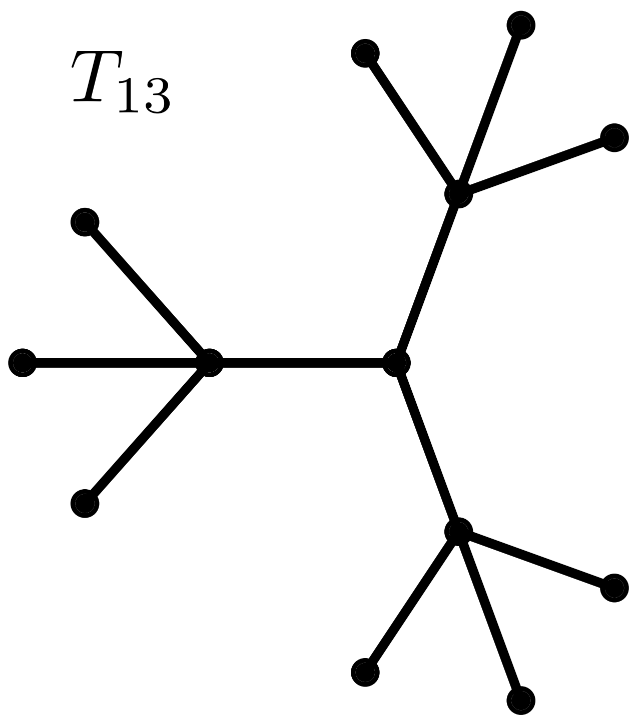


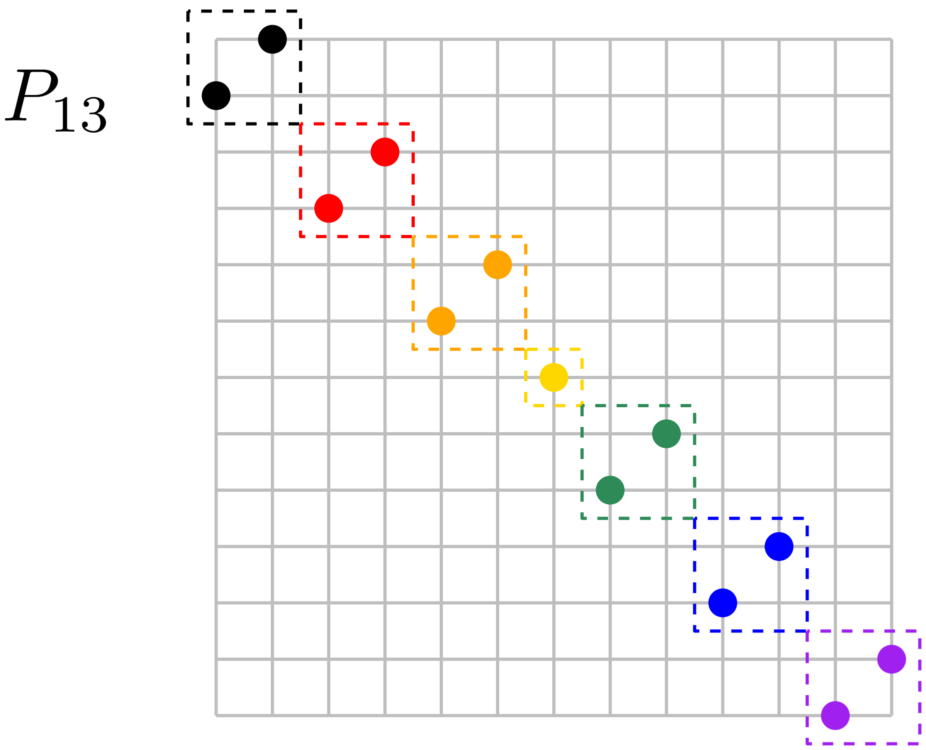
set of 14 points

New Lower Bound

Theorem (Mütze-S. '18, Computer-assisted): $f_4(n) = n$ for $n \leq 11$.

Theorem (Mütze-S. '18): T_{13} has no L-shaped embedding in P_{13}, hence, $f_4(13) \geq 14$.





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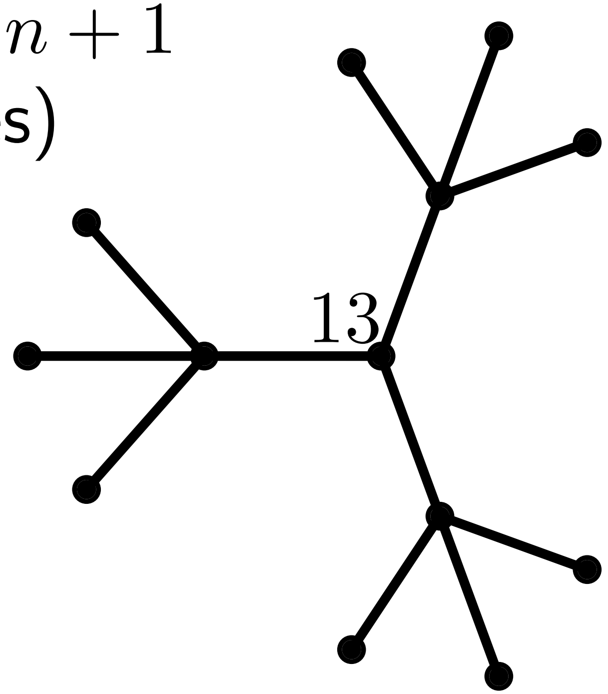
- Further examples for $n \in \{13, 14, 16, 17, 18, 19, 20\}$ (thus $f_4(n) \geq n + 1$ for those values)



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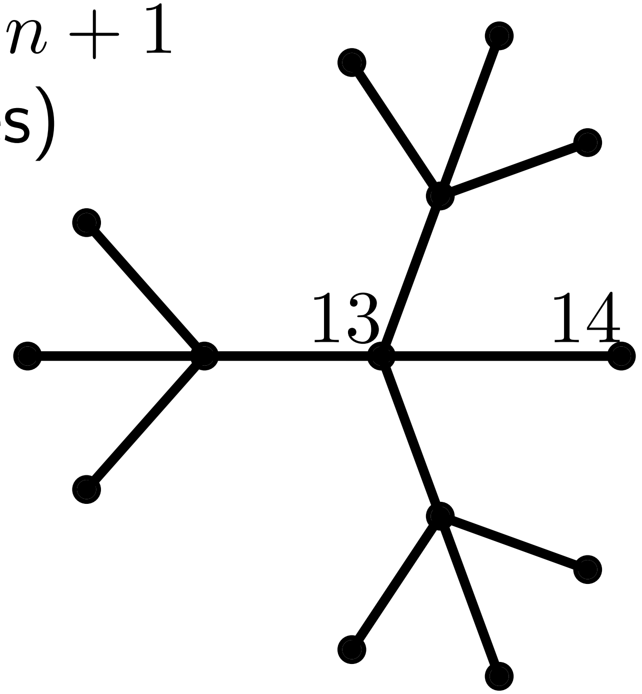
- Further examples for $n \in \{13, 14, 16, 17, 18, 19, 20\}$ (thus $f_4(n) \geq n + 1$ for those values)



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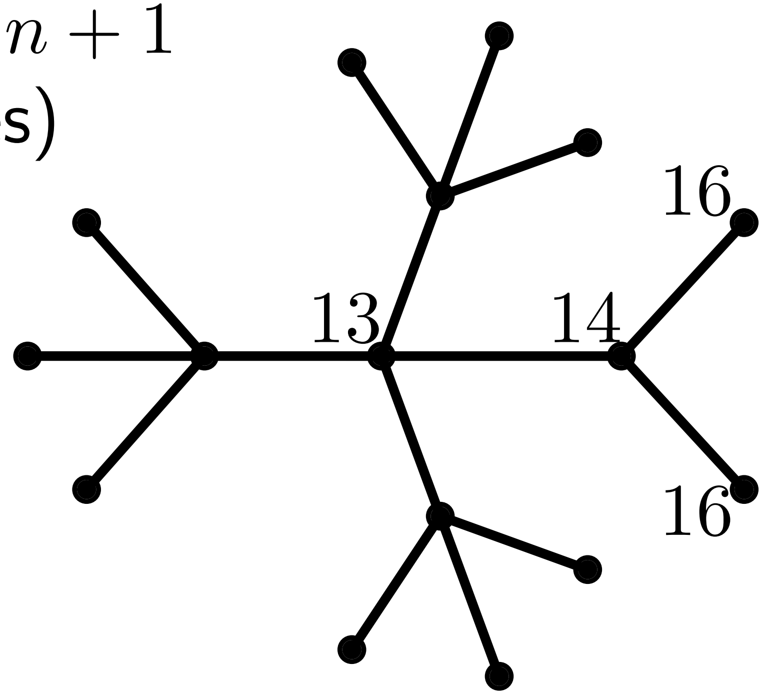

