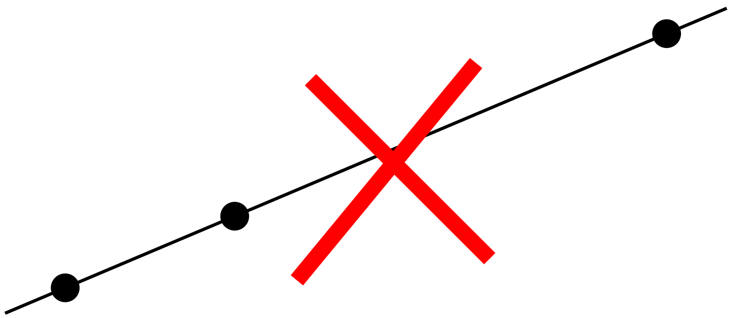
Erdős–Szekeres–type Problems on Planar Point Sets

Manfred Scheucher

General Position

a finite point set S in the plane is
in general position if $\nexists$ collinear points in S



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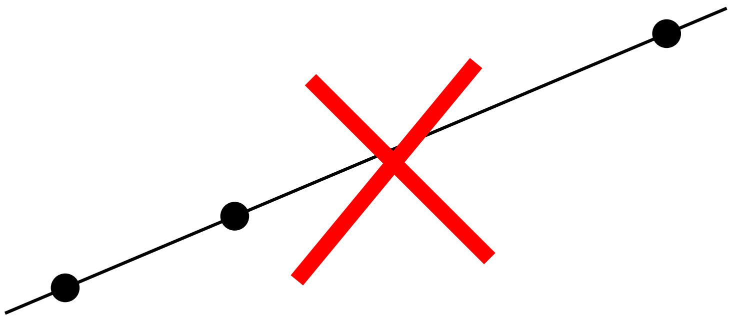
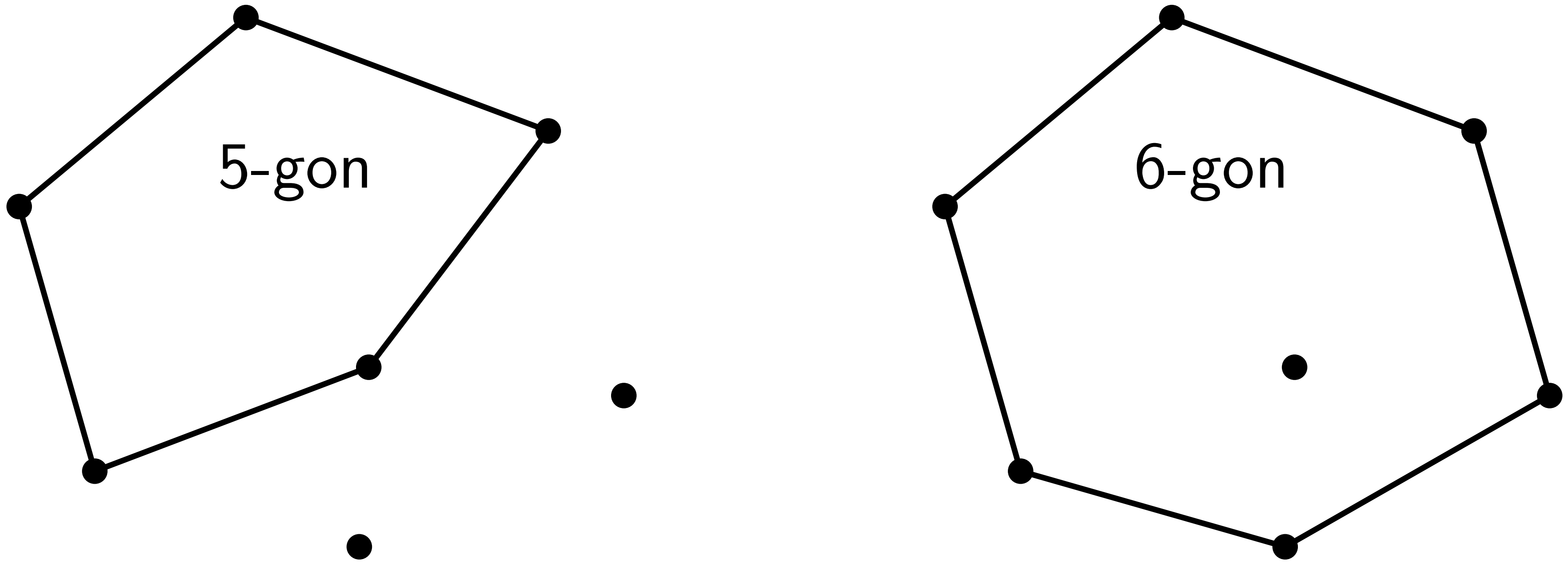


throughout this presentation, every set is in general position

k-Gons

a k-gon (in S) is the vertex set of a convex k-gon



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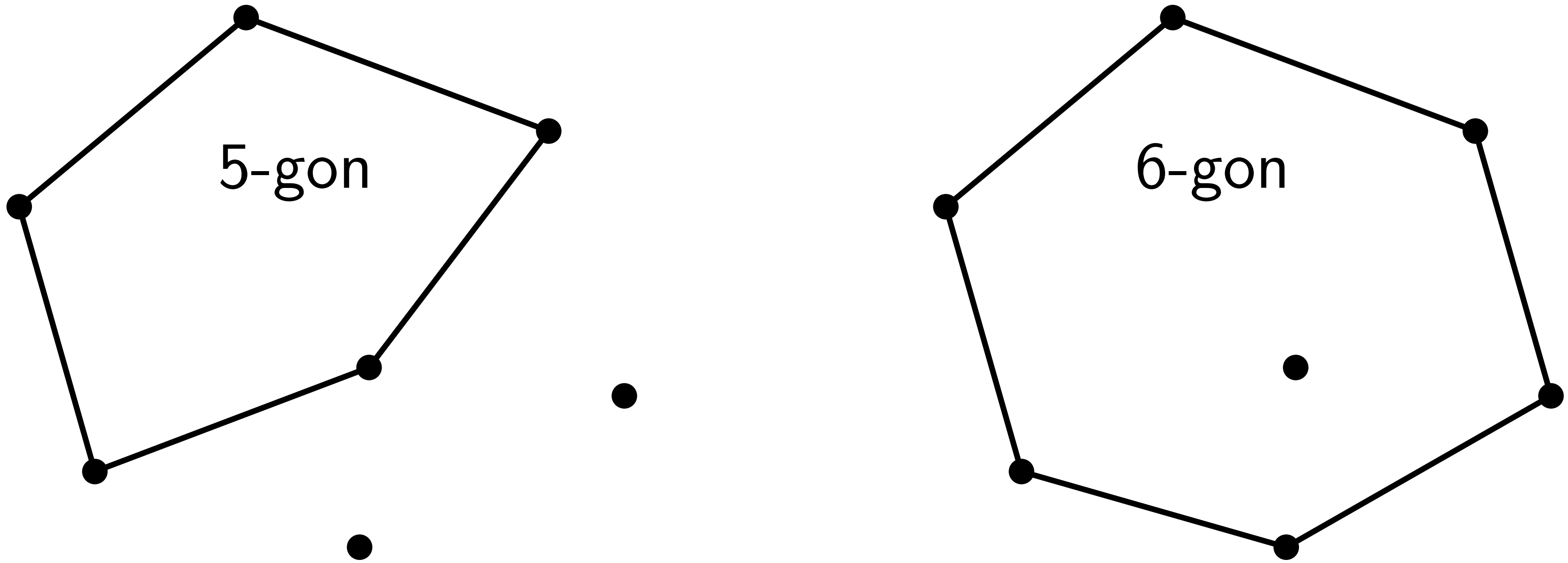

Theorem (Erdős & Szekeres 1935).
$\forall\ k \in \mathbb{N}$, $\exists$ a smallest integer $g(k)$ such that every set of $g(k)$ points contains a k-gon.

k-Gons

Theorem (Erdős & Szekeres '35)

$2^{k-2} + 1 \leq g(k) \leq \binom{2k-4}{k-2} + 1$

↑

equality conjectured by Szekeres, Erdős offered 500$ for a proof

k-Gons

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$\vdots$ several improvements of order $4^{k-o(k)}$

Theorem. $g(k) \leq 2^{k+o(k)}$. [Suk '16]

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- $g(k) \le 2^{k+O(k^{2/3} \log k)}$ [Suk '16]
- $g(k) \le 2^{k+O(\sqrt{k \log k})}$,
  also for pseudo-configurations of points
  [Holmsen, Mojarrad, Pach and Tardos '17]

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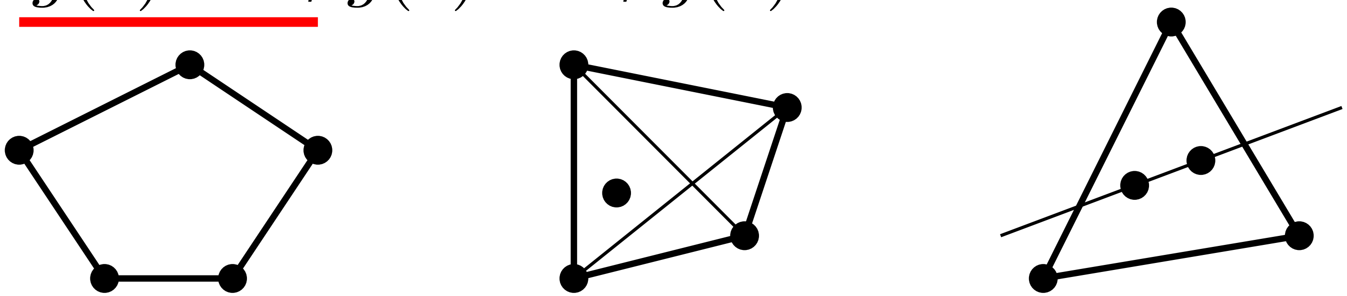
Known: $g(4) = 5$, $g(5) = 9$, $g(6) = 17$



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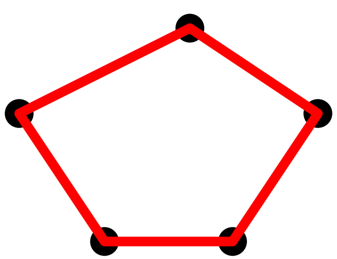
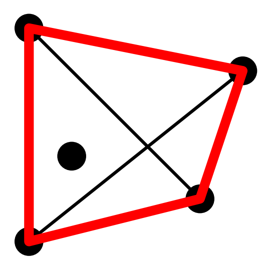
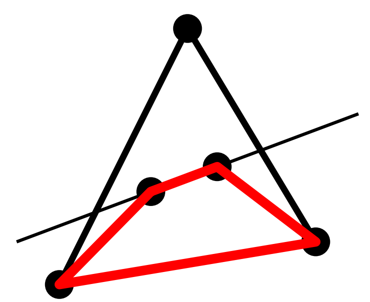

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computer assisted proof, 1500 CPU hours [Szekeres–Peters '06]

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<1 hour using SAT solvers [S.'18, Marić '19]

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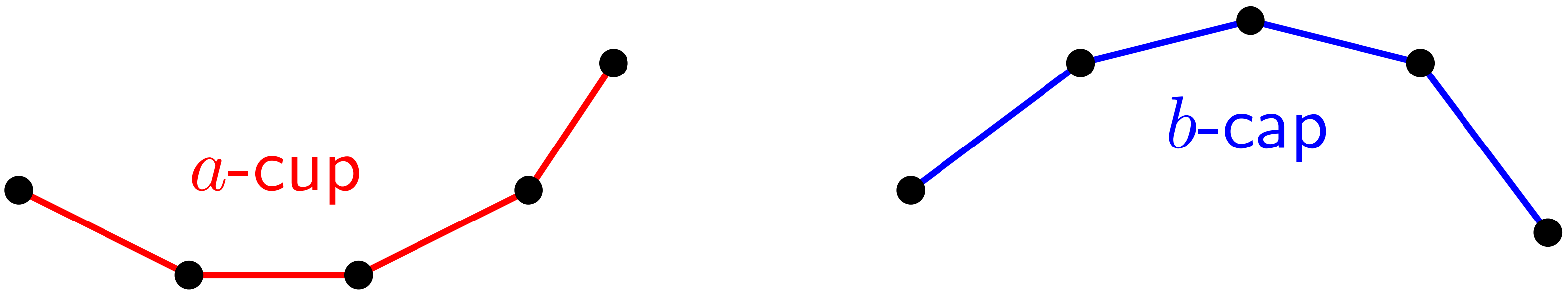
we show that every set of $\phi(a, b) + 1 = \binom{a+b-4}{a-2} + 1$ points contains a-cup or b-cap.



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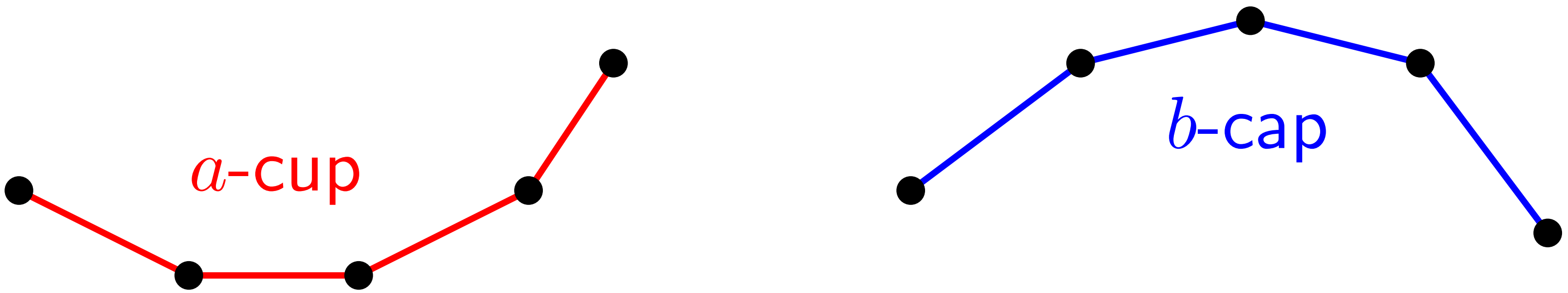

using $a=b=k$, we then get $g(k) \leq \binom{2k-4}{k-2}+1$.

Cups and Caps: Upper Bound

Let S be a set of $\phi(a,b) + 1 = \binom{a+b-4}{a-2} + 1$ points and suppose it does not contain a a-cup.

