A brief introduction to Combinatorial Geometry

Manfred Scheucher

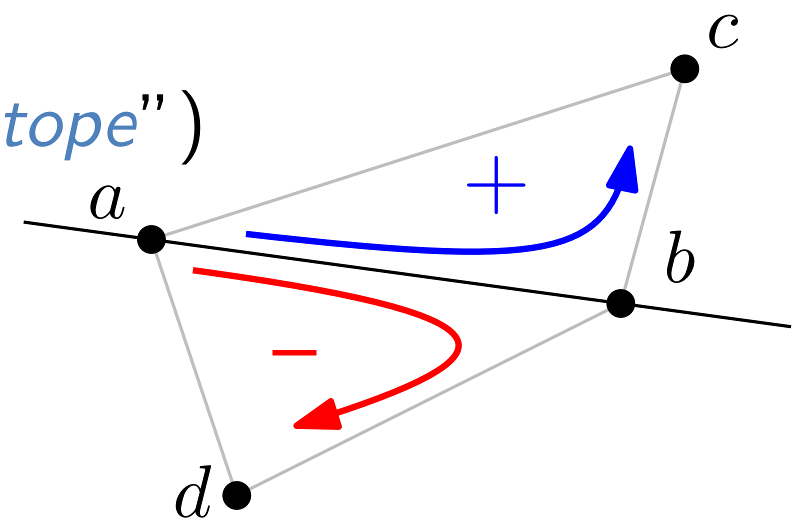
Point Configurations

$P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$, $p_i = (x_i, y_i)$

induces triple-orientations ("*chirotope*")

$\chi : [n]^3 \to \{+, 0, -\}$

"positive, collinear, negative"



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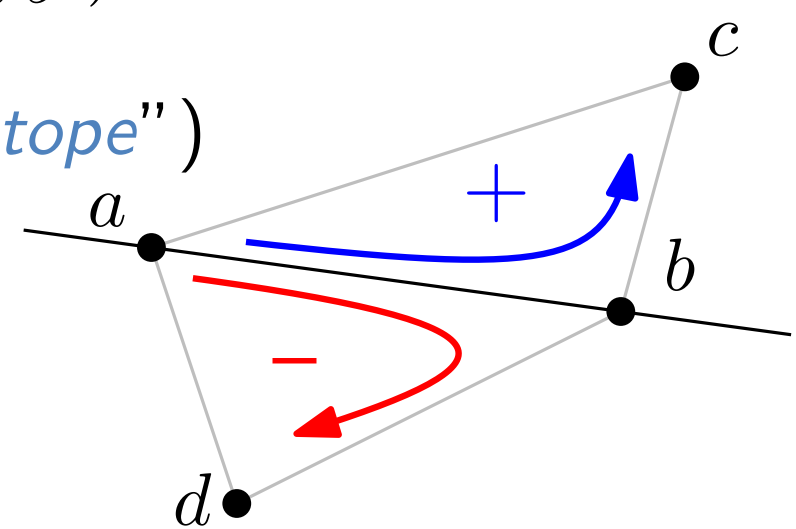

formally:

$$\chi(a,b,c) := \operatorname{sgn}\det \begin{pmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{pmatrix}$$

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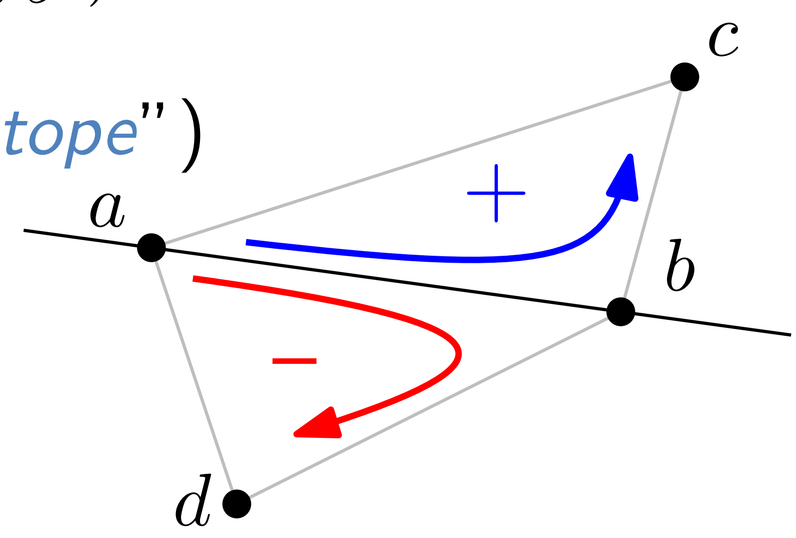

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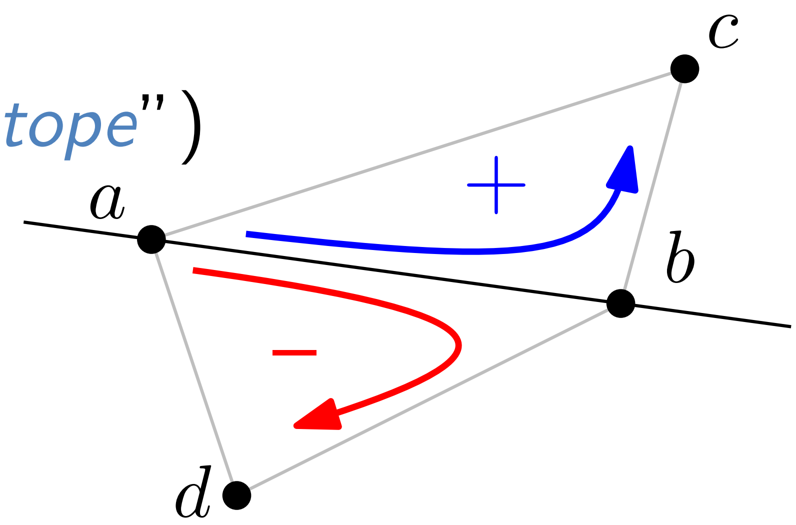

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2.) *exchange axioms*:

$$\left.\begin{array}{l} \chi(dbc) \cdot \chi(aef) > 0 \\ \chi(ebc) \cdot \chi(daf) > 0 \\ \chi(fbc) \cdot \chi(dea) > 0 \end{array}\right\} \Rightarrow \chi(abc) \cdot \chi(def) > 0$$

Point Configurations

$$\det(x_1,\ldots,x_r)\cdot\det(y_1,\ldots,y_r) =$$
$$\sum_{i=1}^{r}\det(y_i,x_2,\ldots,x_r)\cdot\det(y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_r)$$

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(via Grassman-Plücker relations/Lin.Alg.)

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Point Configurations

(via Laplace expansion)

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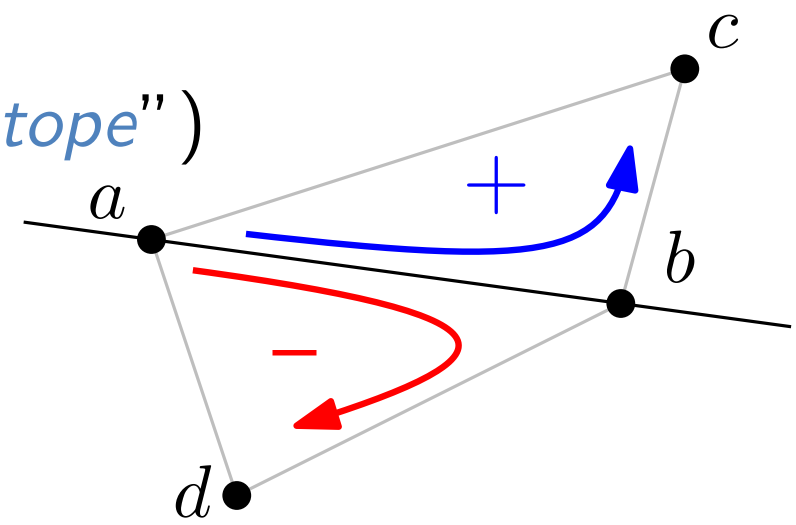

alternating + exchange axioms are necessary conditions

(problem: stretchability $\to$ "*pseudopointsets*")

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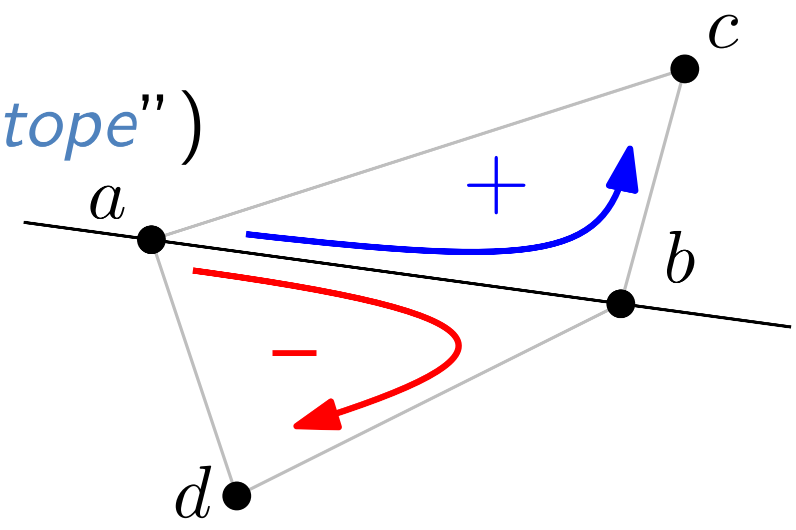

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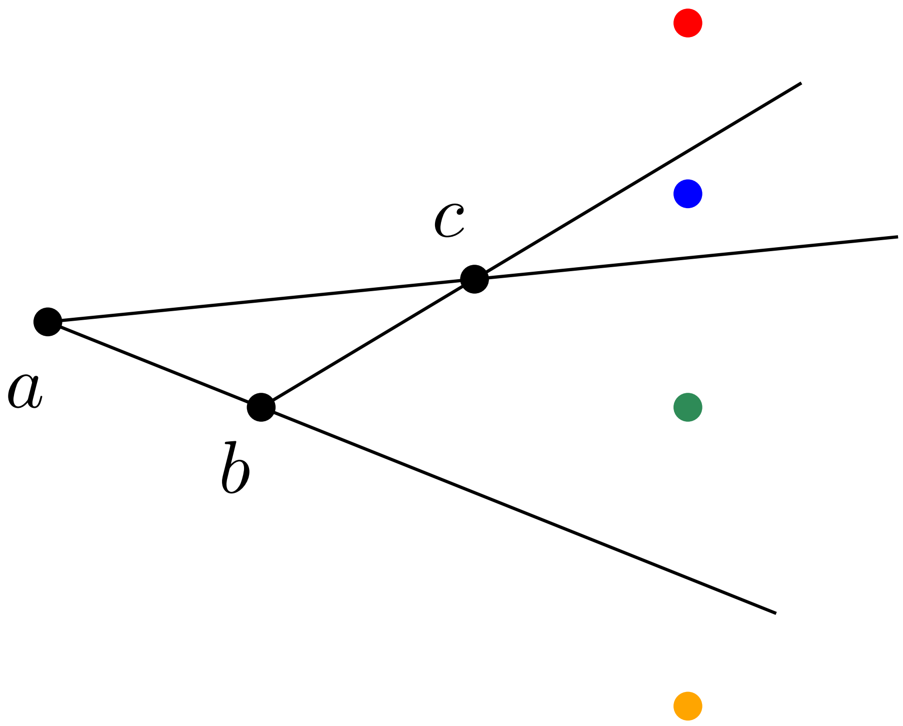
here $\Theta(n^6)$ constraints, but $\Theta(n^5)$ sufficient

Sorted Point Configurations

if point sorted left-to-right



$\forall a, b, c, d$: "$\leq$ 1 sign change"

abc	abd	acd	bcd
$+$	$+$	$+$	$+$
$+$	$+$	$+$	$-$
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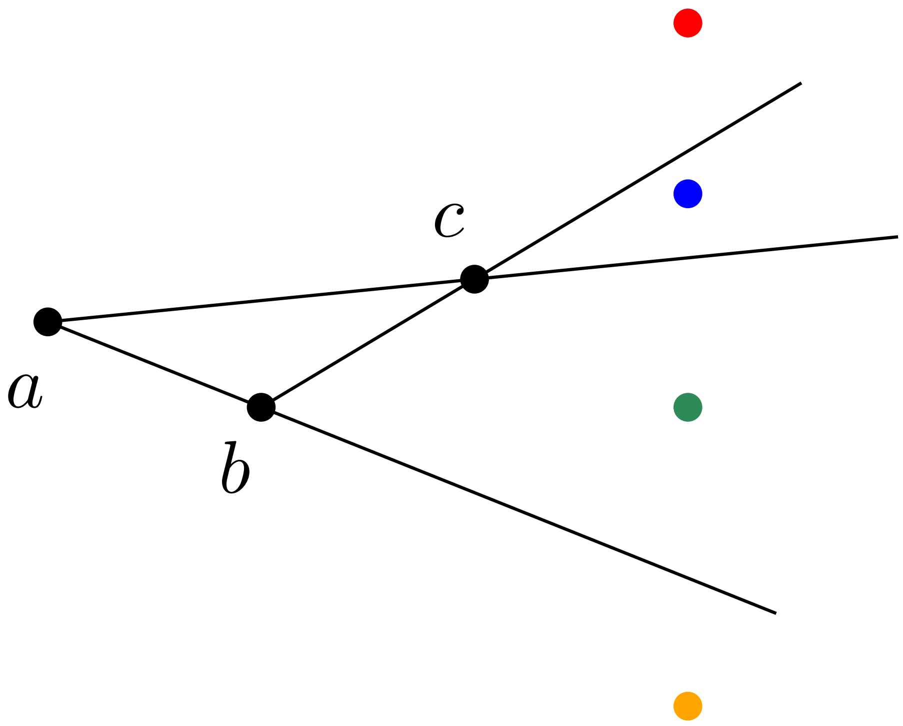

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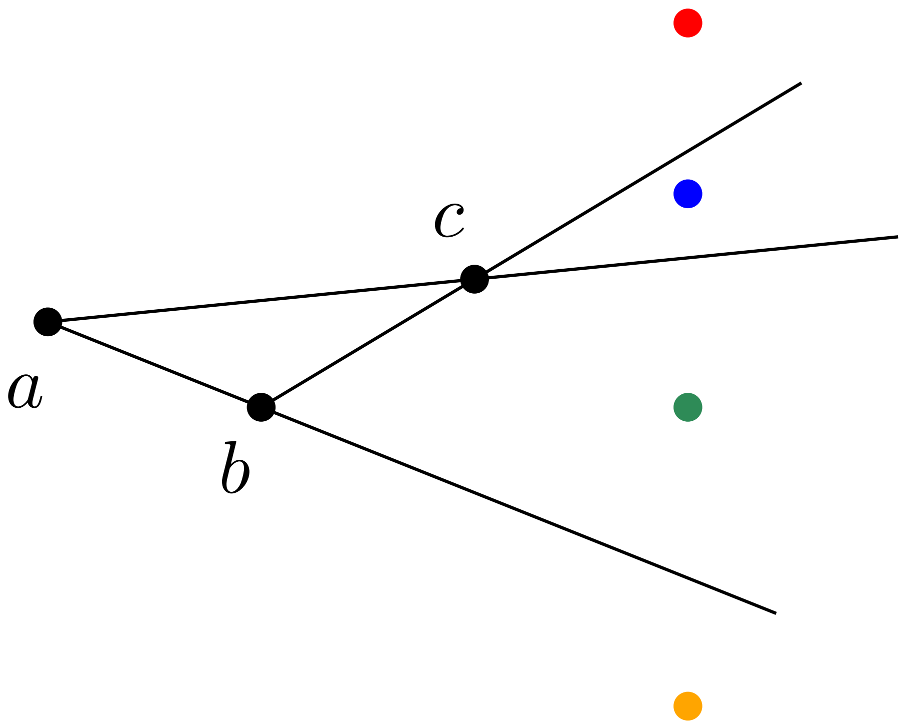

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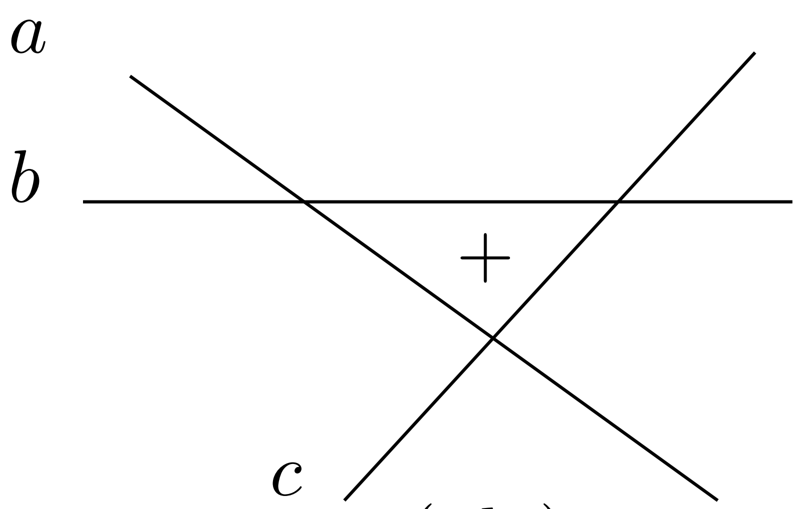
+	+	−	−
+	−	−	−
−	−	−	−
−	−	−	+
−	−	+	+
−	+	+	+