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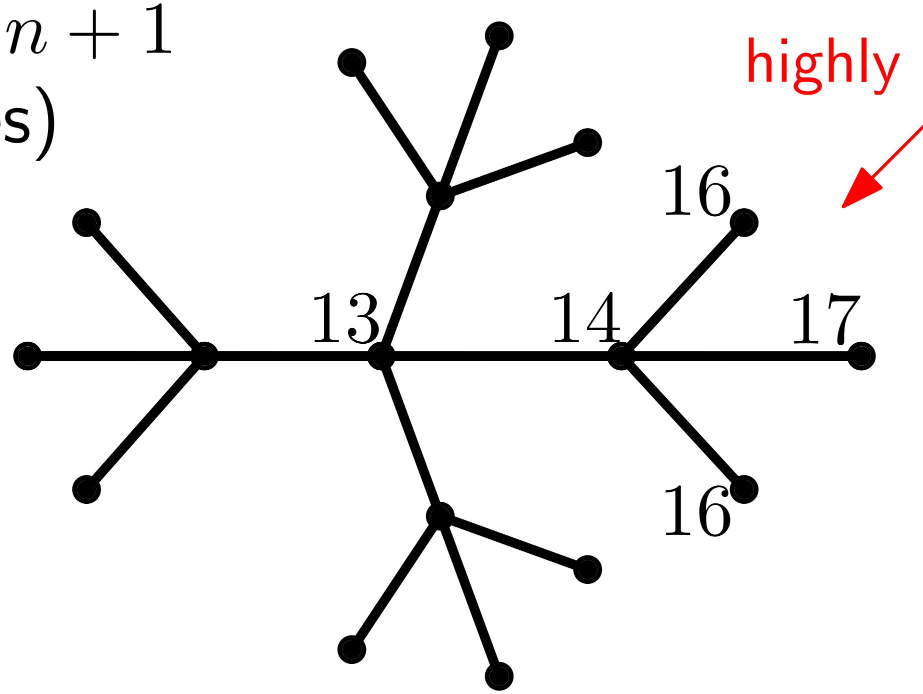
highly symmetric



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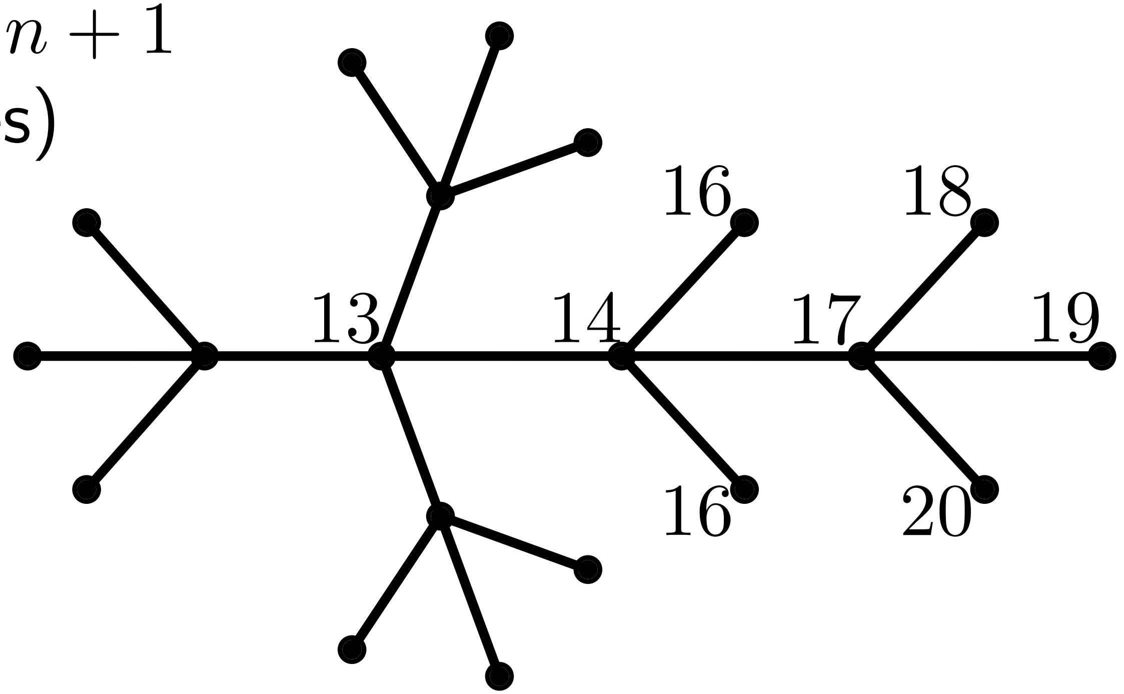

Computer-assisted Proof

- T . . . tree on vertices $\{v_1, \ldots, v_n\}$
- P . . . point set $\{P_1, \ldots, P_n\}$
- formulate Boolean satisfiability instance:
  $\exists$ solution iff. T admits an L-shaped embedding in P

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- test all pairs of trees and point sets

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$\Theta(c^n)$ $\Theta(n!)$

SAT Model: Variables

- $M_{i,j}$ ... vertex v_i is mapped to point P_j
- $H_{a,b}$ ... edge ab is connected horizontally to a

SAT Model: Clauses

- Injective mapping V to P

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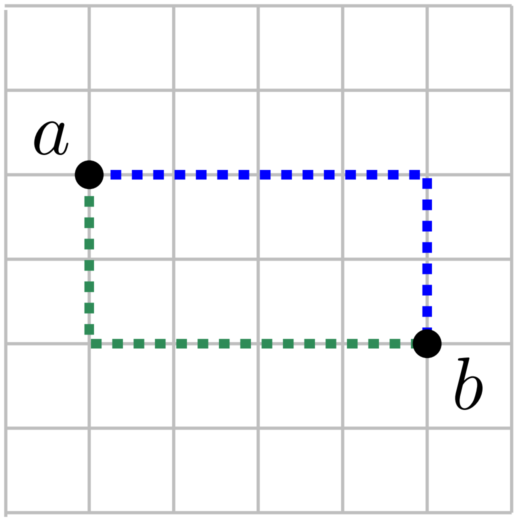

SAT Model: Clauses

- Injective mapping V to P
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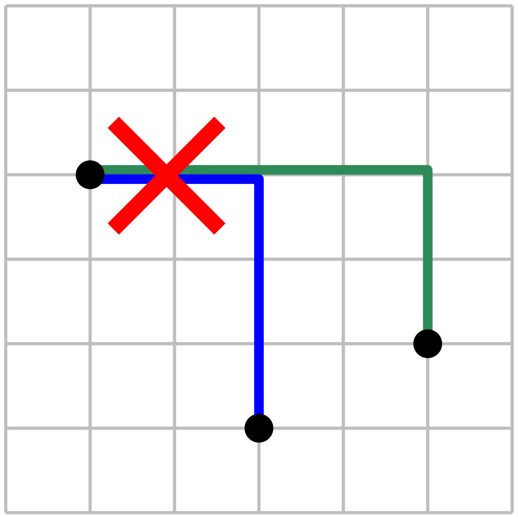
- No overlapping edges



