On the expected number of holes in random point sets

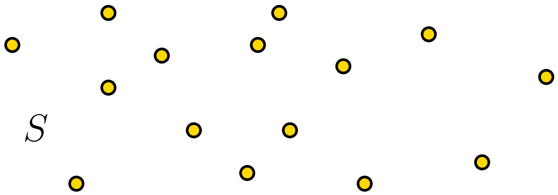
Martin Balko, Manfred Scheucher, and Pavel Valtr

Preliminaries

Preliminaries

Theorem (Erdős, Szekeres, 1935)

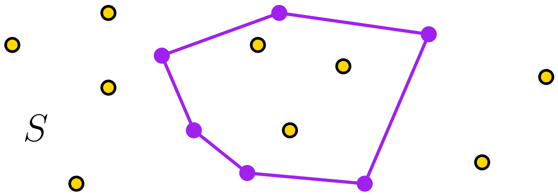
For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains k points in convex position.



Preliminaries

Theorem (Erdős, Szekeres, 1935)

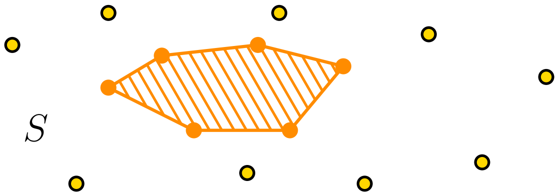
For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains k points in convex position.



Preliminaries

Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains k points in convex position.

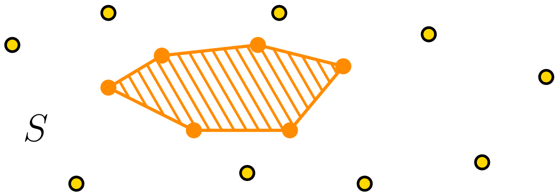


- A k-hole in a point set S is a k-tuple of points from S in convex position with no points of S in the interior of their convex hull.

Preliminaries

Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains k points in convex position.



- A k-hole in a point set S is a k-tuple of points from S in convex position with no points of S in the interior of their convex hull.
- Every set of ≥ 3 points contains a 3-hole. Also, ≥ 5 points $\rightarrow$ 4-hole and ≥ 10 points $\rightarrow$ 5-hole (Harborth, 1978).

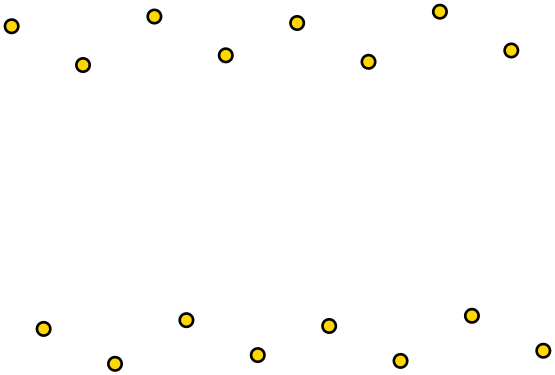
Sets with no large holes

Sets with no large holes

- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a k-hole?

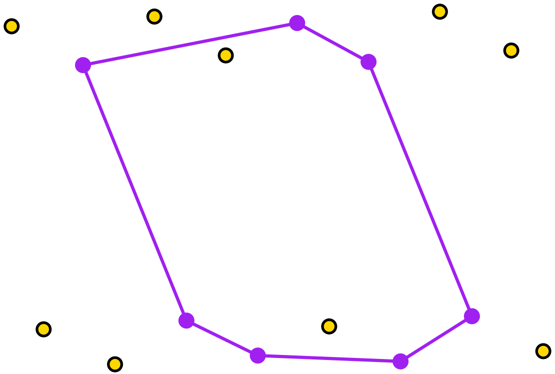
Sets with no large holes

- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a k-hole?
- No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).



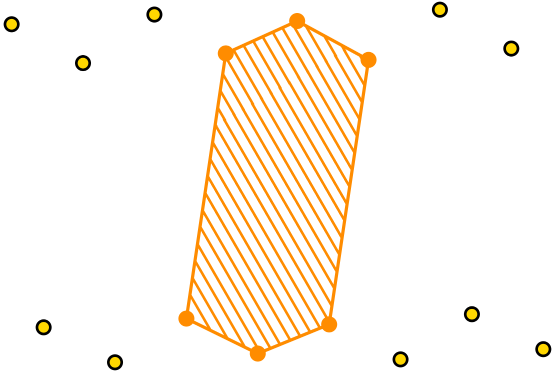
Sets with no large holes

- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a k-hole?
- No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).



Sets with no large holes

- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a k-hole?
- No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).



- Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).

Counting k-holes

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3, 4, 5, 6\}$.

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3, 4, 5, 6\}$.
- How many k-holes do we always have?

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3, 4, 5, 6\}$.
- How many k-holes do we always have?
- Let $h_k(n)$ be the minimum number of k-holes among all sets of n points in the plane in general position.

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3, 4, 5, 6\}$.
- How many k-holes do we always have?
- Let $h_k(n)$ be the minimum number of k-holes among all sets of n points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3, 4, 5, 6\}$.
- How many k-holes do we always have?
- Let $h_k(n)$ be the minimum number of k-holes among all sets of n points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
- Holes were also considered in higher dimensions.

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3,4,5,6\}$.
- How many k-holes do we always have?
- Let $h_k(n)$ be the minimum number of k-holes among all sets of n points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
- Holes were also considered in higher dimensions.
- There are d-dimensional Horton sets not containing k-holes for sufficiently large $k = k(d)$ (Valtr, 1992).

