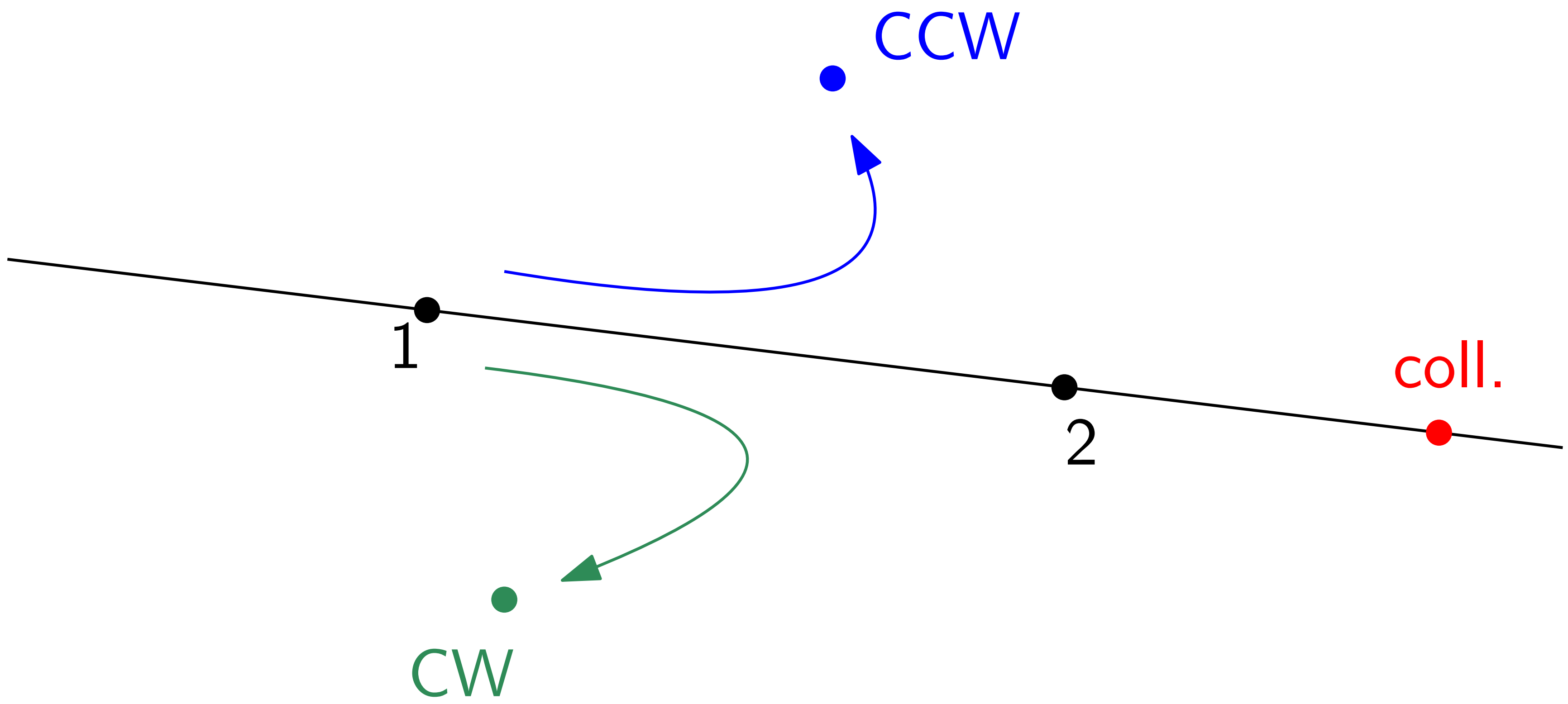
Minimal Geometric Graph Representations of Order Types

Oswin Aichholzer, Martin Balko, Michael Hoffmann, Jan Kynčl, Wolfgang Mulzer, Irene Parada, Alexander Pilz, Manfred Scheucher, Pavel Valtr, Birgit Vogtenhuber, Emo Welzl

Order Types

triple orientations: clockwise, counter clockwise, collinear



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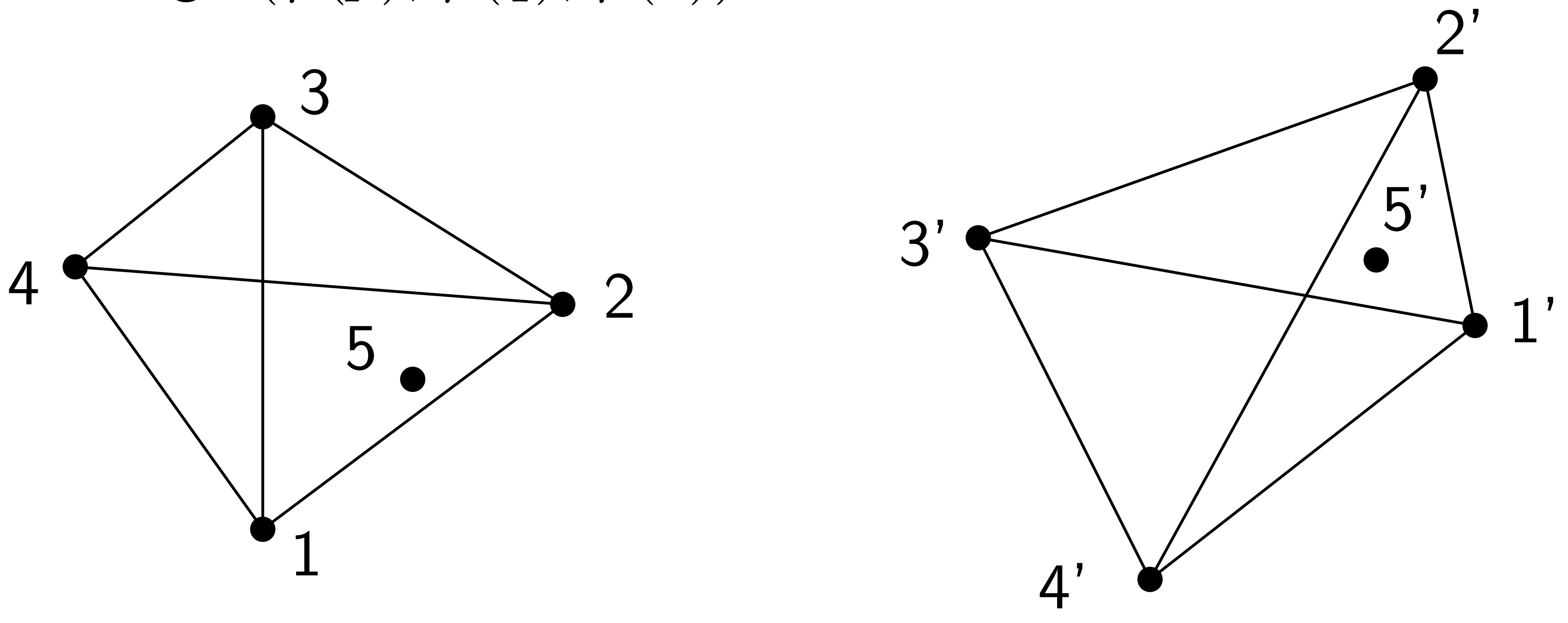
[Goodman and Pollack '83]: two point sets S and T *have the same order type* if there is a bijection $\varphi : S \to T$ such that any triple $(p, q, r) \in S^3$ has the same orientation as the image $(\varphi(p), \varphi(q), \varphi(r)) \in T^3$



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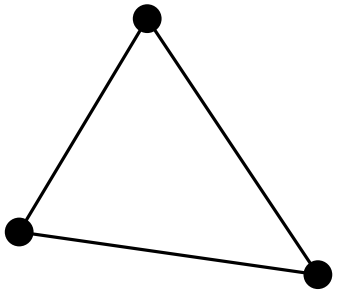
equivalence relation on point sets

equivalence classes: the *order types*

fixed size $\Rightarrow$ finitely many classes

Order Types

$n = 3$:



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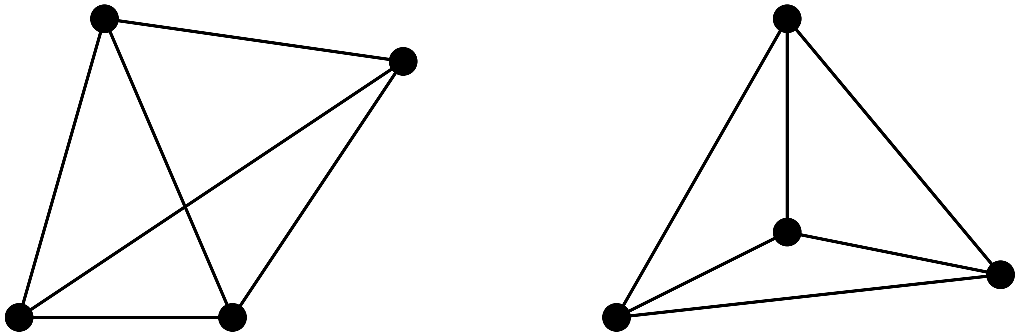

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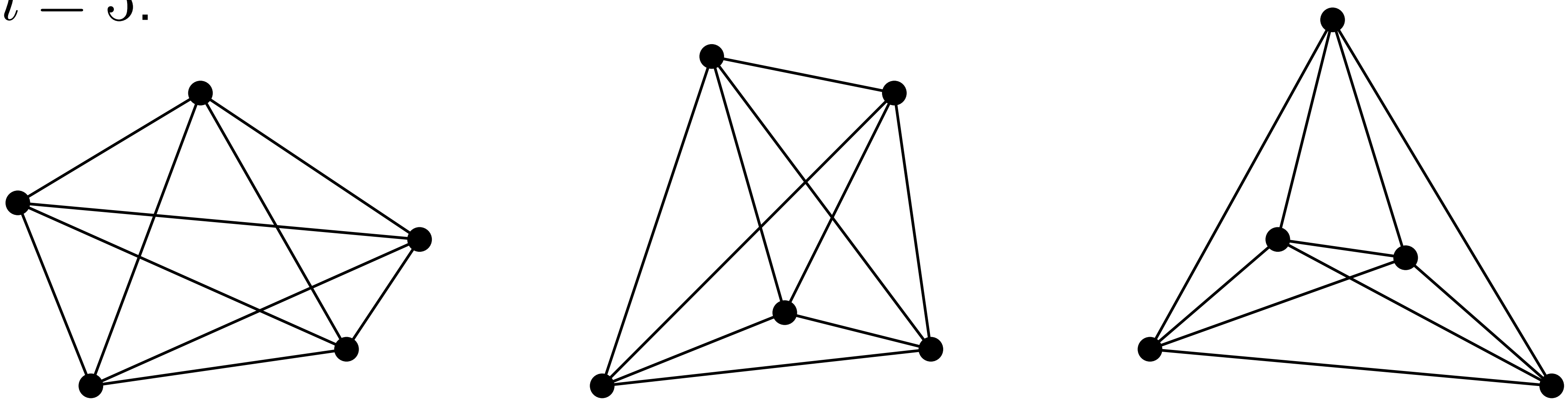