We show that there is a b-cap.

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Base Case: If $a = 2$ or $b = 2$, we have
$\phi(a, b) + 1 = \binom{\geq 0}{0} + 1 = 2$ points $\Rightarrow$ 2-cup / 2-cap

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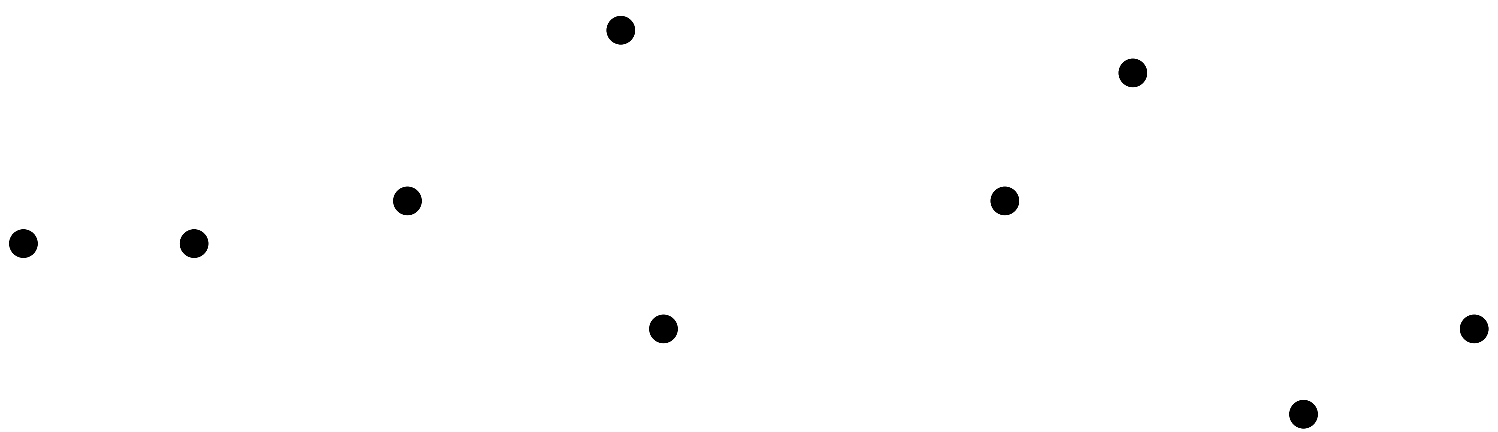
Step: Let E be the set of rightmost (end)points of all $(a-1)$-cups



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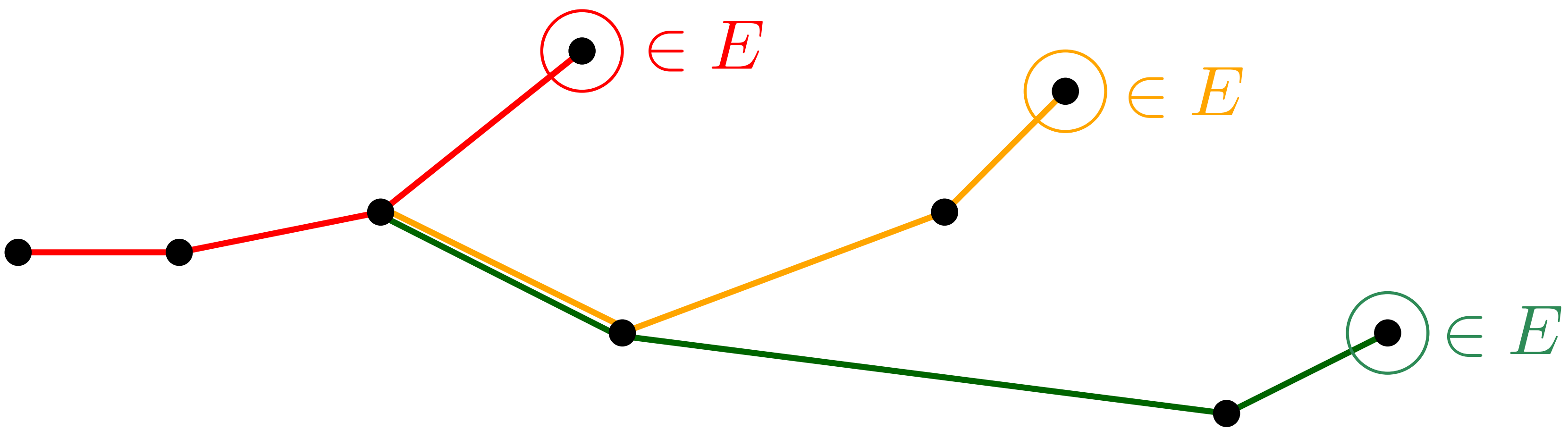

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$|E|=|S|-|S\setminus E|\geq\phi(a,b-1)+1$

because $\phi(a,b)=\phi(a-1,b)+\phi(a,b-1)$

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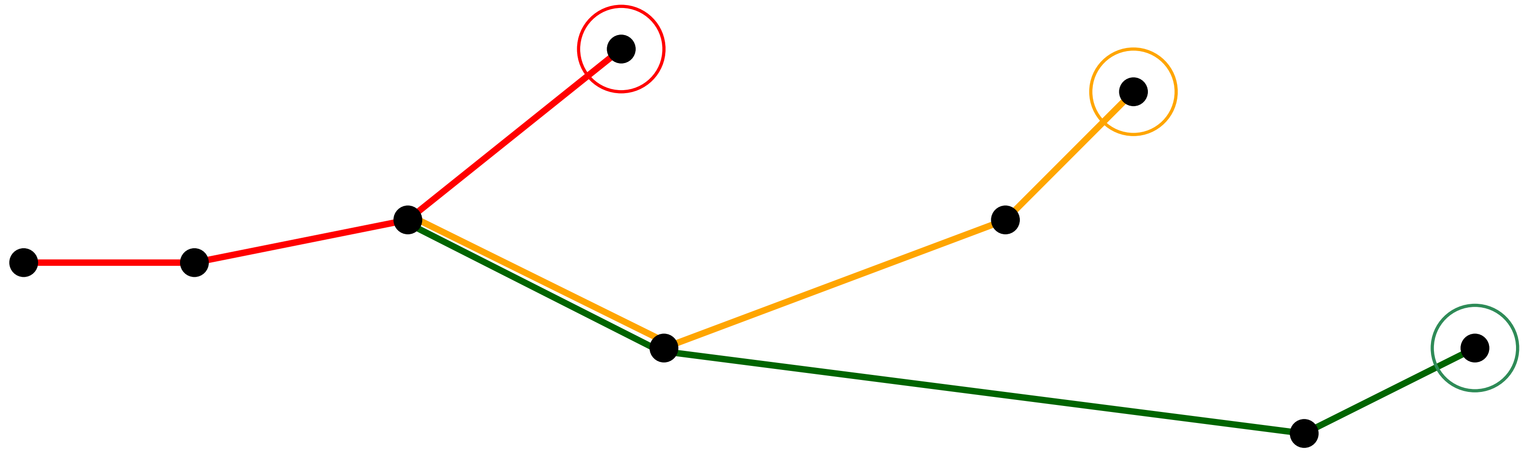

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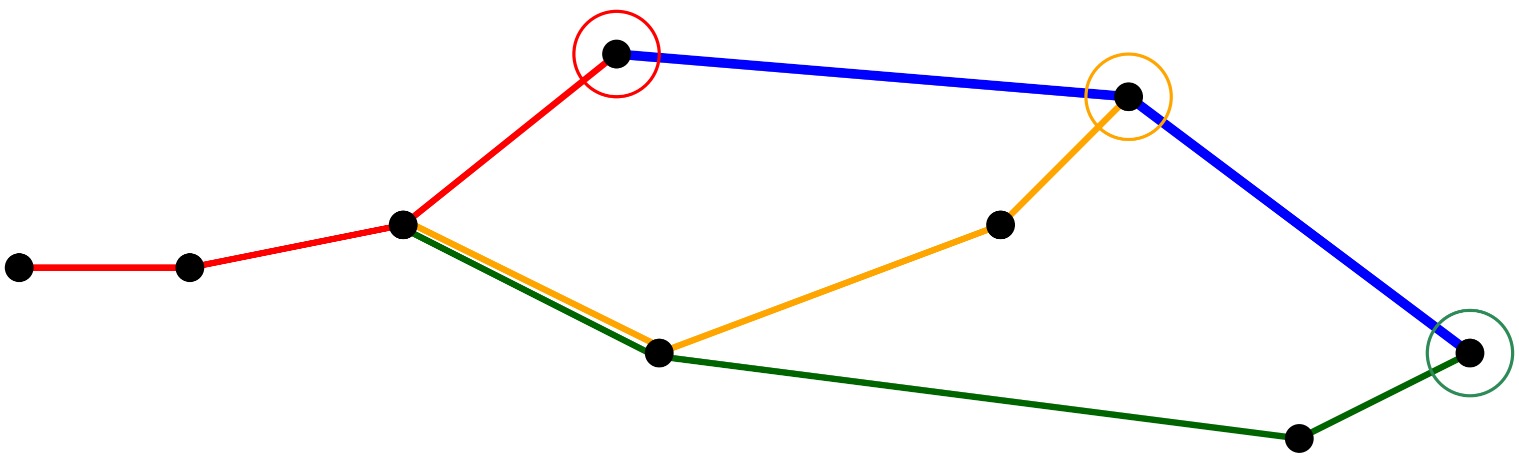

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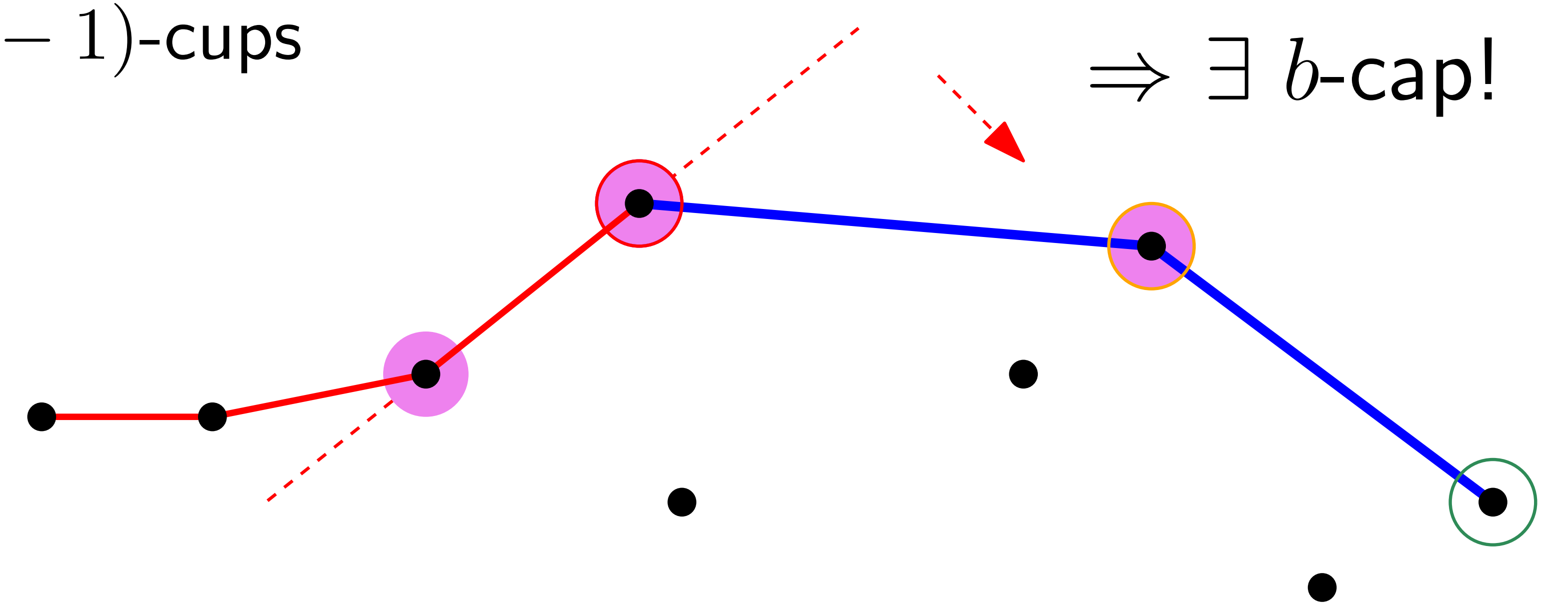
$\Rightarrow \exists$ b-cap! Q.E.D.



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Cups and Caps: Lower Bound

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this bound is actually tight because there exist sets $S_{a,b}$ with $\phi(a,b) = \binom{a+b-4}{a-2}$ points without a-cups and b-caps.