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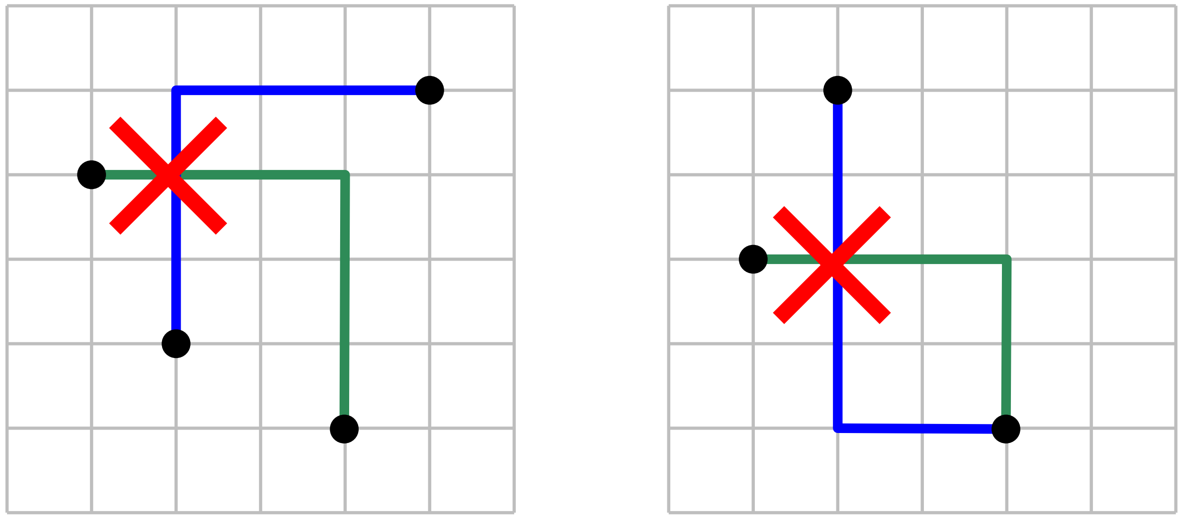

SAT Model: Clauses

- Injective mapping V to P
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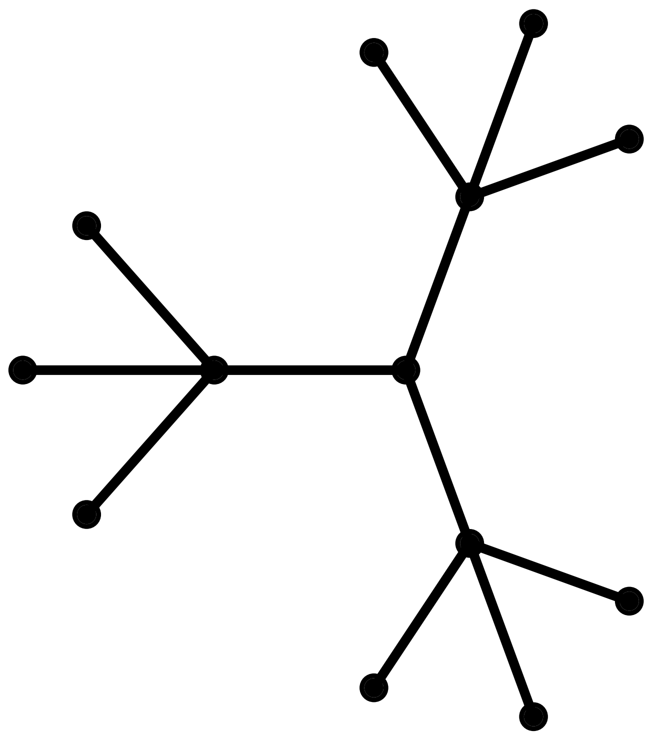
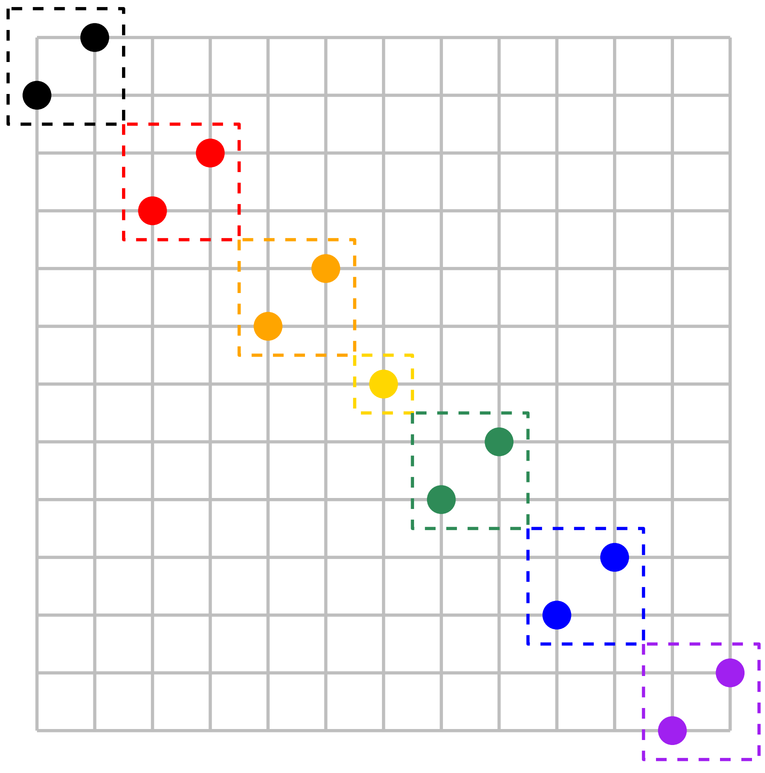
- use SAT solver to decide embedability

Proof of Theorem 2

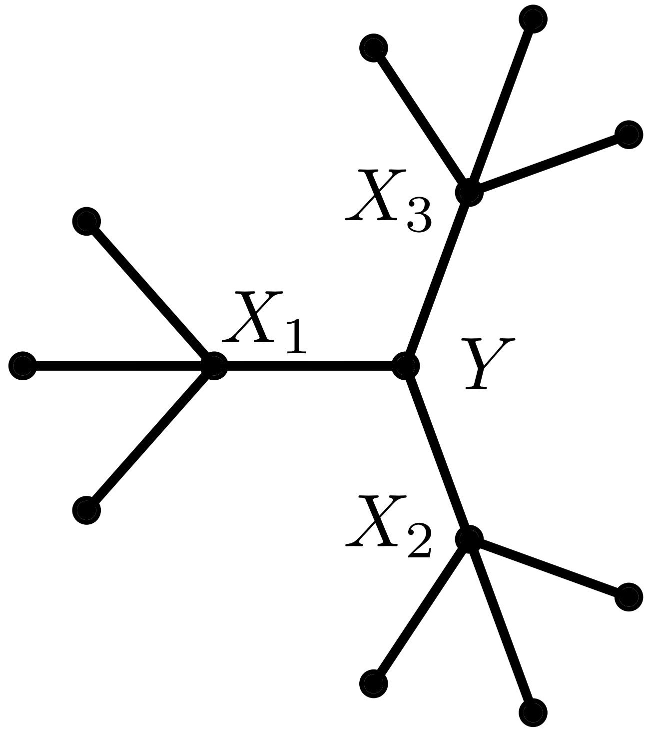
- Assume T_{13} admits an L-shaped embedding in P_{13}