Counting k-holes

- Every sufficiently large set of points in general position contains a k-hole for $k \in \{3, 4, 5, 6\}$.
- How many k-holes do we always have?
- Let $h_k(n)$ be the minimum number of k-holes among all sets of n points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
- Holes were also considered in higher dimensions.
- There are d-dimensional Horton sets not containing k-holes for sufficiently large $k = k(d)$ (Valtr, 1992).
- The minimum number of $(d+1)$-holes (empty simplices) in an n-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ (Bárány, Füredi, 1987).

Random point sets

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH_{d,k}^K(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.
- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.
- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

- Valtr proved $EH^K_{2,3}(n) \leq 4\binom{n}{2}$.

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.
- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

- Valtr proved $EH^K_{2,3}(n) \leq 4\binom{n}{2}$. This is tight, as Reitzner and Temesvari recently showed $\lim_{n\to\infty} n^{-2} EH^K_{2,3}(n) = 2$.

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.
- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

- Valtr proved $EH^K_{2,3}(n) \leq 4\binom{n}{2}$. This is tight, as Reitzner and Temesvari recently showed $\lim_{n\to\infty} n^{-2} EH^K_{2,3}(n) = 2$.
- They also proved

$$\frac{2}{d!} \leq \lim_{n\to\infty} n^{-d} EH^K_{d,d+1}(n) \leq \frac{d}{(d+1)} \frac{\kappa_{d-1}^{d+1}\kappa_{d^2}}{\kappa_d^{d-1}\kappa_{(d-1)(d+1)}}$$

for $d \geq 3$, where κ_d is the volume of the d-dimensional unit ball.

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.
- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

- Valtr proved $EH^K_{2,3}(n) \leq 4\binom{n}{2}$. This is tight, as Reitzner and Temesvari recently showed $\lim_{n\to\infty} n^{-2} EH^K_{2,3}(n) = 2$.
- They also proved

$$d^{-c_1 \cdot d} \leq \frac{2}{d!} \leq \lim_{n\to\infty} n^{-d} EH^K_{d,d+1}(n) \leq \frac{d}{(d+1)} \frac{\kappa_{d-1}^{d+1}\kappa_{d^2}}{\kappa_d^{d-1}\kappa_{(d-1)(d+1)}} \leq d^{-c_2 \cdot d}$$

for $d \geq 3$, where κ_d is the volume of the d-dimensional unit ball.

Random point sets

- Let $k \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.
- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

- Valtr proved $EH^K_{2,3}(n) \leq 4\binom{n}{2}$. This is tight, as Reitzner and Temesvari recently showed $\lim_{n\to\infty} n^{-2} EH^K_{2,3}(n) = 2$.
- They also proved

$$d^{-c_1 \cdot d} \leq \frac{2}{d!} \leq \lim_{n\to\infty} n^{-d} EH^K_{d,d+1}(n) \leq \frac{d}{(d+1)} \frac{\kappa_{d-1}^{d+1}\kappa_{d^2}}{\kappa_d^{d-1}\kappa_{(d-1)(d+1)}} \leq d^{-c_2 \cdot d}$$

for $d \geq 3$, where κ_d is the volume of the d-dimensional unit ball. The upper bound holds with equality if K is an ellipsoid.

Our results: asymptotic growth rate of $EH^K_{d,k}(n)$

Our results: asymptotic growth rate of $EH^K_{d,k}(n)$

- We extend previous bounds to larger holes.

Our results: asymptotic growth rate of $EH^K_{d,k}(n)$

- We extend previous bounds to larger holes.
- We have an upper bound $EH^K_{d,k}(n) \leq O(n^d)$ for any fixed k.

Our results: asymptotic growth rate of $EH^K_{d,k}(n)$

- We extend previous bounds to larger holes.
- We have an upper bound $EH^K_{d,k}(n) \le O(n^d)$ for any fixed k.

Theorem 1 (2020)

Let $d \ge 2$ and $k \ge d+1$ be integers and let $K \subseteq \mathbb{R}^d$ be a convex body of unit volume. If $n \ge k$, then the expected number $EH^K_{d,k}(n)$ is at most

$$2^{d-1} \cdot \left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot \frac{n(n-1)\cdots(n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$

Our results: asymptotic growth rate of $EH^K_{d,k}(n)$

- We extend previous bounds to larger holes.
- We have an upper bound $EH^K_{d,k}(n) \leq O(n^d)$ for any fixed k.

Theorem 1 (2020)

Let $d \geq 2$ and $k \geq d+1$ be integers and let $K \subseteq \mathbb{R}^d$ be a convex body of unit volume. If $n \geq k$, then the expected number $EH^K_{d,k}(n)$ is at most

$$2^{d-1} \cdot \left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot \frac{n(n-1)\cdots(n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$

- We can also show that this is asymptotically tight.

Our results: asymptotic growth rate of $EH^K_{d,k}(n)$

- We extend previous bounds to larger holes.
- We have an upper bound $EH^K_{d,k}(n) \leq O(n^d)$ for any fixed k.

Theorem 1 (2020)

Let $d \geq 2$ and $k \geq d+1$ be integers and let $K \subseteq \mathbb{R}^d$ be a convex body of unit volume. If $n \geq k$, then the expected number $EH^K_{d,k}(n)$ is at most

$$2^{d-1} \cdot \left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot \frac{n(n-1)\cdots(n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$

- We can also show that this is asymptotically tight.

Theorem 2 (2021)

For all fixed integers $d \geq 2$ and $k \geq d+1$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have $EH^K_{d,k}(n) \geq \Omega(n^d)$.

Our results: leading constants in $EH^{K}_{d,d+1}(n)$

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.
- Let p^K_d be the probability that the convex hull of $d+2$ points chosen unif. and ind. at random from K is a d-simplex.