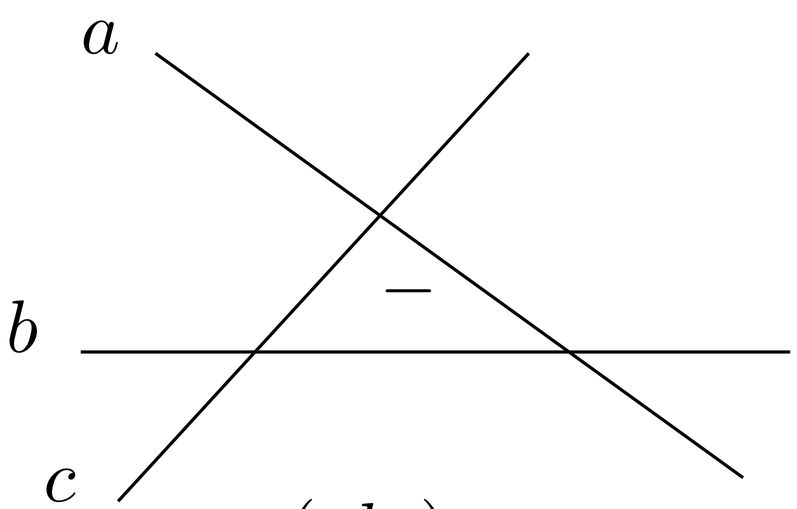
$\Theta(n^4)$ conditions ("*signotope*")

Line Arrangement

L set of lines $\ell_1, \ldots, \ell_n$ with $\ell : y = \alpha x + \beta$



$\chi(abc) = +$



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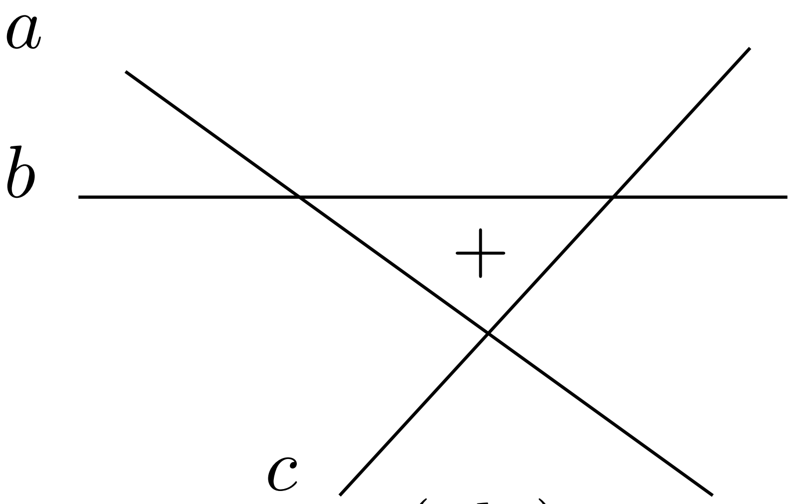

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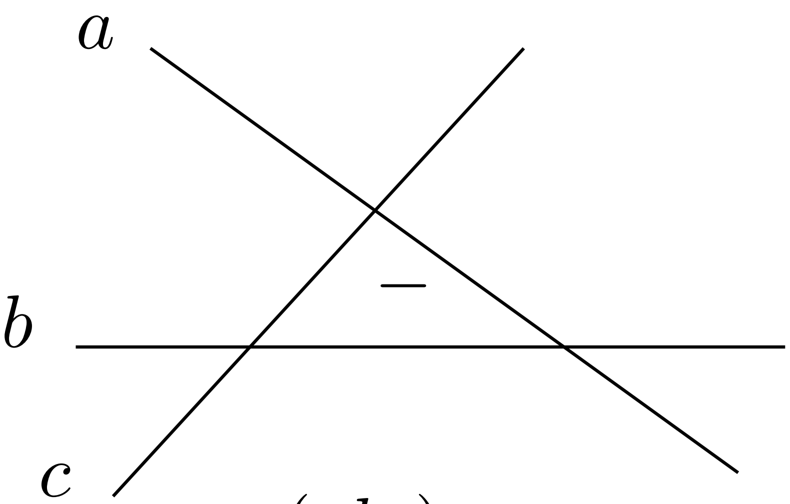

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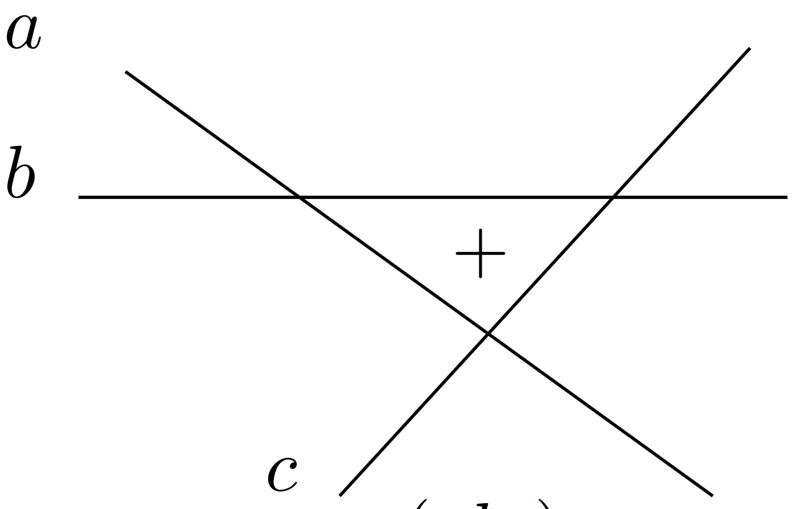

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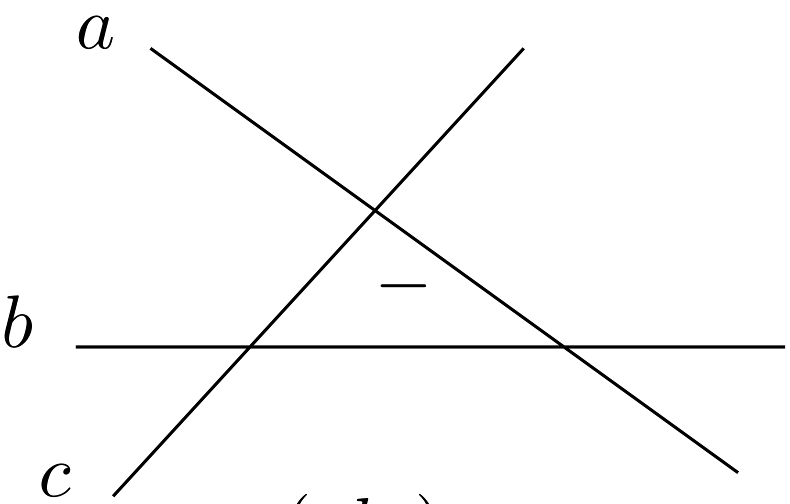

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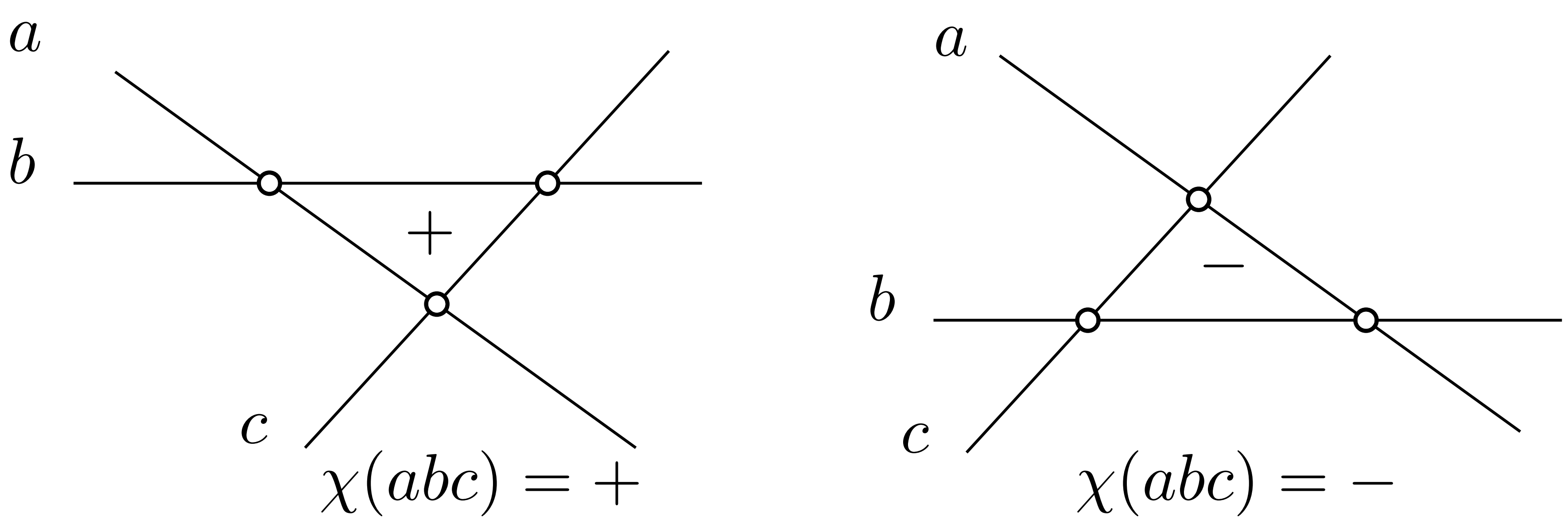

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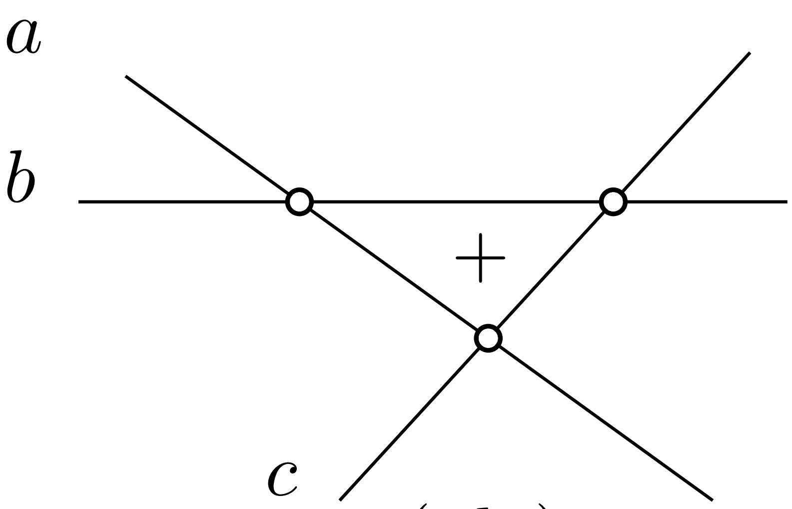

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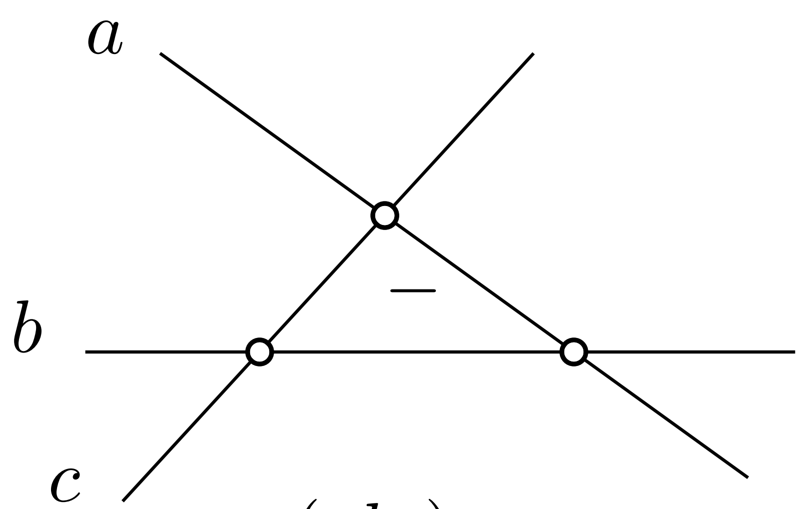

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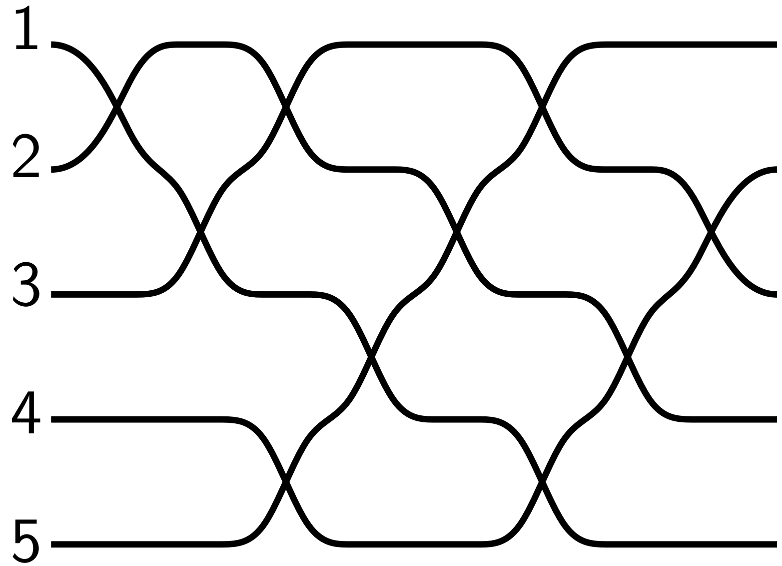
no big surprise because of point-line duality: $(\alpha, \beta) \to y = \alpha x + \beta$

Pseudoline Arrangements

Folkman–Lawrence Topological Repres. Theorem 1978:
every chirotope has a *pseudoline arrangement* representation

(x-monotone curves, ≤ 1 common intersection)



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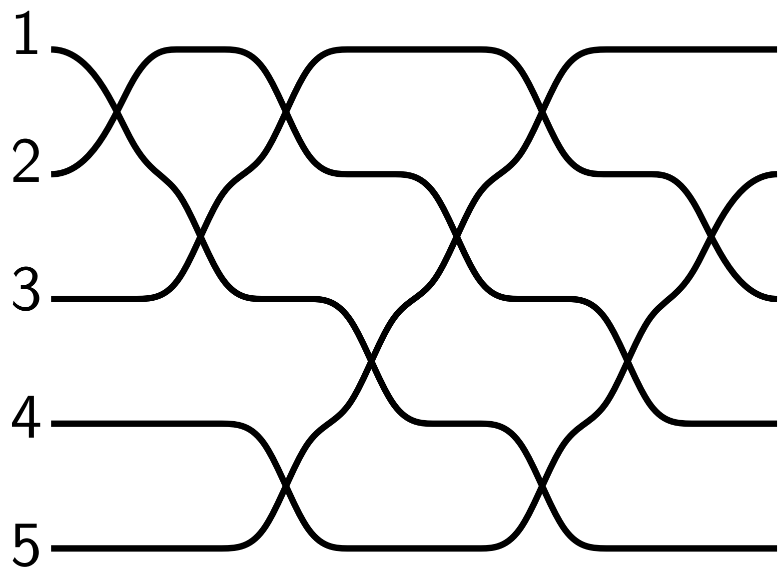

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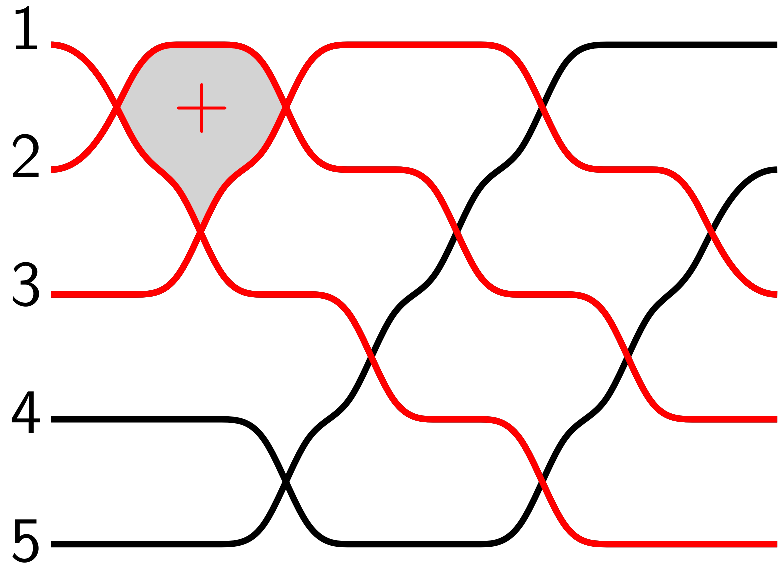