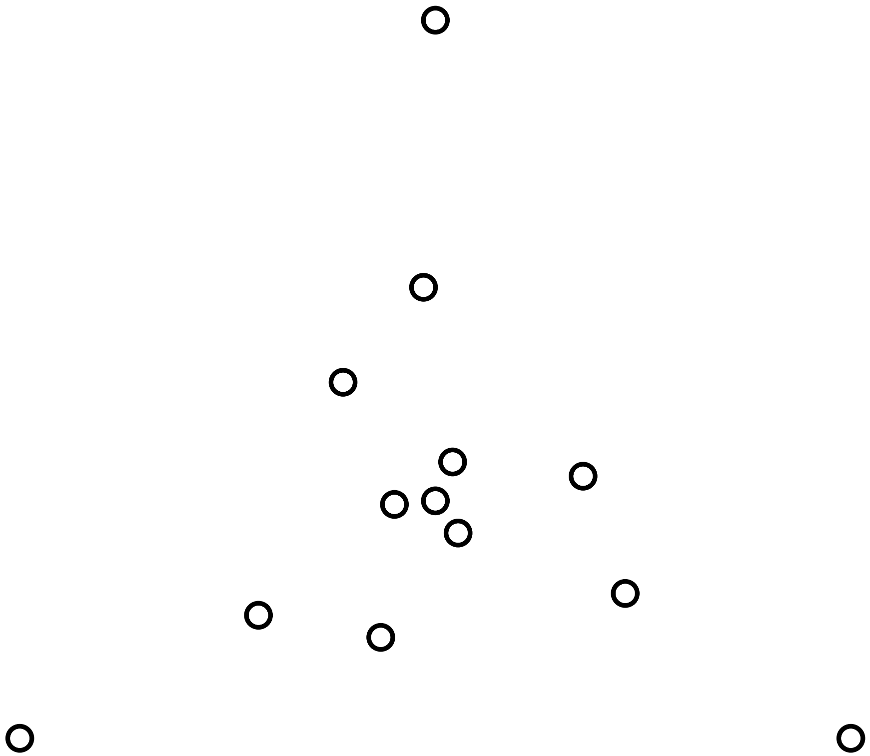
Point Set Representation

- List of coordinates

0160 7359
1768 6530
2592 6679
4239 6383
3955 5593
2960 5759
2338 4960
2880 4320
2960 2520
5759 7359
3076 5497
2684 5783
3113 5976

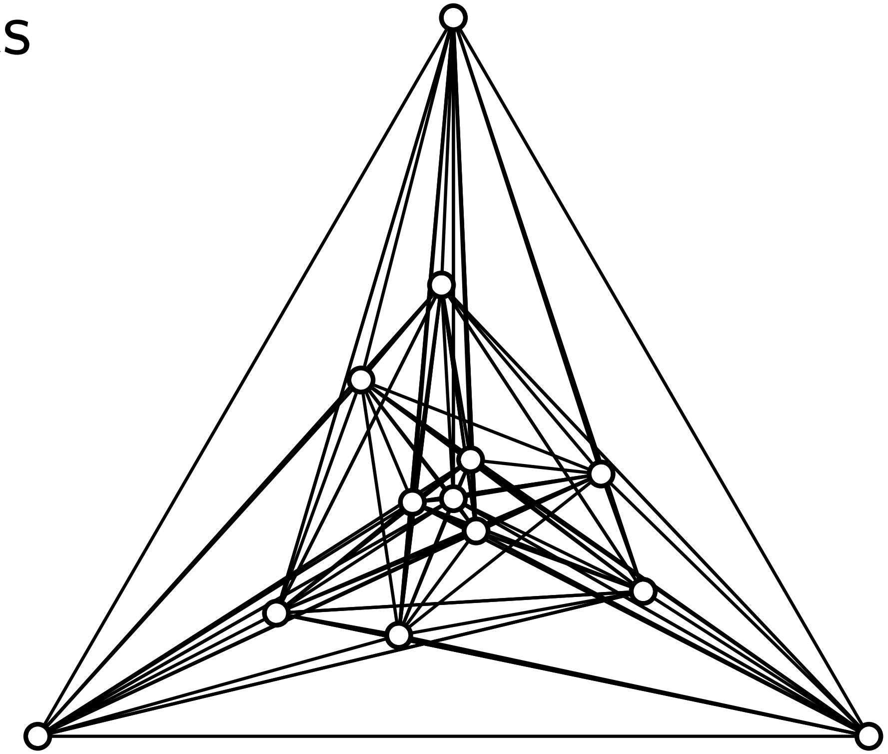
Point Set Representation

- List of coordinates
- Figure of the point set



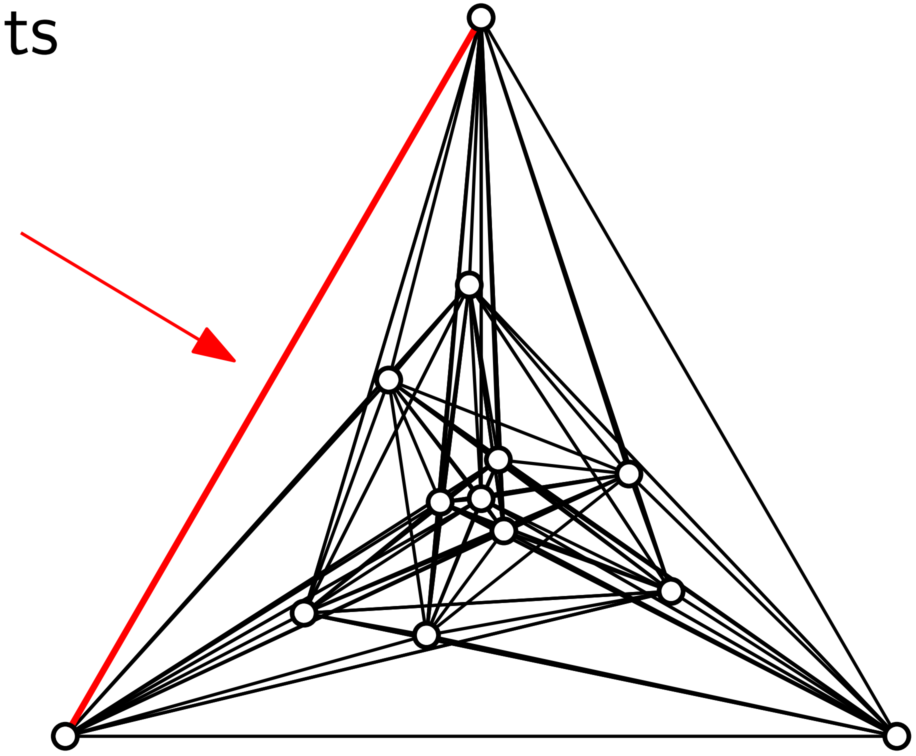
Point Set Representation

- List of coordinates
- Figure of the point set
- + spanned lines / segments



Point Set Representation

- List of coordinates
- Figure of the point set
- + spanned lines / segments
- ⇒ identification of (non)redundant edges!



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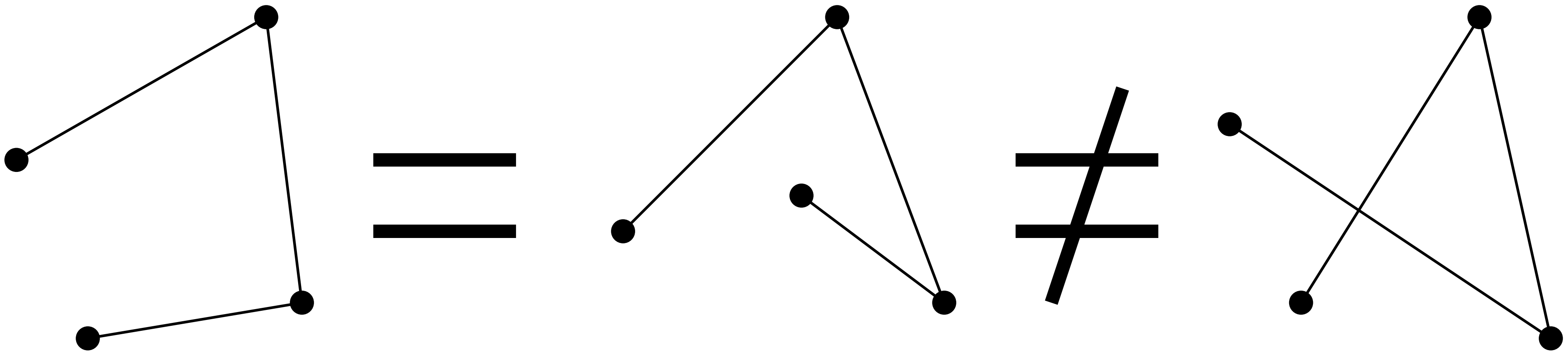
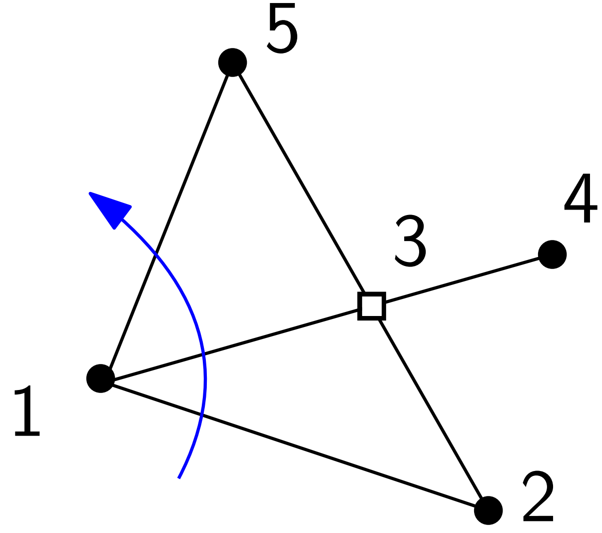


Geometric Graphs

- *geometric graph (on S)*: vertices mapped to set S, edges drawn as straight-line segments
- geometric graphs G, H *topologically equivalent* if $\exists$ homeomorphism of the plane transforming G into H
- equivalence class describable by cyclic order around vertices and crossings



Geometric Graphs

- we consider "topology-preserving deformations"

Definition: A geometric graph G *supports* a set S of points if every "continuous deformation" that

- keeps edges straight and
- preserves topological equiv.

also preserves the order type of the vertex set.