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.
- Let p^K_d be the probability that the convex hull of $d+2$ points chosen unif. and ind. at random from K is a d-simplex. Let $p_d = \max_K p^K_d$.

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.
- Let p^K_d be the probability that the convex hull of $d+2$ points chosen unif. and ind. at random from K is a d-simplex. Let $p_d = \max_K p^K_d$.

Theorem 3 (2021)

For every $d \geq 2$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!\,p_{d-1}}$.

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.
- Let p^K_d be the probability that the convex hull of $d+2$ points chosen unif. and ind. at random from K is a d-simplex. Let $p_d = \max_K p^K_d$.

Theorem 3 (2021)

For every $d \geq 2$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

Proof uses heavy machinery from stochastic geometry (Blaschke–Petkantschin formula & a result by Reitzner and Temesvari)

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.
- Let p^K_d be the probability that the convex hull of $d+2$ points chosen unif. and ind. at random from K is a d-simplex. Let $p_d = \max_K p^K_d$.

Theorem 3 (2021)

For every $d \geq 2$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!\,p_{d-1}}$.

- This is tight in the plane and also for simplices in $\mathbb{R}^3$.

Corollary (2021)

For every convex body $K \subseteq \mathbb{R}^3$ of unit volume, we have $3 \leq C^K_{3,4} \leq \frac{12\pi^2}{35} \approx 3.38$. Moreover, the left inequality is tight if K is a tetrahedron and the right inequality is tight if K is an ellipsoid.

Our results: leading constants in $EH^K_{d,d+1}(n)$

- Since we have asymptotically sharp estimates on $EH^K_{d,k}(n)$, we started studying the leading constants $C^K_{d,k} = \lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.
- We improved Reitzner's and Temesvari's bound $C^K_{d,d+1} \geq \frac{2}{d!}$.
- Let p^K_d be the probability that the convex hull of $d+2$ points chosen unif. and ind. at random from K is a d-simplex. Let $p_d = \max_K p^K_d$.

Theorem 3 (2021)

For every $d \geq 2$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

- This is tight in the plane and also for simplices in $\mathbb{R}^3$.
- We also believe that our lower bound on $C^K_{d,d+1}$ from Theorem 3 is tight for simplices in any dimension d.

Our results: leading constants in $EH^K_{d,k}(n)$

Our results: leading constants in $EH^K_{d,k}(n)$

- We can also prove new result about $C^K_{d,k}$ for $k > d + 1$.

Our results: leading constants in $EH_{d,k}^{K}(n)$

- We can also prove new result about $C_{d,k}^{K}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C_{2,4}^{K}$, which does not give better estimate than 72.

Our results: leading constants in $EH_{d,k}^{K}(n)$

- We can also prove new result about $C_{d,k}^{K}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C_{2,4}^{K}$, which does not give better estimate than 72. We improved it to $C_{2,4}^{K} \leq 12$ in 2020.

Our results: leading constants in $EH^K_{d,k}(n)$

- We can also prove new result about $C^K_{d,k}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C^K_{2,4}$, which does not give better estimate than 72. We improved it to $C^K_{2,4} \leq 12$ in 2020. Now, we can determine $C^K_{2,4}$ exactly.

Our results: leading constants in $EH_{d,k}^{K}(n)$

- We can also prove new result about $C_{d,k}^{K}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C_{2,4}^{K}$, which does not give better estimate than 72. We improved it to $C_{2,4}^{K} \leq 12$ in 2020. Now, we can determine $C_{2,4}^{K}$ exactly.

Theorem 4 (2021)

For every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have

$$C_{2,4}^{K} =$$

Our results: leading constants in $EH^K_{d,k}(n)$

- We can also prove new result about $C^K_{d,k}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C^K_{2,4}$, which does not give better estimate than 72. We improved it to $C^K_{2,4} \leq 12$ in 2020. Now, we can determine $C^K_{2,4}$ exactly.

Theorem 4 (2021)

For every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have

$$C^K_{2,4} = 10 - \frac{2\pi^2}{3} \approx 3.420.$$

Our results: leading constants in $EH^K_{d,k}(n)$

- We can also prove new result about $C^K_{d,k}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C^K_{2,4}$, which does not give better estimate than 72. We improved it to $C^K_{2,4} \leq 12$ in 2020. Now, we can determine $C^K_{2,4}$ exactly.

Theorem 4 (2021)

For every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have

$$C^K_{2,4} = 10 - \frac{2\pi^2}{3} \approx 3.420.$$

- In general, we can show that $C^K_{2,k}$ does not depend on K.

Our results: leading constants in $EH^K_{d,k}(n)$

- We can also prove new result about $C^K_{d,k}$ for $k > d + 1$.
- For 4-holes in the plane, Fabila-Monroy, Huemer, and Mitsche proved an upper bound on $C^K_{2,4}$, which does not give better estimate than 72. We improved it to $C^K_{2,4} \leq 12$ in 2020. Now, we can determine $C^K_{2,4}$ exactly.

Theorem 4 (2021)

For every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have

$$C^K_{2,4} = 10 - \frac{2\pi^2}{3} \approx 3.420.$$

- In general, we can show that $C^K_{2,k}$ does not depend on K.

Theorem 5 (2021)

For every integer $k \geq 3$, there is a constant $C = C(k)$ such that, for every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have $C^K_{2,k} = C$.

Sketch of the proof of $C_{2,3}^K = 2$

Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.

Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$.