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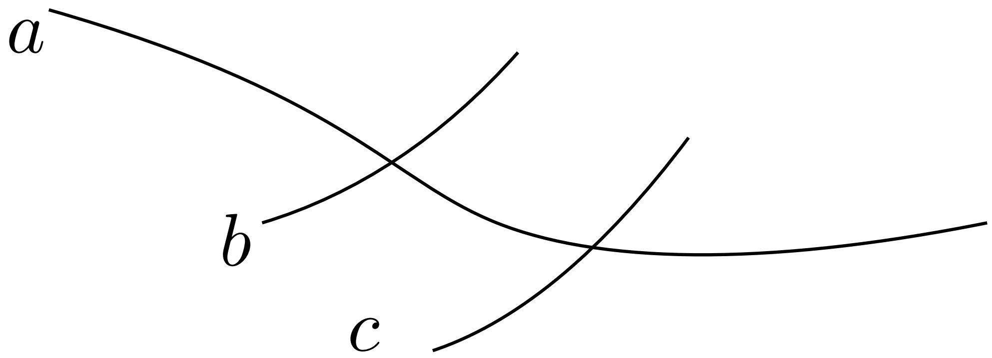
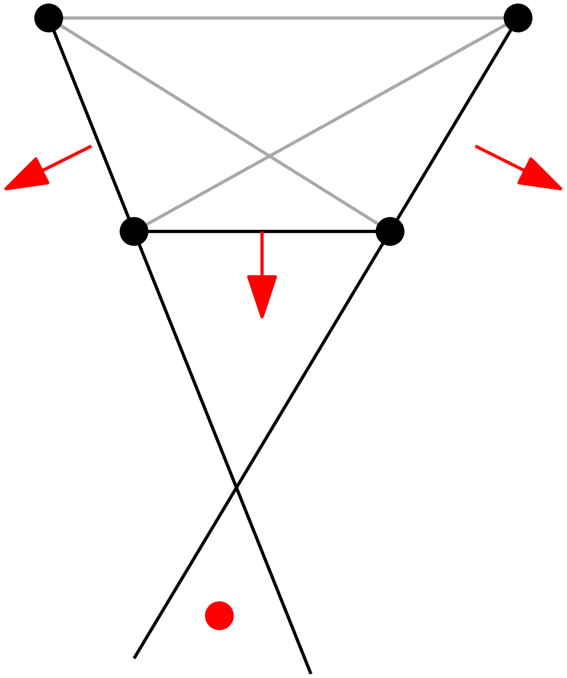
proof idea: χ determines order of intersections
($\chi(abc) = +$ iff a intersects b before c)



Pseudopoint Configurations

dual "pseudoline" setting more illustrative because of geometric problems when adding points in the primal



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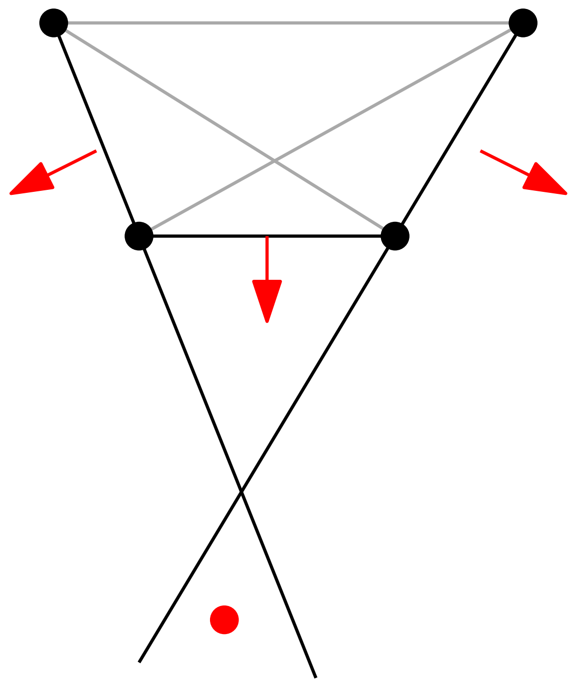

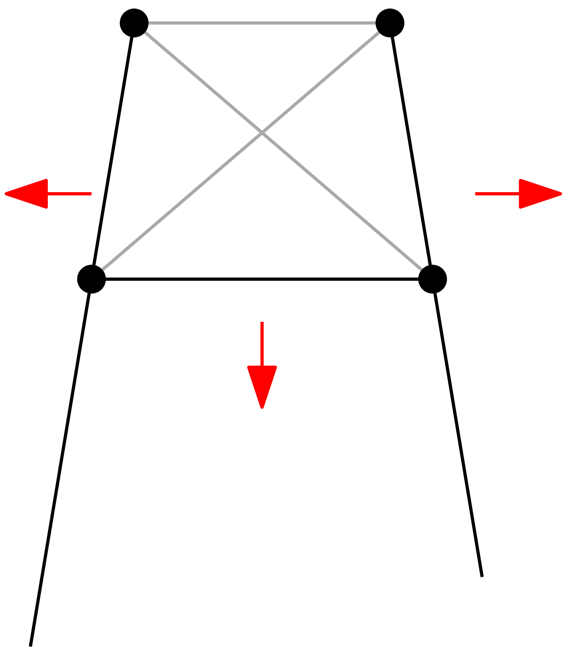

problem: no such cell!

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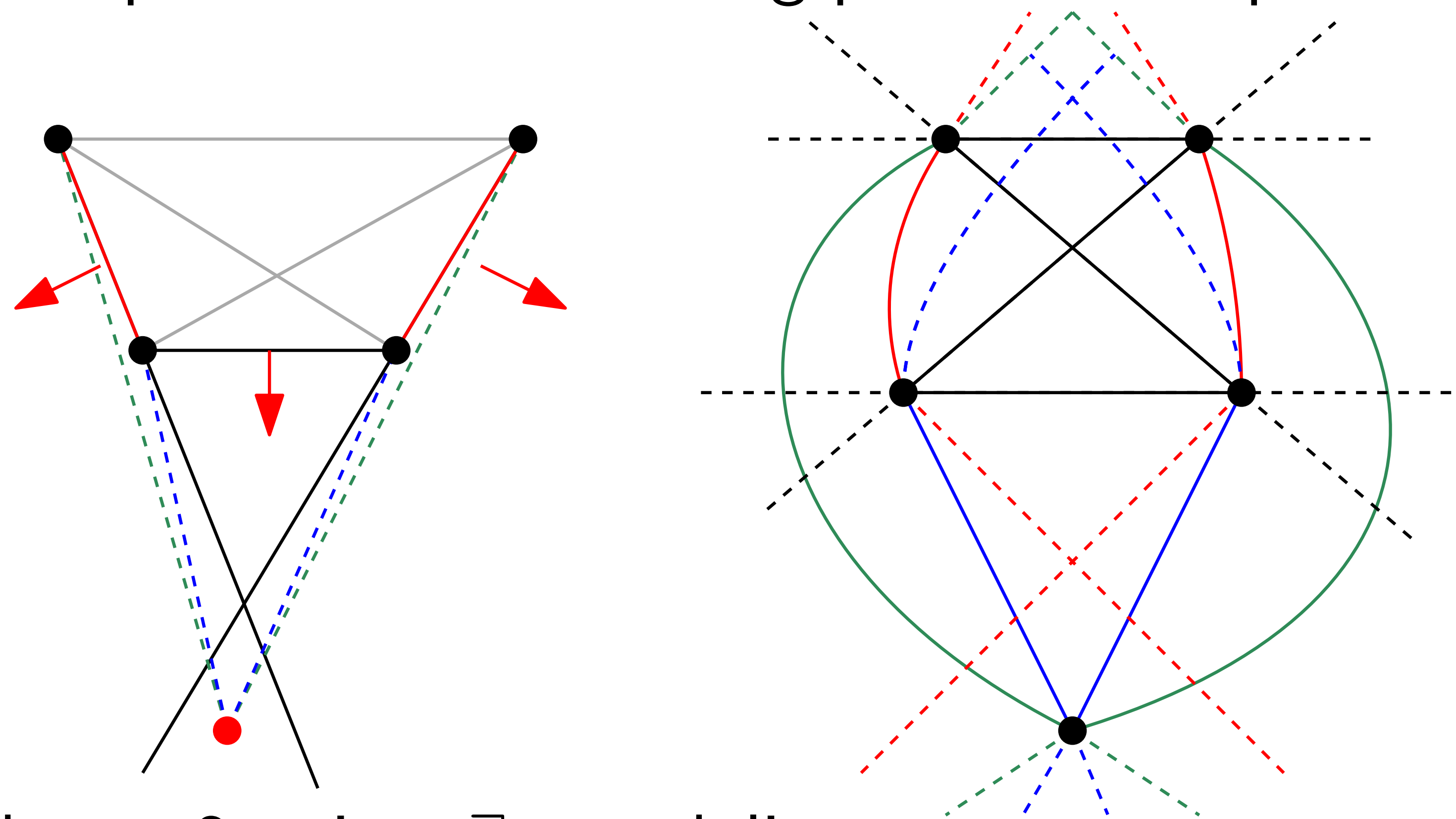

through any 2 points ∃ pseudoline

(*pseudopoint config.*, *pseudolinear drawing of K_n*)

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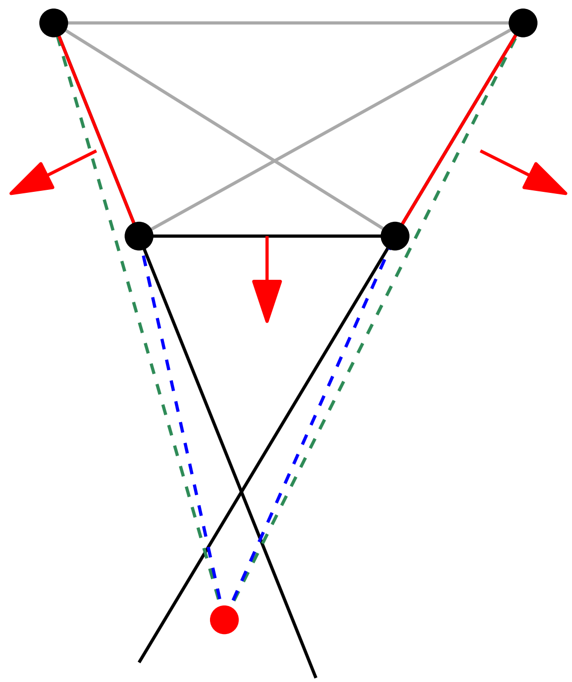

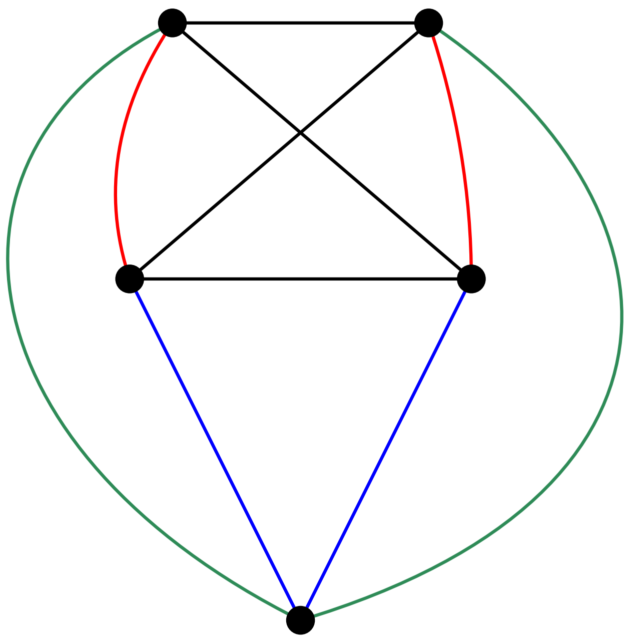

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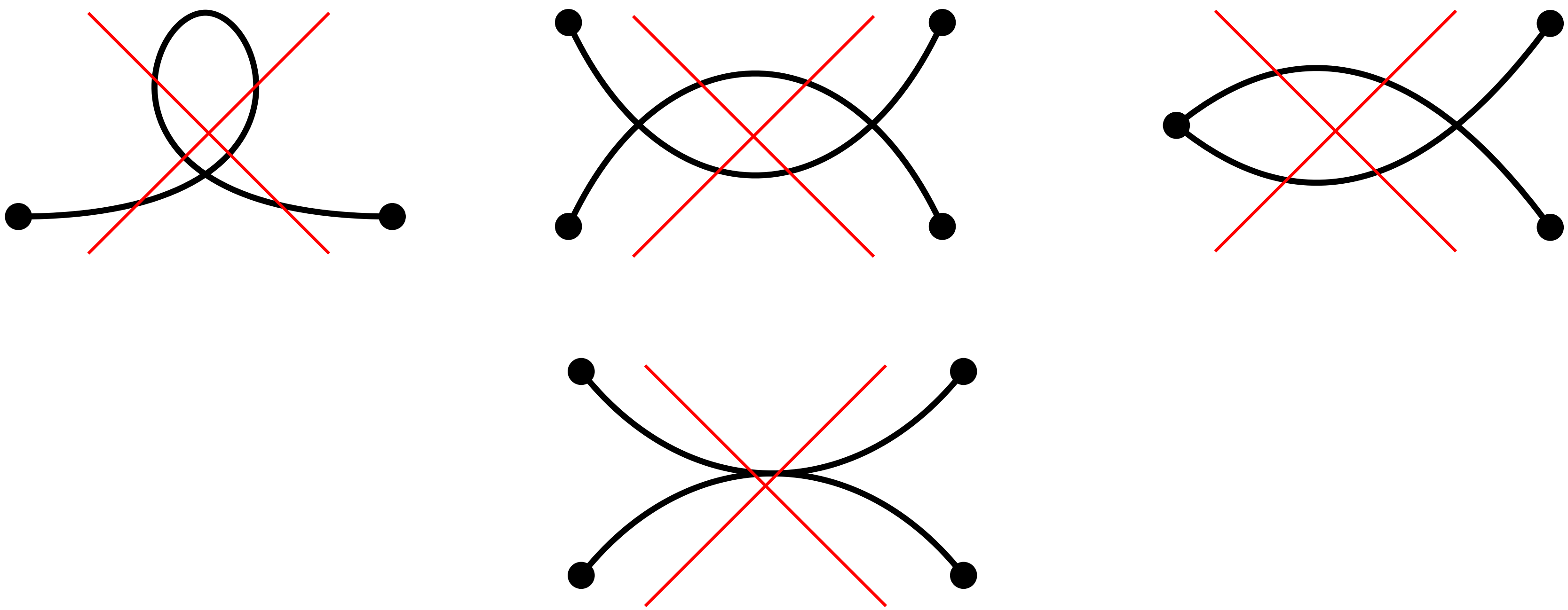
Topological Drawings of K_n

A *(simple) topological drawing* is a drawing of a complete graph without



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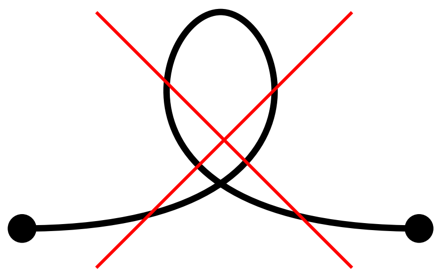