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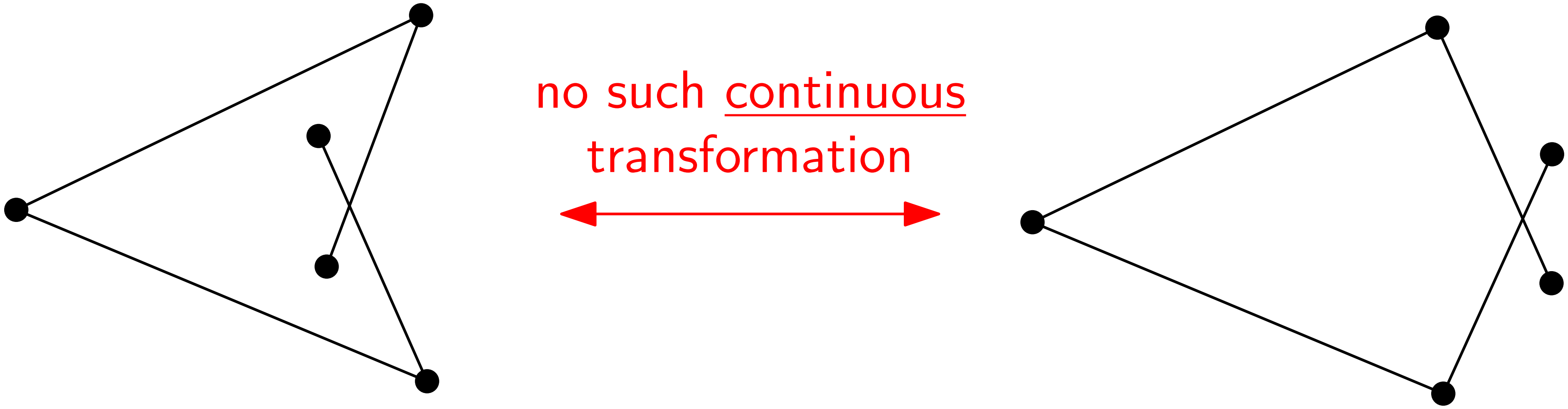


Geometric Graphs

Definition: A geometric graph G *supports* a set S of points if every ambient isotopy that

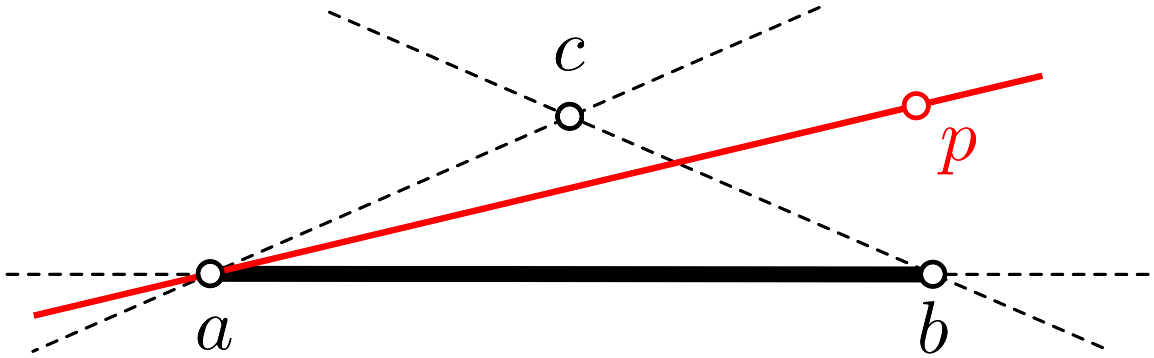
- keeps edges straight and
- preserves topological equiv.

also preserves the order type of the vertex set.

continuous map $f\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ is *ambient isotopy* if $f(\cdot,t)$ is homeomorphism $\forall t \in [0,1]$ and $f(\cdot,0) = Id$

Exit Edges

- S finite point set in general position
- ab *exit edge with witness* c if $\nexists p \in S$ s.t. line $\overline{ap}$ separates b from c or $\overline{bp}$ separates a from c



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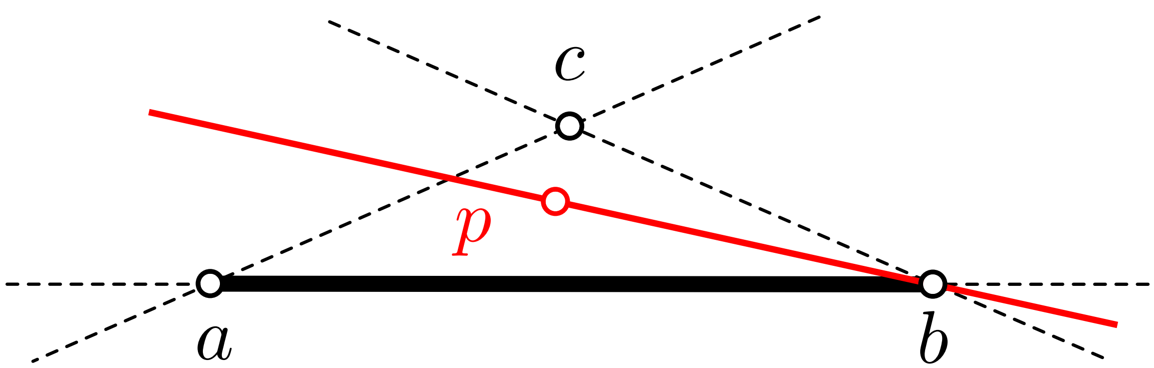

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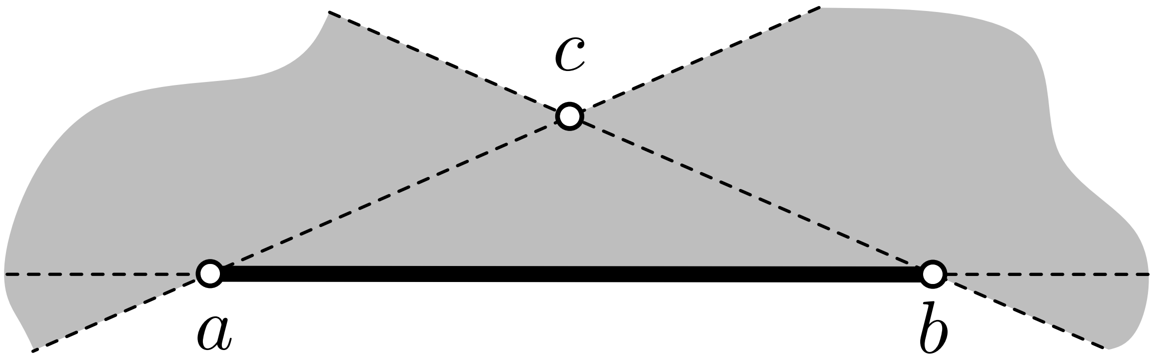

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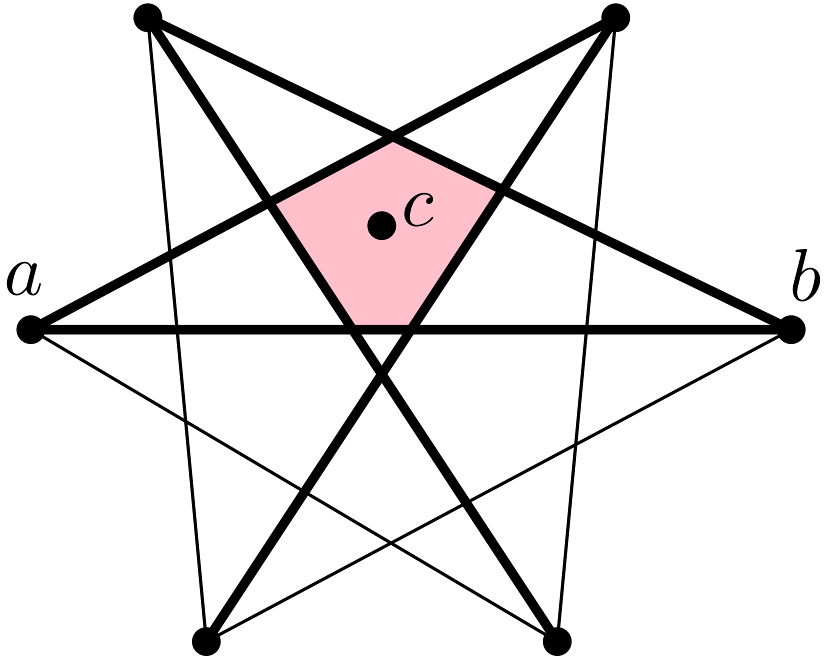
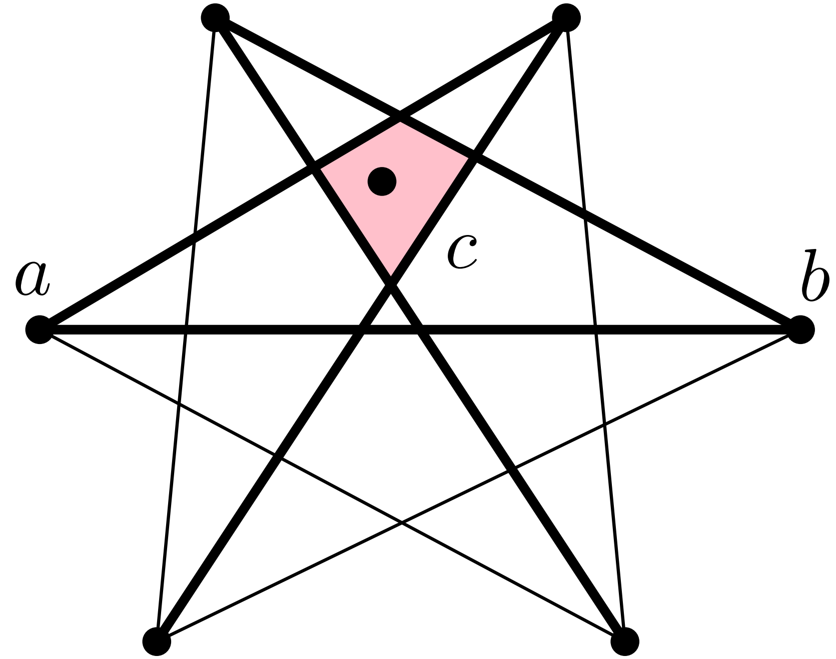
- $\Rightarrow$ *exit graph of* S

Exit Edges

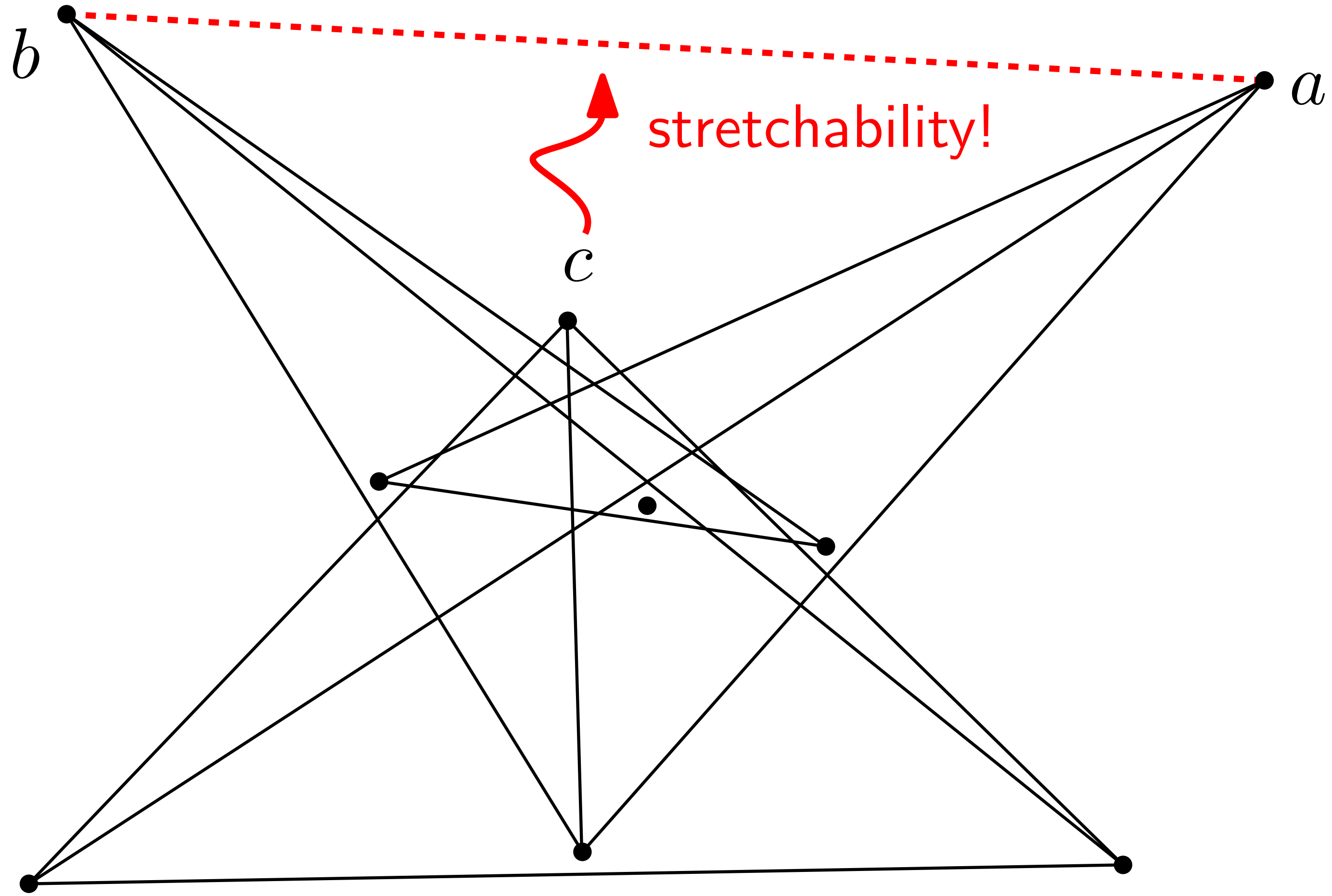
- other lines might prevent witness from passing exit edge