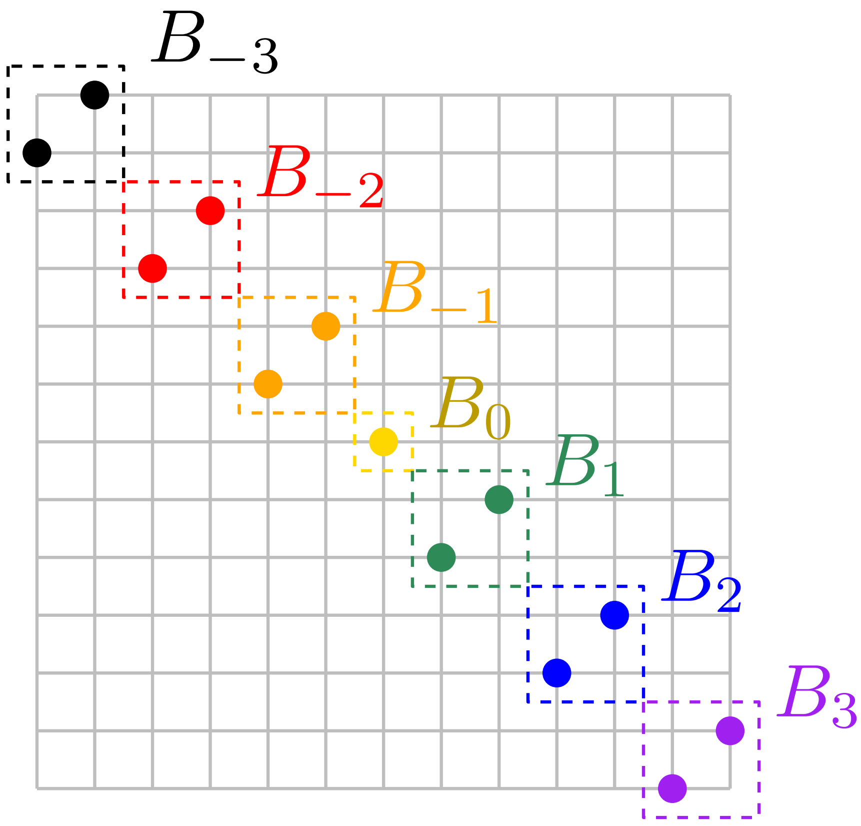




Proof of Theorem 2

- Assume T_{13} admits an L-shaped embedding in P_{13}
- T_{13} and P_{13} have symmetries



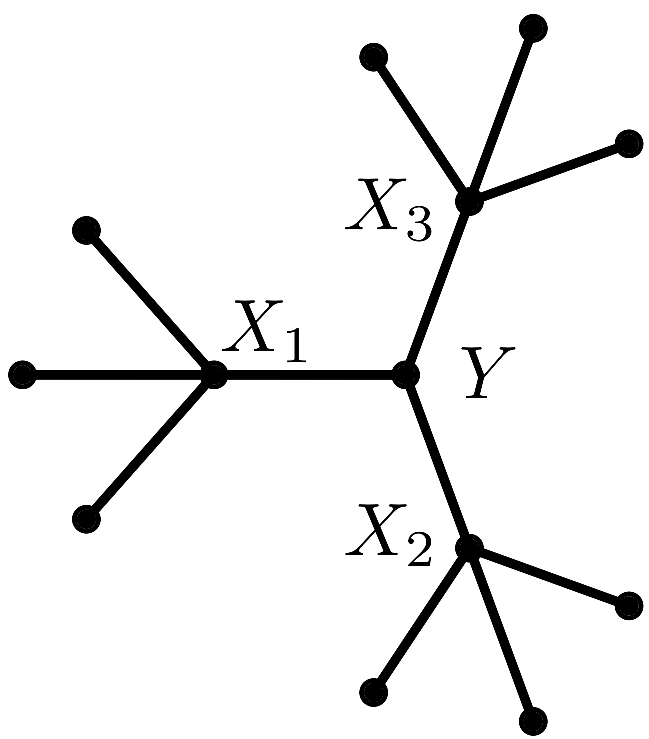


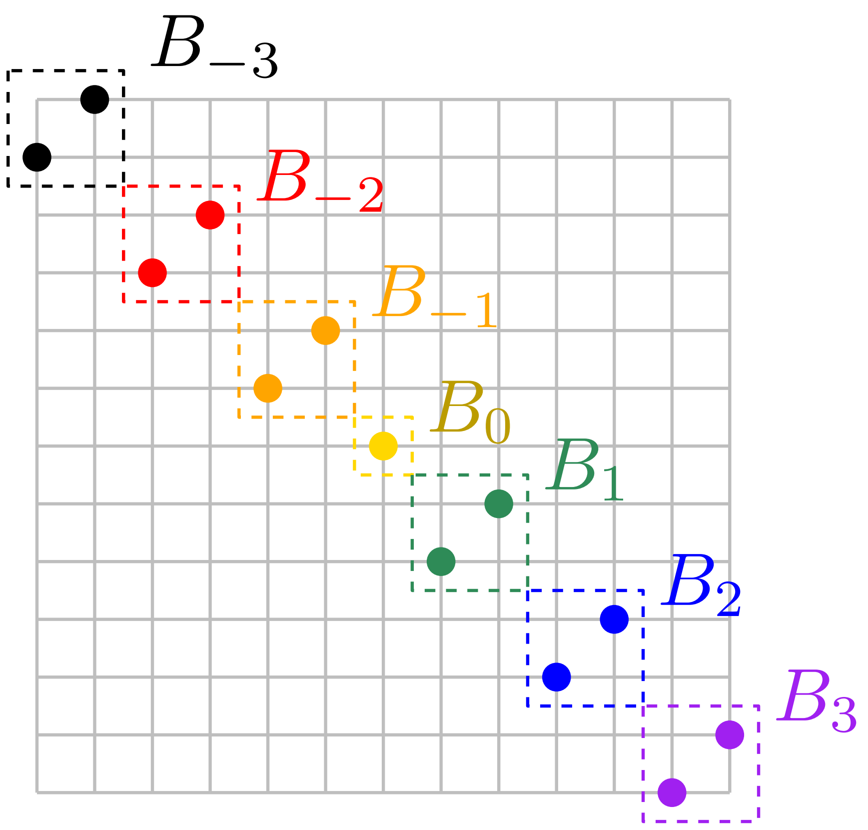
Proof of Theorem 2

- neither of X_1, X_2, X_3, Y is mapped to $B_{\pm 3}$

degree 4 vertices

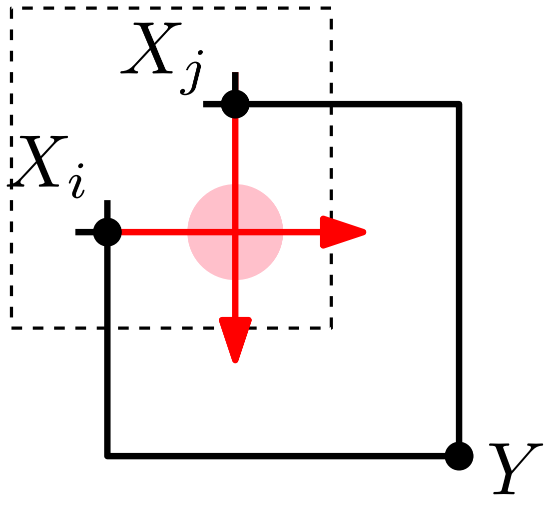
boundary points

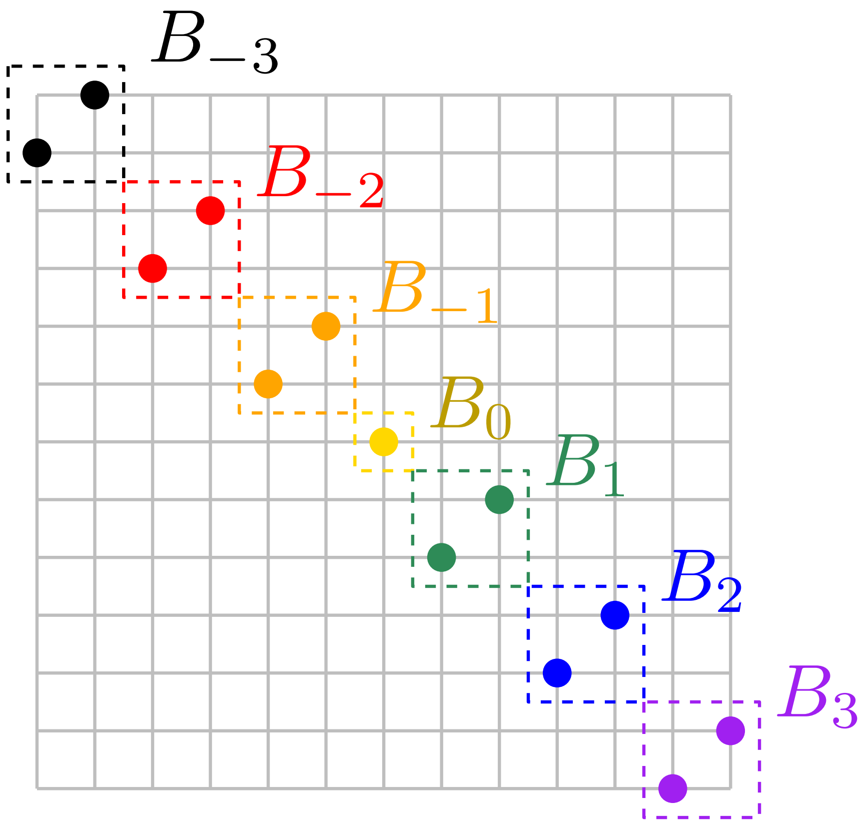




Proof of Theorem 2

- neither of X_1, X_2, X_3, Y is mapped to $B_{\pm 3}$
- each X_i is mapped to a distinct block