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- this will give
  $|S_{a,b}| = \phi(a,b) = \phi(a,b-1) + \phi(a-1,b) = \binom{a+b-4}{a-2,b-2}$

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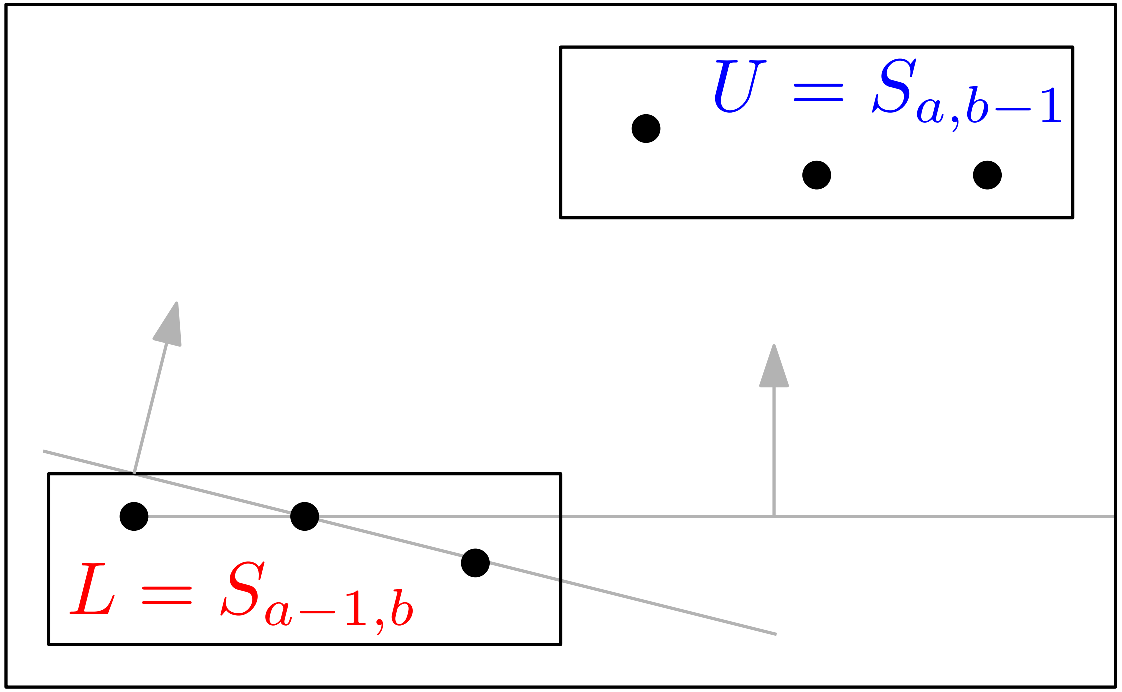
- U right of L
- U "high above" L



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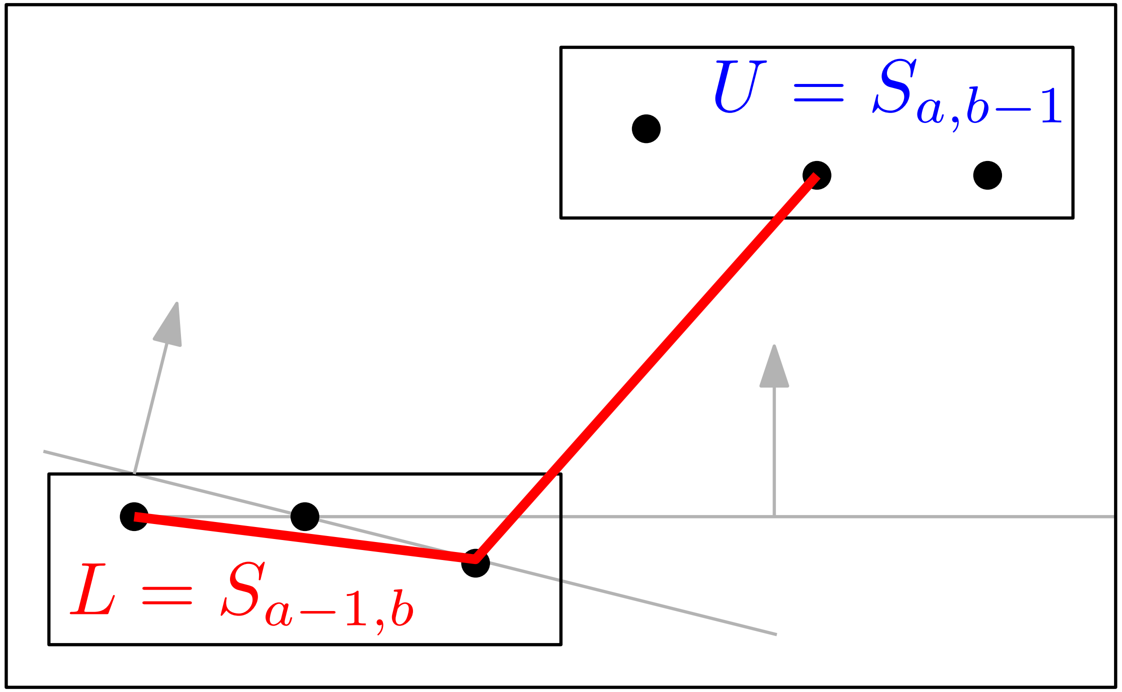

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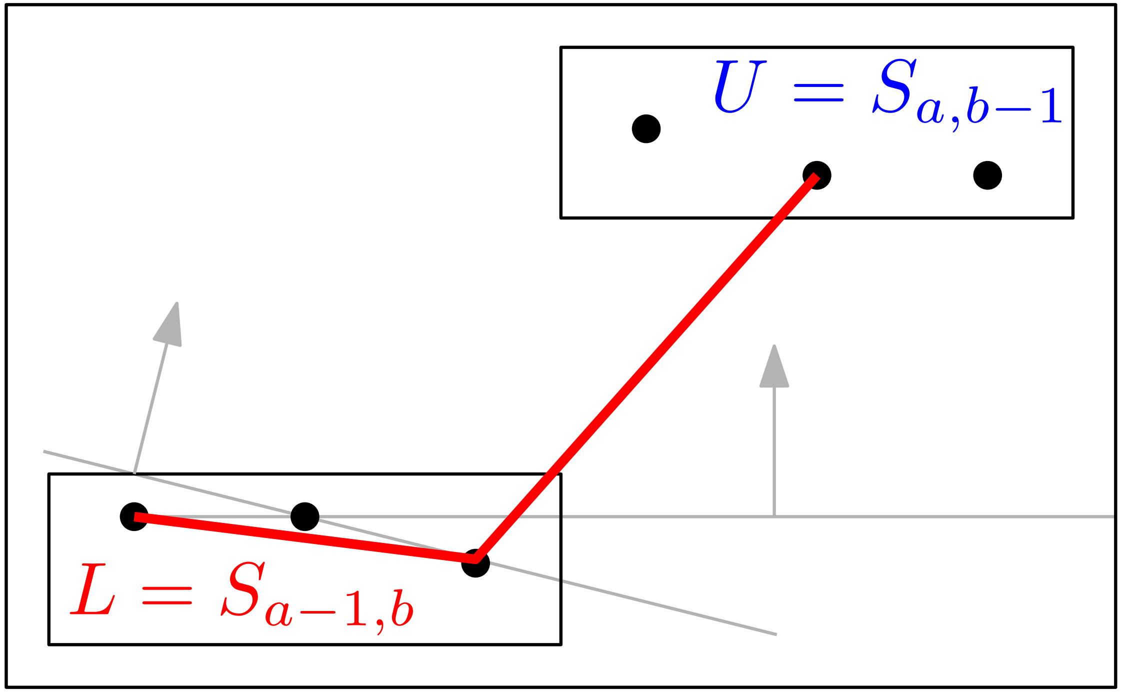

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k-Gons: $2^{k-2}+1$ Lower Bound

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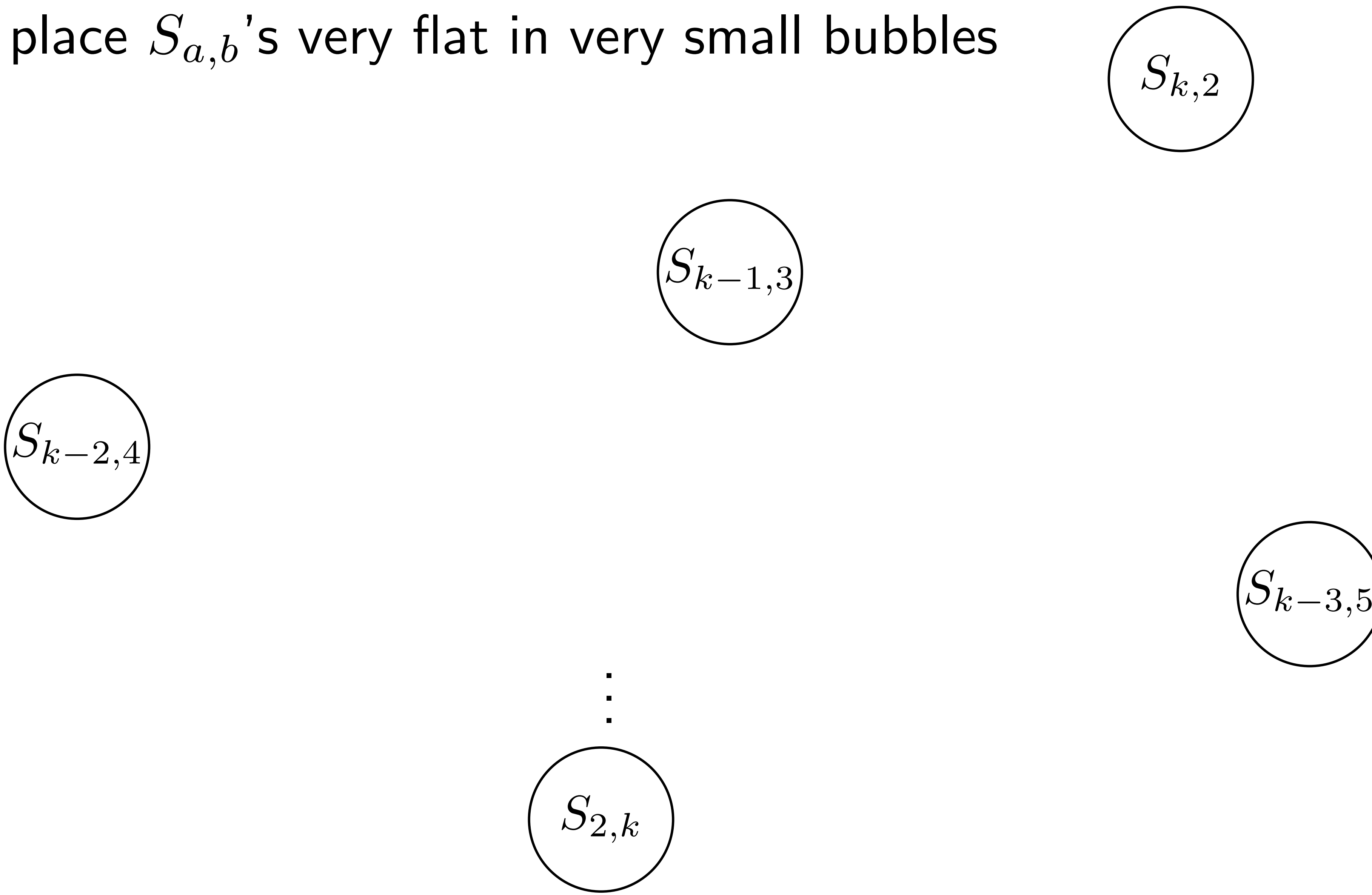
- use $S_{2,k}$, $S_{3,k-1}$, . . . , $S_{k,2}$ as gadgets to construct a large set without k-gons

k-Gons: $2^{k-2}+1$ Lower Bound

- place $S_{a,b}$'s very flat in very small bubbles



k-Gons: $2^{k-2}+1$ Lower Bound

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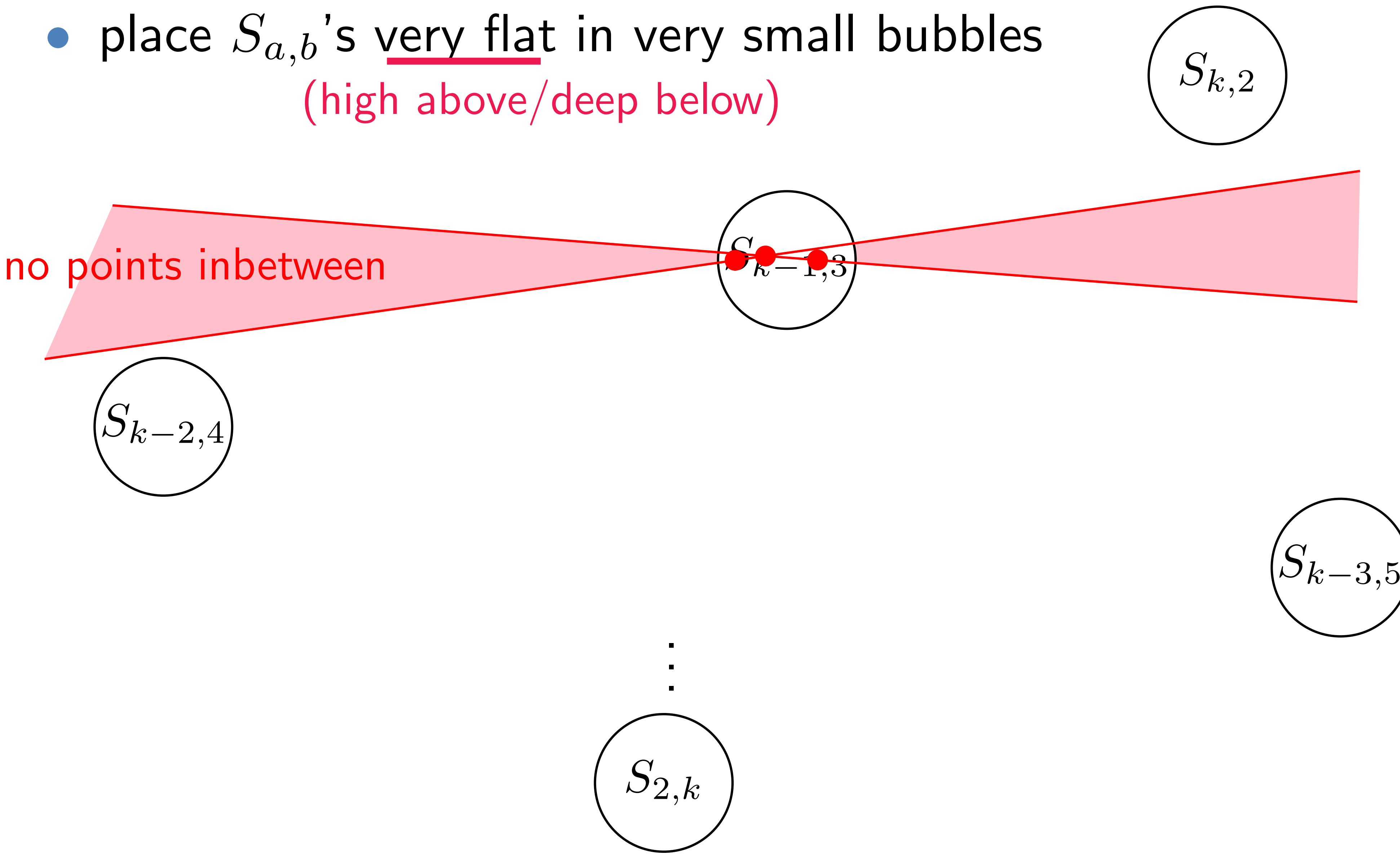
(high above/deep below)