Sketch of the proof of $C_{2,3}^{K} = 2$

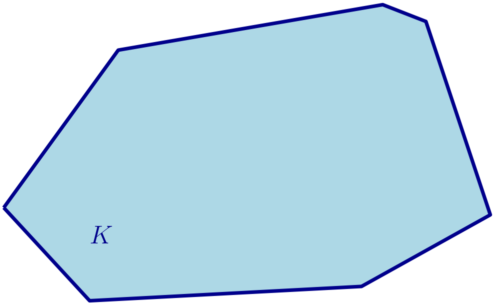
- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.

Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.

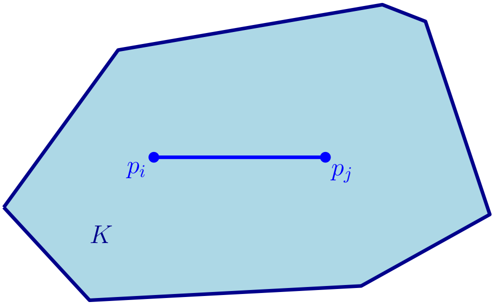
Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



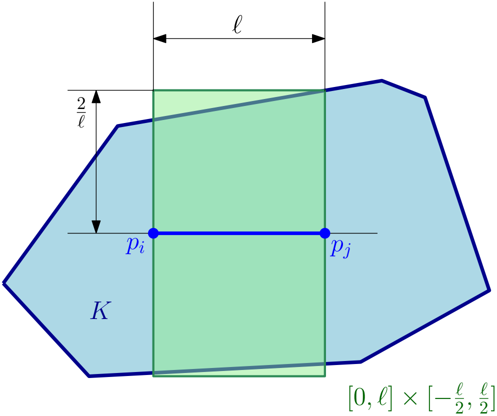
Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



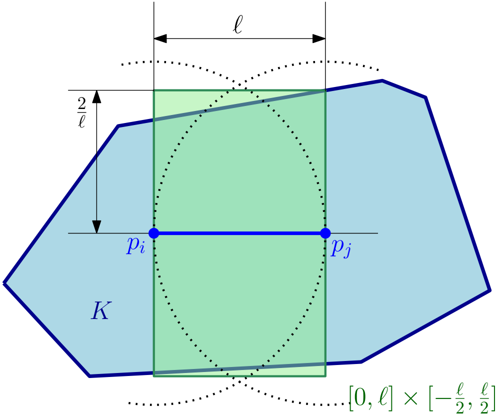
Sketch of the proof of $C^K_{2,3} = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



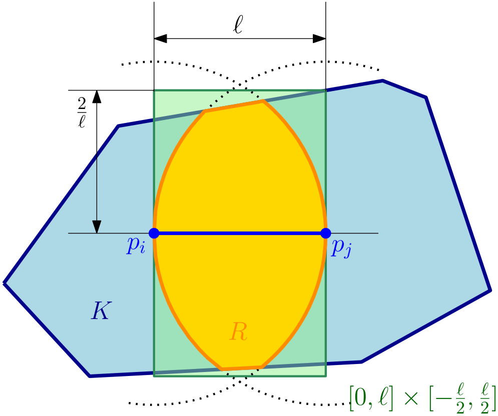
Sketch of the proof of $C^K_{2,3} = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



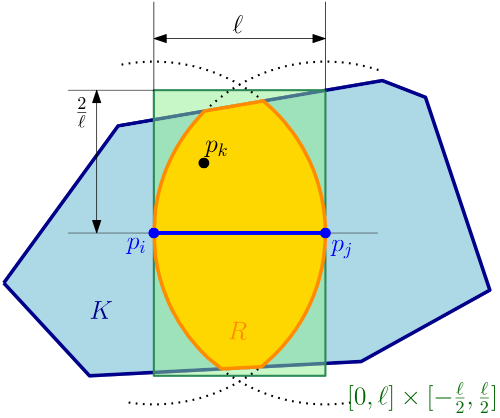
Sketch of the proof of $C^K_{2,3} = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



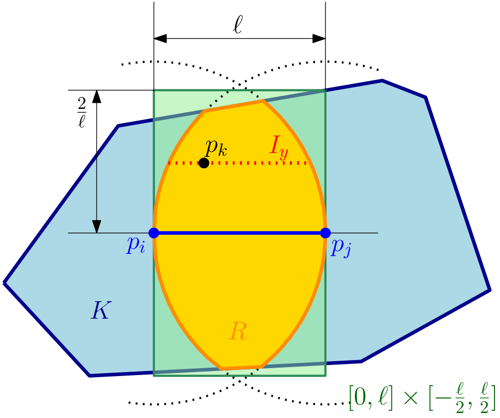
Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



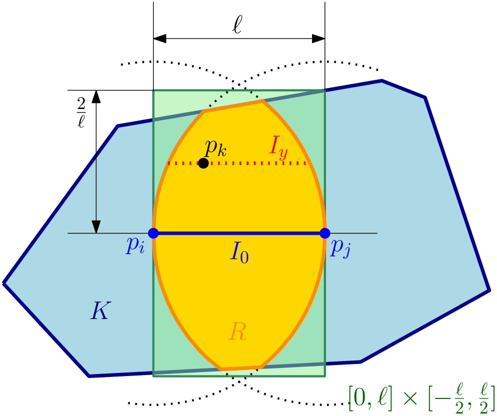
Sketch of the proof of $C^K_{2,3} = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