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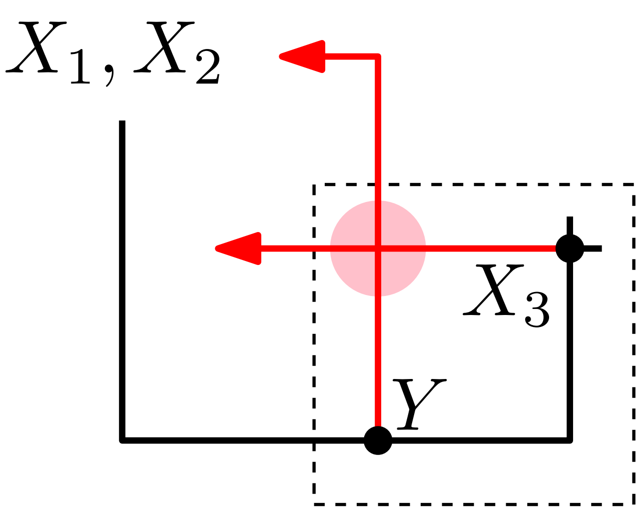

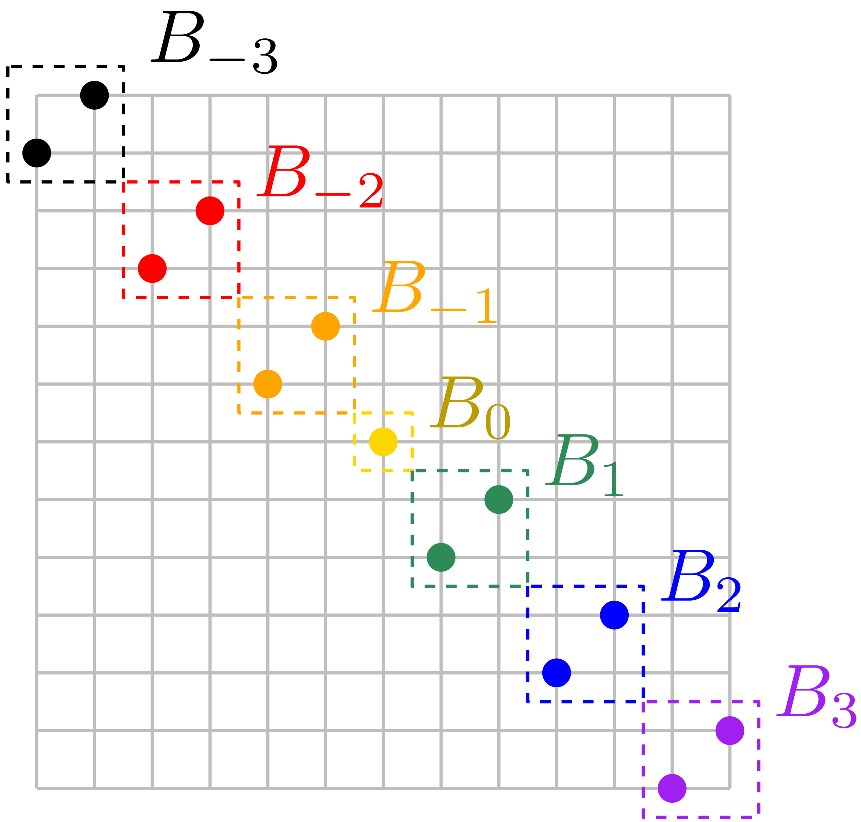

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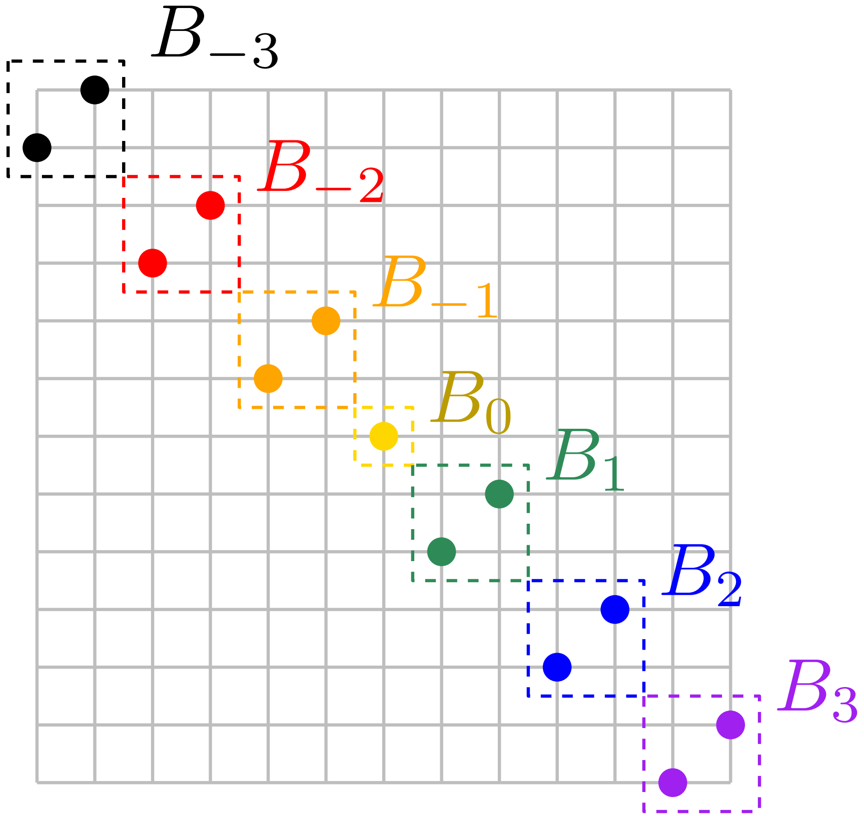

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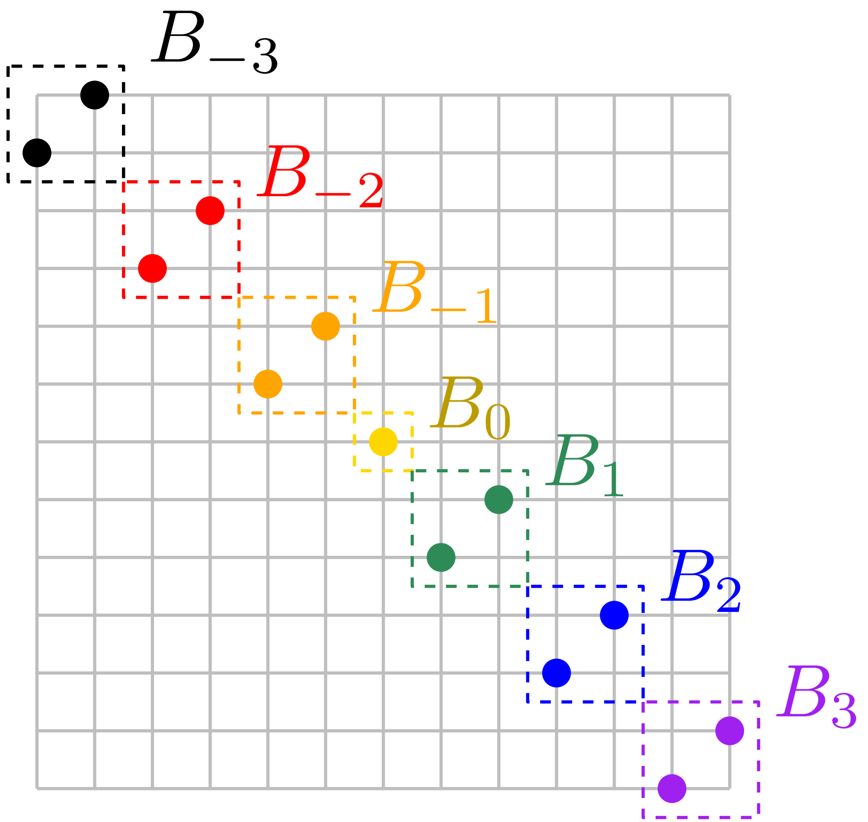
(w.l.o.g.)