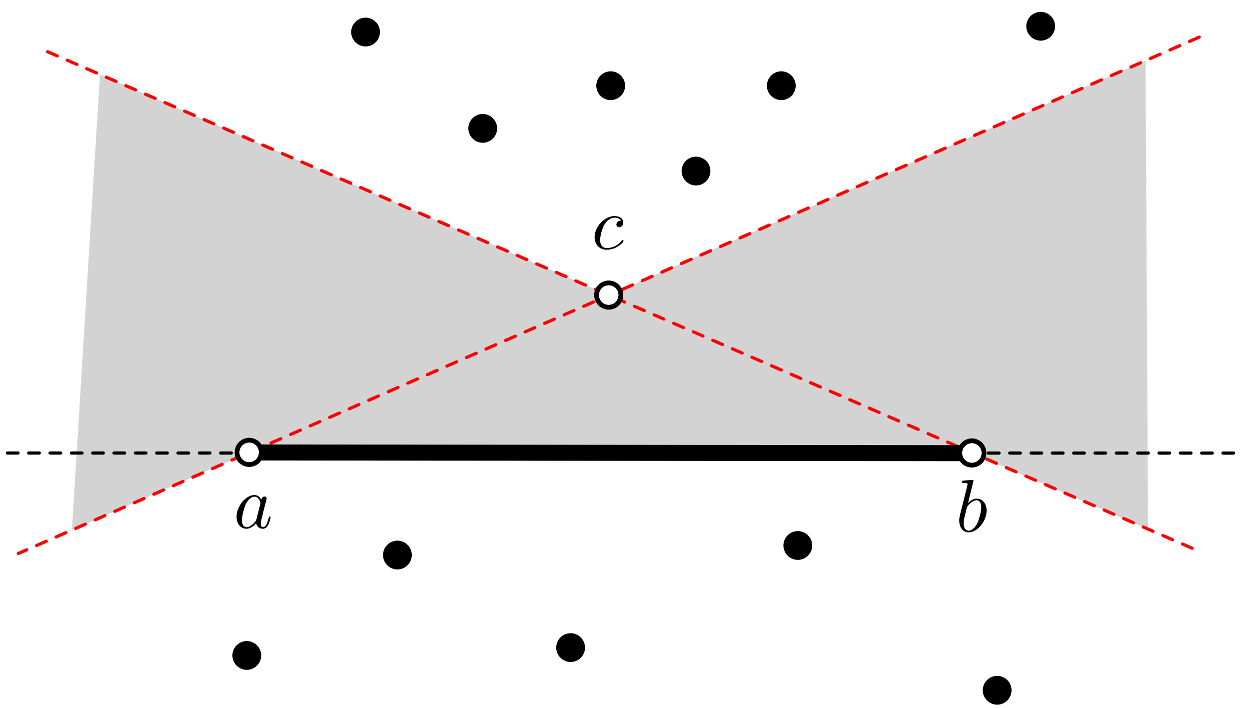
Exit Edges

- … and even worse…



Exit Edges

Proposition. S ... point set in general position
$S(t)$ ... continuous deformation of S
(a, b, c) ... first triple to become collinear at time $t_0 > 0$
If c lies on segment ab in $S(t_0)$,
then ab is an exit edge in $S(0)$ with witness c



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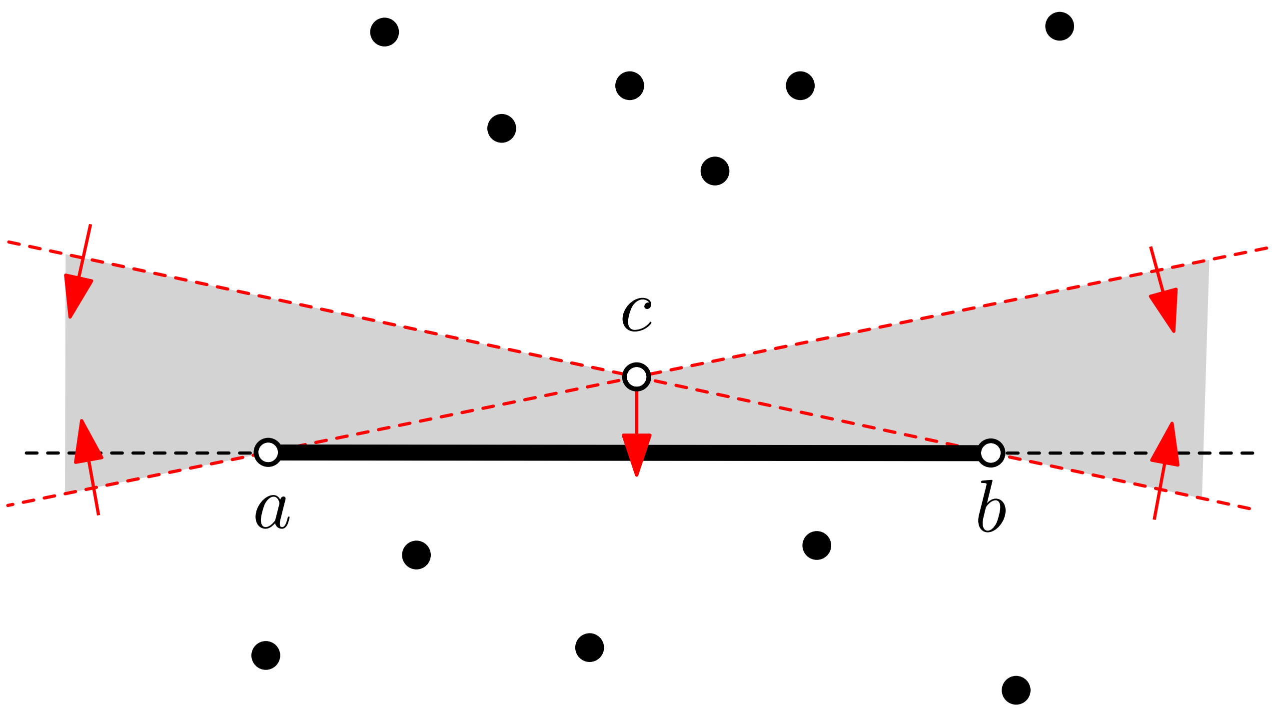

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Corollary. The exit graph of every point set is supporting.

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Corollary. The exit graph of every point set is supporting.

- the inversion of the statement is not true in general – exit edges might not be necessary for a supporting graph
- strongly related to "minimal reduced systems" [Bokowski and Sturmfels '86]

Exit Edges

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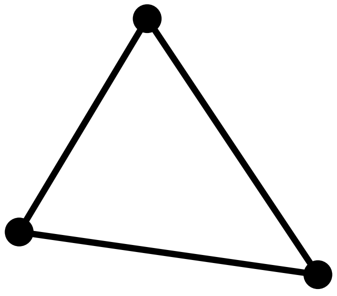

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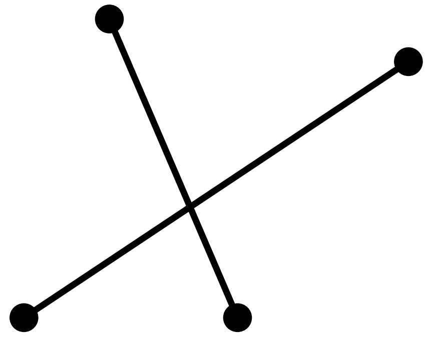

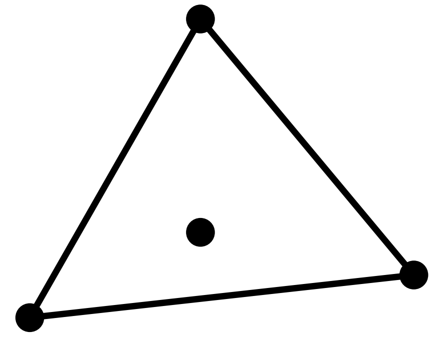

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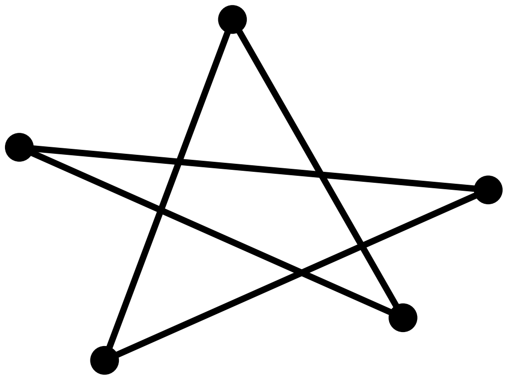