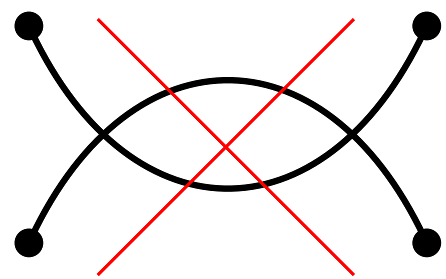

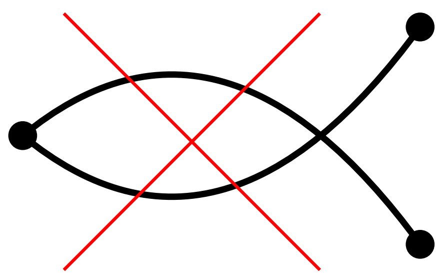

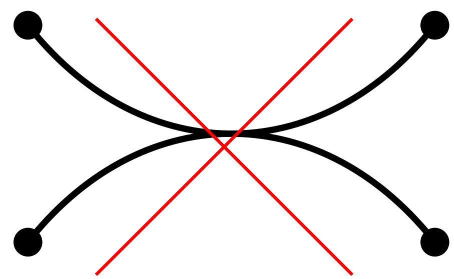

similar to straight-line and crossing-minimal drawings

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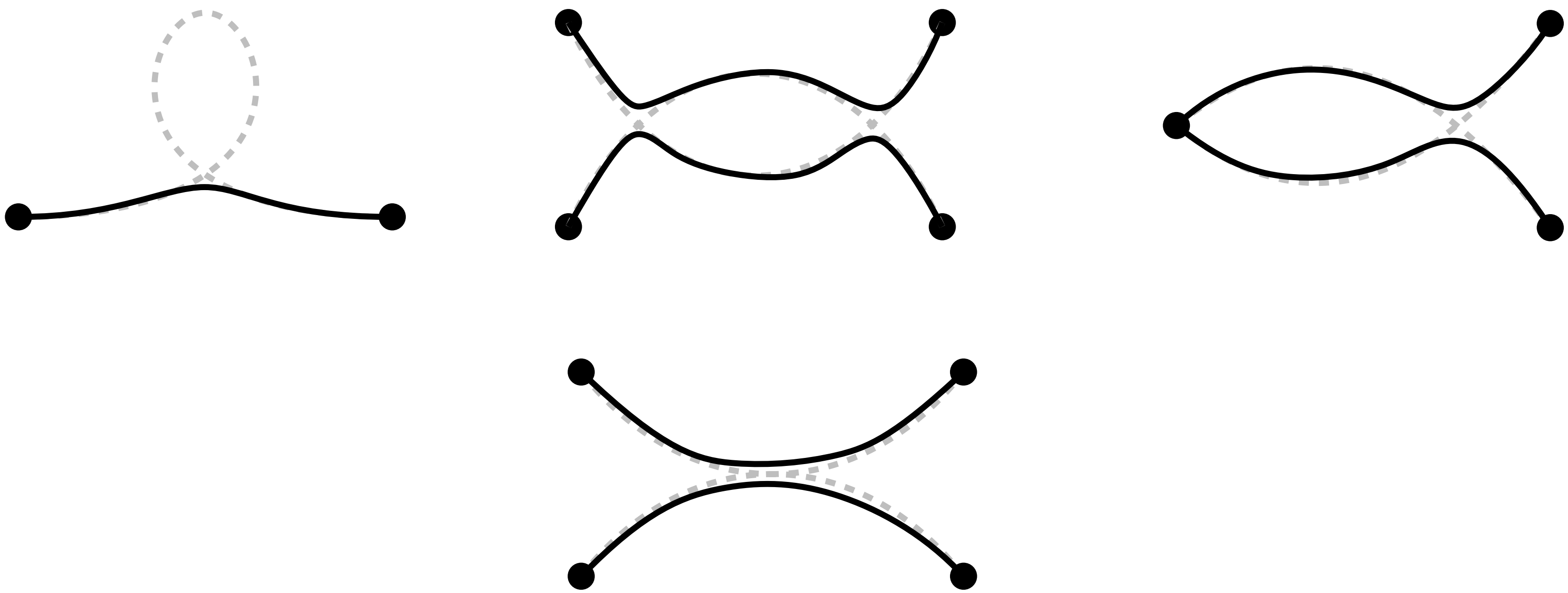

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Topological Drawings of K_n

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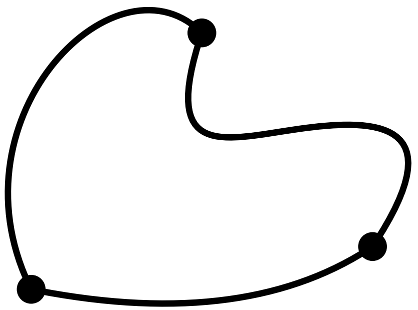

unique: triangle!

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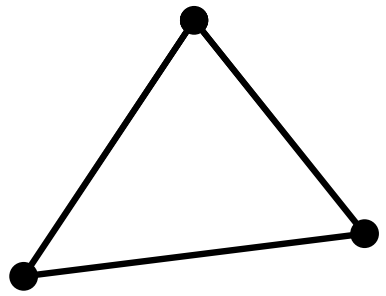

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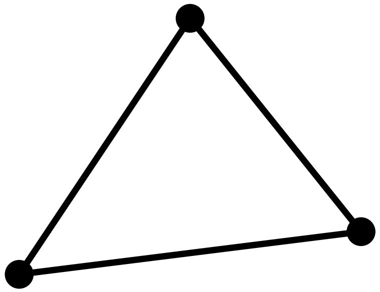

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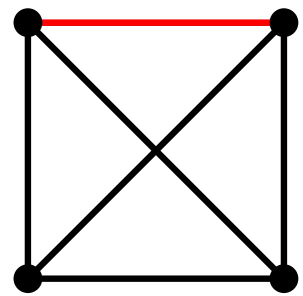
$n = 4$

(1 crossing)



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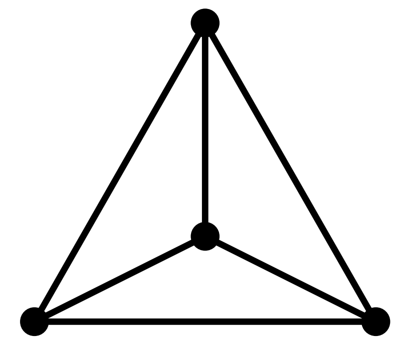

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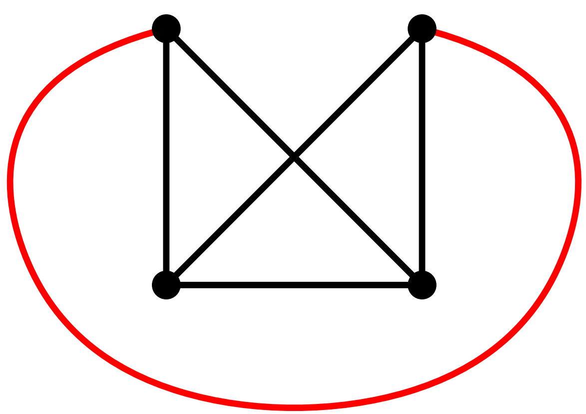

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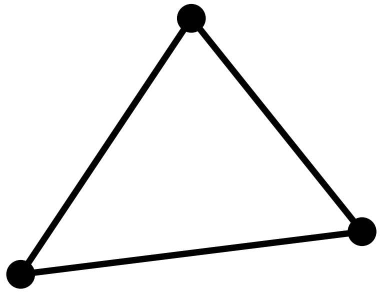

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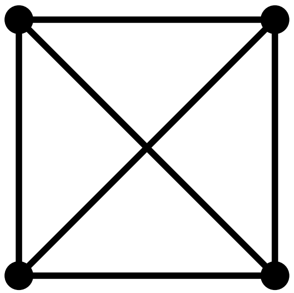

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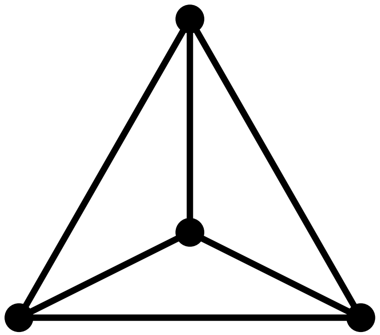
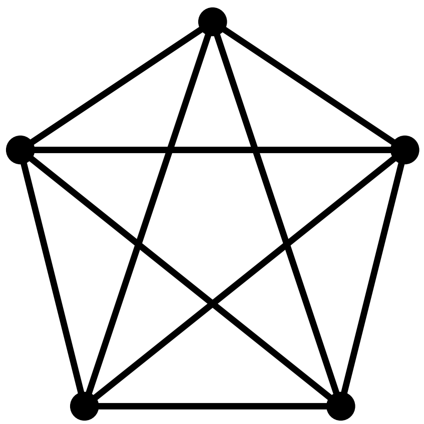
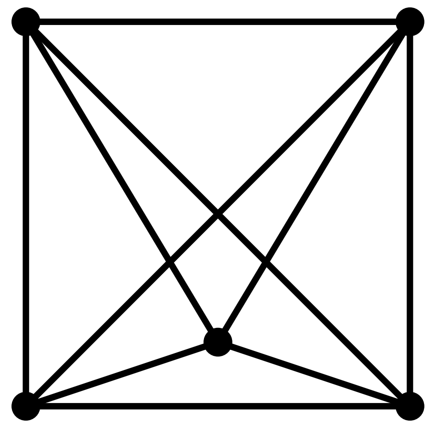


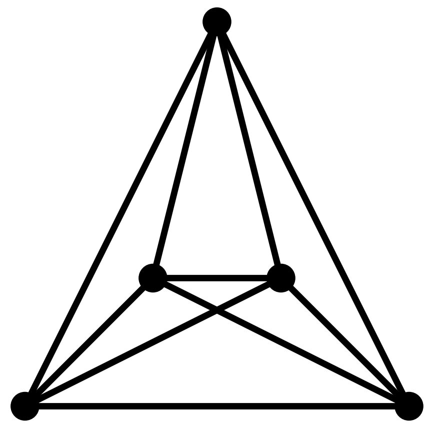
Topological Drawings of K_n

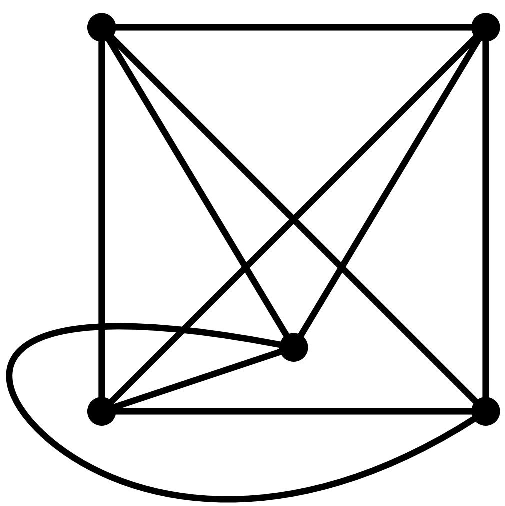
$n = 5$

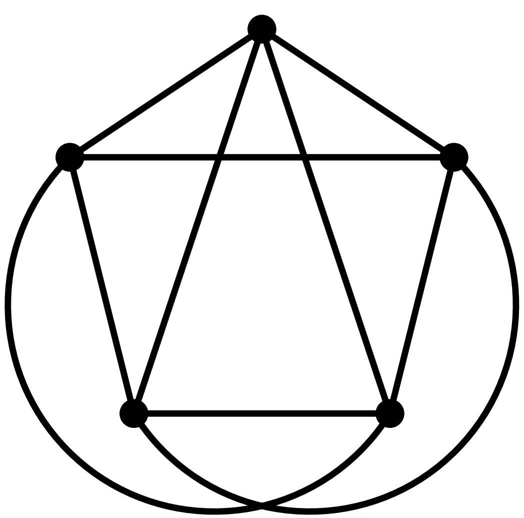
(5 crossings) (3 crossings) (1 crossing)











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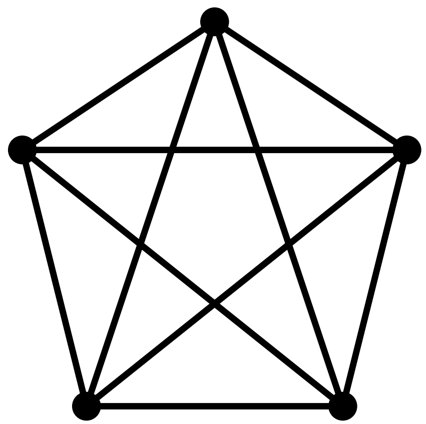
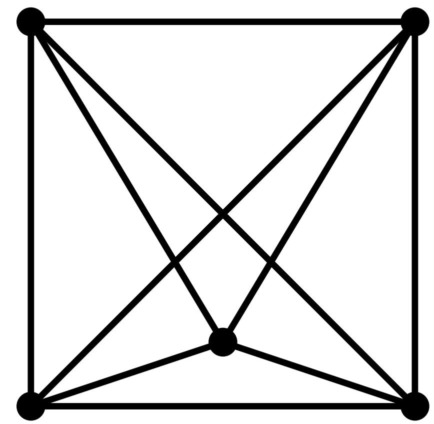
$n = 5$

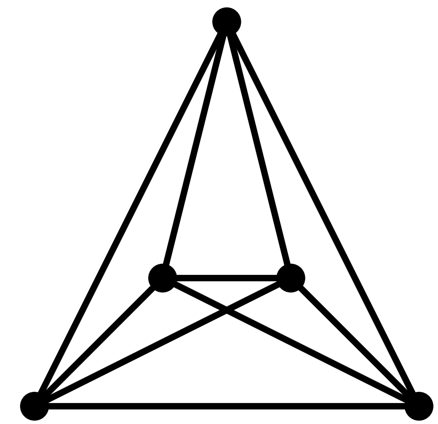
(5 crossings)



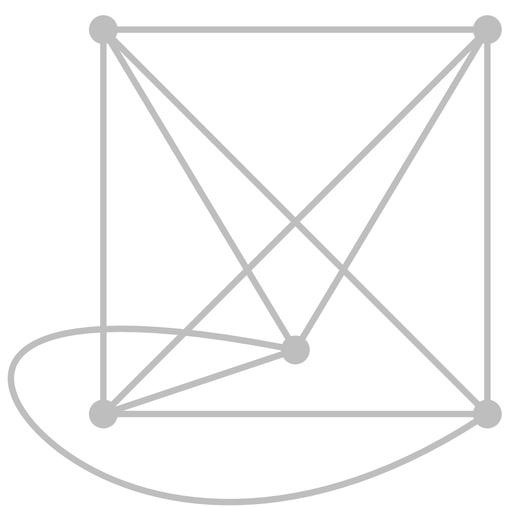
(3 crossings)

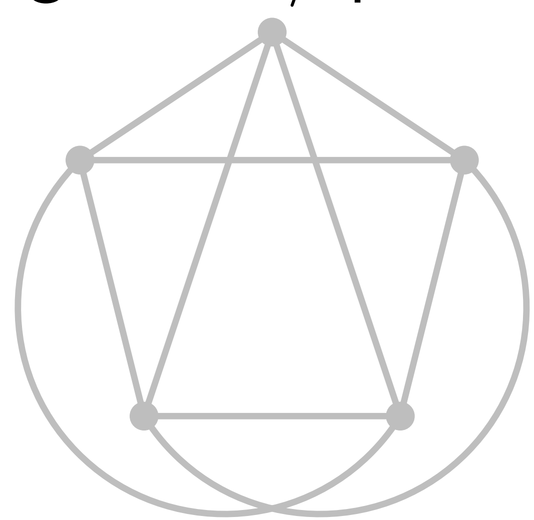


(1 crossing)