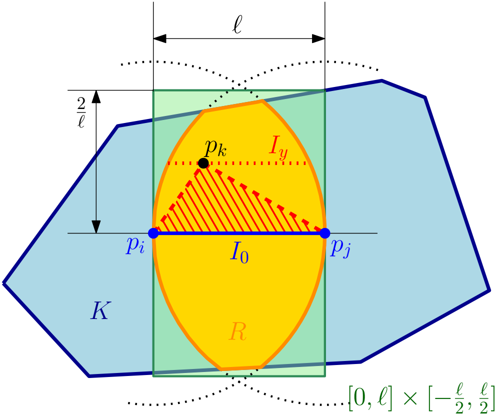
Sketch of the proof of $C_{2,3}^K = 2$

- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



Sketch of the proof of $C_{2,3}^K = 2$

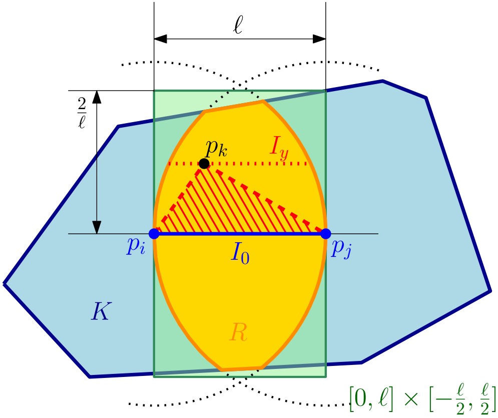
- We show that the expected number of 3-holes in a set S of n points selected unif. and indep. at random from $K \subseteq \mathbb{R}^2$ is $2n^2 + o(n^2)$.
- For p_i and p_j from S, we count the expected number of 3-holes in S with the longest edge $p_i p_j$. We assume $p_i = (0,0)$ and $p_j = (\ell, 0)$.
- The third point $p_k \in S$ of the 3-hole lies in the set R of points from $K \cap [0,\ell] \times [\frac{-2}{\ell}, \frac{2}{\ell}]$ that are at distance at most ℓ from p_i and p_j.



Sketch of the proof of $C_{2,3}^K = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

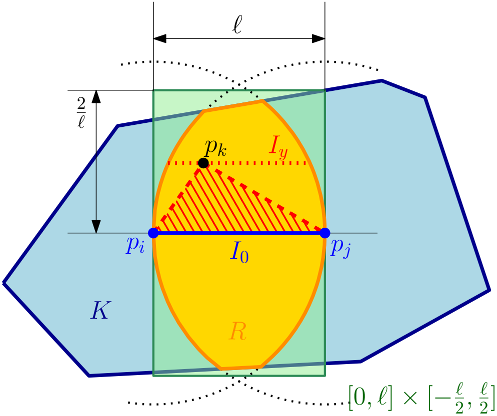


Sketch of the proof of $C_{2,3}^K = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k \text{ is empty in } S]$ equals $\left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3}$.



Sketch of the proof of $C_{2,3}^K = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k \text{ is empty in } S]$ equals $\left(1 - \frac{|y|\cdot\ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3} \mathrm{d}y$$

Sketch of the proof of $C^K_{2,3} = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k$ is empty in $S]$ equals $\left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3} \mathrm{d}y = \frac{n-2}{n} \int_{\frac{-2n}{\ell}}^{\frac{2n}{\ell}} \left|I_{\frac{Y}{n}}\right| \left(1 - \frac{|Y|\ell}{2n}\right)^{n-3} \mathrm{d}Y.$$

Sketch of the proof of $C^K_{2,3} = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k$ is empty in $S]$ equals $\left(1 - \frac{|y|\cdot\ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y|\cdot\ell}{2}\right)^{n-3} \mathrm{d}y = \frac{n-2}{n} \int_{\frac{-2n}{\ell}}^{\frac{2n}{\ell}} \left|I_{\frac{Y}{n}}\right| \left(1 - \frac{|Y|\ell}{2n}\right)^{n-3} \mathrm{d}Y.$$

- By the Monotone Convergence Theorem, we get for $n \to \infty$

$$2 \cdot \int_0^\infty |I_0| \cdot e^{-Y\cdot\ell/2} \mathrm{d}Y =$$

Sketch of the proof of $C^K_{2,3} = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k$ is empty in $S]$ equals $\left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3} \mathrm{d}y = \frac{n-2}{n} \int_{\frac{-2n}{\ell}}^{\frac{2n}{\ell}} \left|I_{\frac{Y}{n}}\right| \left(1 - \frac{|Y|\ell}{2n}\right)^{n-3} \mathrm{d}Y.$$

- By the Monotone Convergence Theorem, we get for $n \to \infty$

$$2 \cdot \int_0^\infty |I_0| \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 2 \cdot \int_0^\infty \ell \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y$$

Sketch of the proof of $C^K_{2,3} = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k$ is empty in $S]$ equals $\left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3} \mathrm{d}y = \frac{n-2}{n} \int_{\frac{-2n}{\ell}}^{\frac{2n}{\ell}} \left|I_{\frac{Y}{n}}\right| \left(1 - \frac{|Y|\ell}{2n}\right)^{n-3} \mathrm{d}Y.$$

- By the Monotone Convergence Theorem, we get for $n \to \infty$

$$2 \cdot \int_0^\infty |I_0| \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 2 \cdot \int_0^\infty \ell \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 4.$$

Sketch of the proof of $C^K_{2,3} = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k$ is empty in $S]$ equals $\left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3} \mathrm{d}y = \frac{n-2}{n} \int_{\frac{-2n}{\ell}}^{\frac{2n}{\ell}} \left|I_{\frac{Y}{n}}\right| \left(1 - \frac{|Y|\ell}{2n}\right)^{n-3} \mathrm{d}Y.$$

- By the Monotone Convergence Theorem, we get for $n \to \infty$

$$2 \cdot \int_0^{\infty} |I_0| \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 2 \cdot \int_0^{\infty} \ell \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 4.$$

- Since there are $\binom{n}{2}$ pairs $\{p_i, p_j\}$, the expected number of 3-holes in S is $4(1 + o(1)) \cdot \binom{n}{2} = 2n^2 + o(n^2)$.