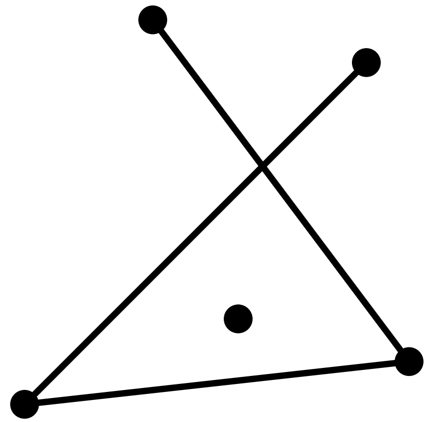

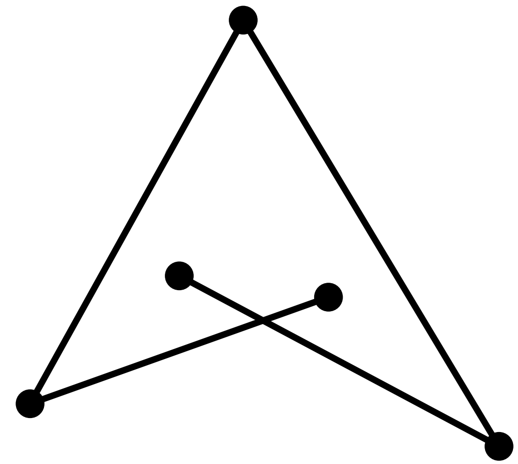

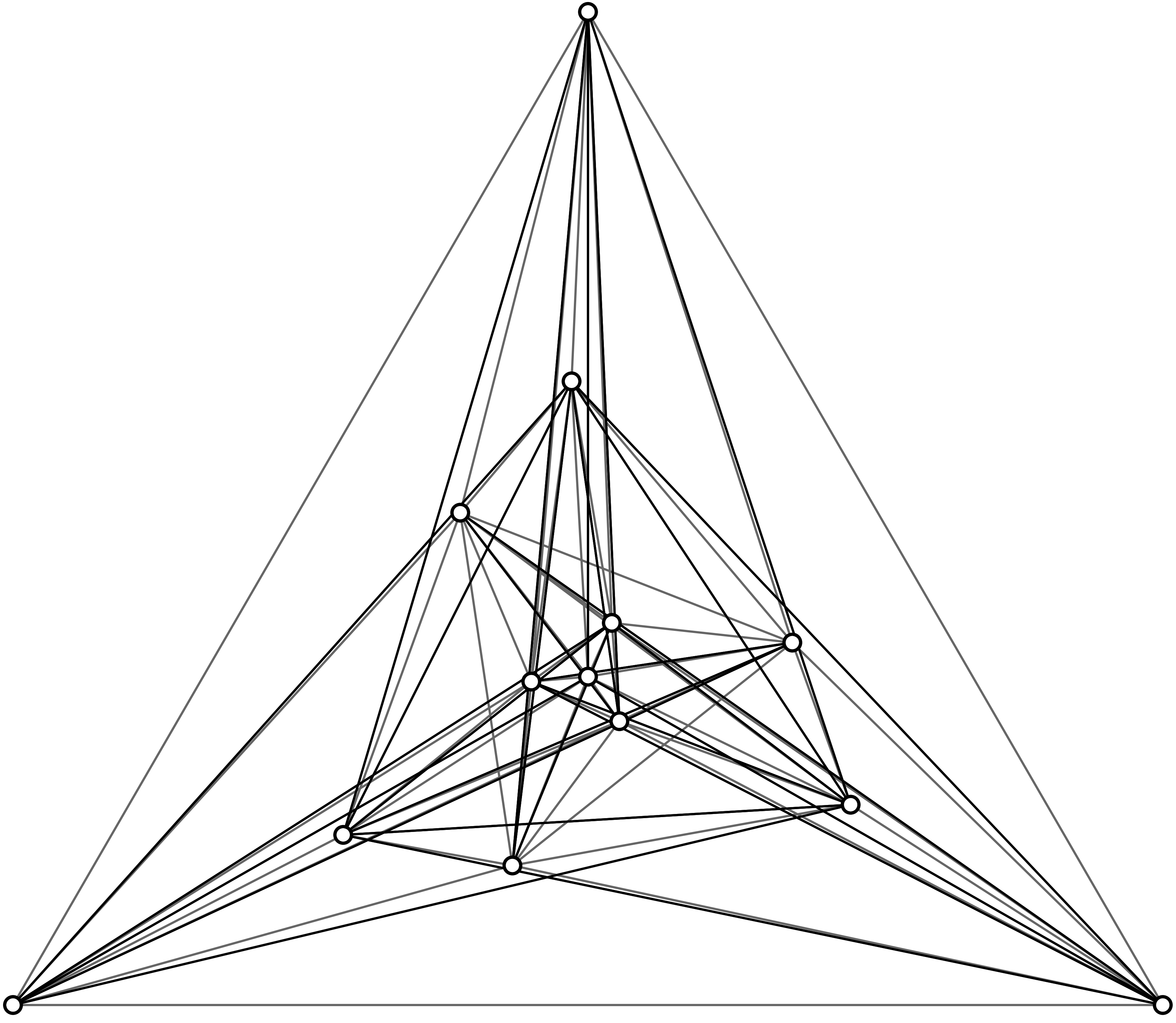

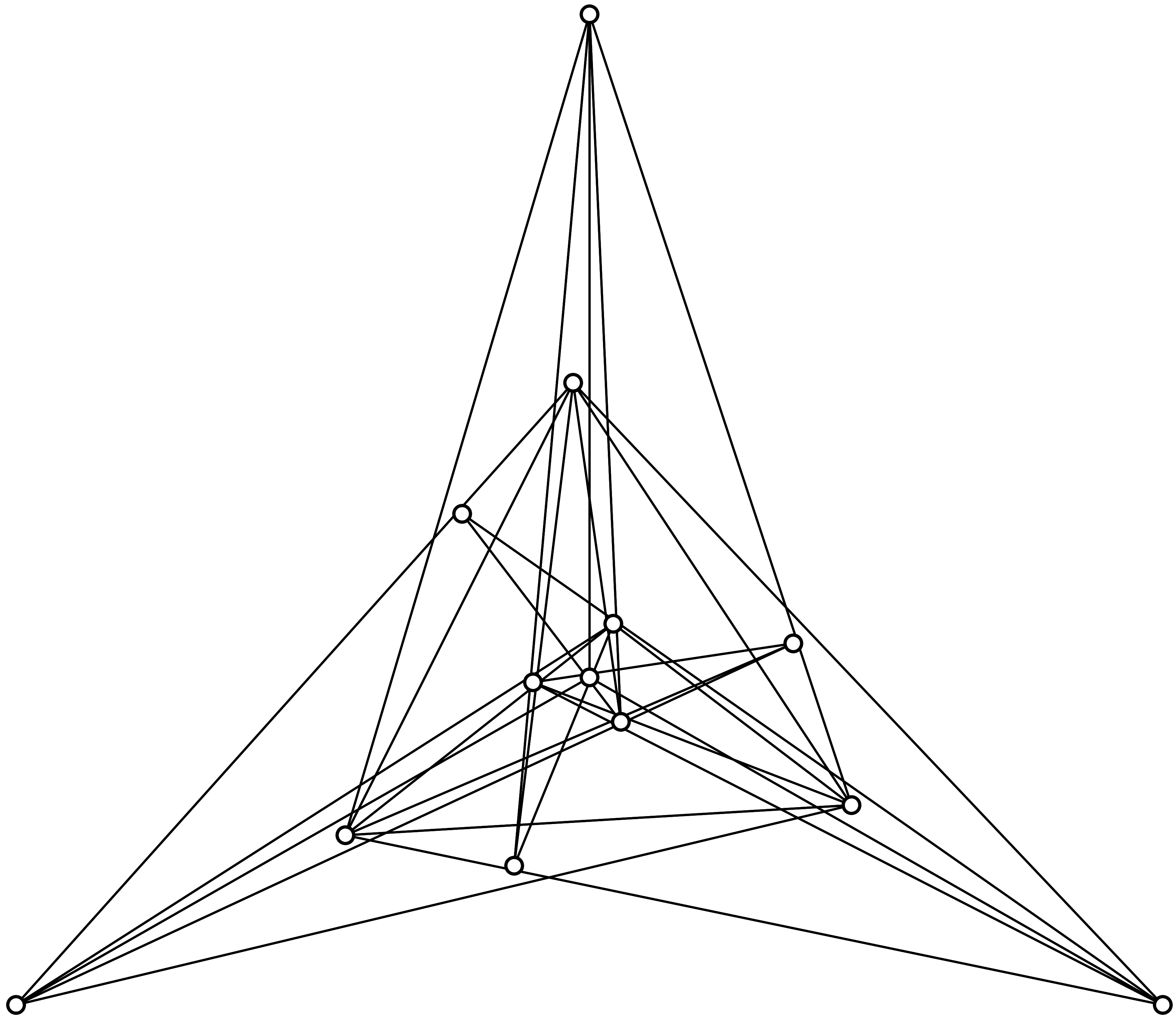

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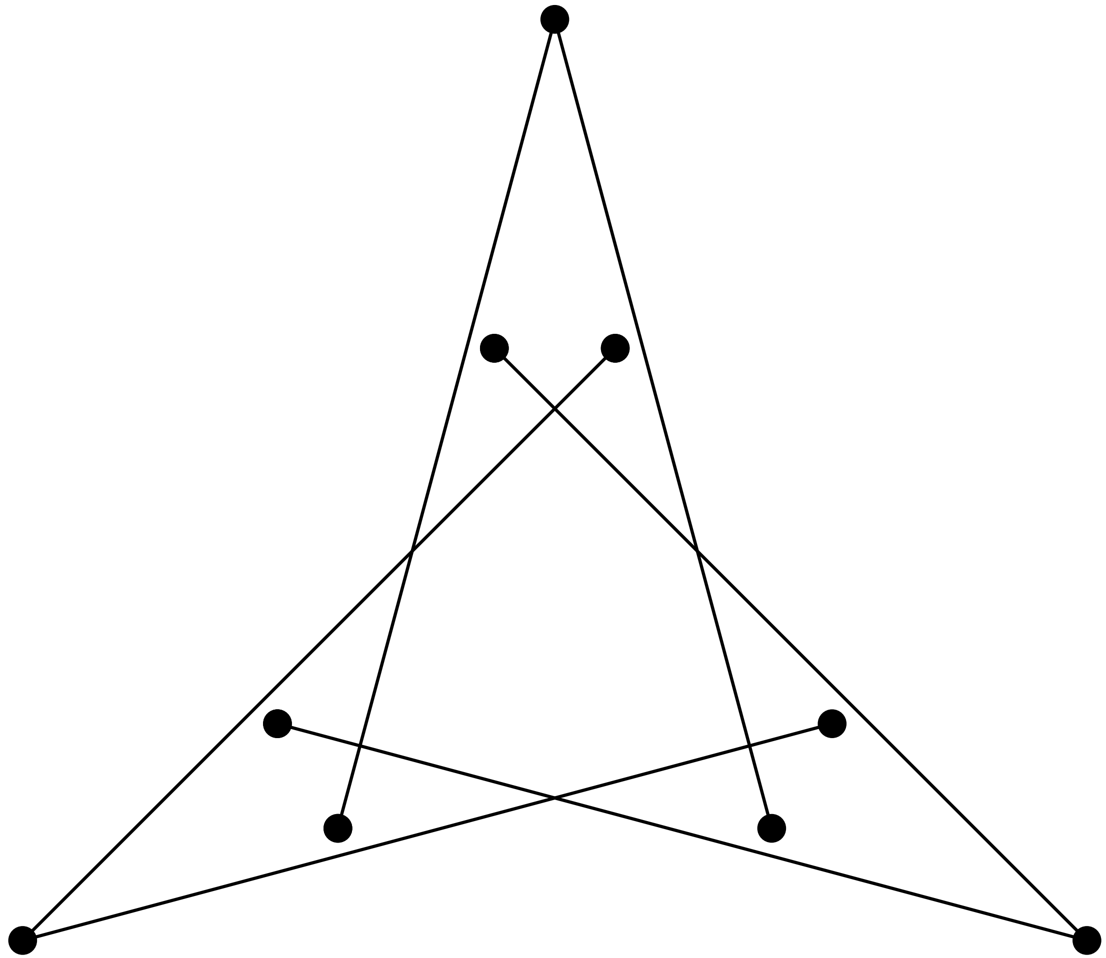
Properties