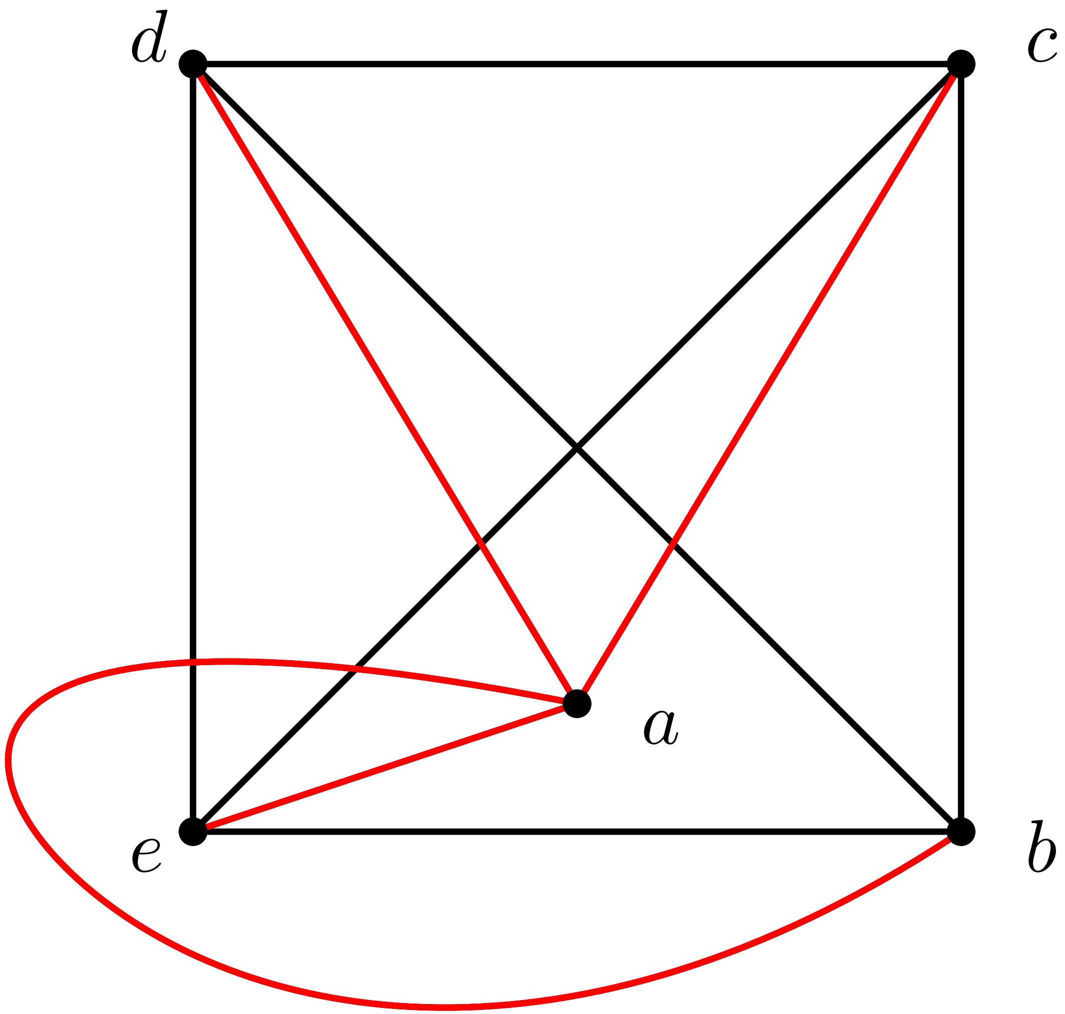
straight-line / pseudolinear





Rotation Systems

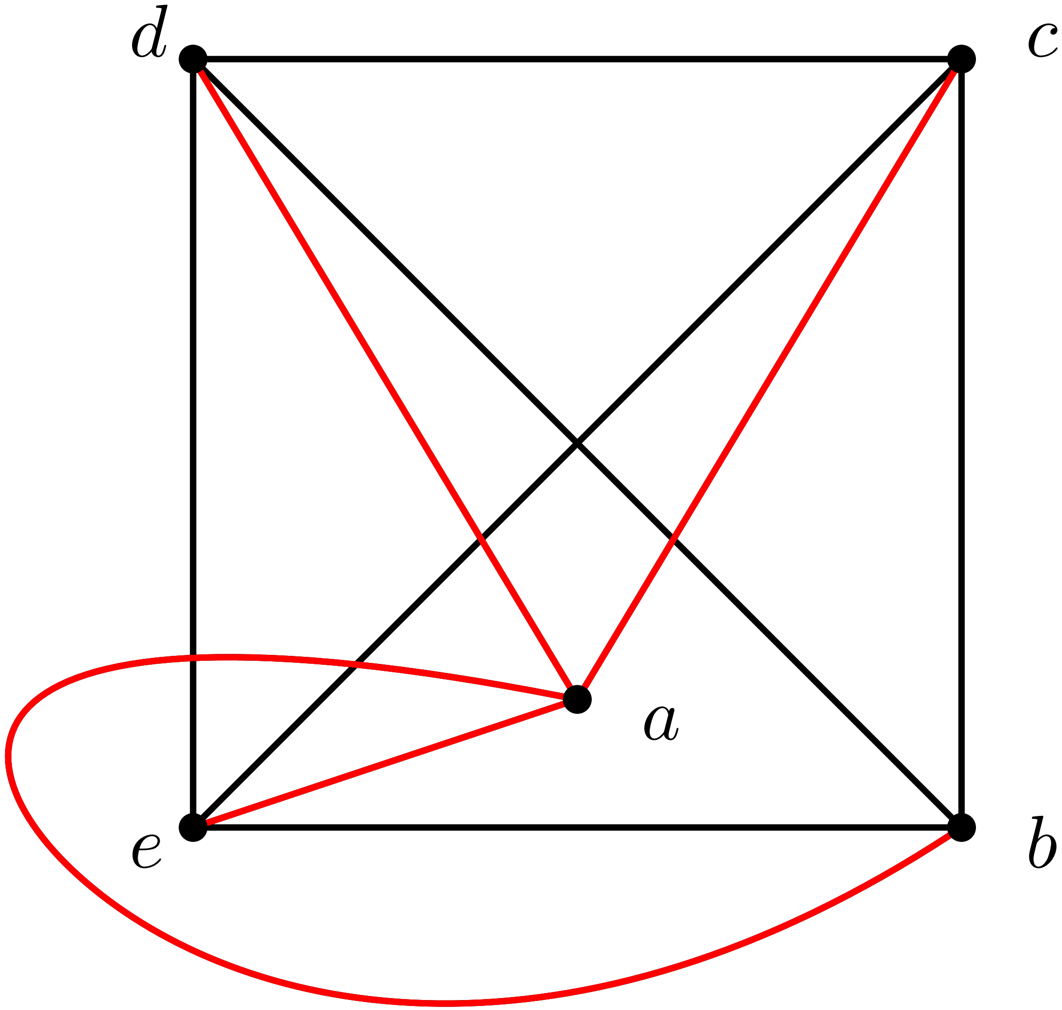
rotation system: cyclic order around each vertex



a sees b, e, c, d

Rotation Systems

rotation system: cyclic order around each vertex



a sees b, e, c, d

b sees a, c, d, e

c sees a, b, d, e

d sees a, b, c, e

e sees e, a, b, c

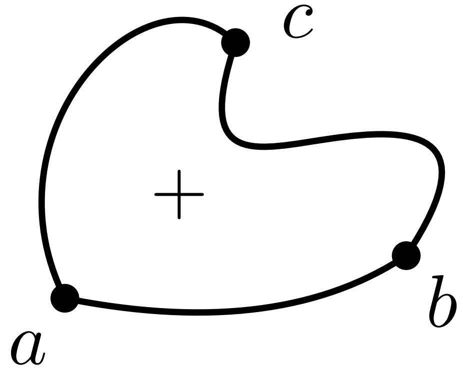
Rotation Systems

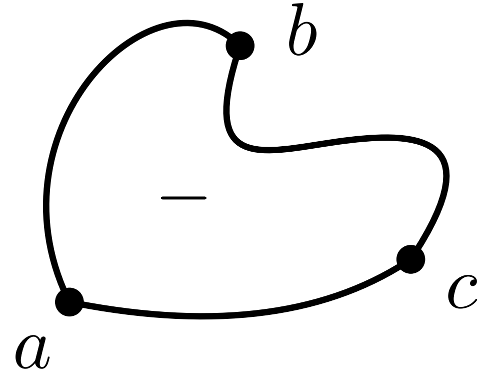
rotation system: cyclic order around each vertex

Kynčl 2015: rotation system representable by topological drawing $\Leftrightarrow$ all 6-tuples (5-tuples) are

Generalized Signotopes

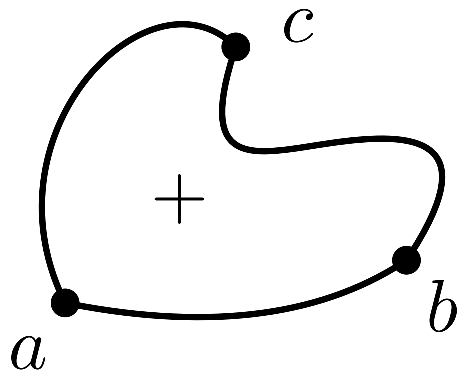
each K_3 has orientation

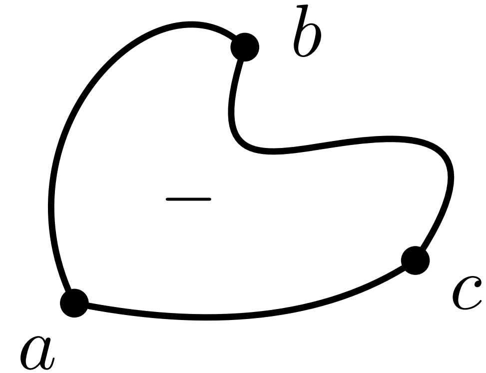




Generalized Signotopes

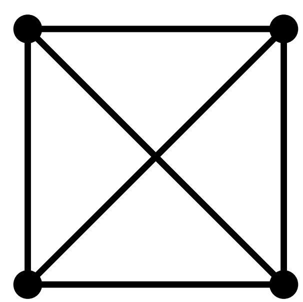
each K_3 has orientation

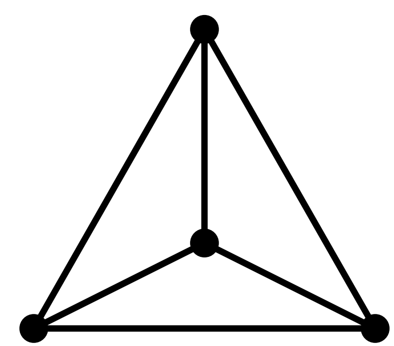




each K_4 gives sequence, except

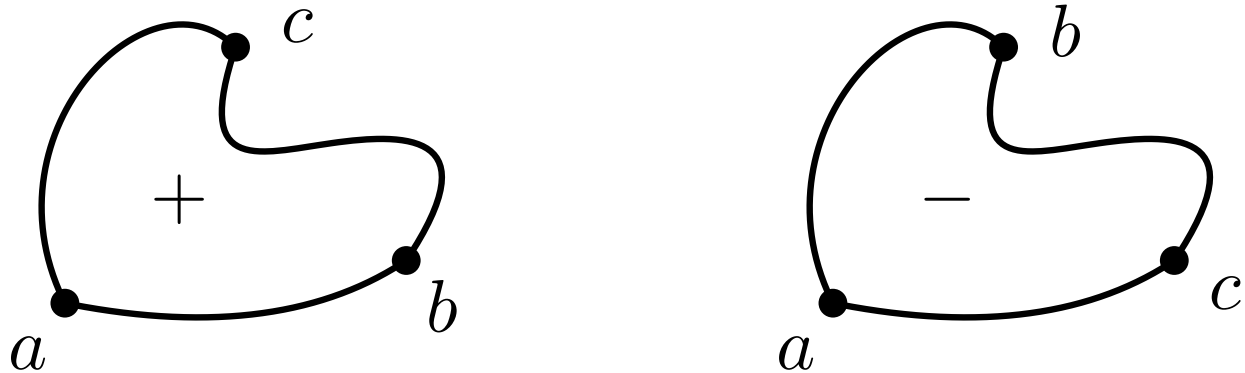
abc	abd	acd	bcd
$+$	$-$	$+$	$-$
$-$	$+$	$-$	$+$





Generalized Signotopes

each K_3 has orientation



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abc	abd	acd	bcd
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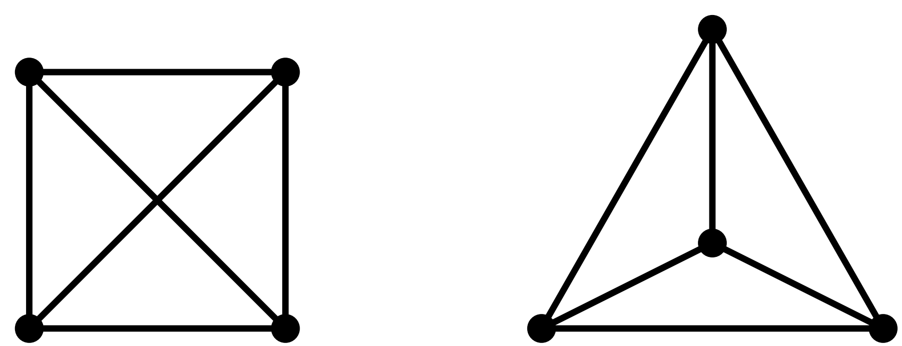
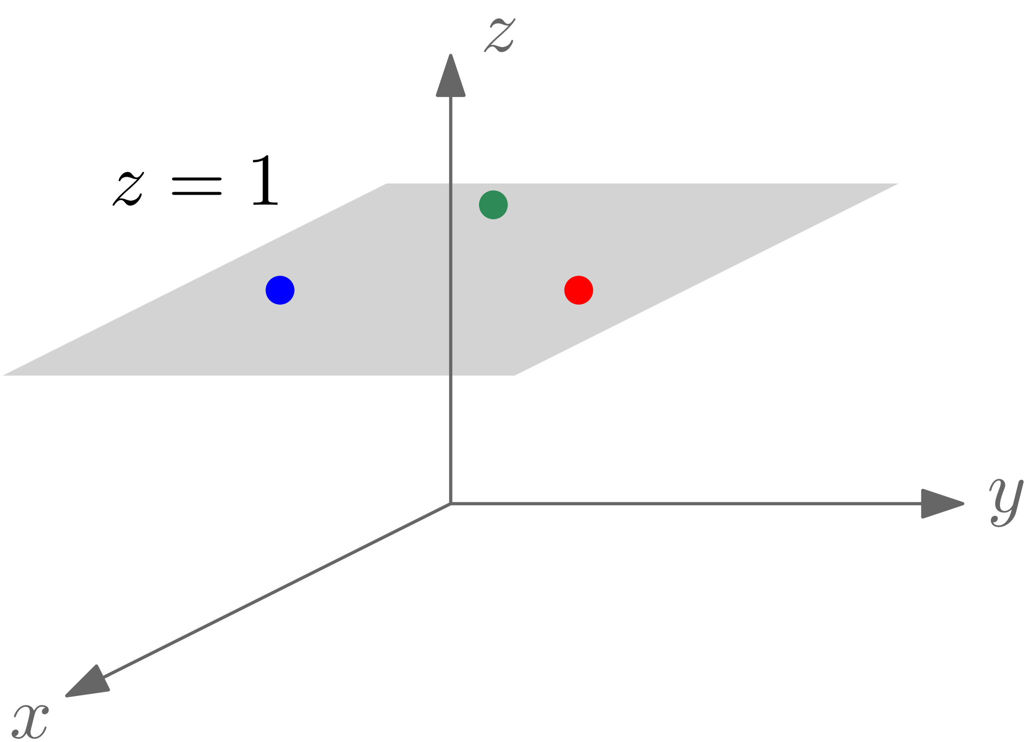
$-$	$+$	$-$	$+$



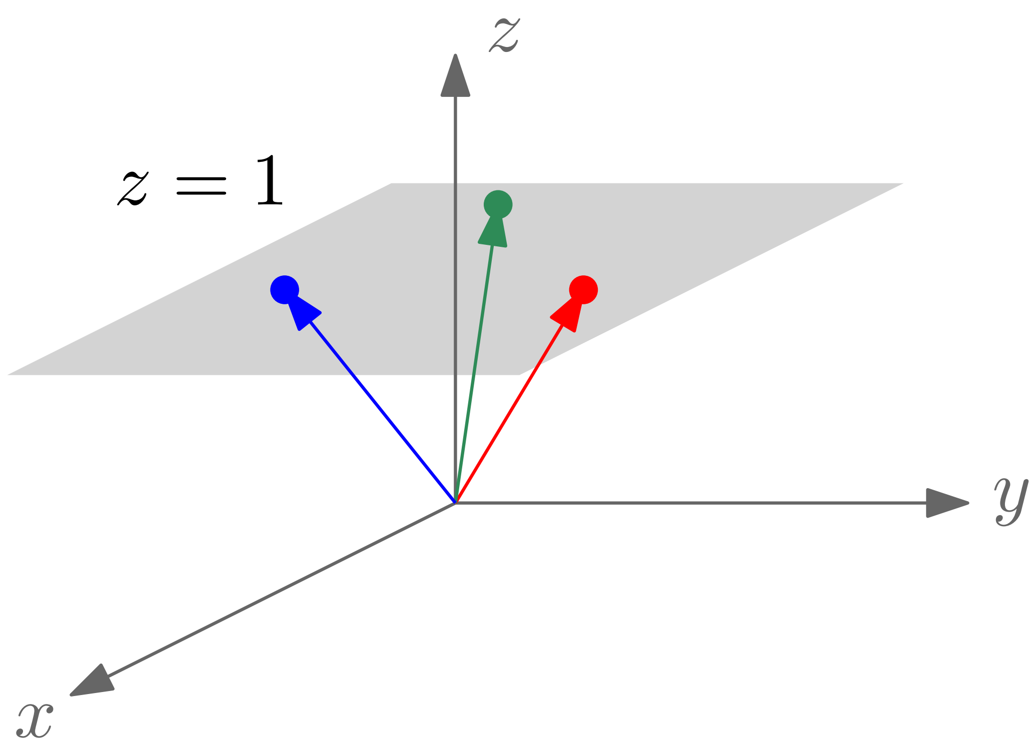
no characterization known yet
(possibly infinitely many obstructions / NP-hard?)