Sketch of the proof of $C^K_{2,3} = 2$

- Thus the expected number of 3-holes with longest edge $p_i p_j$ equals

$$(n-2) \cdot \int_{-2/\ell}^{2/\ell} |I_y| \cdot \Pr[p_i p_j p_k \text{ is empty in } S] \, \mathrm{d}y.$$

- The probability $\Pr[p_i p_j p_k$ is empty in $S]$ equals $\left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3}$.
- Therefore the expected number is

$$(n-2) \int_{\frac{-2}{\ell}}^{\frac{2}{\ell}} |I_y| \left(1 - \frac{|y| \cdot \ell}{2}\right)^{n-3} \mathrm{d}y = \frac{n-2}{n} \int_{\frac{-2n}{\ell}}^{\frac{2n}{\ell}} \left|I_{\frac{Y}{n}}\right| \left(1 - \frac{|Y|\ell}{2n}\right)^{n-3} \mathrm{d}Y.$$

- By the Monotone Convergence Theorem, we get for $n \to \infty$

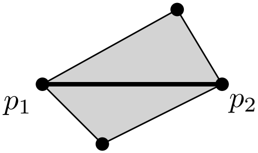
$$2 \cdot \int_0^\infty |I_0| \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 2 \cdot \int_0^\infty \ell \cdot e^{-Y \cdot \ell/2} \mathrm{d}Y = 4.$$

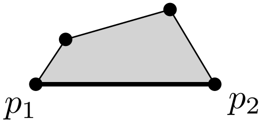
- Since there are $\binom{n}{2}$ pairs $\{p_i, p_j\}$, the expected number of 3-holes in S is $4(1 + o(1)) \cdot \binom{n}{2} = 2n^2 + o(n^2)$. □

Sketch of the proof of $C^K_{2,4} = 10 - \frac{2\pi^2}{3}$

Sketch of the proof of $C_{2,4}^K = 10 - \frac{2\pi^2}{3}$

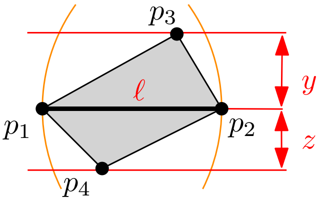
2 types of 4-holes:





Sketch of the proof of $C^K_{2,4} = 10 - \frac{2\pi^2}{3}$

type 1:



$$\int_{y=0}^{2/\ell} \int_{z=0}^{2/\ell - y} |I_y| \cdot |I_{-z}| \cdot \left(1 - \frac{\ell \cdot (y+z)}{2}\right)^{n-4} \, \mathrm{d}z\mathrm{d}y$$

Sketch of the proof of $C^K_{2,4} = 10 - \frac{2\pi^2}{3}$

type 1: $$\int_{y=0}^{2/\ell} \int_{z=0}^{2/\ell - y} |I_y| \cdot |I_{-z}| \cdot \left(1 - \frac{\ell \cdot (y+z)}{2}\right)^{n-4} \, \mathrm{d}z \mathrm{d}y$$

substitute $Y = yn$ and $Z = zn$

$$\frac{1}{n^2} \int_{Y=0}^{2n/\ell} \int_{Z=0}^{2n/\ell - Y} |I_{Y/n}| \cdot |I_{-Z/n}| \cdot \left(1 - \frac{\ell \cdot (Y+Z)}{2n}\right)^{n-4} \, \mathrm{d}Z \mathrm{d}Y.$$

Sketch of the proof of $C^K_{2,4} = 10 - \frac{2\pi^2}{3}$

type 1: $$\int_{y=0}^{2/\ell} \int_{z=0}^{2/\ell - y} |I_y| \cdot |I_{-z}| \cdot \left(1 - \frac{\ell \cdot (y+z)}{2}\right)^{n-4} \, \mathrm{d}z \mathrm{d}y$$

substitute $Y = yn$ and $Z = zn$

$$\frac{1}{n^2} \int_{Y=0}^{2n/\ell} \int_{Z=0}^{2n/\ell - Y} |I_{Y/n}| \cdot |I_{-Z/n}| \cdot \left(1 - \frac{\ell \cdot (Y+Z)}{2n}\right)^{n-4} \, \mathrm{d}Z \mathrm{d}Y.$$

we integrate over the set $\{(Y,Z) \in \mathbb{R}^2 \colon Y + Z \leq 2n/\ell, Y, Z \geq 0\}$, which becomes $\{(Y,Z) \in \mathbb{R}^2 \colon Y, Z \geq 0\}$ for n going to infinity

$|I_{Y/n}|$ and $|I_{Z/n}|$ both become $|I_0| = \ell$

Sketch of the proof of $C^K_{2,4} = 10 - \frac{2\pi^2}{3}$

type 1: $$\int_{y=0}^{2/\ell} \int_{z=0}^{2/\ell - y} |I_y| \cdot |I_{-z}| \cdot \left(1 - \frac{\ell \cdot (y+z)}{2}\right)^{n-4} \, \mathrm{d}z \mathrm{d}y$$

substitute $Y = yn$ and $Z = zn$

$$\frac{1}{n^2} \int_{Y=0}^{2n/\ell} \int_{Z=0}^{2n/\ell - Y} |I_{Y/n}| \cdot |I_{-Z/n}| \cdot \left(1 - \frac{\ell \cdot (Y+Z)}{2n}\right)^{n-4} \, \mathrm{d}Z \mathrm{d}Y.$$

we integrate over the set $\{(Y,Z) \in \mathbb{R}^2 \colon Y + Z \leq 2n/\ell, Y, Z \geq 0\}$, which becomes $\{(Y,Z) \in \mathbb{R}^2 \colon Y, Z \geq 0\}$ for n going to infinity