Lower Bound: $\frac{3n}{5} + O(1)$ bound . . . $n - 3$ construction



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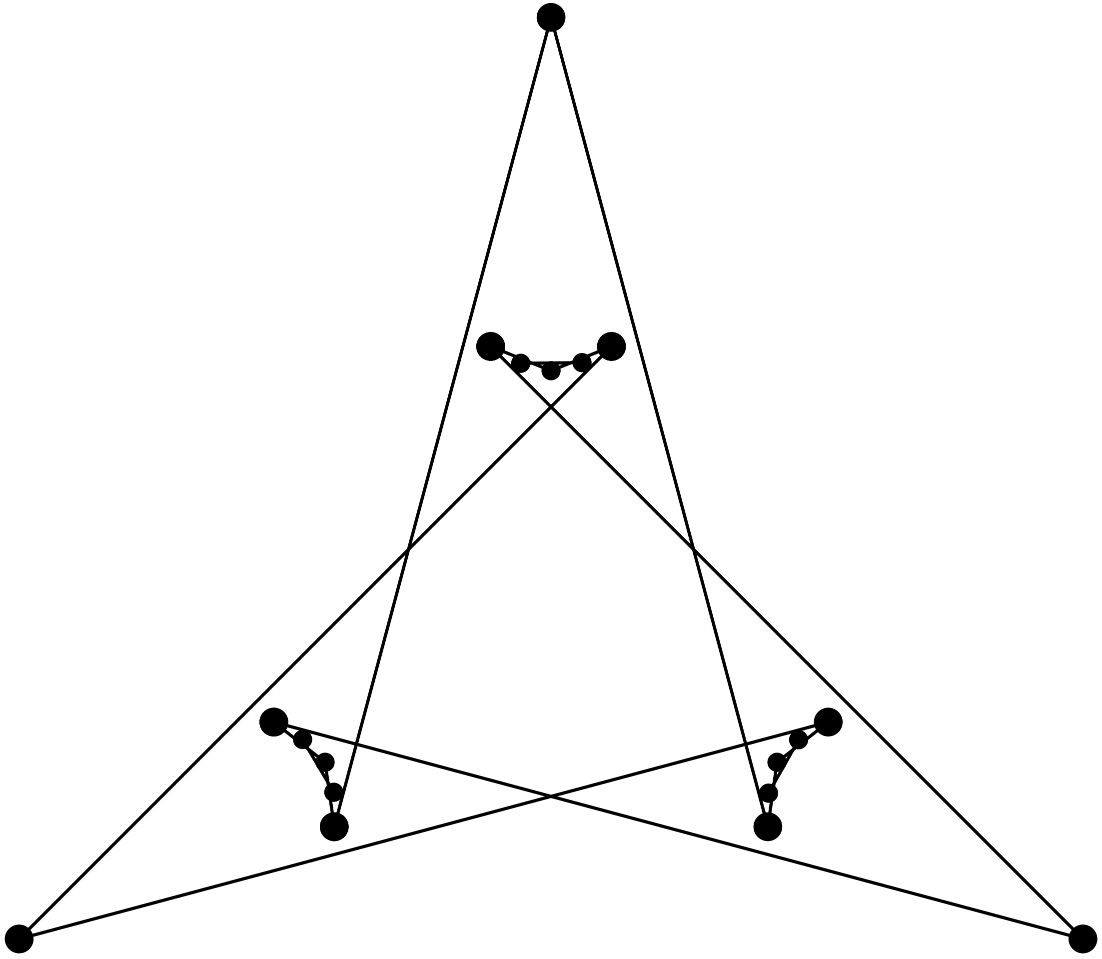
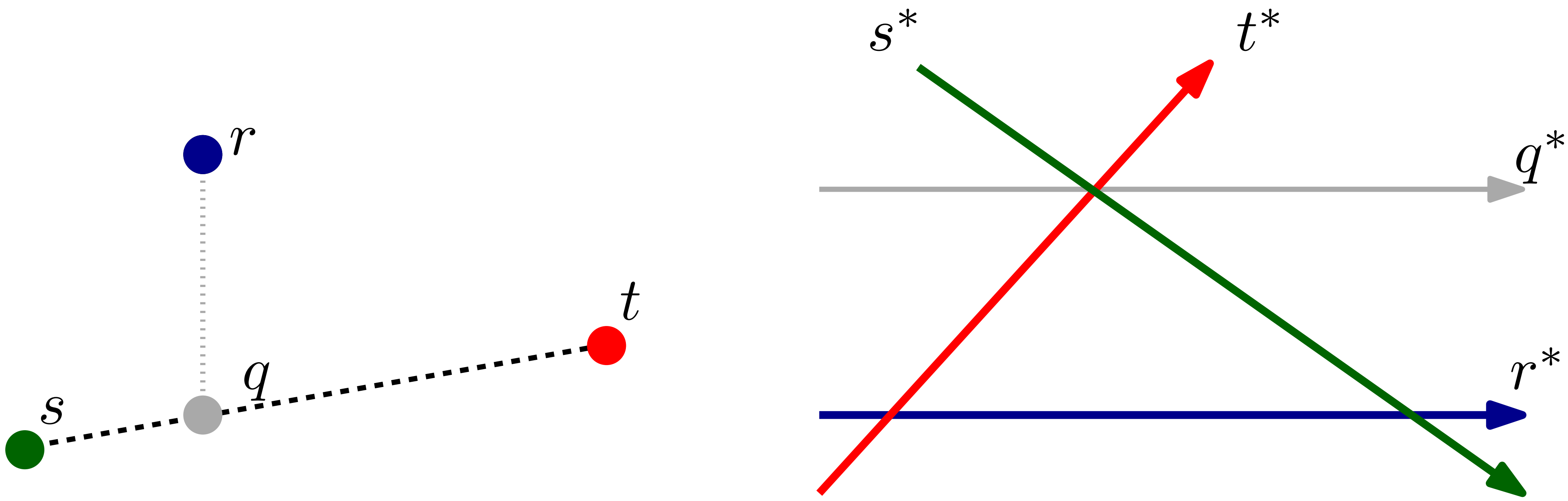


Properties

Lower Bound: $\frac{3n}{5} + O(1)$ bound $\ldots n-3$ construction

Upper Bound: $\Theta(n^2)$
(empty $\triangle$ in line arr., $\leq \frac{n(n-1)}{3}$ [Roudneff '72, Blanc '11])



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Continuously morph S into S', keeping planarity and topologically equivalence to G.
[Alamdari, Angelini, Barrera-Cruz, Chan, Da Lozzo, Di Battista, Frati, Haxell, Lubiw, Patrignani, Roselli, Singla, Wilkinson '17]

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Continuously morph S into S', keeping planarity and topologically equivalence to G [Alamdari et al. '17]

$\Rightarrow$ G does not support S. $\square$

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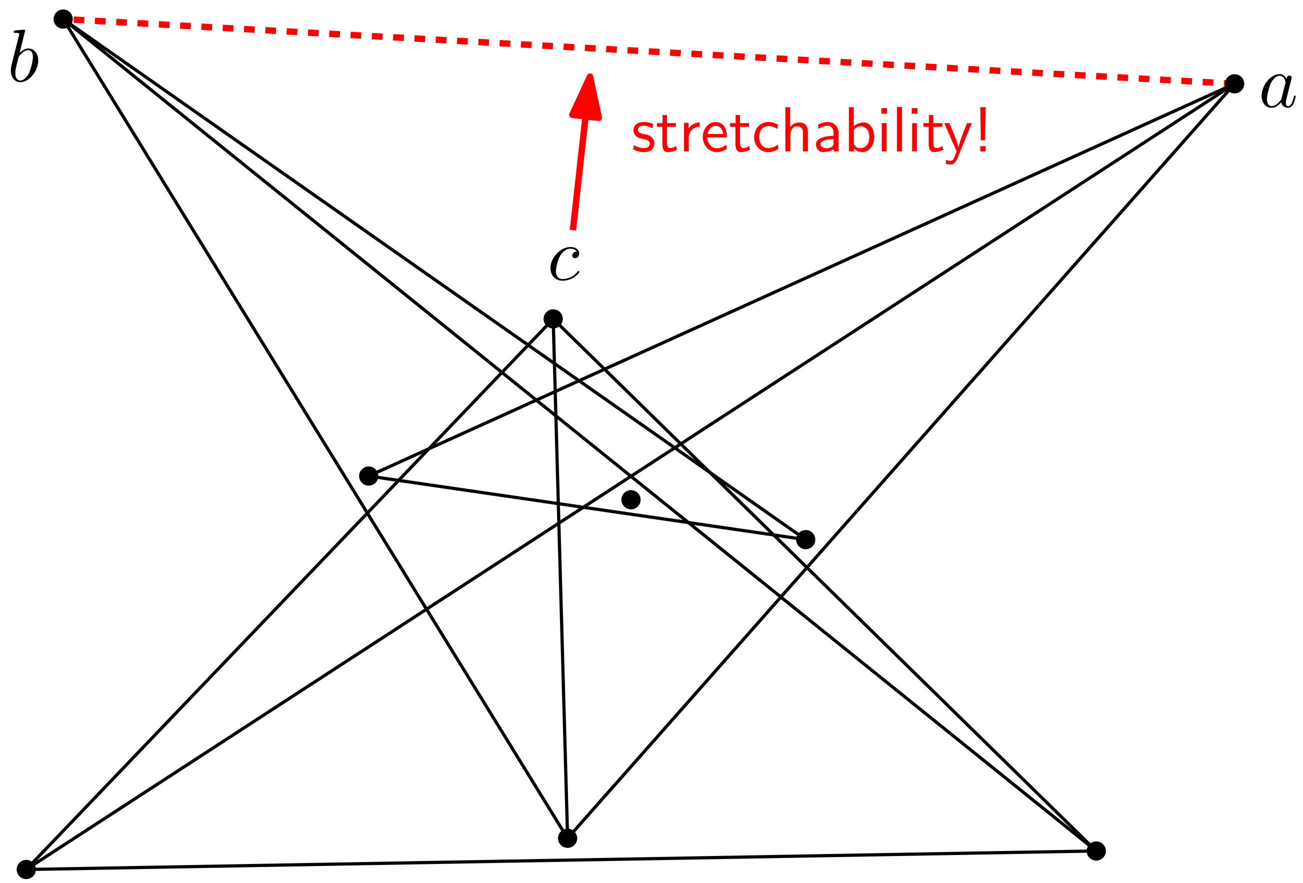