Point-Line Duality Revised



points in plane $(z = 1)$

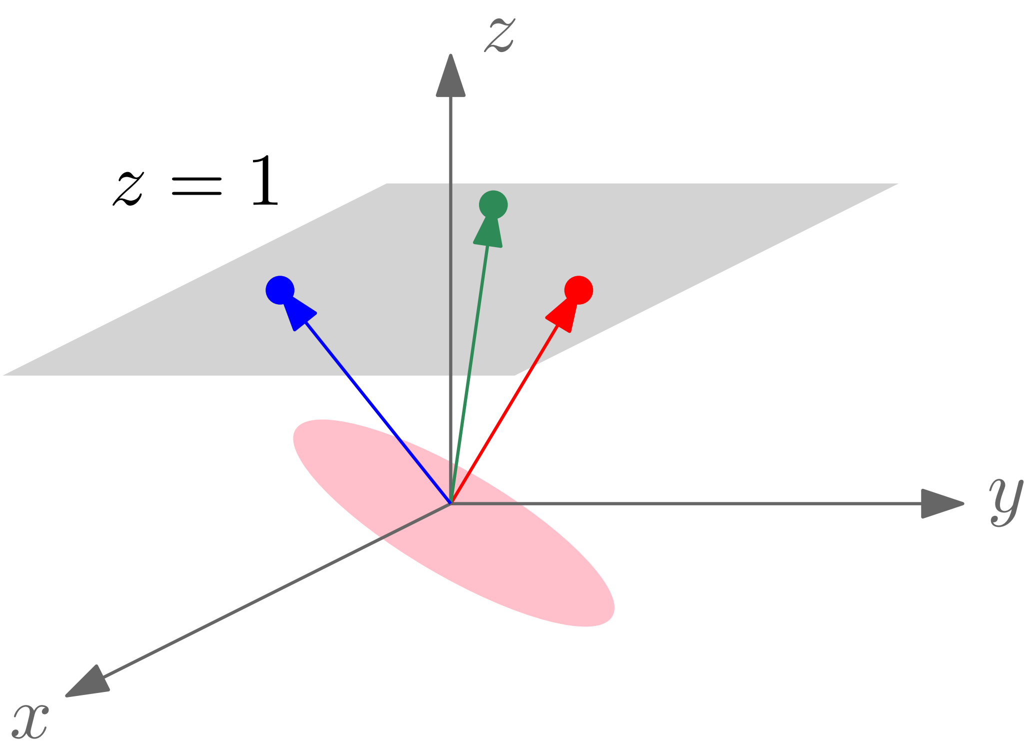
Point-Line Duality Revised



points in plane ($z = 1$)

lines throuh origin

Point-Line Duality Revised

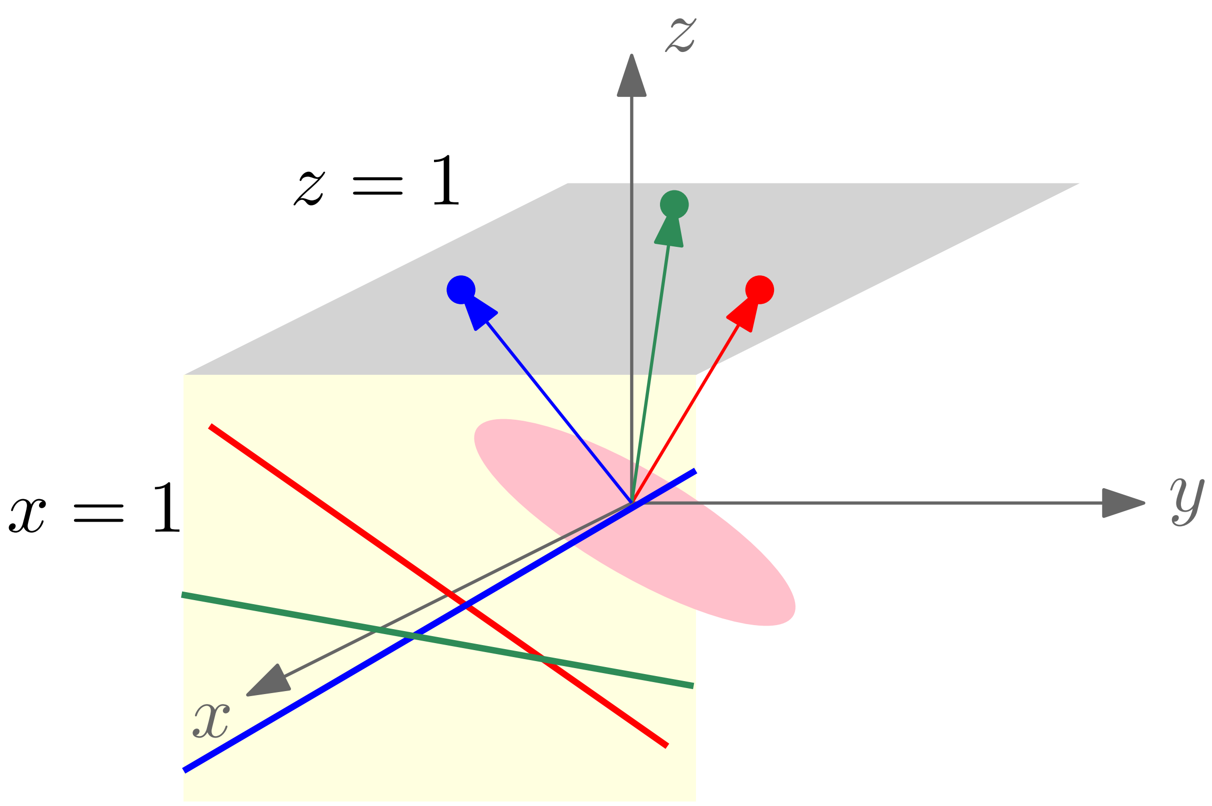


points in plane ($z = 1$)

lines throuh origin

orthogonal planes

Point-Line Duality Revised



points in plane $(z = 1)$

lines throuh origin

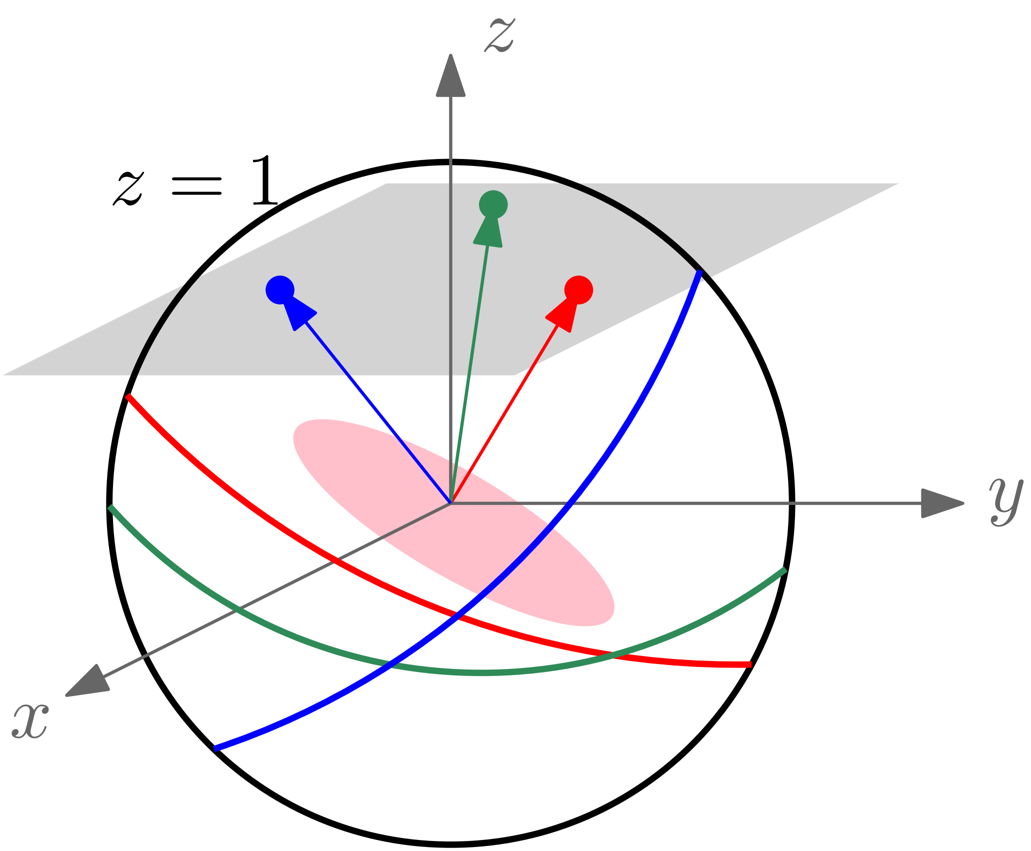
orthogonal planes

duality I:

intersect with plane $x = 1$

line arrangement in $x = 1$

Point-Line Duality Revised



points in plane $(z = 1)$

lines throuh origin

orthogonal planes

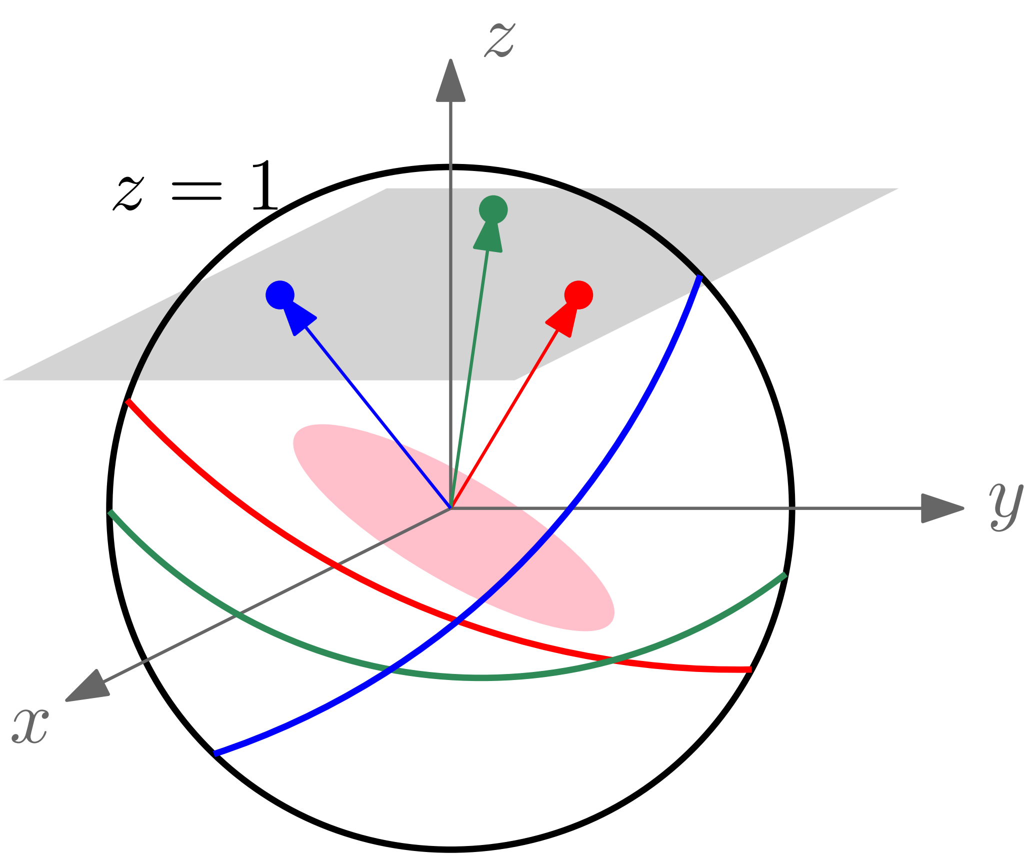
duality I:
intersect with plane $x = 1$

line arrangement in $x = 1$

duality II:
intersect with unit sphere

great-circle arrangement

Point-Line Duality Revised



(front hemisphere like plane)