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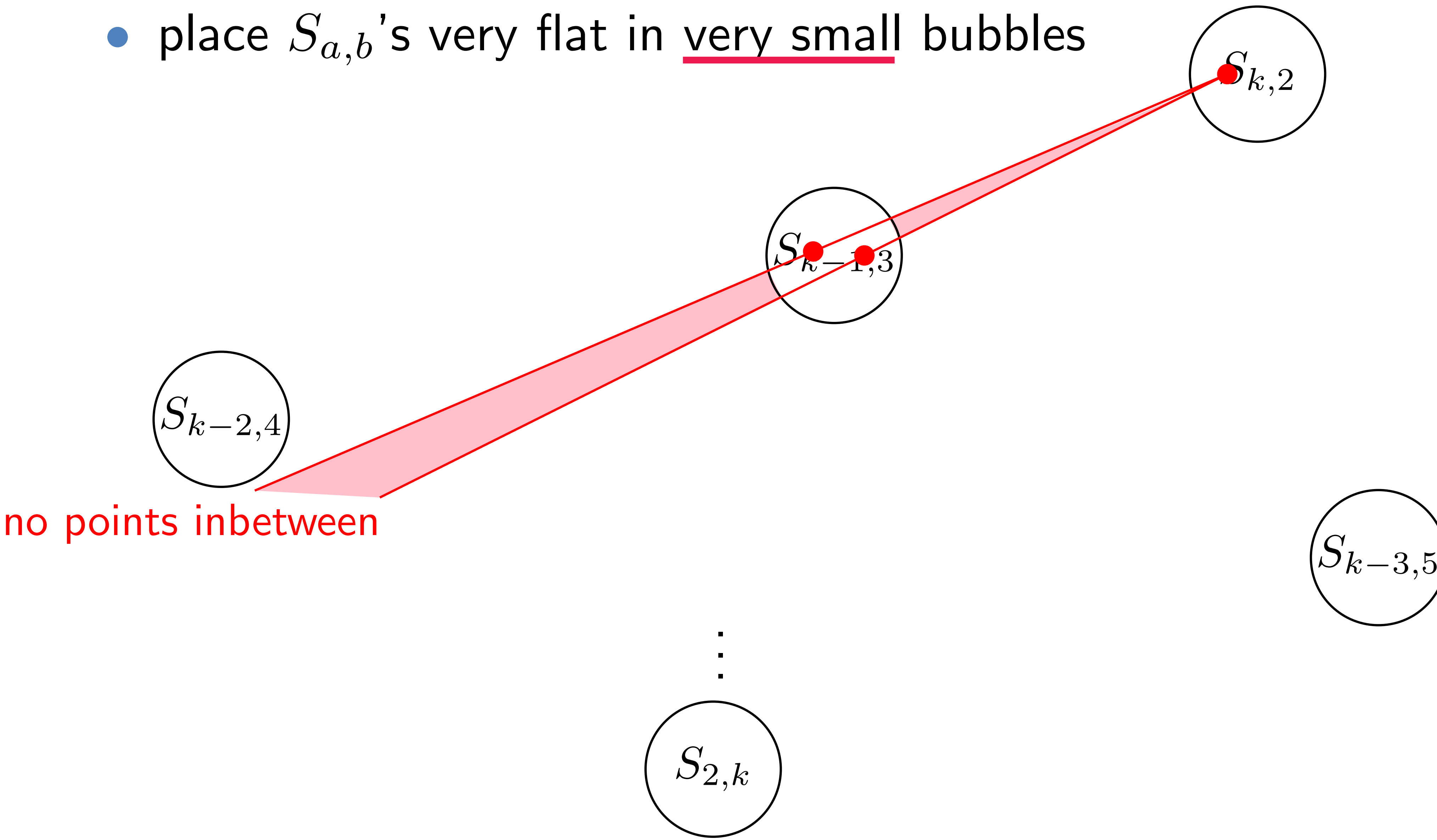
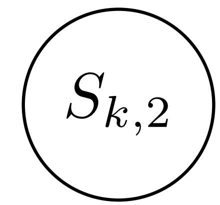
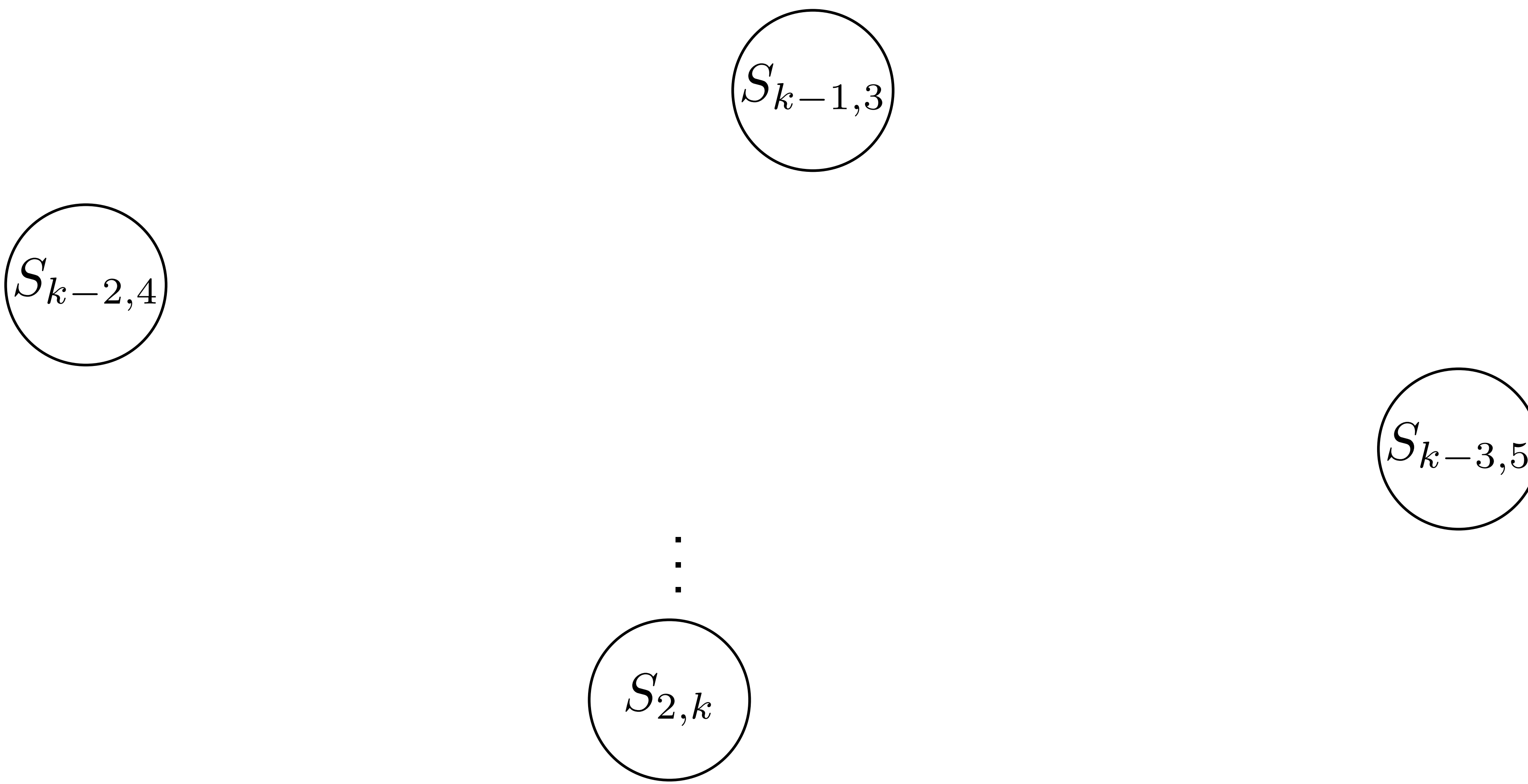


k-Gons: $2^{k-2} + 1$ Lower Bound

- place $S_{a,b}$'s very flat in very small bubbles
- bubbles can have arbitrary relative positions





k-Gons: $2^{k-2}+1$ Lower Bound

$S_{k,2}$

$S_{k-1,3}$

$S_{k-2,4}$

$S_{k-3,5}$

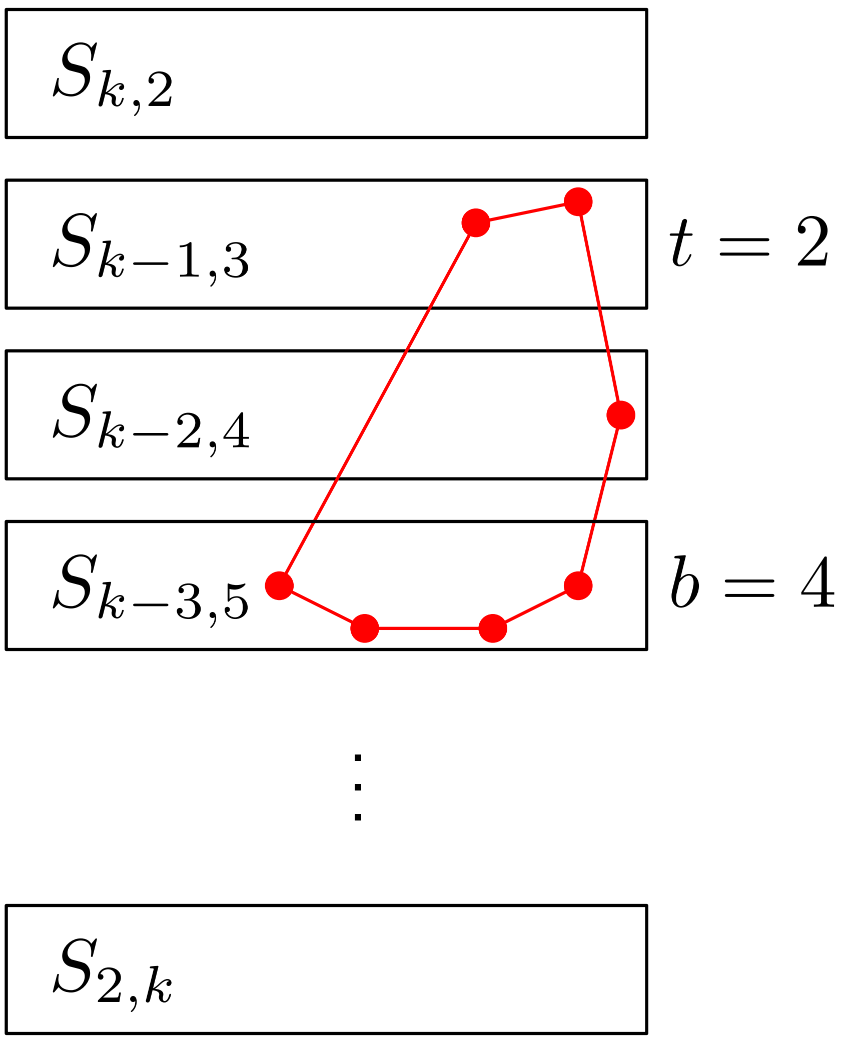
$\vdots$

$S_{2,k}$

k-Gons: $2^{k-2}+1$ Lower Bound

- each ℓ-gon has:

  $(\leq t)$-cap in top-layer t,
  $(\leq k-b)$-cup in bottom-layer b,
  ≤ 1 point per interm. layer
  $\Rightarrow \ell < k$



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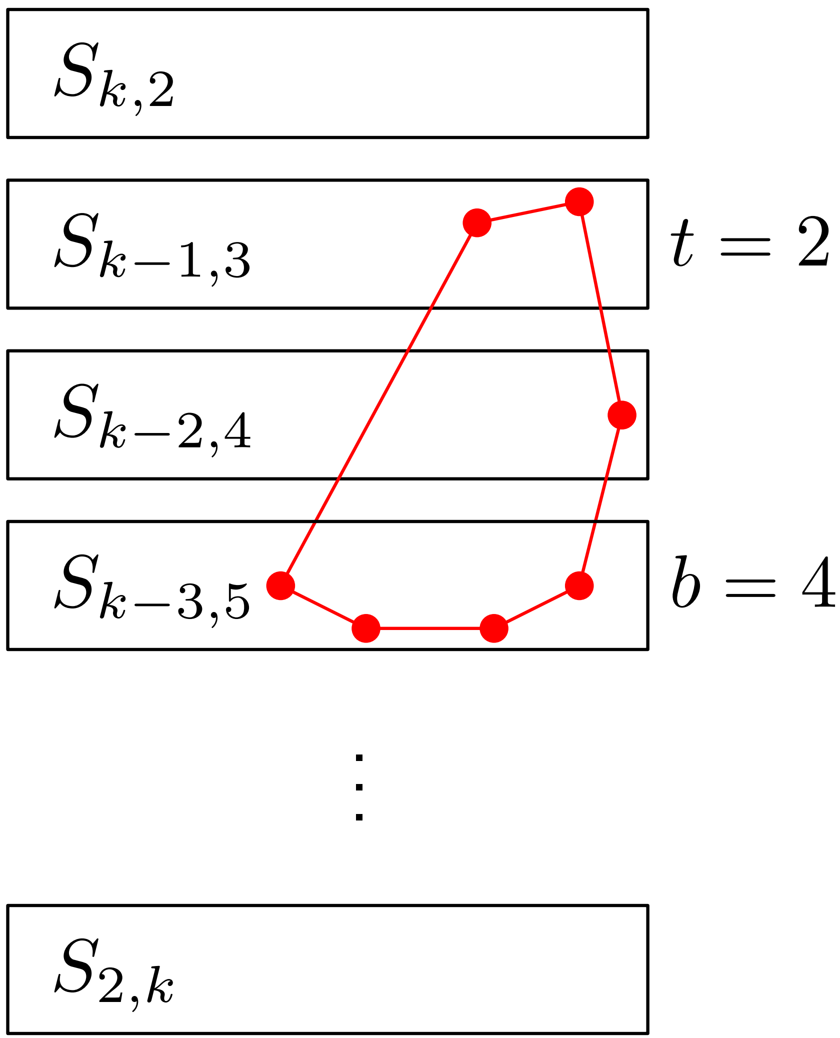