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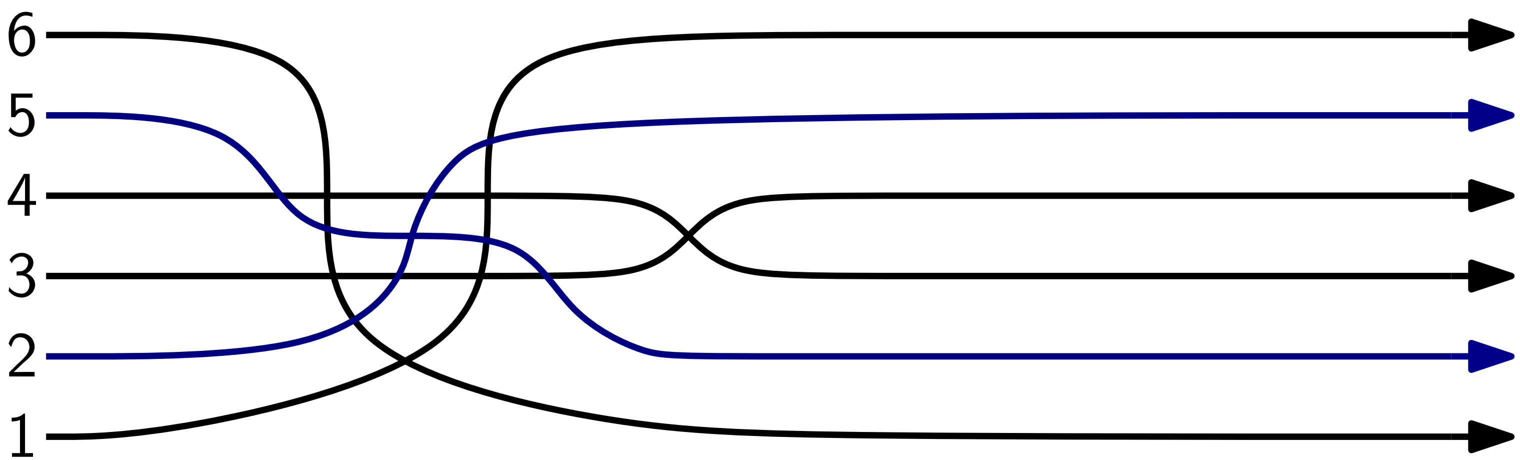
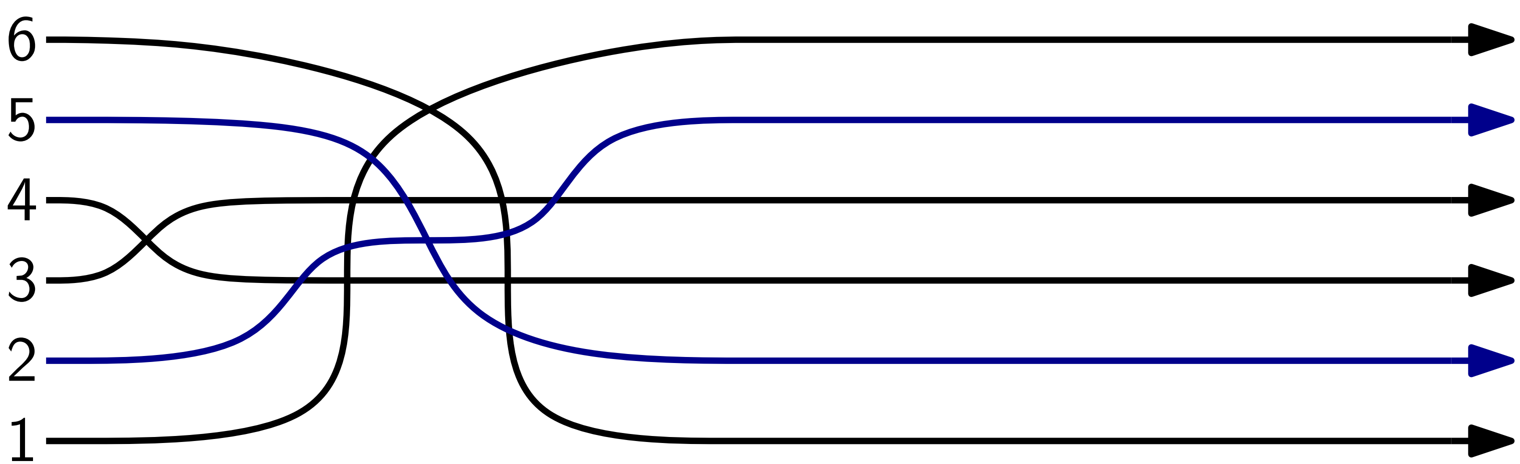


the construction based on example of two line arrangements with the "same" triangles [Felsner and Weil '00]



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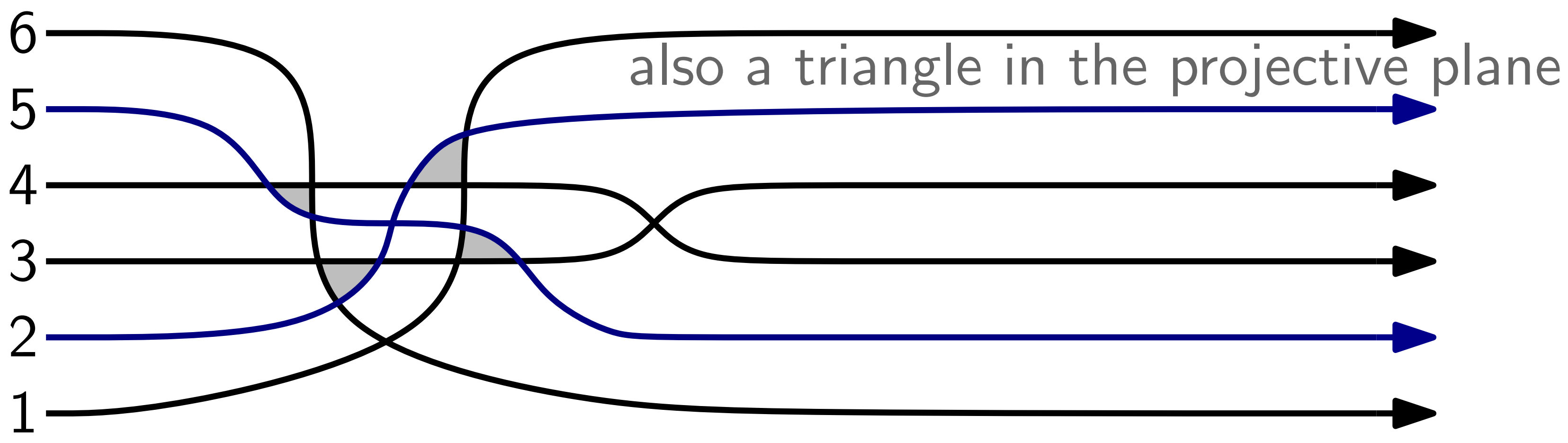

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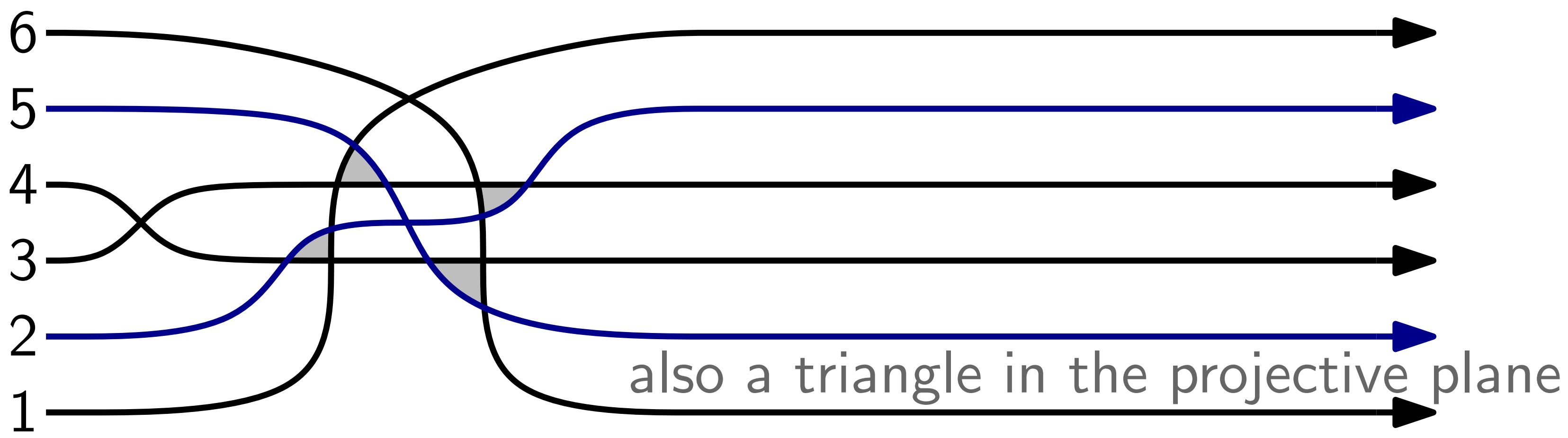