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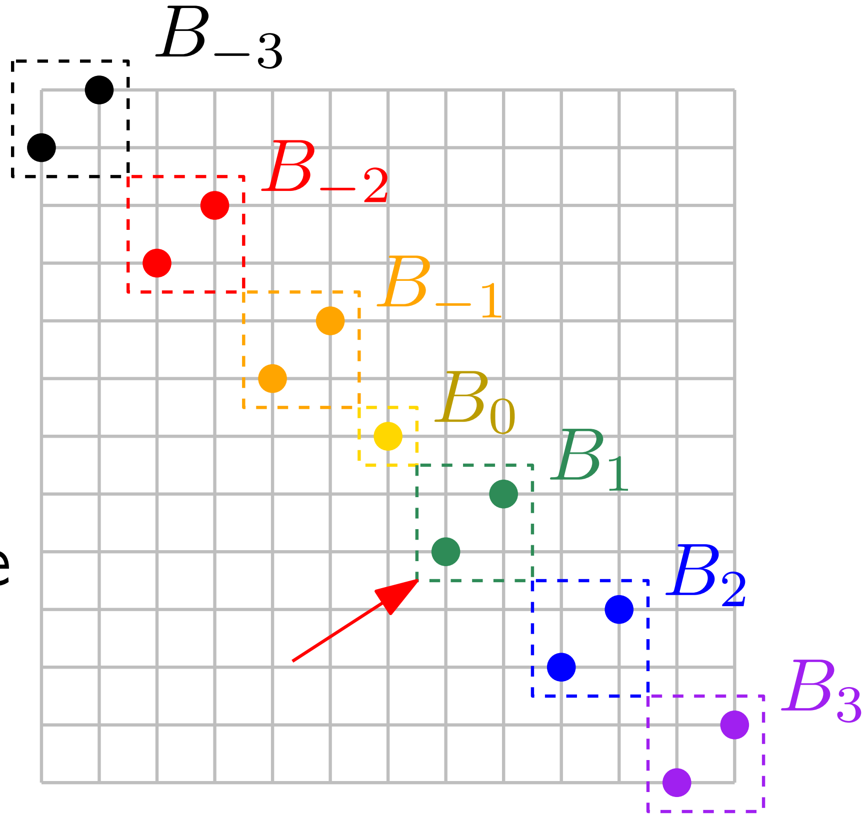
- By symmetry, we may assume they are mapped to B_1



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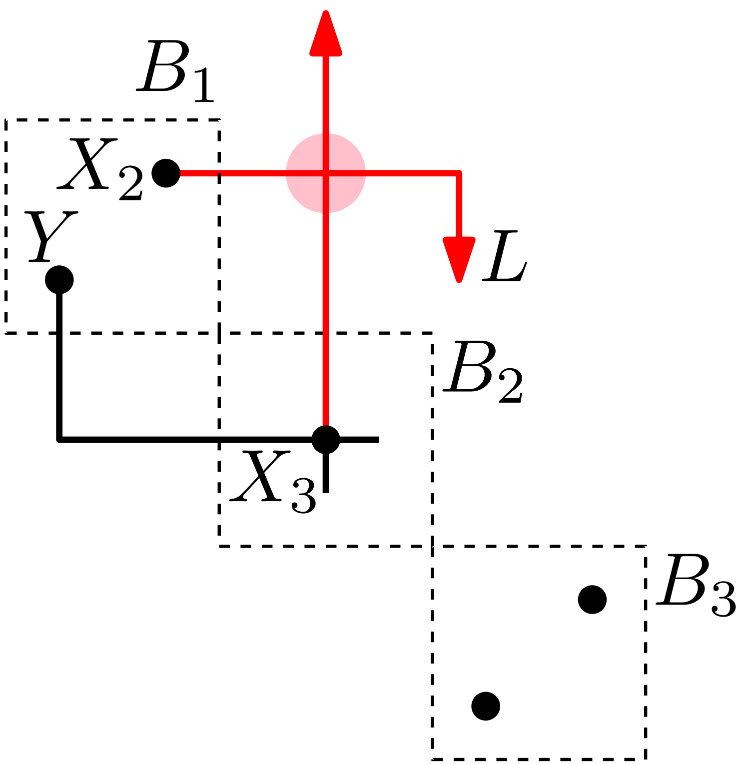

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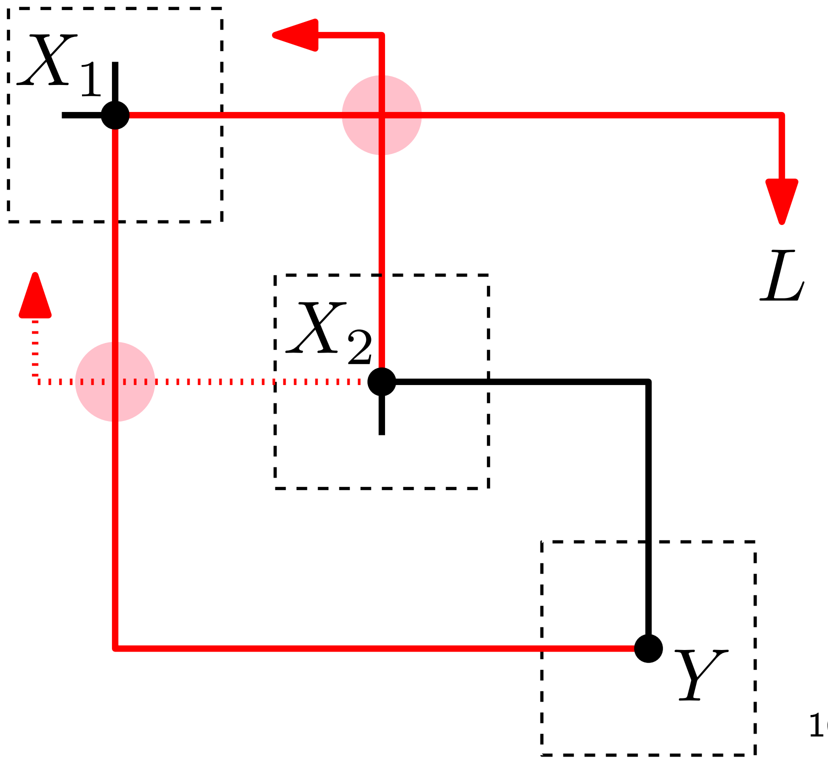
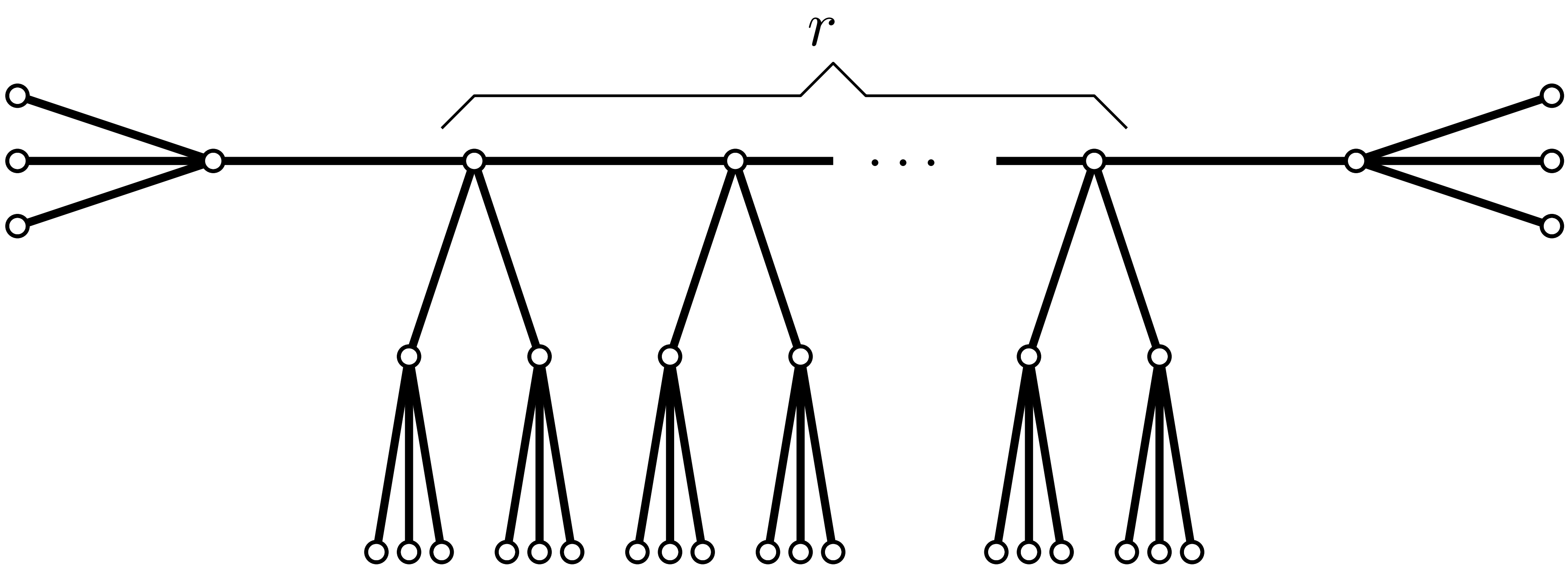
- X_1, X_2, X_3 from left to right
- Case 2: Y and X_2 mapped to distinct blocks