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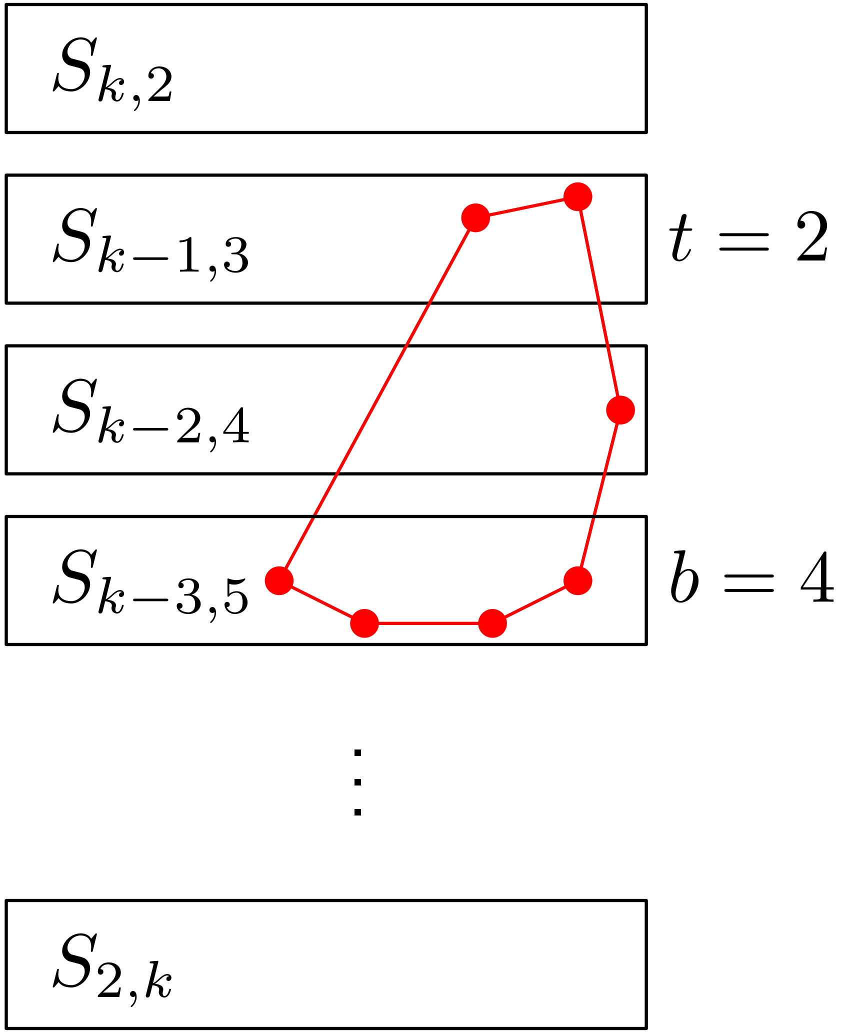
- therefore $g(k) \geq 2^{k-2}+1$



Sketch of Suk's $2^{k+o(k)}$ Upper Bound

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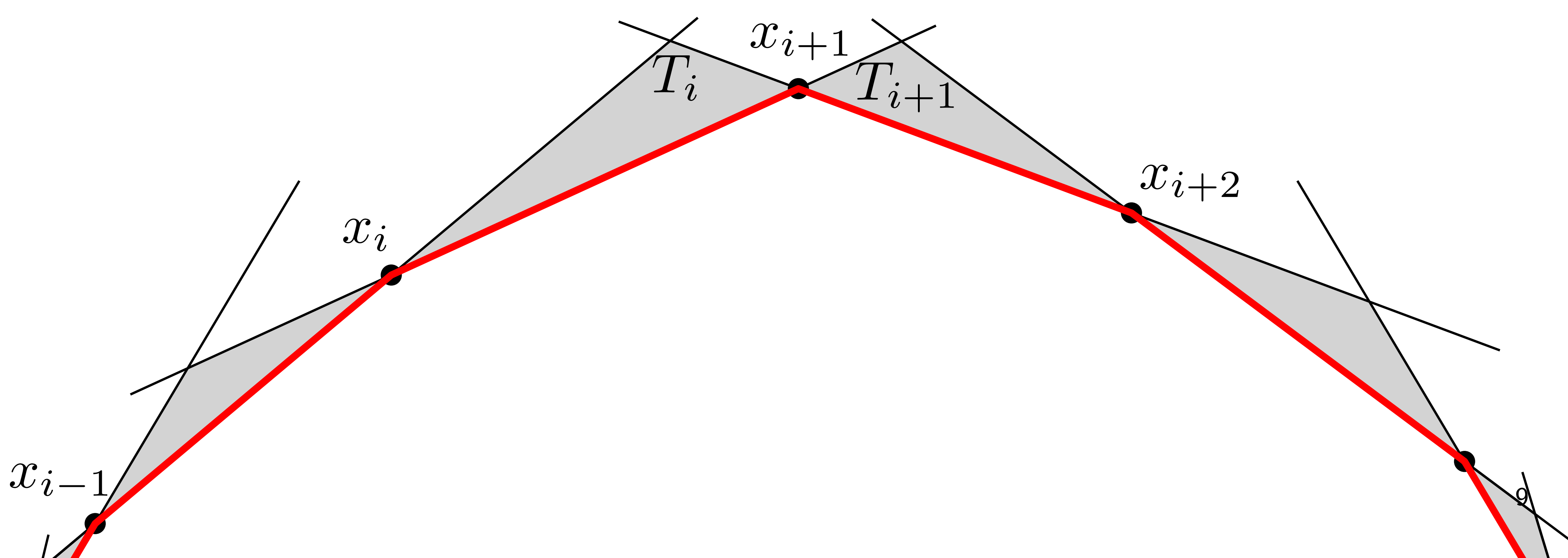
Lemma (Fractional EST, Pór & Valtr '02) Let S be a point set with $|S| \geq 2^{32k}$ points. Then there exists k-cup/cap $X \subset S$ satisfying $|T_i \cap S| \geq \frac{|S|}{2^{32k}}$ for every i.



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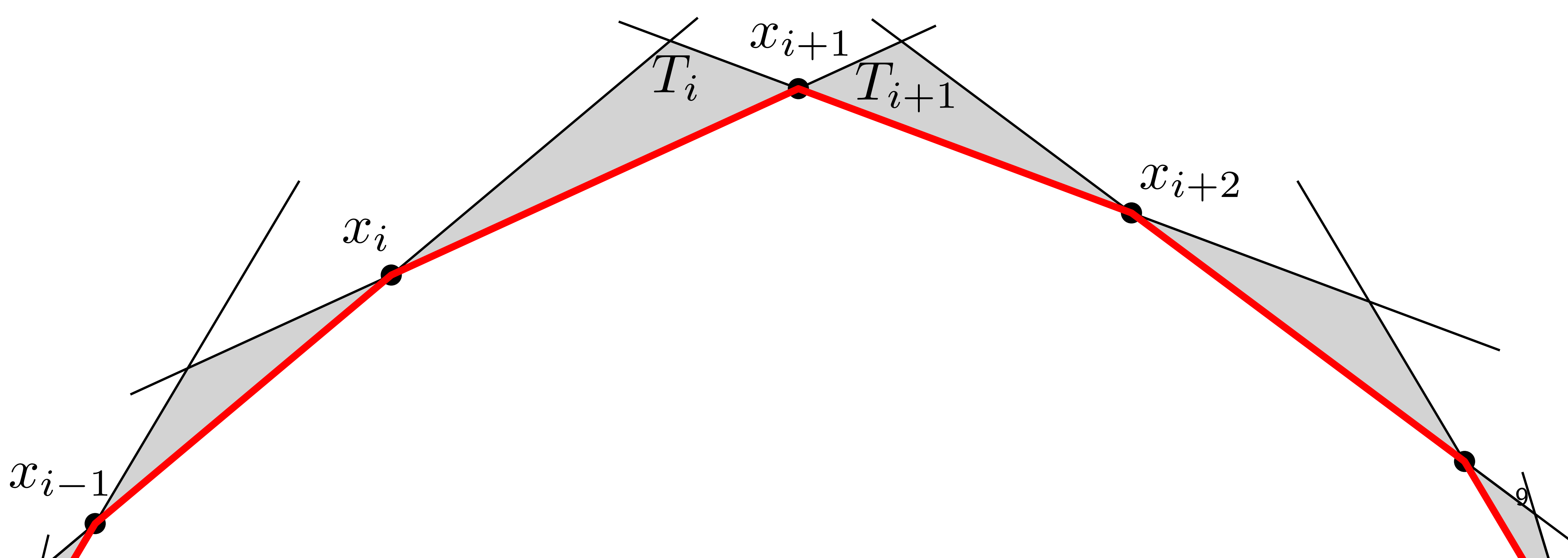
linear in S!



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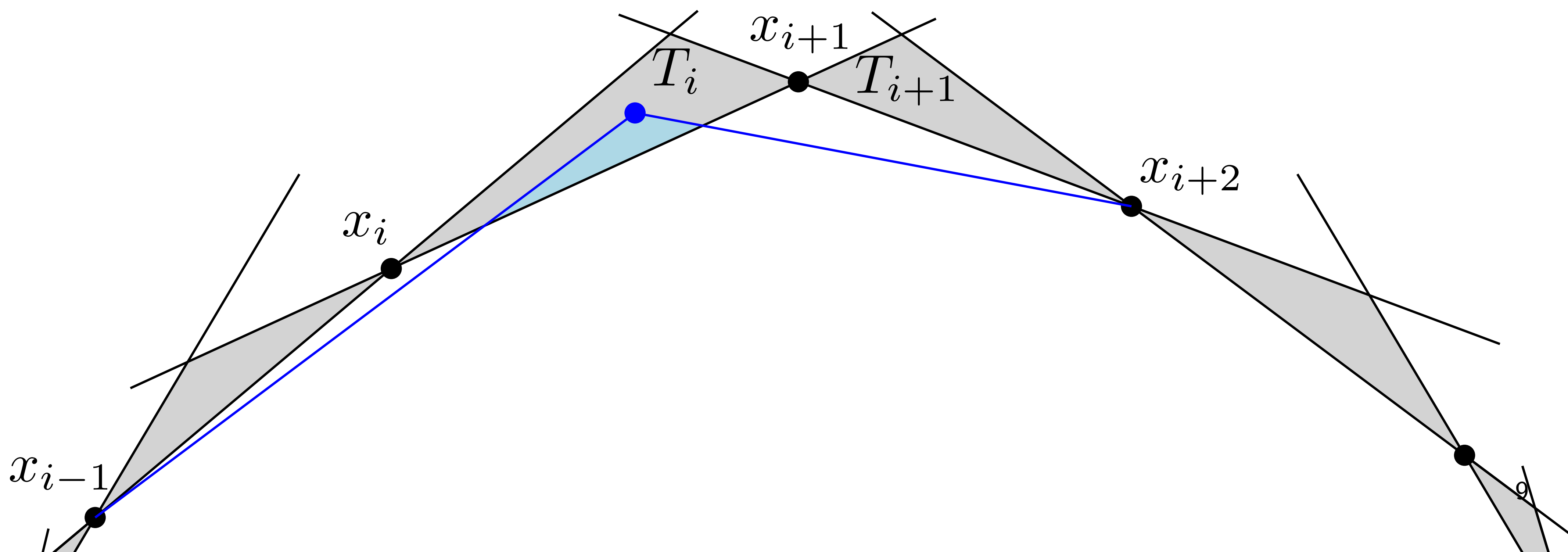
- Fractional EST
- auxiliary poset on each T_i



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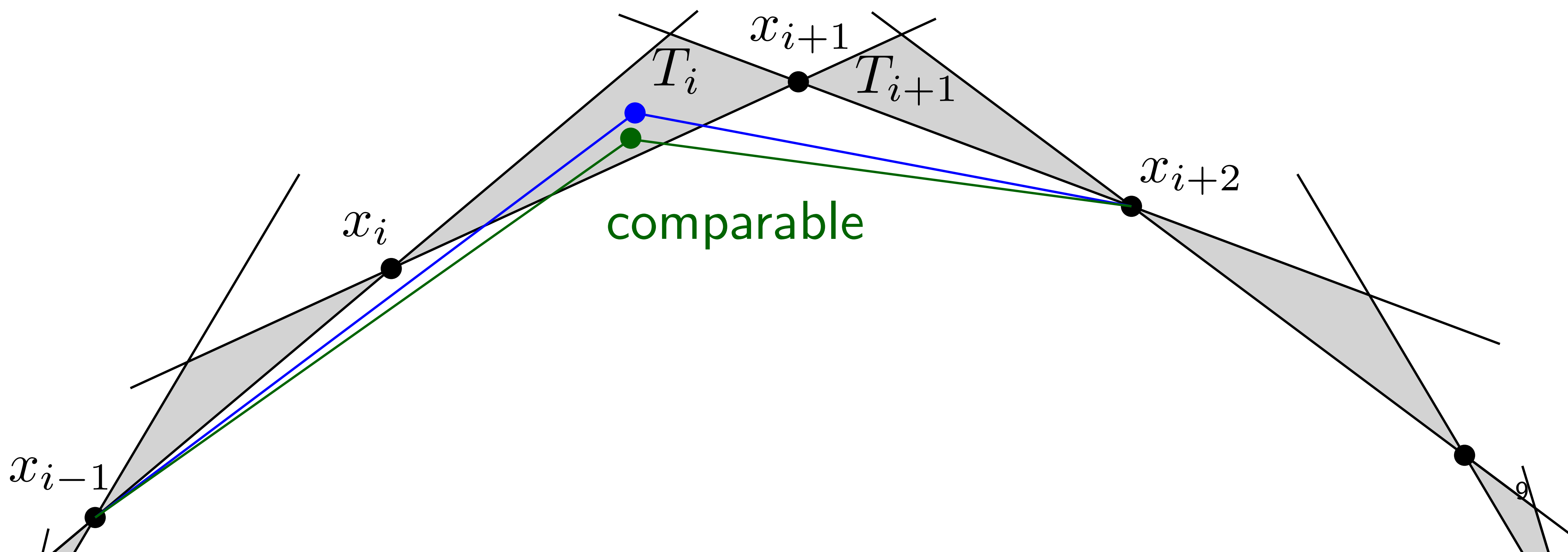