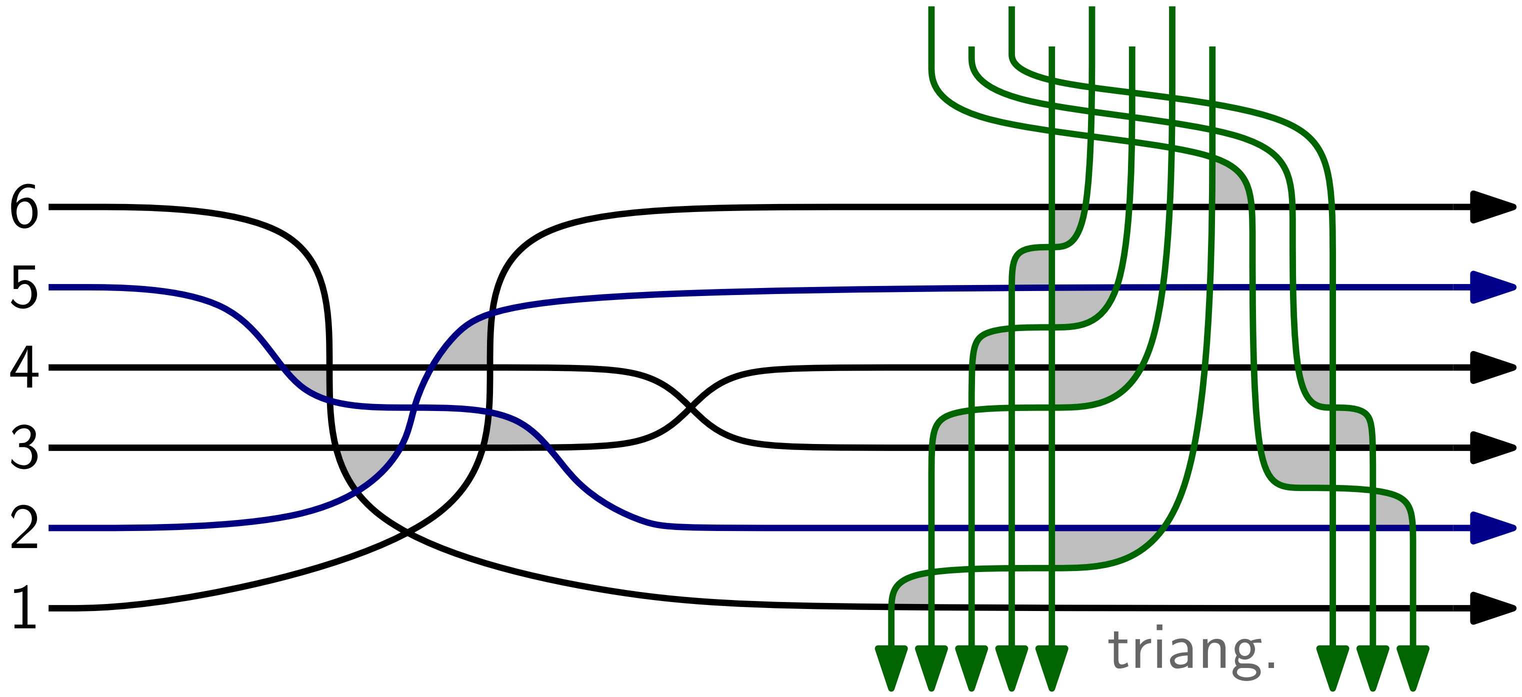
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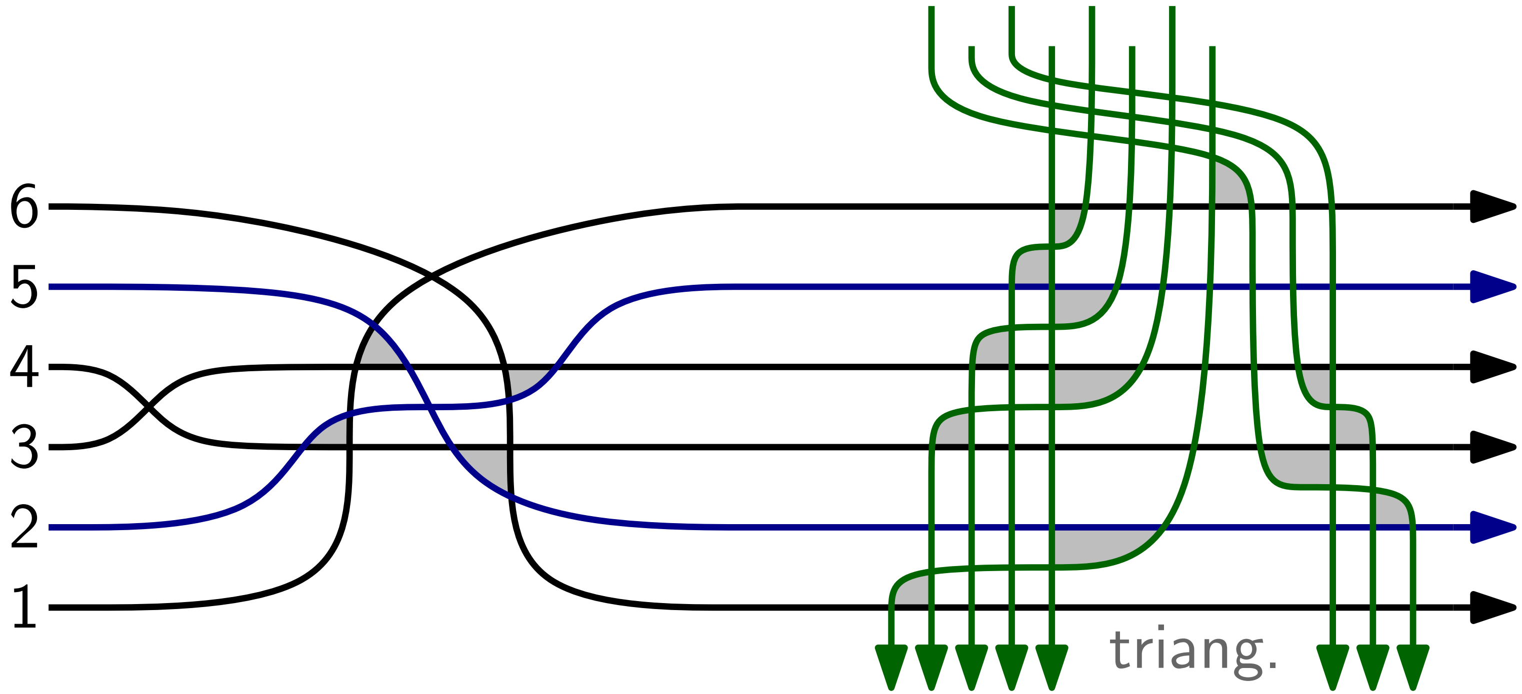
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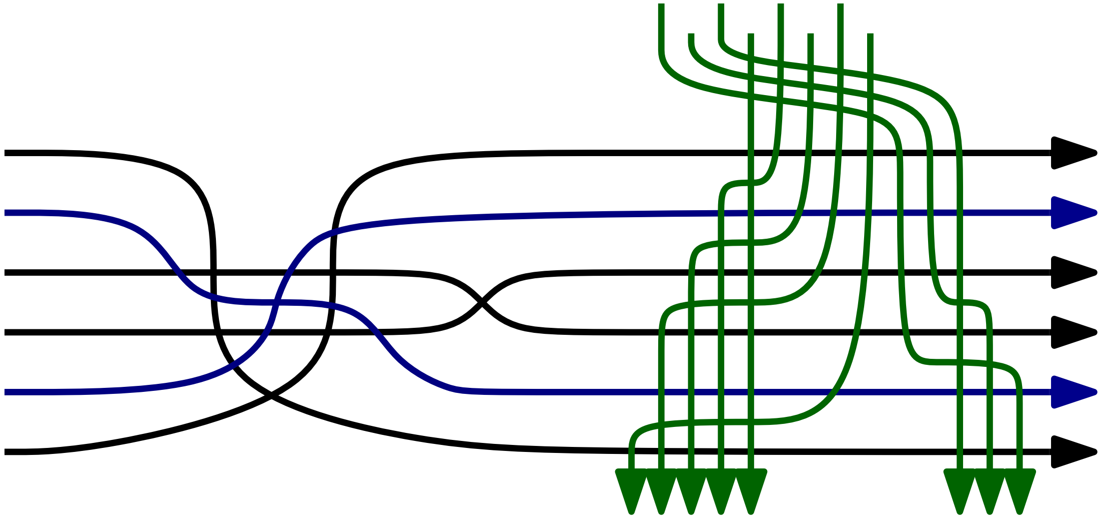
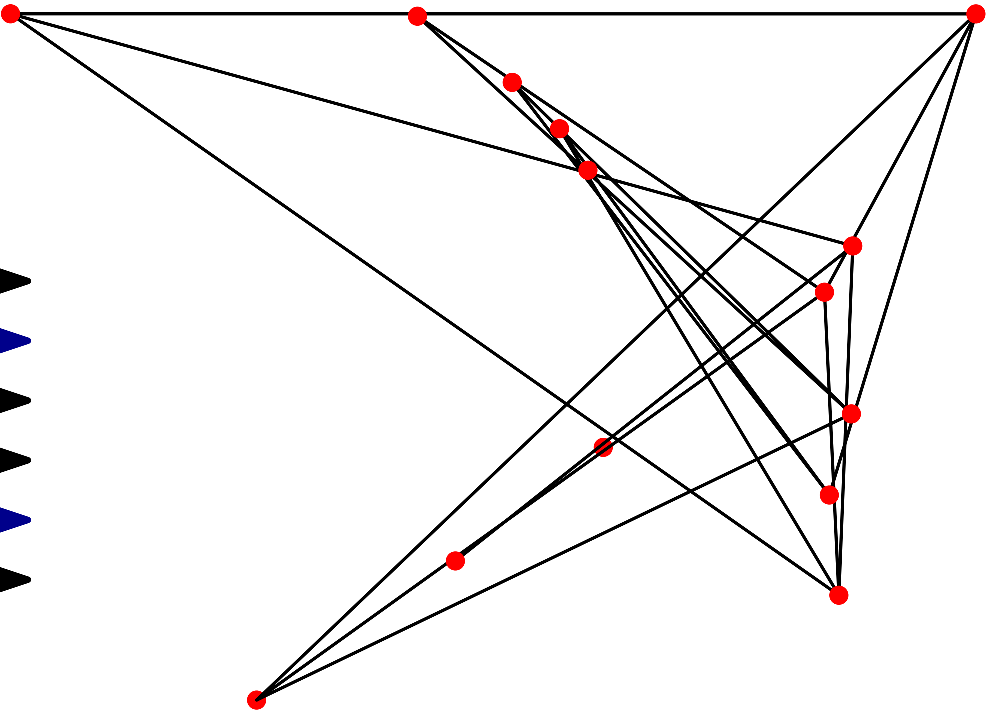
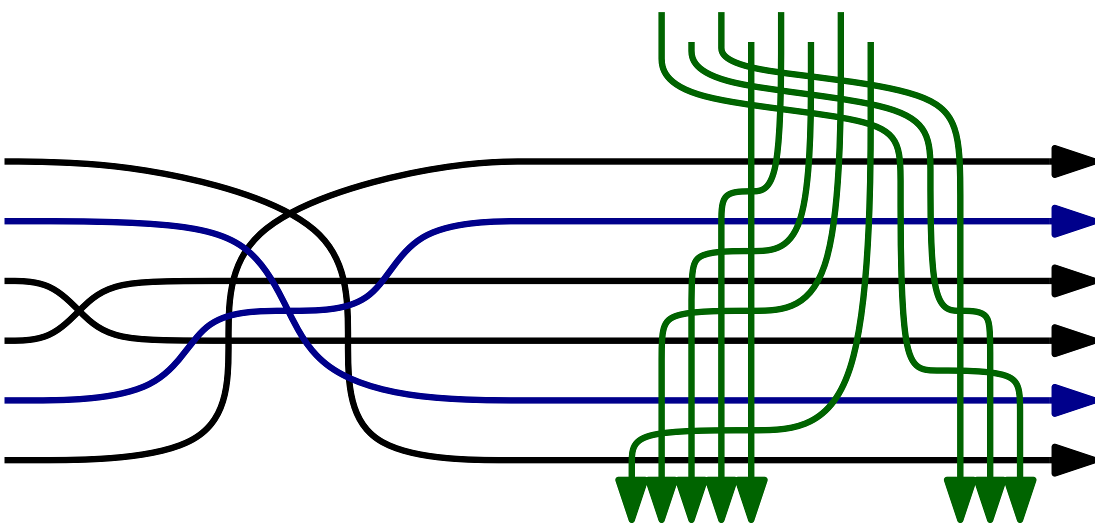
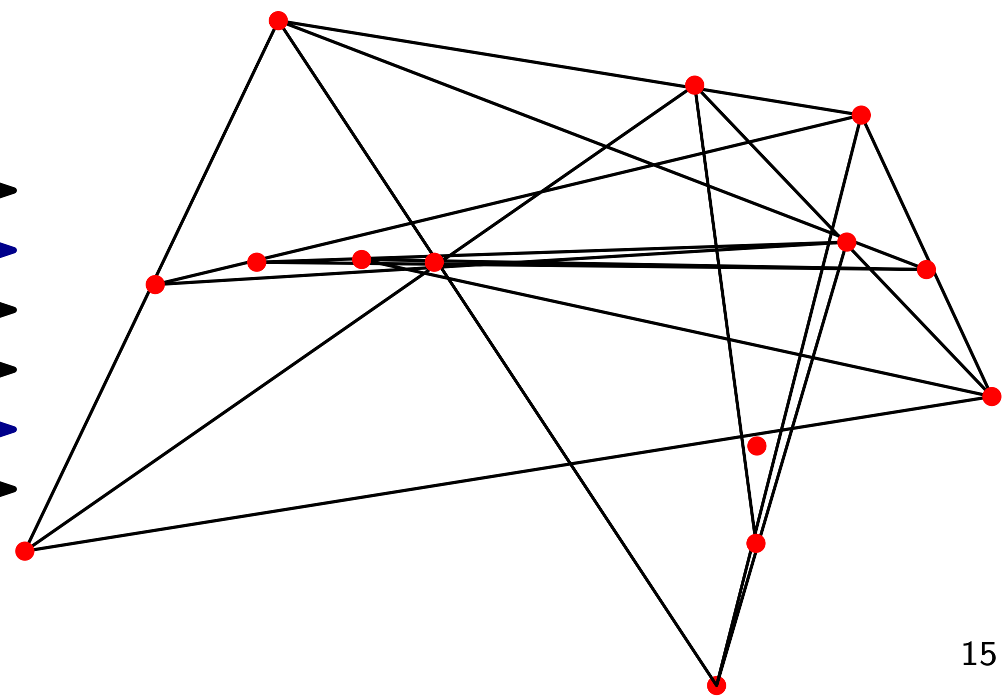

Thank you for your attention!