q.e.d.



Lower Bound: Ordered Trees

- $\exists$ infinite family which do not always admit an L-shaped embedding if **cyclic order** around each vertex is **fixed.**



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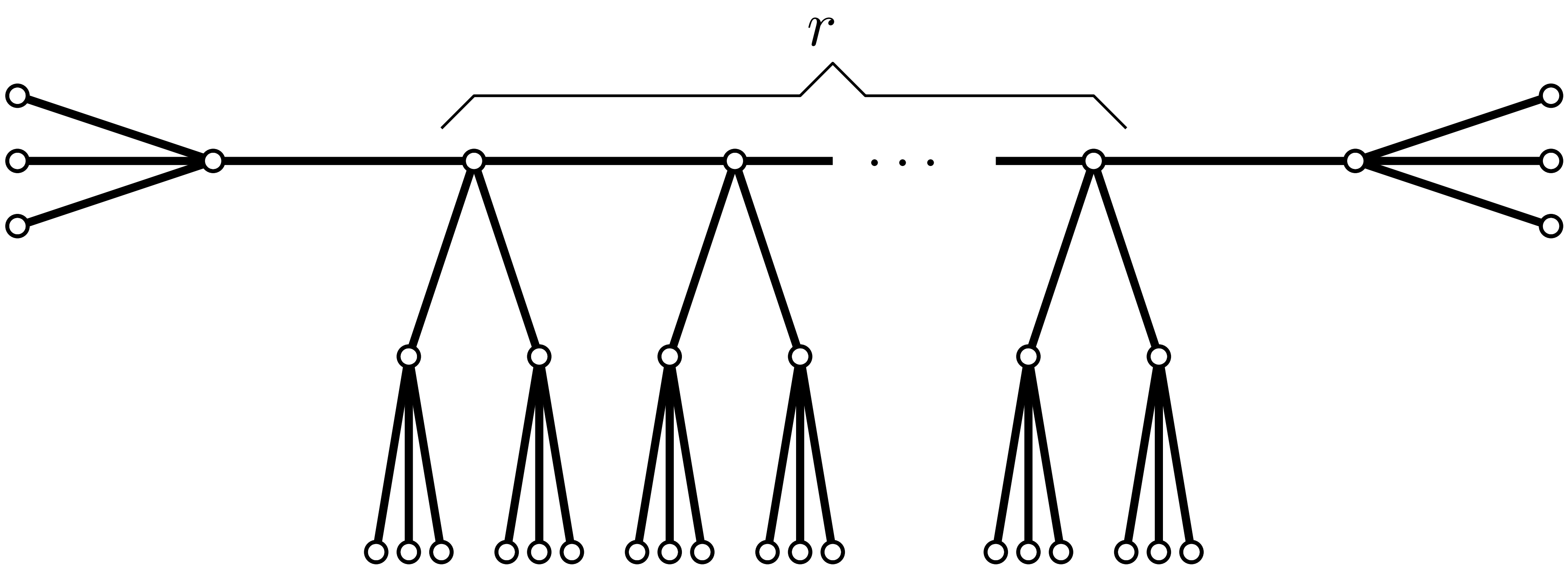


- Conjecture: non-embedable also in the original setting.

Further Results

- Randomly chosen point sets
- Randomly chosen trees

Point Sets Chosen Uniformly at Random

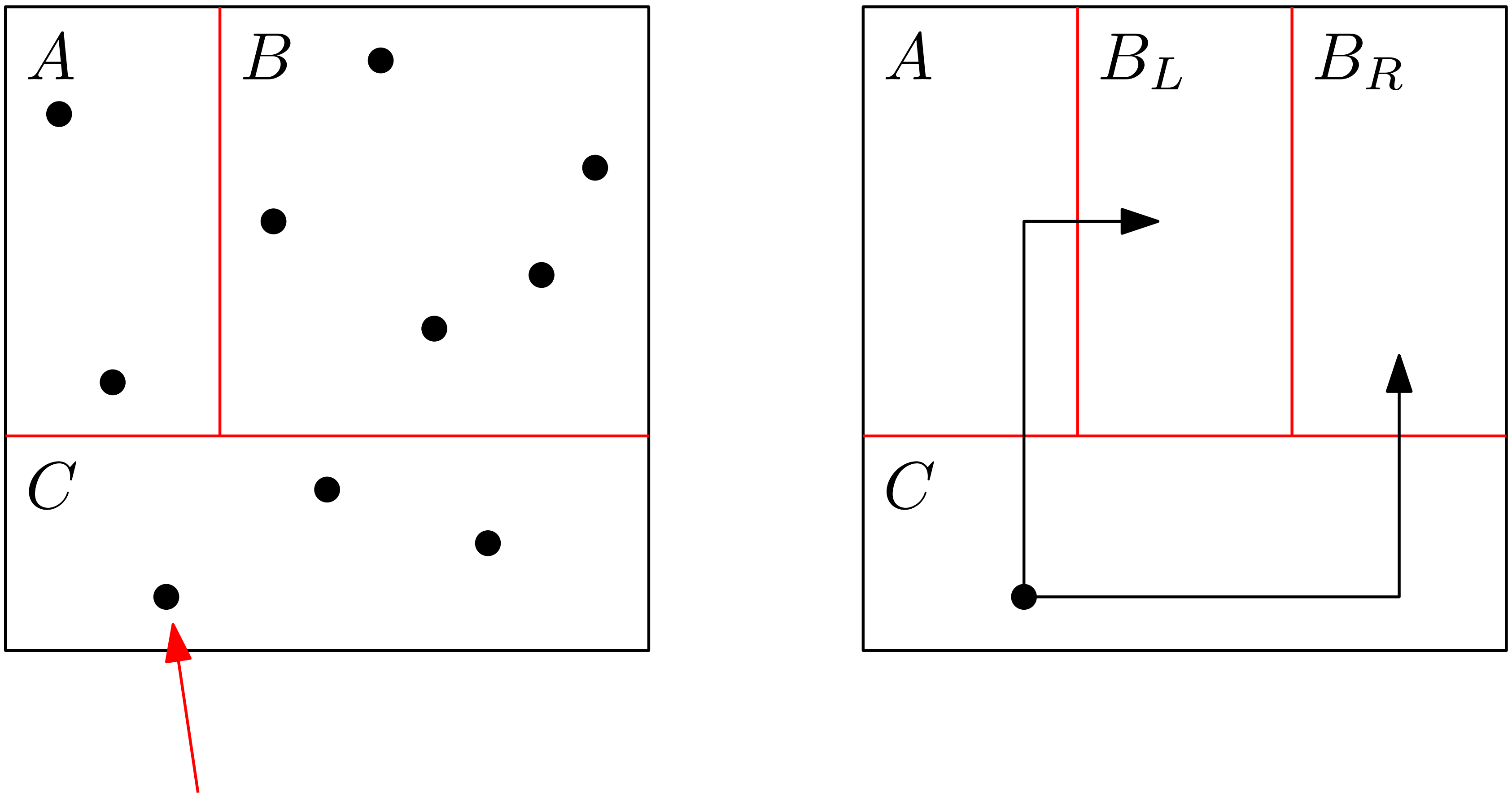
$f^{1/2}(T)$ ... minimum number m s.t. tree T admits a planar L-shaped embedding in at least half of all sets of m points