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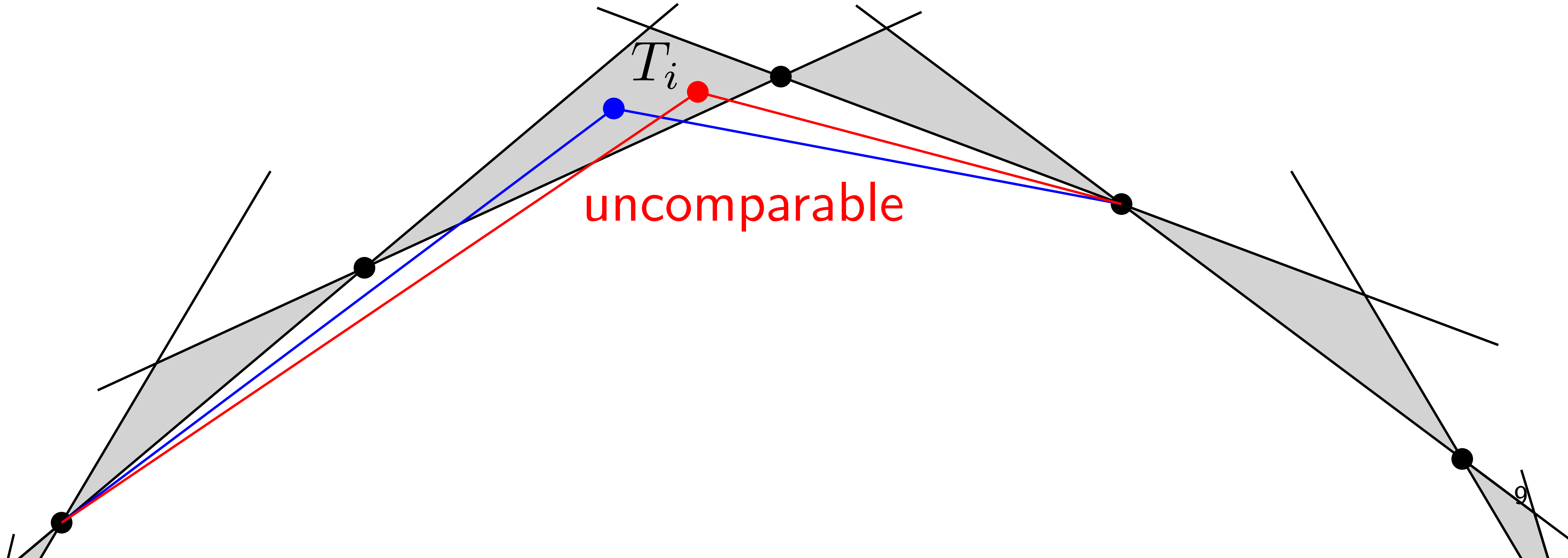

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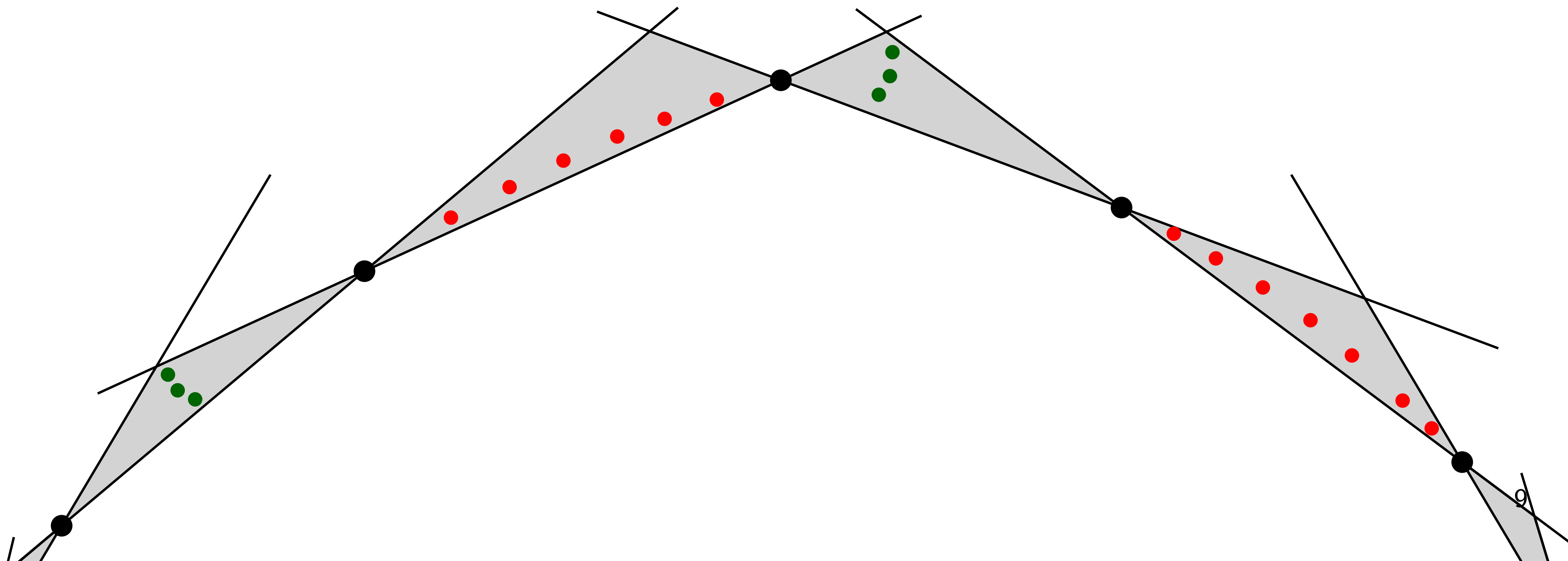
- auxiliary poset on each T_i
- Dilworth's theorem: large antichain or chain in each T_i



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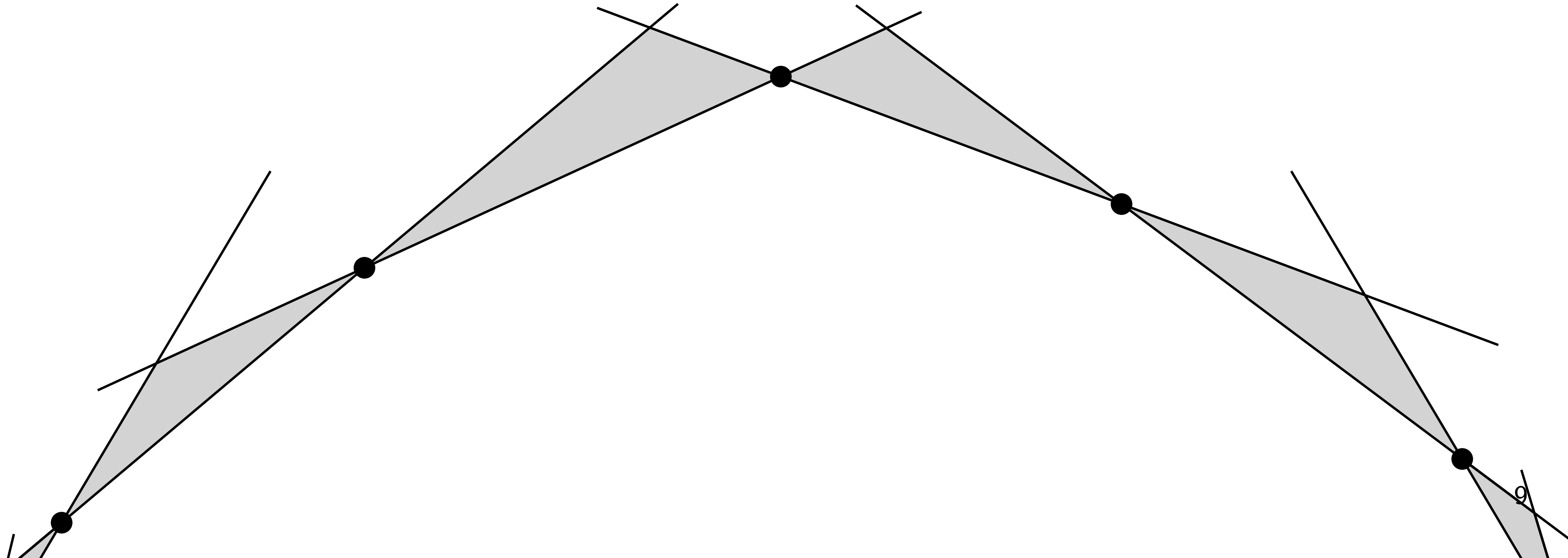
- Fractional EST
- auxiliary poset on each T_i
- Dilworth's theorem: large antichain or chain in each T_i
- pigeonhole principle:



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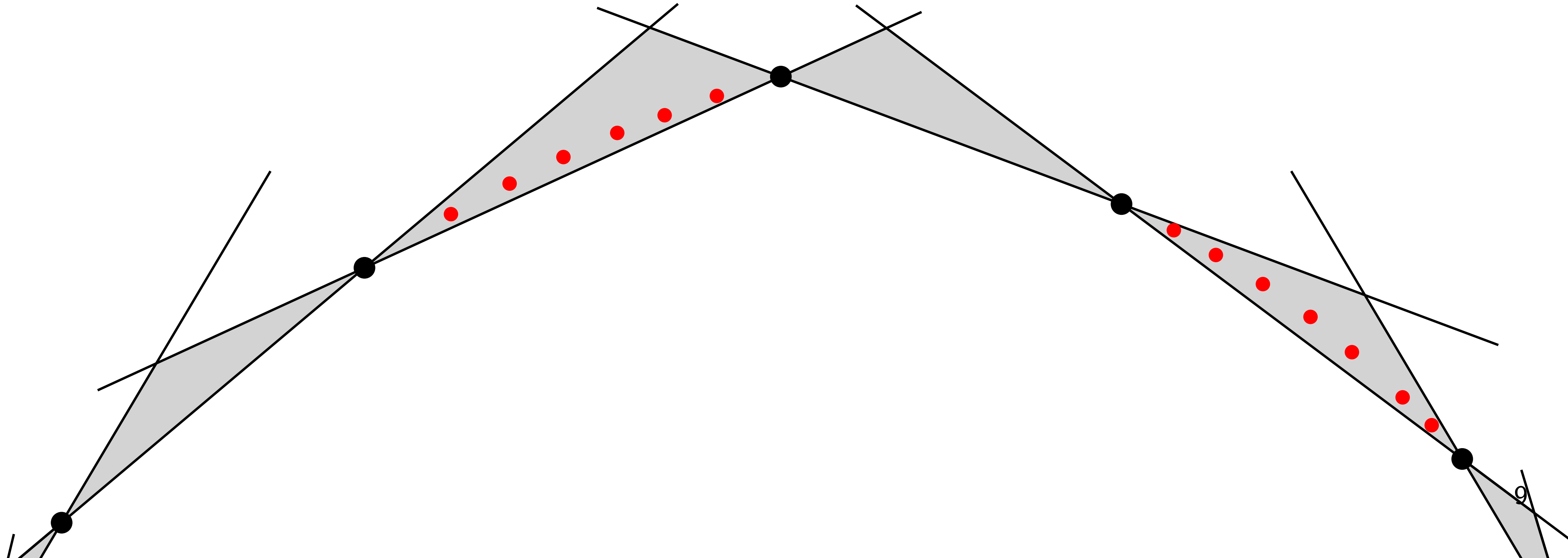

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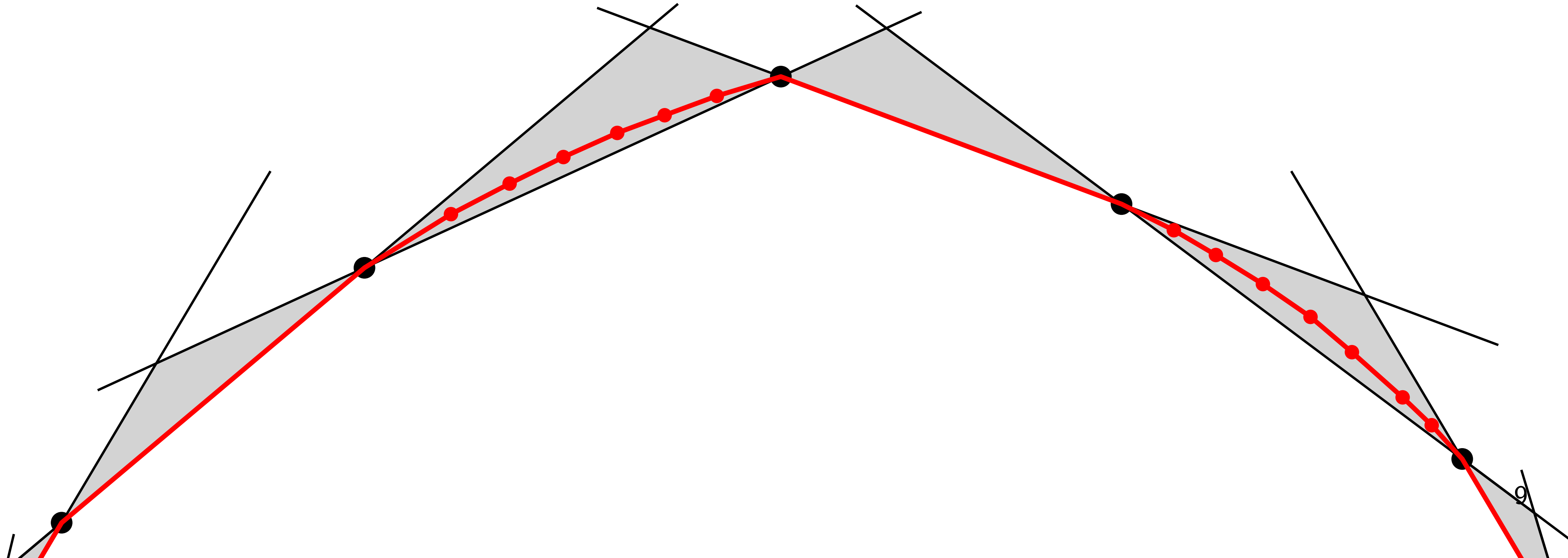