points in plane $(z = 1)$

lines throuh origin

orthogonal planes

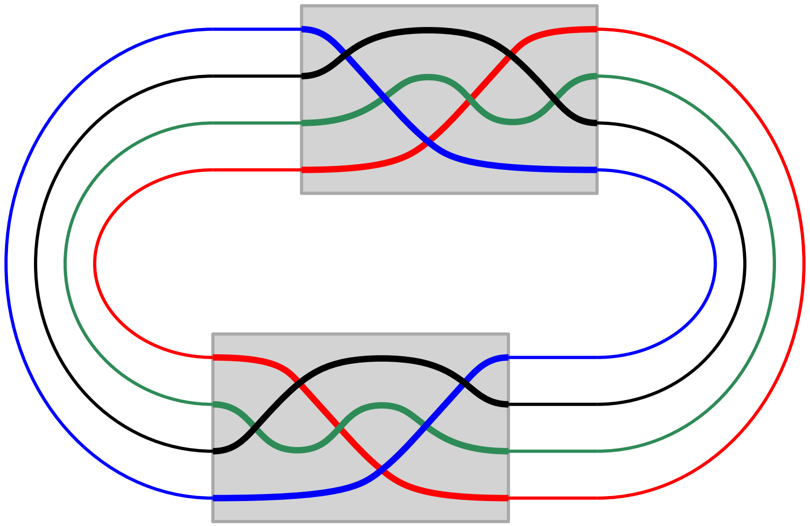
duality I:
intersect with plane $x = 1$

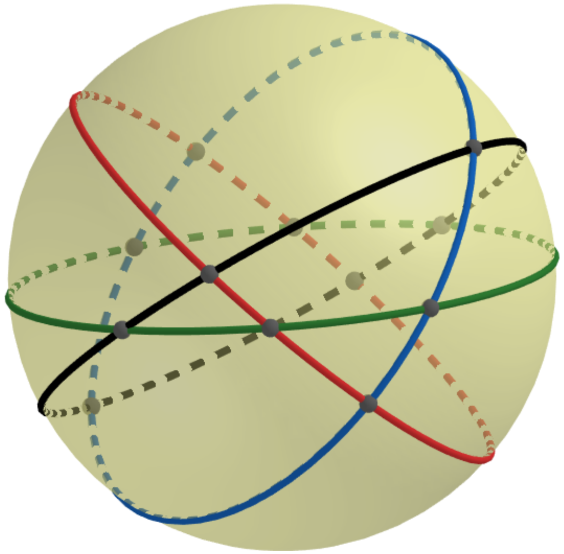
line arrangement in $x = 1$

duality II:
intersect with unit sphere

great-circle arrangement

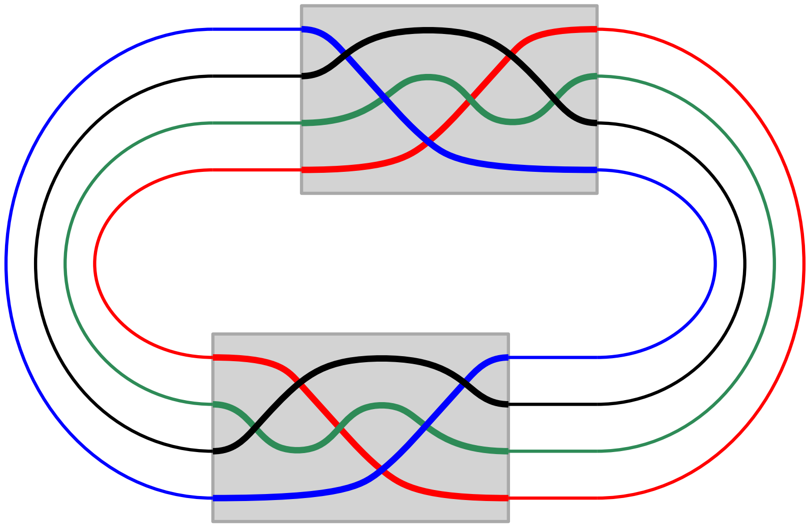
(Great)Circle Arrangements

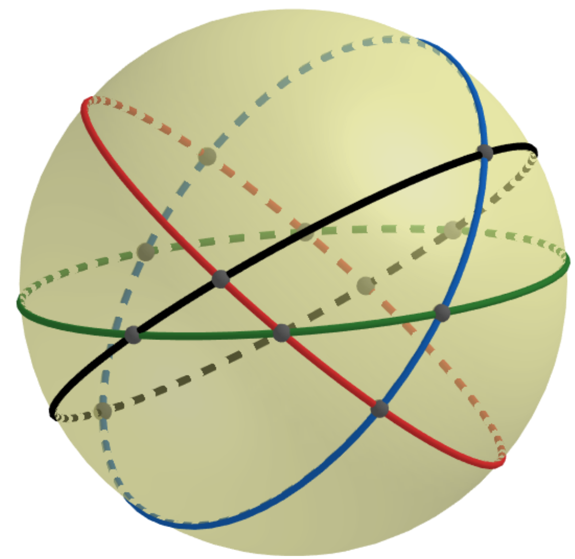




1-to-1 correspondence to pseudoline arrangements

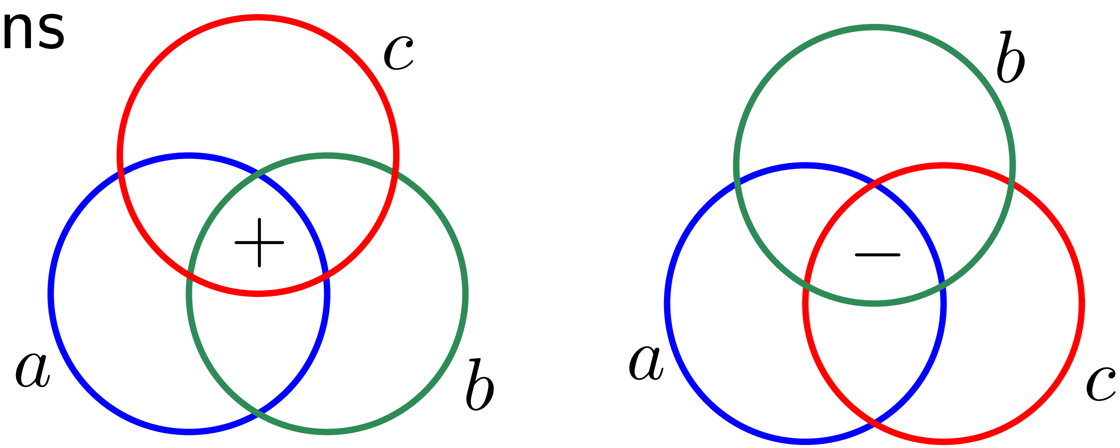
(Great)Circle Arrangements



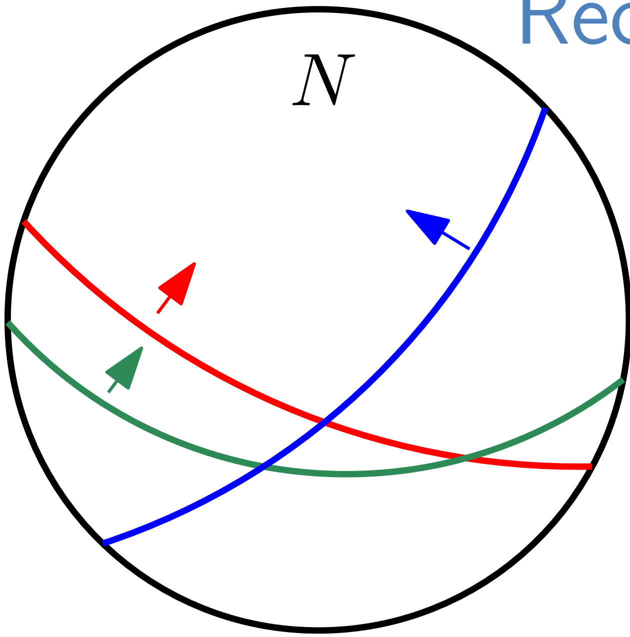


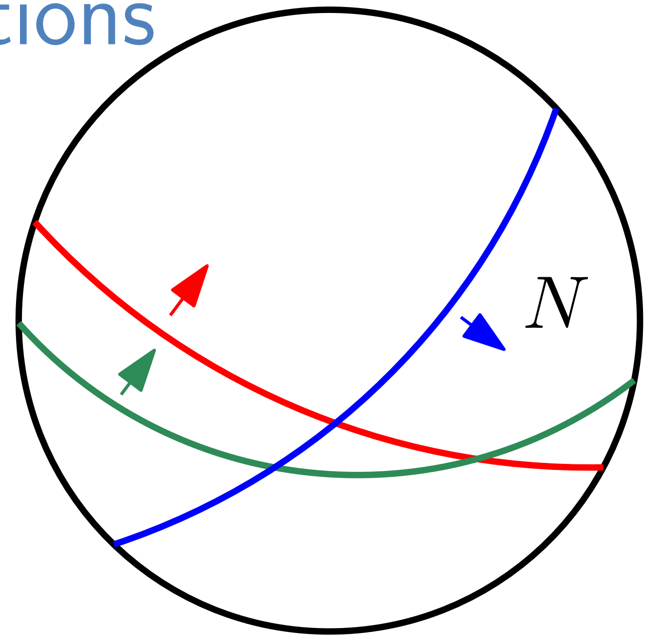
1-to-1 correspondence to pseudoline arrangements

all triples have orientations



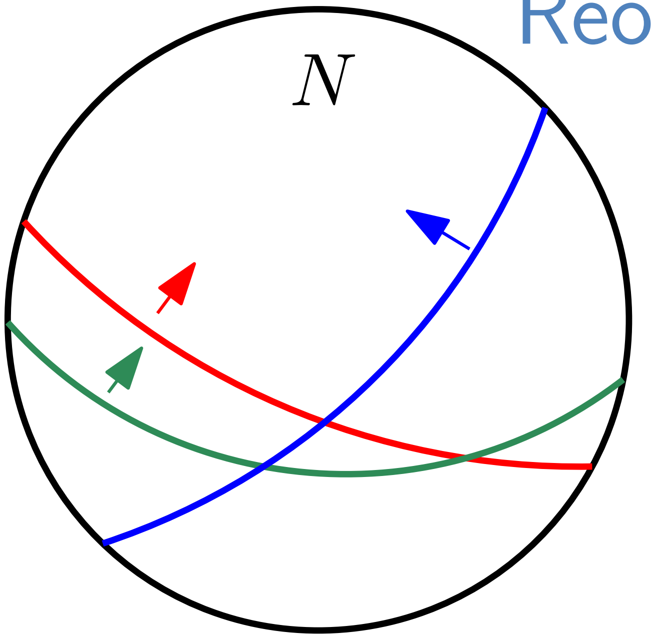
Reorientations

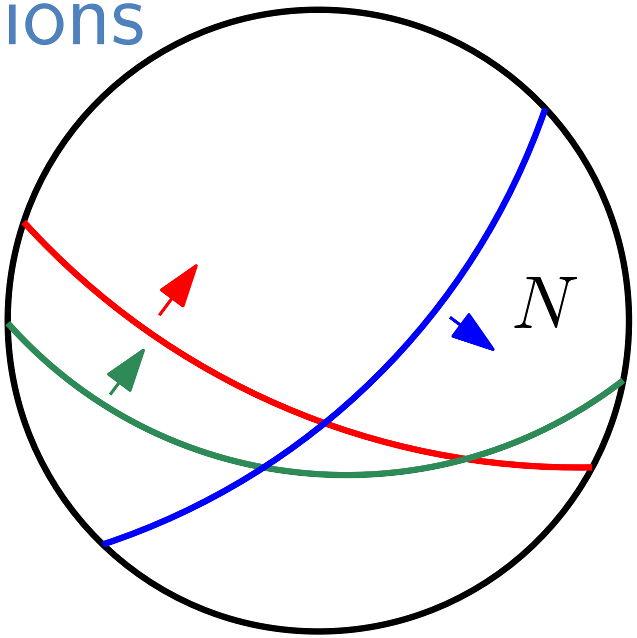




circles flip over when choosing other north pole

Reorientations

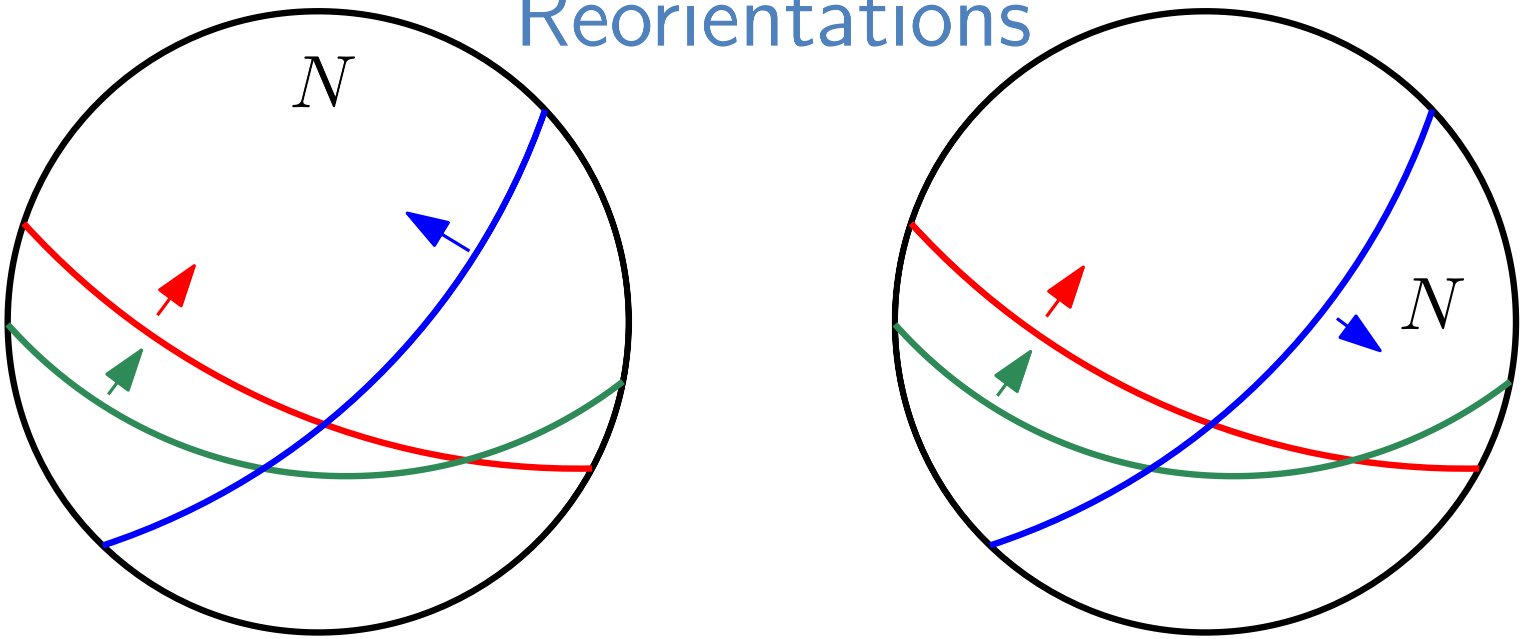




circles flip over when choosing other north pole

2^n ways to reorient n circles, each gives valid chirotope

Reorientations

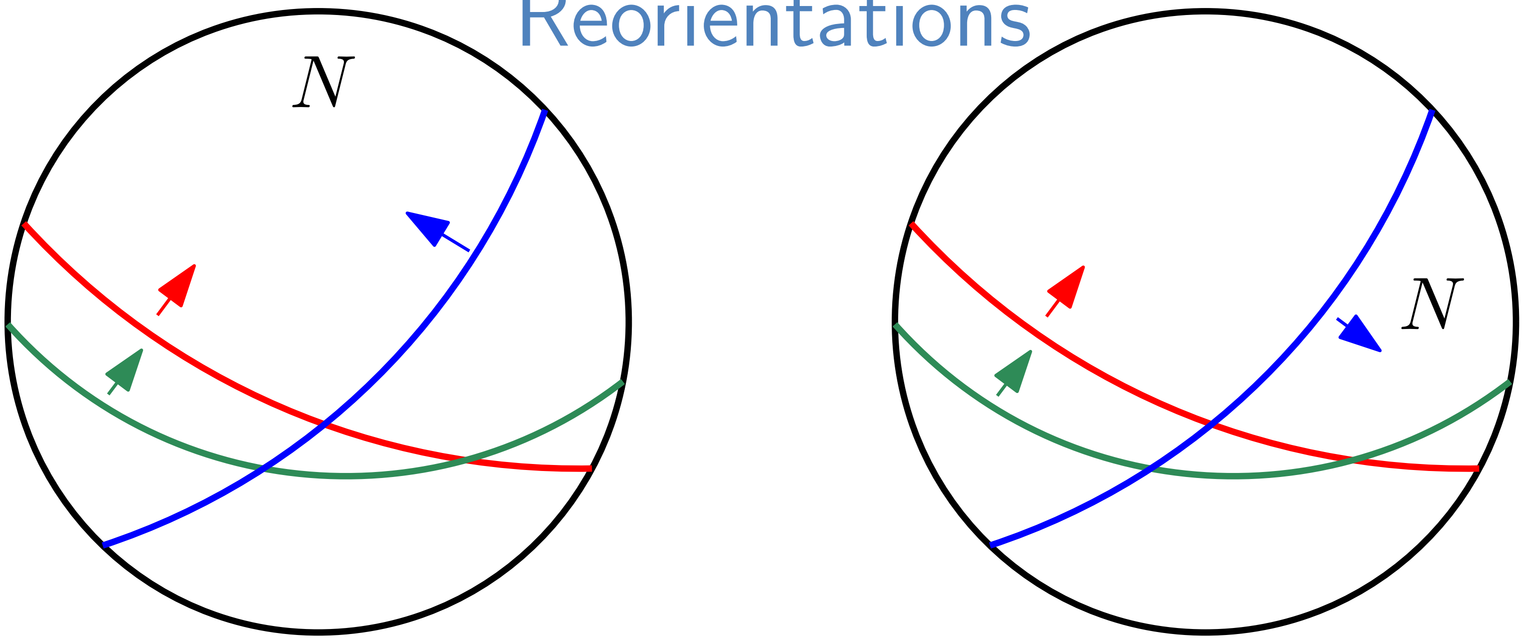


circles flip over when choosing other north pole

2^n ways to reorient n circles, each gives valid chirotope

but only $\#$ of cells $= \Theta(n^2)$ orientations determine a north pole

Reorientations



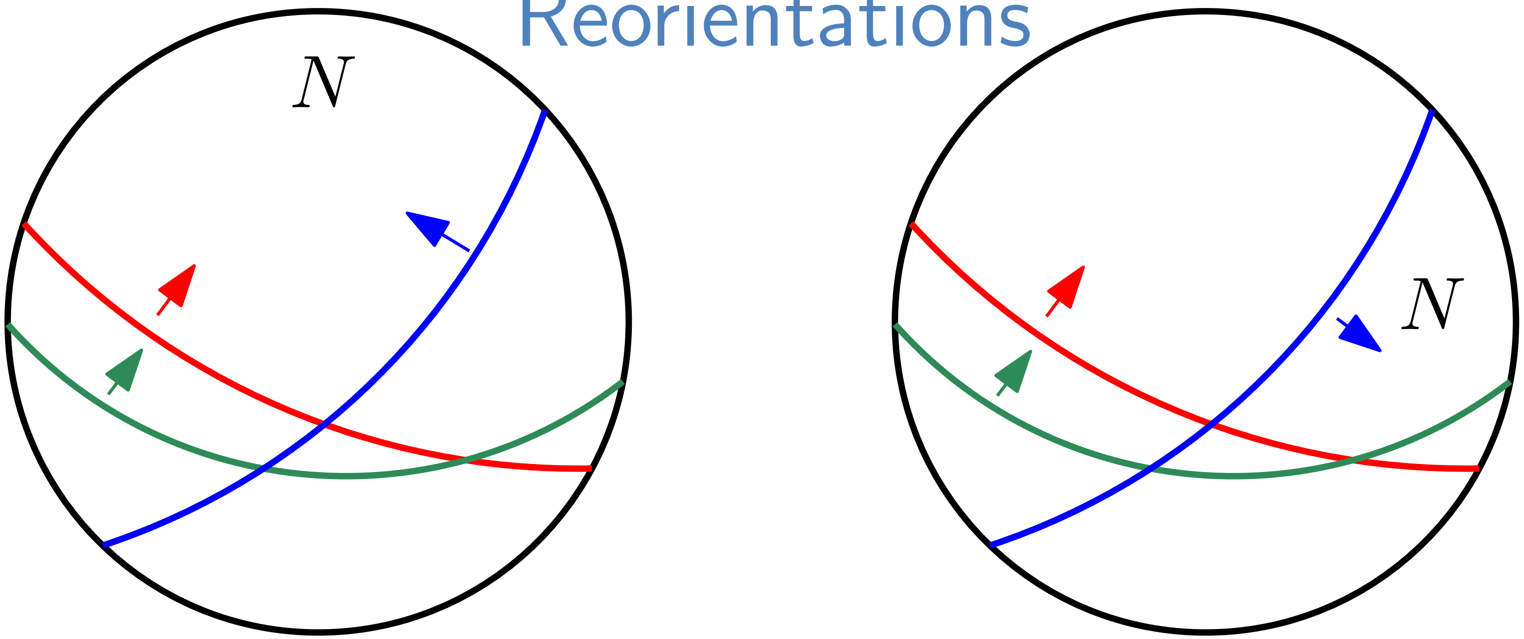
circles flip over when choosing other north pole