$$f_d^{1/2}(n) := \max_{\substack{T\text{ : tree on } n \text{ vertices} \\ \text{max. deg. } \Delta(T) \leq d}} f^{1/2}(T)$$

Theorem: $f_3^{1/2}(n) = O(n \log n (\log \log n)^2)$

Theorem: $f_4^{1/2}(n) = O(n^{1.333})$

Proof Idea



point exists with probability $1 - (1 - |A|/|P|)^{|C|}$

Proof Idea

- $|P| = \alpha n \log_2 n$
- $|A| := |C| := \alpha \frac{n}{2}$ and $|B| := 2\alpha \frac{n}{2} \log_2 \frac{n}{2}$
- Point exists with probability at least

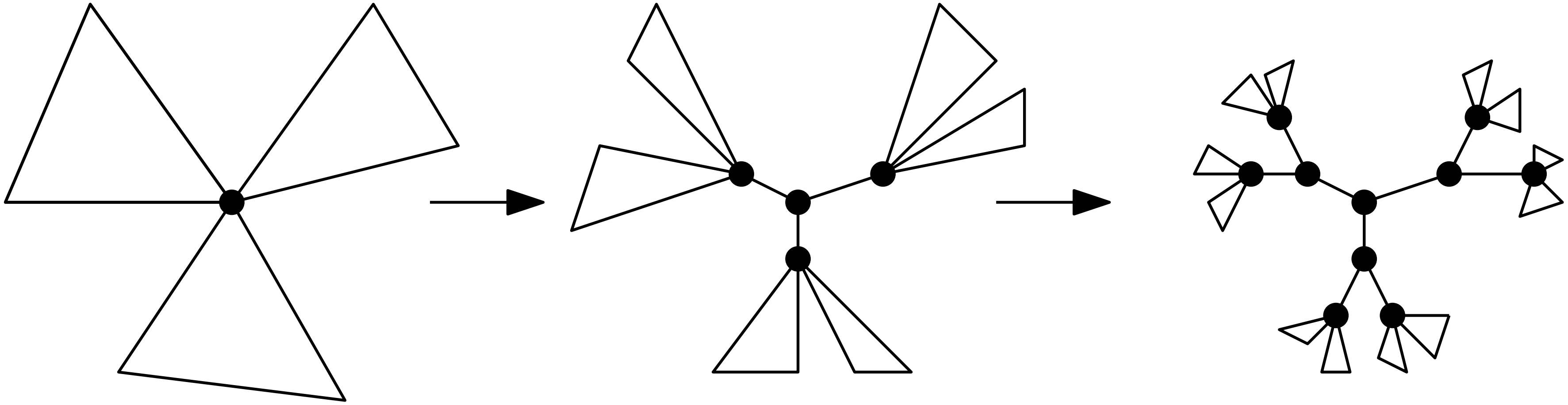
$$1 - \left(1 - \frac{1}{2\log_2 n}\right)^{\alpha \frac{n}{2}} \geq 1 - \left(\frac{1}{e}\right)^{\alpha \frac{n}{4 \log_2 n}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha \ln 2}{2}}$$

  since $(1 - \frac{1}{x})^x \leq \frac{1}{e}$ on $[1, \infty)$ and $\frac{x}{\log x} \geq e$ on $(1, \infty)$
- $\ldots$
- $O(n \log^2 n)$, success with probability at least $\frac{1}{2}$
- Actually, $O(n \log n (\log \log n)^2)$ possible

Proof Idea

Recursion works fine with balanced trees ...

... but problem with inbalanced trees!!

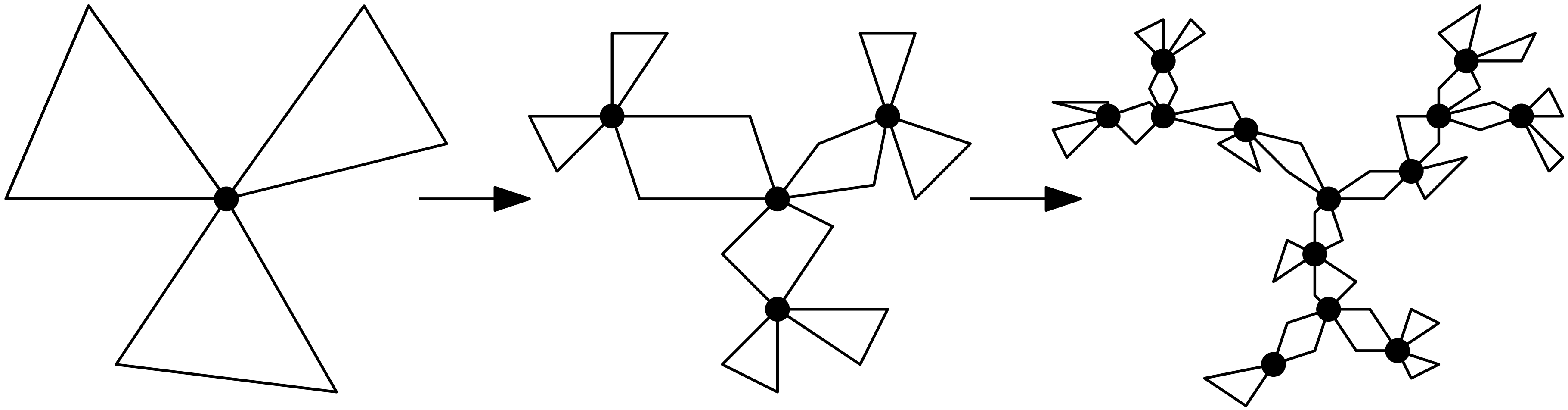


recursion layer 1   recursion layer 2   recursion layer 3

Proof Idea

Solution (finalizing the idea): $1/2$-separators!

... and plugging things together



recursion layer 1 recursion layer 2 recursion layer 3

(Rooted) Trees Chosen Uniformly at Random

T_n . . . set of all rooted trees with n vertices and $\Delta \leq 4$

Consider T_n as random variable
(every tree in T_n is equally likely)

Theorem: Let $\varepsilon > 0$. Then the probability
$\mathbb{P}[f(T_n) = O(n^{1.5+\varepsilon})]$ is at least $p = \frac{2\varepsilon}{1+2\varepsilon} > 0$.

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Proof Outline:

- *register function* ρ (upper bound of σ)

$$\rho_r(v) := \max\{0, \rho_r(u_1), \rho_r(u_2) + 1, \ldots, \rho_r(u_k) + k - 1\}$$

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Proof Outline:

- *register function* ρ (upper bound of σ) has expected value $\mathbb{E}(\rho(T_n)) = {}^1\!/_2 \log_2 n + O(1)$ [Drmota and Prodinger, 2006]
- Follows from $f(T) \leq 2^{\sigma(T)} n$ and Markov's inequality

Thank you

for your

attention!