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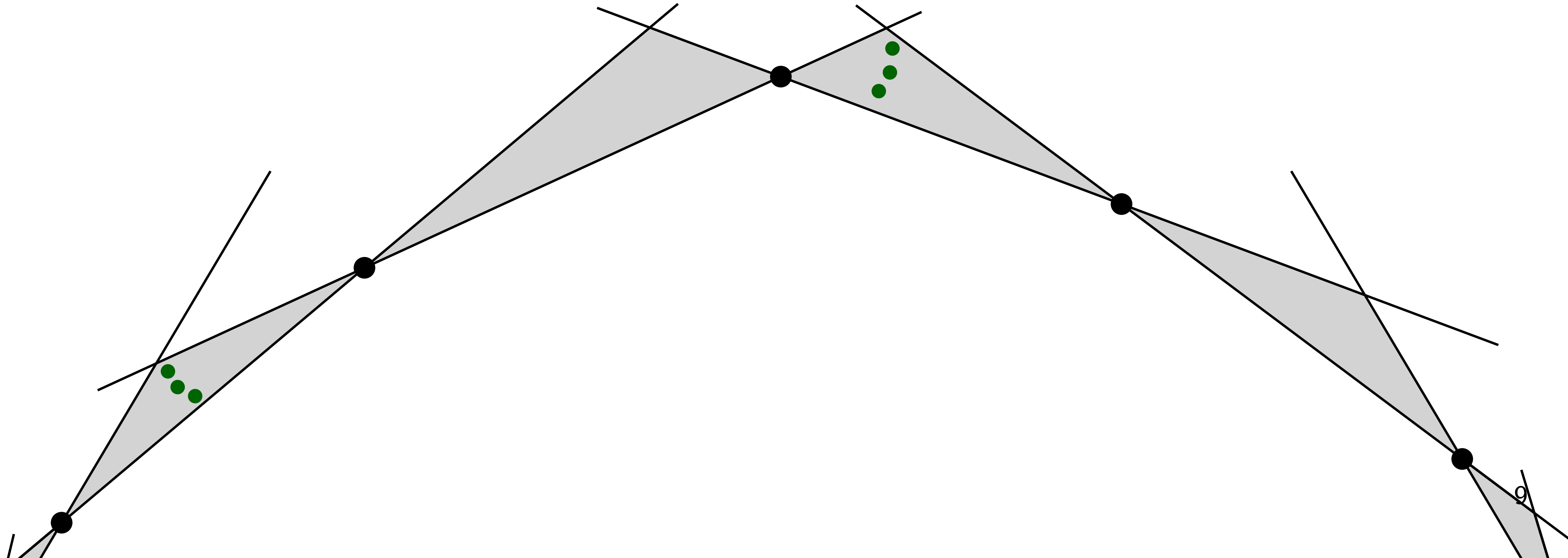

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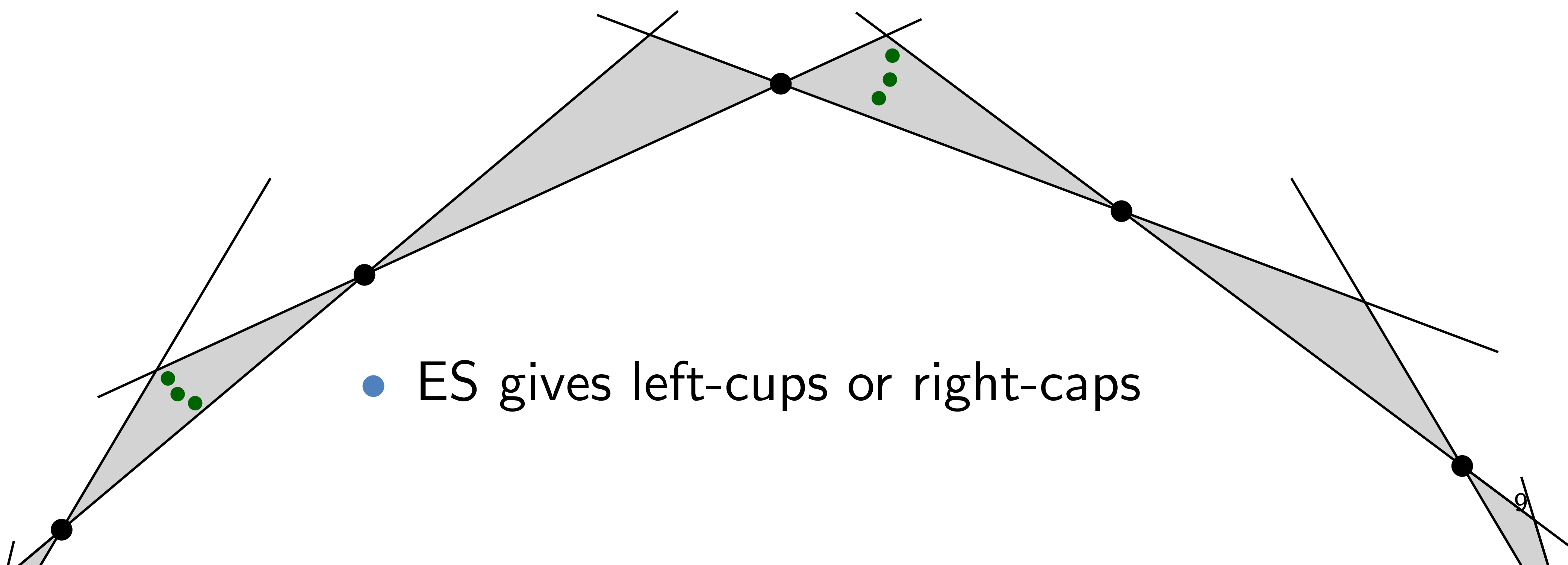

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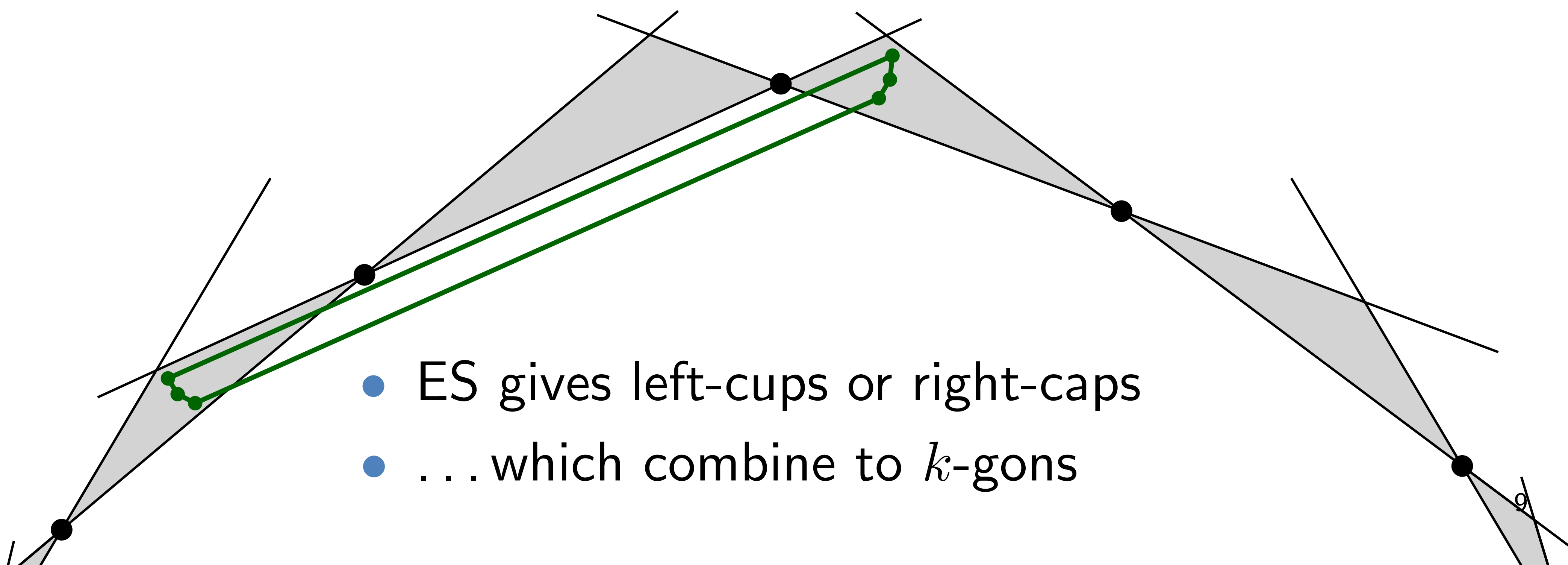

- ES gives left-cups or right-caps
- . . . which combine to k-gons

Quantity of k-Gons

maximum $\#$ of k-gons among all sets of n points?

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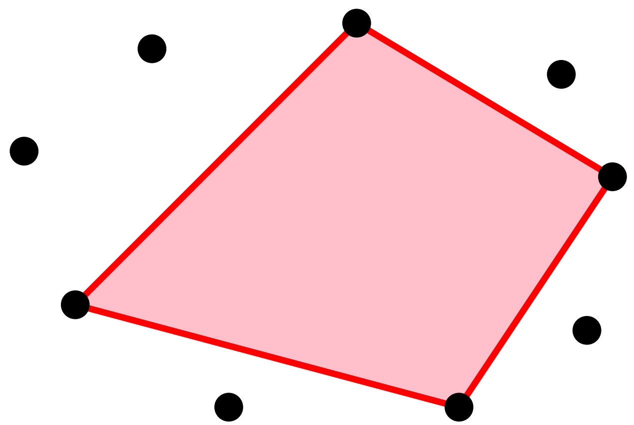

n points in convex position

any k-subset is k-gon

$$\Rightarrow \max = \binom{n}{k}$$

Quantity of k-Gons

$g_k(n)$:= minimum # of k-gons among all sets of n points

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- $g_k(n) = \Theta(n^k)$

each $g(k)$-subset gives k-gon

$$\text{because } g_k(n) \geq \frac{\binom{n}{g(k)}}{\binom{n-k}{g(k)-k}} \geq c \cdot \frac{n^{g(k)}}{n^{g(k)-k}} = c \cdot n^k$$

each k-gon counted by at most that many $g(k)$-subsets

Quantity of k-Gons

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- $g_k(n) = \Theta(n^k)$
- $k = 4$: rectilinear crossing number of K_n:
  $g_4(n) = \overline{cr}(K_n) \sim c_4 \cdot \binom{n}{4}$ with $0.37999 < c_4 < 0.38055$
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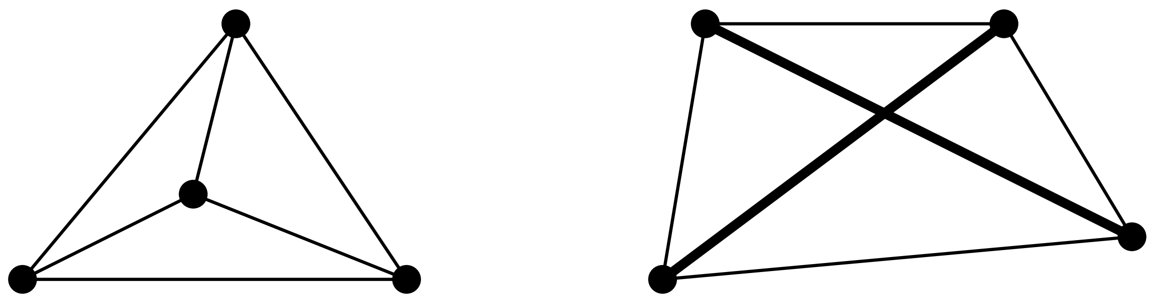

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- various notions of crossing numbers have been studied intensively (not necessarily straight-line drawings, not necessarily complete graphs)

An Invariant: crossings – k-edges relation

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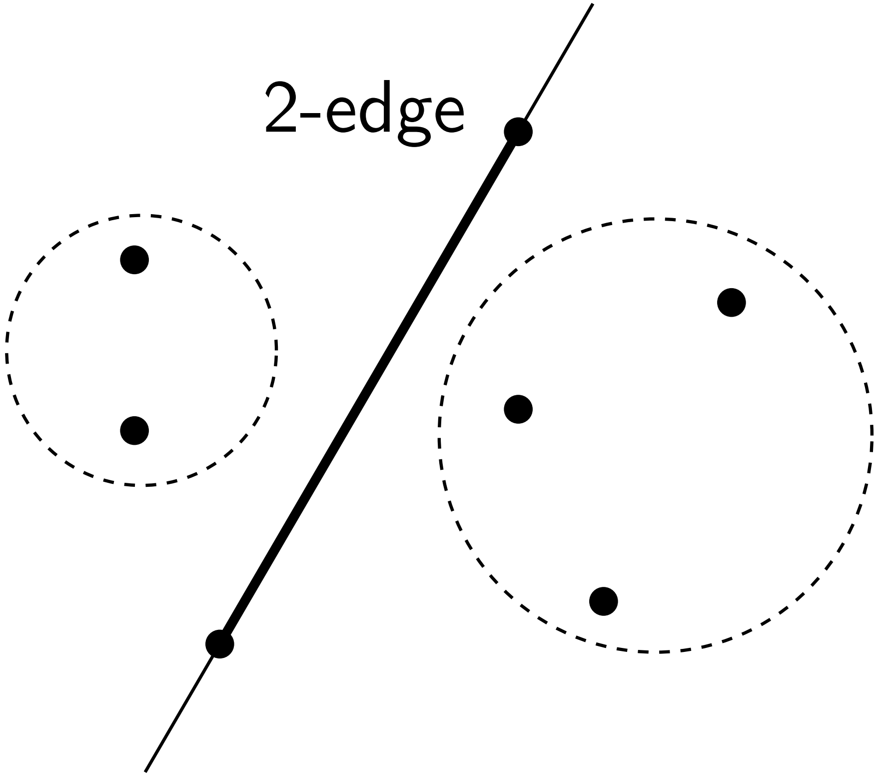
- $(a, b) \in \binom{S}{2}$ is k-edge if $\overline{ab}$ partitions $S \setminus \{a, b\}$ into k and $n - k - 2$ points



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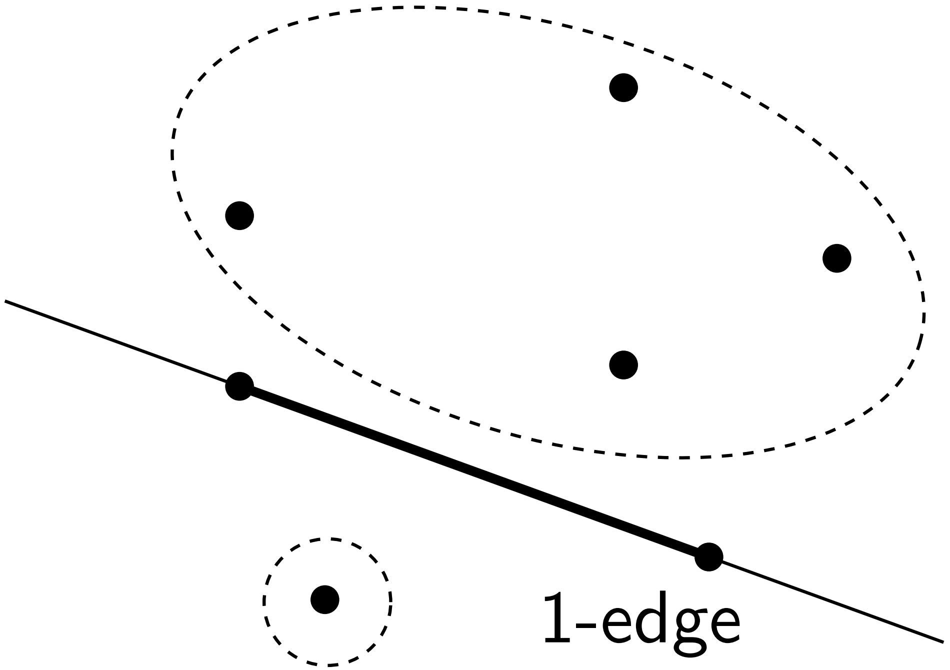