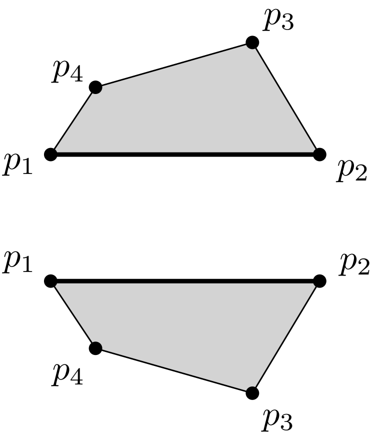
$|I_{Y/n}|$ and $|I_{Z/n}|$ both become $|I_0| = \ell$

By the Monotone Convergence Theorem, we have

$$\int_{Y=0}^{\infty} \int_{Z=0}^{\infty} \ell^2 e^{-\ell \cdot (Y+Z)/2} \, \mathrm{d}Z \mathrm{d}Y = 4.$$

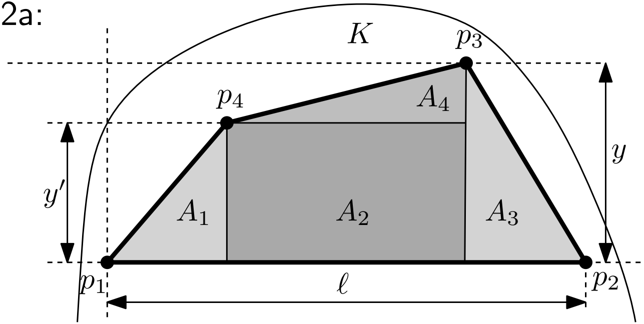
Sketch of the proof of $C_{2,4}^{K} = 10 - \frac{2\pi^2}{3}$

type 2: 4 symmetric subcases



Sketch of the proof of $C_{2,4}^K = 10 - \frac{2\pi^2}{3}$

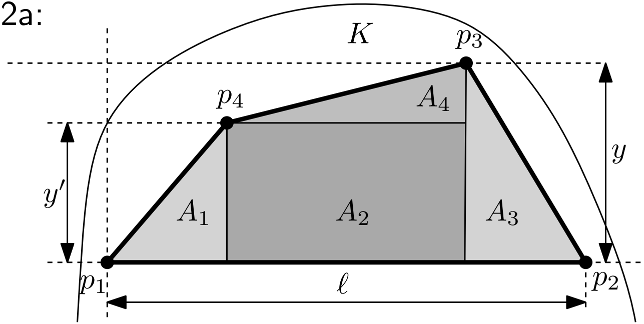
type 2a:



$$\text{area} = \frac{x'y'}{2} + (x - x')y' + \frac{(\ell - x)y}{2} + \frac{(x - x')(y - y')}{2} = \frac{(\ell - x')y + xy'}{2}.$$

Sketch of the proof of $C_{2,4}^K = 10 - \frac{2\pi^2}{3}$

type 2a:



$$\text{area} = \frac{x'y'}{2} + (x - x')y' + \frac{(\ell - x)y}{2} + \frac{(x - x')(y - y')}{2} = \frac{(\ell - x')y + xy'}{2}.$$

$$\int_0^{2/\ell} \int_{l(y)}^{r(y)} \int_0^{y} \int_{l(y')}^{xy'/y} \left(1 - \frac{(\ell - x')y + xy'}{2}\right)^{n-4} \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}x \, \mathrm{d}y = \ldots = 4 - \frac{\pi^2}{3}$$

Sketch of the proof of $C_{2,4}^K = 10 - \frac{2\pi^2}{3}$

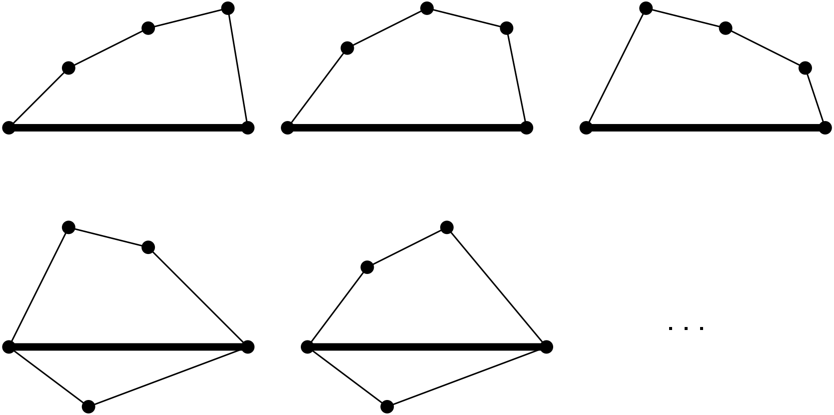
in total:

$$\left(\underbrace{4}_{\text{type 1}} + \underbrace{4 \cdot \left(4 - \frac{\pi^2}{3}\right)}_{\text{type 2}} \right) \cdot \binom{n}{2} = \left(20 - \frac{4}{3}\pi^2\right) \cdot \binom{n}{2}$$