2^n ways to reorient n circles, each gives valid chirotope

but only # of cells $= \Theta(n^2)$ orientations determine a north pole

these are *acyclic* chirotopes and correspond to (pseudo)point conf. and PLA

Reorientations



circles flip over when choosing other north pole

2^n ways to reorient n circles, each gives valid chirotope

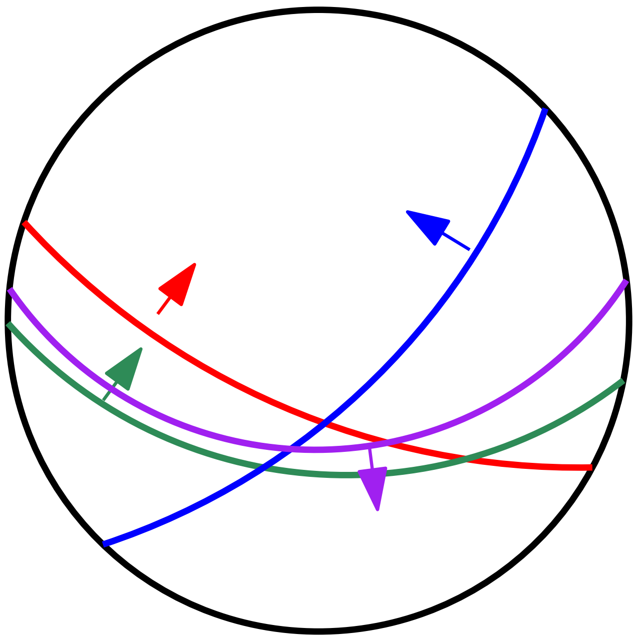
but only # of cells $= \Theta(n^2)$ orientations determine a north pole

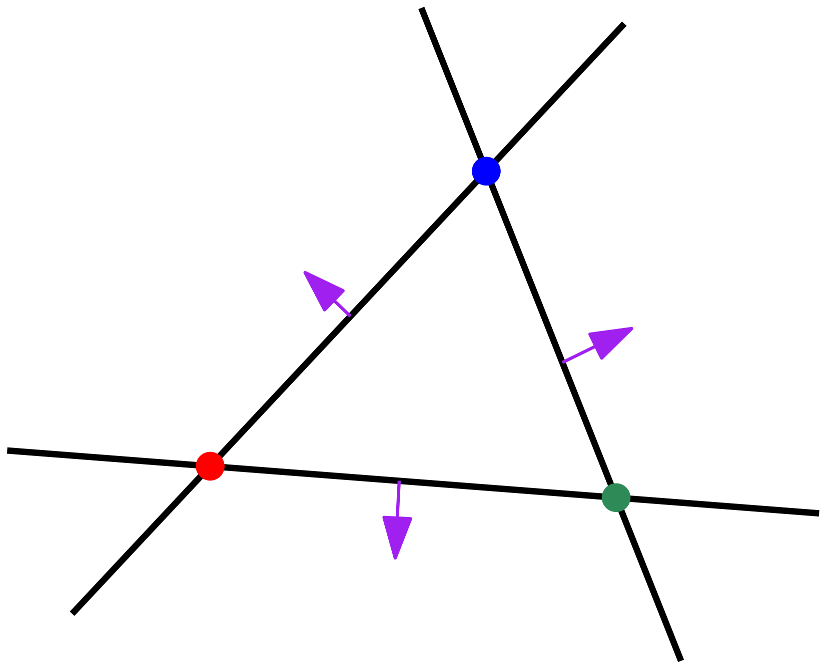
reorientation class a.k.a.

oriented matroid or *projective order type*

Reorientations

obstruction to acyclic orientations:

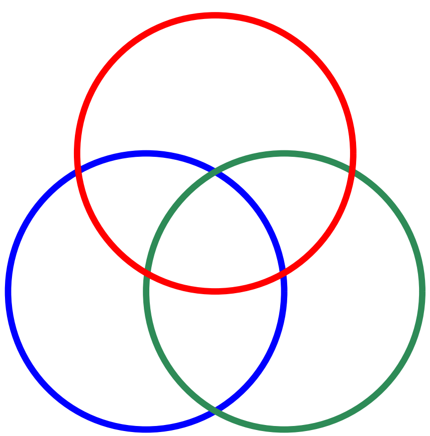




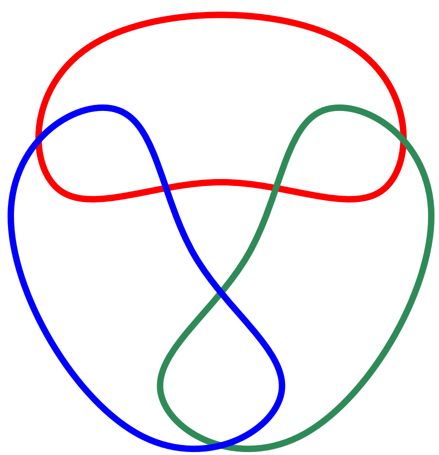
(General) Pseudocircle Arrangements

pseudocircle . . . simple closed curve

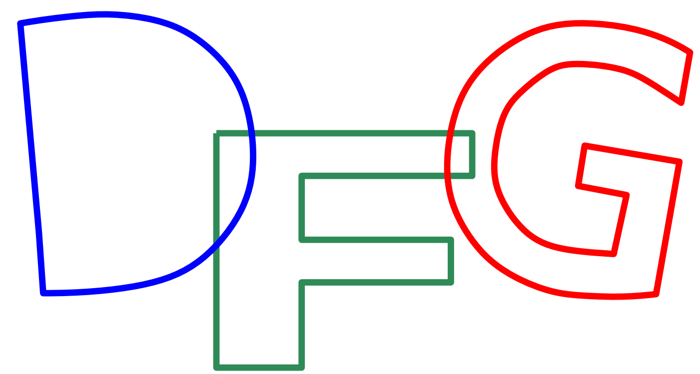
arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross



Krupp

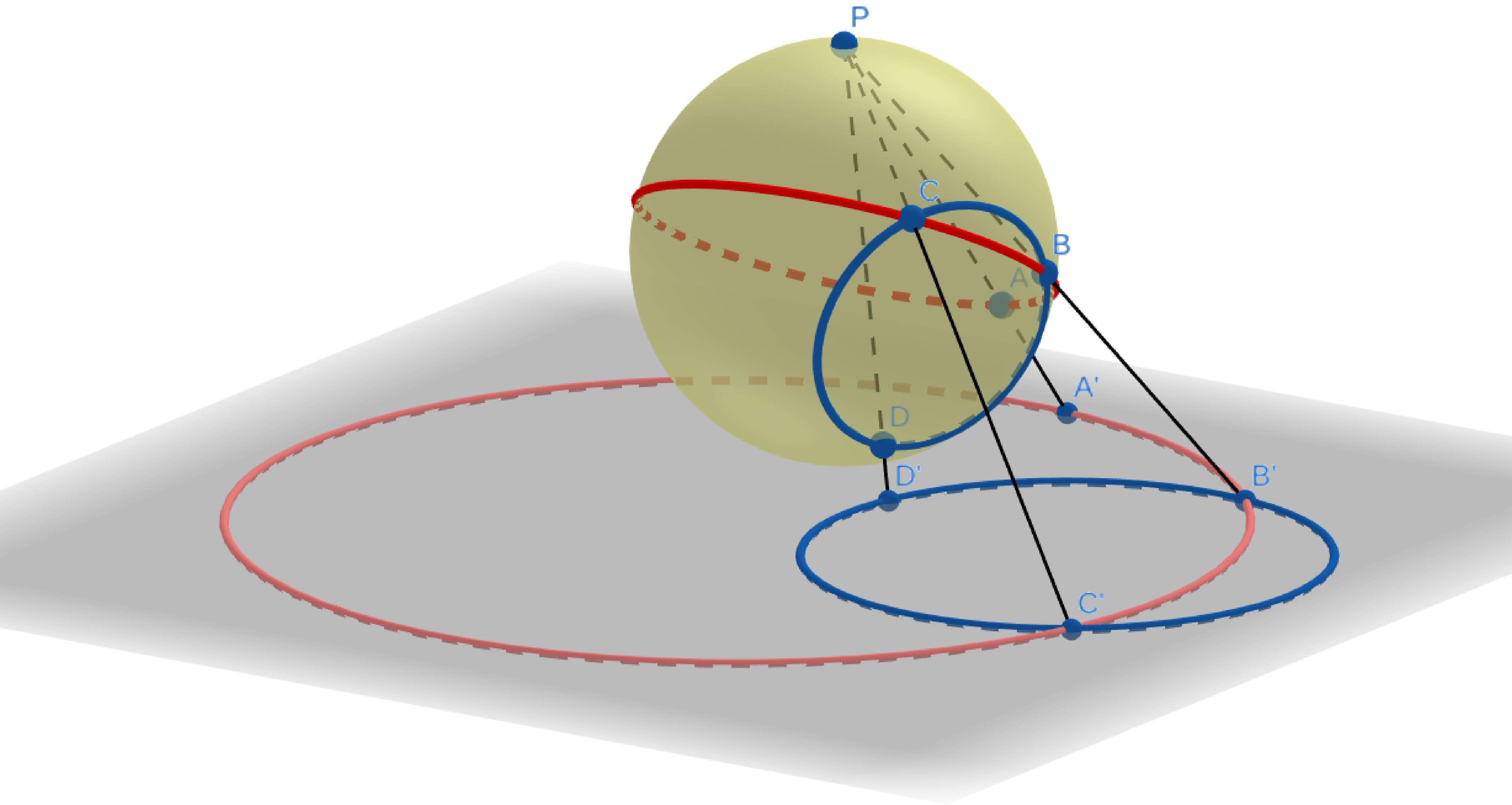


NonKrupp



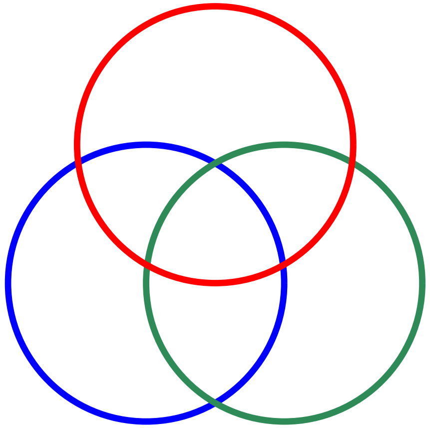
3-Chain

Plane VS Sphere

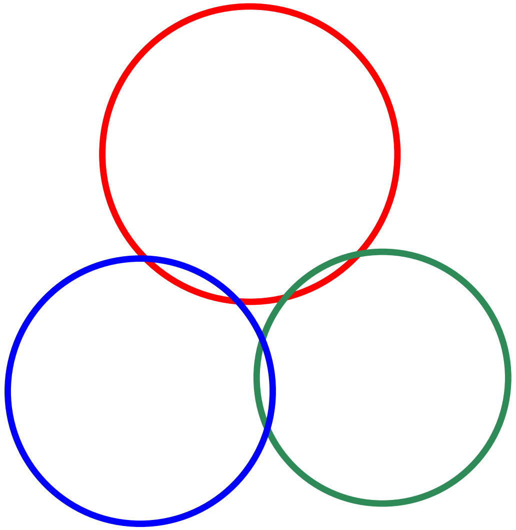


Hierarchy of Pseudocircle Arrangements

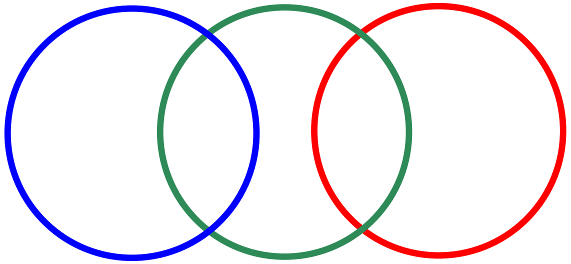
connected . . . graph of arrangement is connected



Krupp



NonKrupp

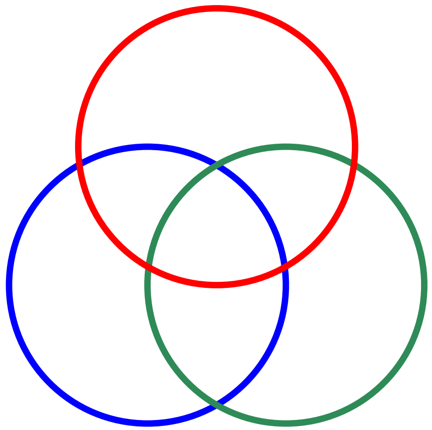


3-Chain

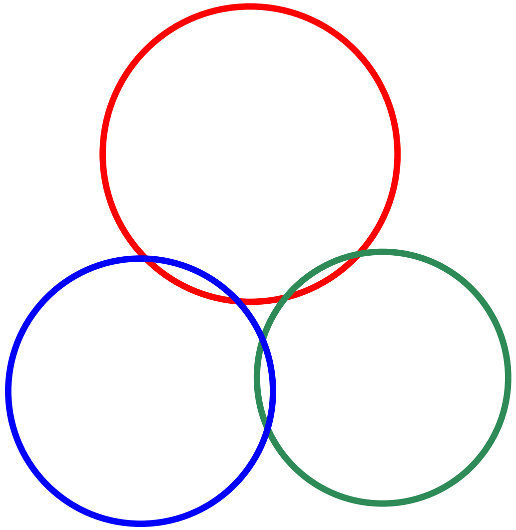
Hierarchy of Pseudocircle Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice



Krupp



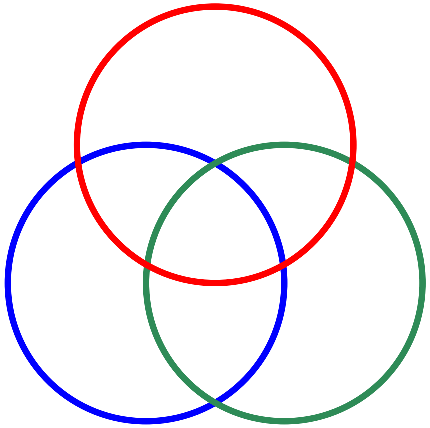
NonKrupp

Hierarchy of Pseudocircle Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp



Krupp

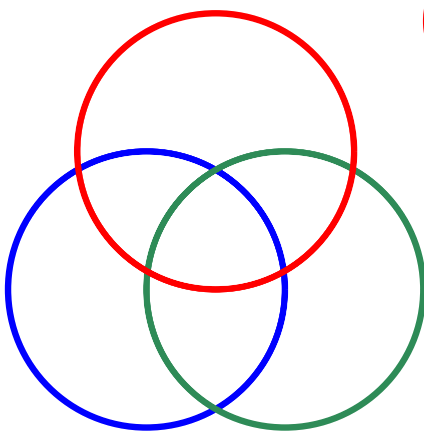
Hierarchy of Pseudocircle Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp

(characterization via chirotopes, 5-tuples)



Krupp

Hierarchy of Pseudocircle Arrangements

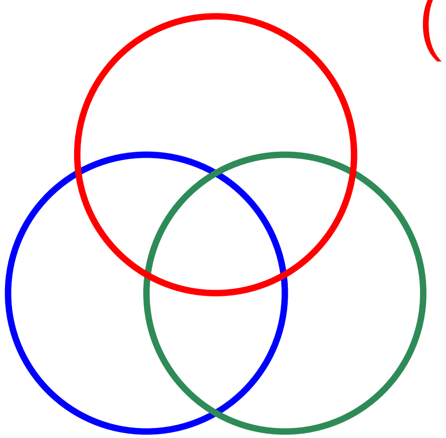
connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

(characterization via 4-tuples, Ortner 2008)

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp

(characterization via chirotopes, 5-tuples)



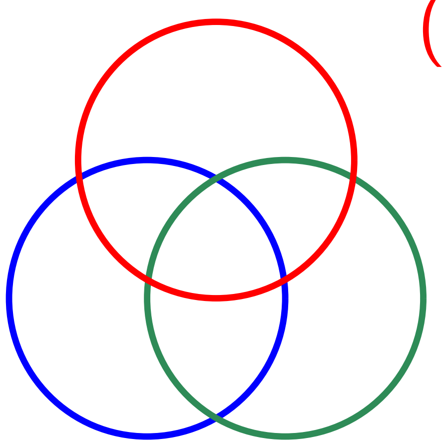
Krupp

Hierarchy of Pseudocircle Arrangements

connected . . . graph of arrangement is connected
(characterization?)

intersecting . . . any 2 pseudocircles cross twice
(characterization via 4-tuples, Ortner 2008)

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp
(characterization via chirotopes, 5-tuples)



Krupp