Sketch of the proof of $C_{2,k}^{K} = C(k)$

Sketch of the proof of $C^K_{2,k} = C(k)$

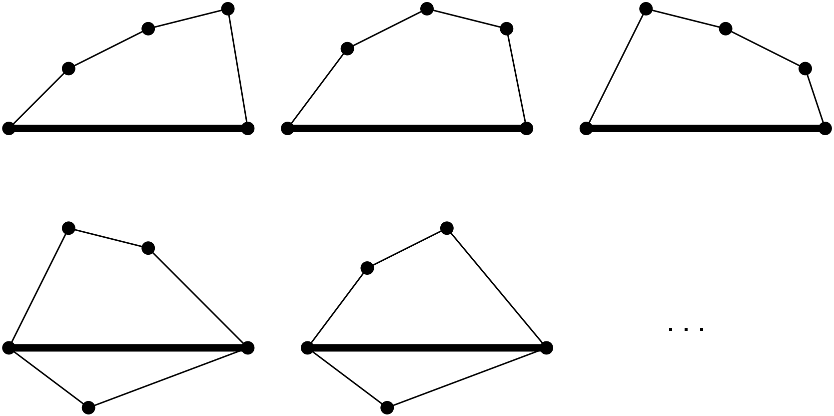
- finitely many types



$\cdots$

Sketch of the proof of $C_{2,k}^{K} = C(k)$

- finitely many types



- each type gives $c(1 + o(1))n^2$ where c does not depent on K

Thank you for your attention.

Open problems I

Open problems I

Theorem 3 (2021)

For every $d \geq 2$ and every convex $K \subseteq \mathbb{R}^d$, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)! p_{d-1}}$.

- By Theorem 3, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $C^K_{d,d+1}$.

Open problems I

Theorem 3 (2021)

For every $d \geq 2$ and every convex $K \subseteq \mathbb{R}^d$, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

- By Theorem 3, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $C^K_{d,d+1}$. However, we are not aware of any general bound on p_{d-1}.

Open problems I

Theorem 3 (2021)

For every $d \geq 2$ and every convex $K \subseteq \mathbb{R}^d$, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

- By Theorem 3, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $C^K_{d,d+1}$. However, we are not aware of any general bound on p_{d-1}. It is a major problem in convex geometry to decide whether p^K_d is maximized if K is a simplex (Sylvester's convex hull problem).

Open problems I

Theorem 3 (2021)

For every $d \geq 2$ and every convex $K \subseteq \mathbb{R}^d$, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

- By Theorem 3, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $C^K_{d,d+1}$. However, we are not aware of any general bound on p_{d-1}. It is a major problem in convex geometry to decide whether p^K_d is maximized if K is a simplex (Sylvester's convex hull problem).
- We have an estimate on p^K_d for bodies of K of small diameter though.

Open problems I

Theorem 3 (2021)

For every $d \geq 2$ and every convex $K \subseteq \mathbb{R}^d$, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

- By Theorem 3, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $C^K_{d,d+1}$. However, we are not aware of any general bound on p_{d-1}. It is a major problem in convex geometry to decide whether p^K_d is maximized if K is a simplex (Sylvester's convex hull problem).
- We have an estimate on p^K_d for bodies of K of small diameter though.

Proposition (2021)

For $\varepsilon > 0$ and $d \geq 1$, let $K \subseteq \mathbb{R}^d$ be a convex body of unit volume. If K has diameter at most $d^{1-\varepsilon}$, then $p^K_d \leq \frac{(d+2)d^{(1-\varepsilon)d}}{d!}$.

Open problems I

Theorem 3 (2021)

For every $d \geq 2$ and every convex $K \subseteq \mathbb{R}^d$, we have $C^K_{d,d+1} \geq \frac{2}{(d-1)!p_{d-1}}$.

- By Theorem 3, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $C^K_{d,d+1}$. However, we are not aware of any general bound on p_{d-1}. It is a major problem in convex geometry to decide whether p^K_d is maximized if K is a simplex (Sylvester's convex hull problem).
- We have an estimate on p^K_d for bodies of K of small diameter though.

Proposition (2021)

For $\varepsilon > 0$ and $d \geq 1$, let $K \subseteq \mathbb{R}^d$ be a convex body of unit volume. If K has diameter at most $d^{1-\varepsilon}$, then $p^K_d \leq \frac{(d+2)d^{(1-\varepsilon)d}}{d!}$.

- Kingman proved exact formula for $p^{B^d}_d$, which gives $p^{B^d}_d = d^{-\Theta(d)}$.

Open problems II

Open problems II

- We conjecture that this is the right growth rate of p_d^K for any K.

Open problems II

- We conjecture that this is the right growth rate of p_d^K for any K.

Conjecture

There is a constant $c > 0$ such that, for every $d \geq 2$, we have $p_d \leq d^{-cd}$.

Open problems II

- We conjecture that this is the right growth rate of p_d^K for any K.

Conjecture

There is a constant $c > 0$ such that, for every $d \geq 2$, we have $p_d \leq d^{-cd}$.

- We also believe that our lower bound on $C^K_{d,d+1}$ from Theorem 3 is tight for simplices in any dimension d.

Open problems II

- We conjecture that this is the right growth rate of p_d^K for any K.

Conjecture

There is a constant $c > 0$ such that, for every $d \geq 2$, we have $p_d \leq d^{-cd}$.

- We also believe that our lower bound on $C^K_{d,d+1}$ from Theorem 3 is tight for simplices in any dimension d.

Conjecture

For every $d \geq 2$, if K is a d-dimensional simplex of unit volume, then

$$C^K_{d,d+1} = \frac{2}{(d-1)!\,p_{d-1}}.$$

Open problems II

- We conjecture that this is the right growth rate of p_d^K for any K.

Conjecture

There is a constant $c > 0$ such that, for every $d \geq 2$, we have $p_d \leq d^{-cd}$.

- We also believe that our lower bound on $C^K_{d,d+1}$ from Theorem 3 is tight for simplices in any dimension d.

Conjecture

For every $d \geq 2$, if K is a d-dimensional simplex of unit volume, then

$$C^K_{d,d+1} = \frac{2}{(d-1)!p_{d-1}}.$$

Thank you for your attention.