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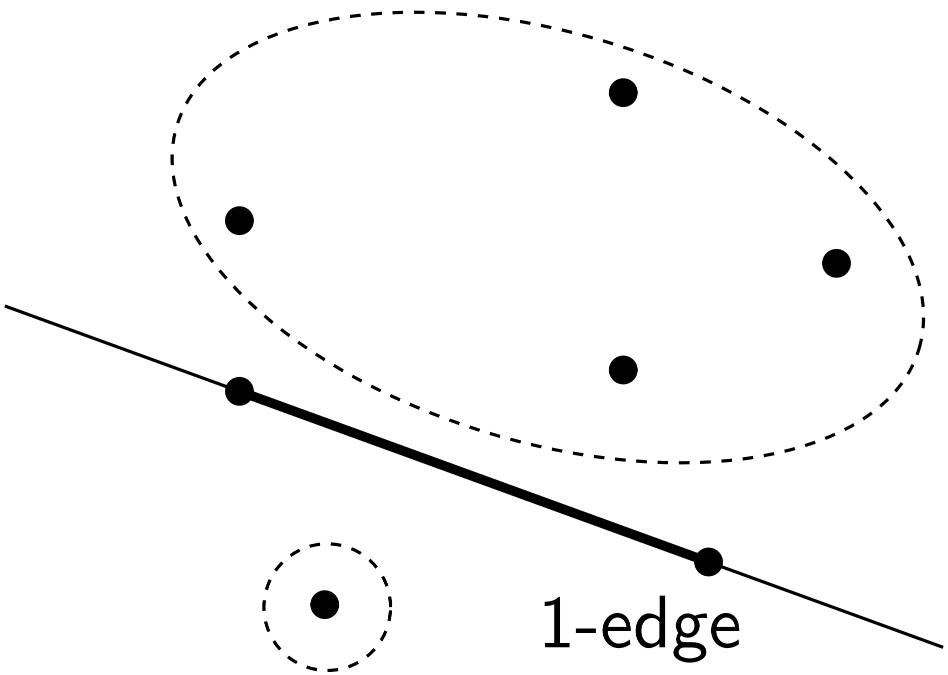

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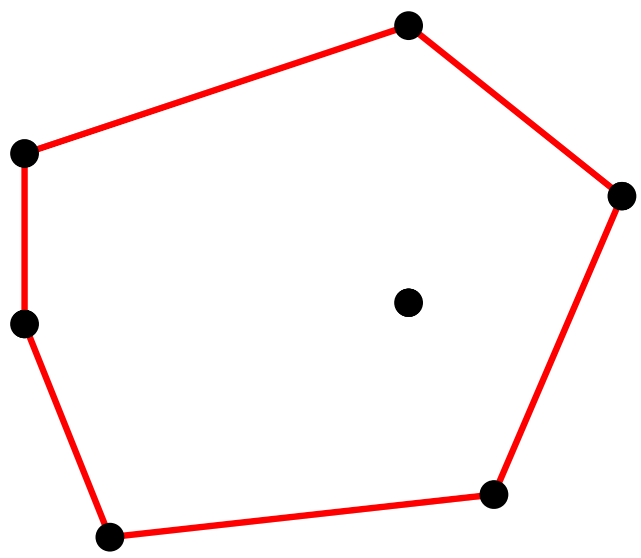

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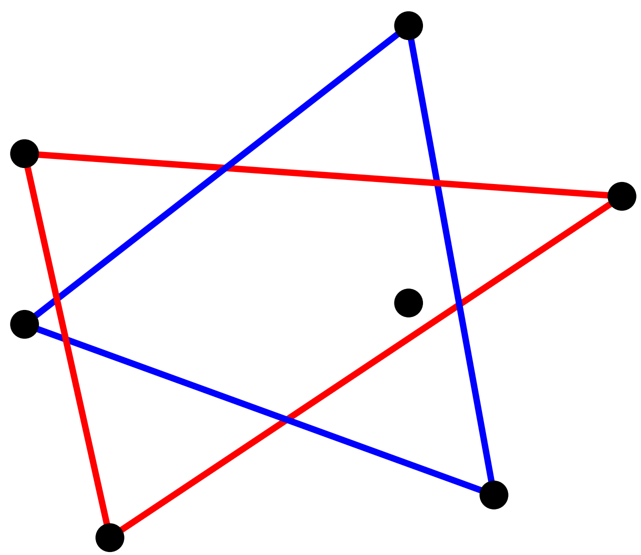

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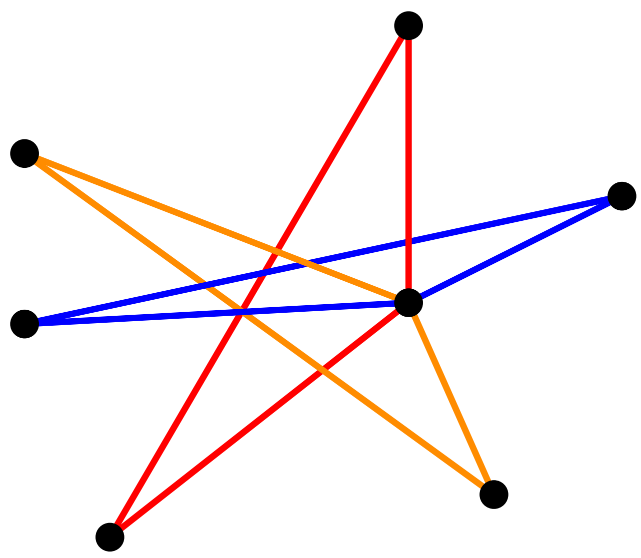

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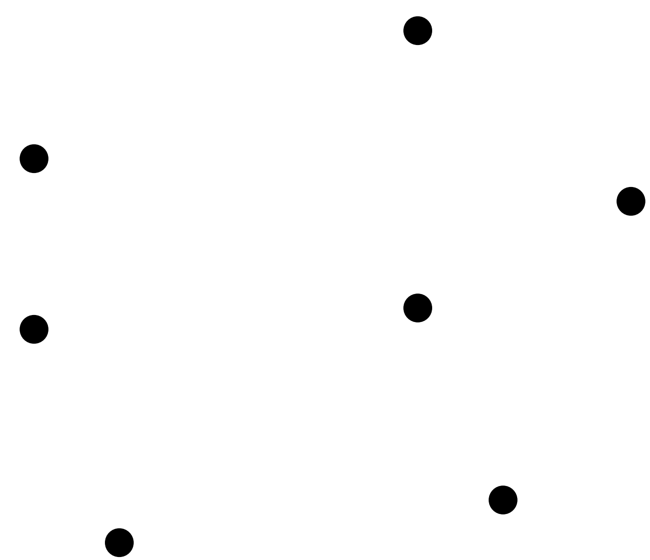

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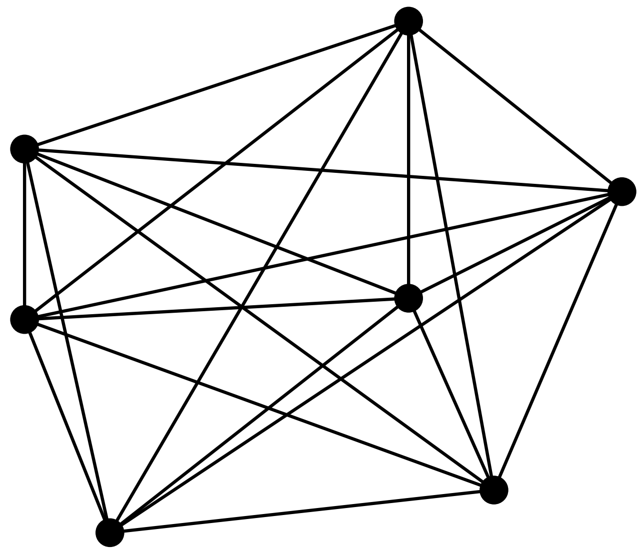

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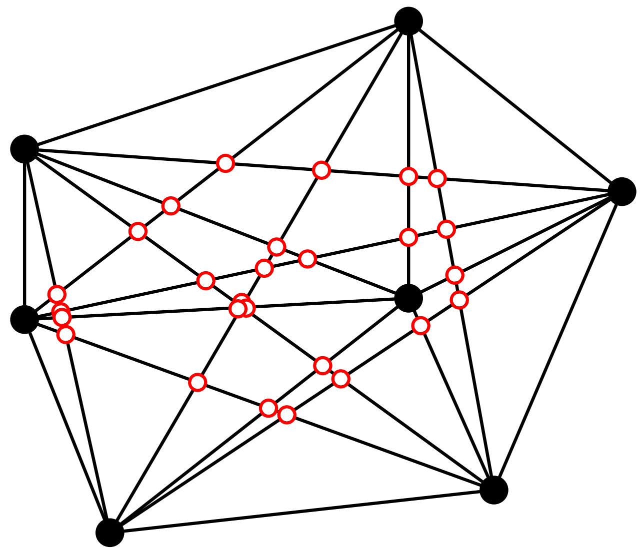

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Remark: $g_k(S)$ can be computed in $O(k \cdot n^3)$ time [Mitchell Rote Sundaram Woeginger '95]

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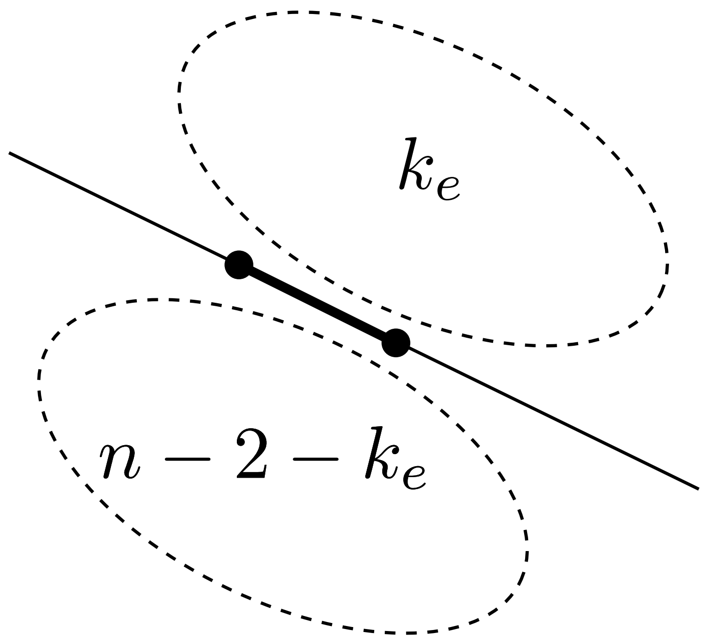

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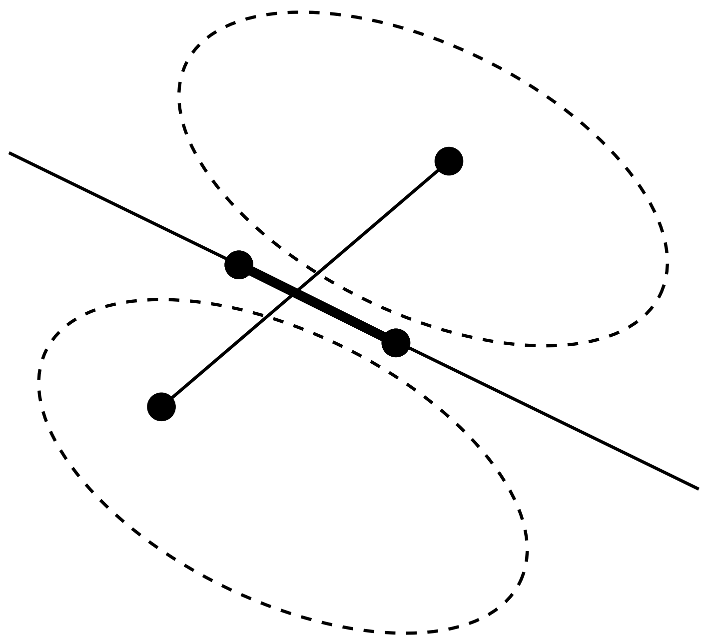

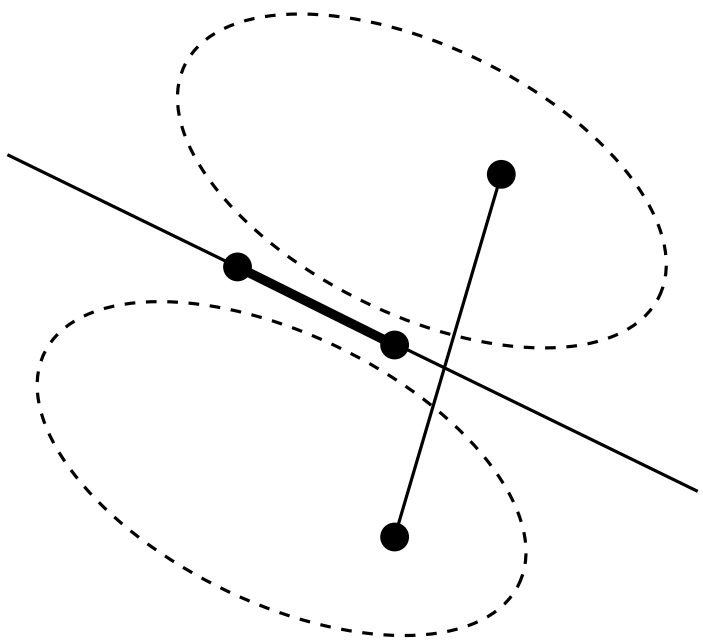

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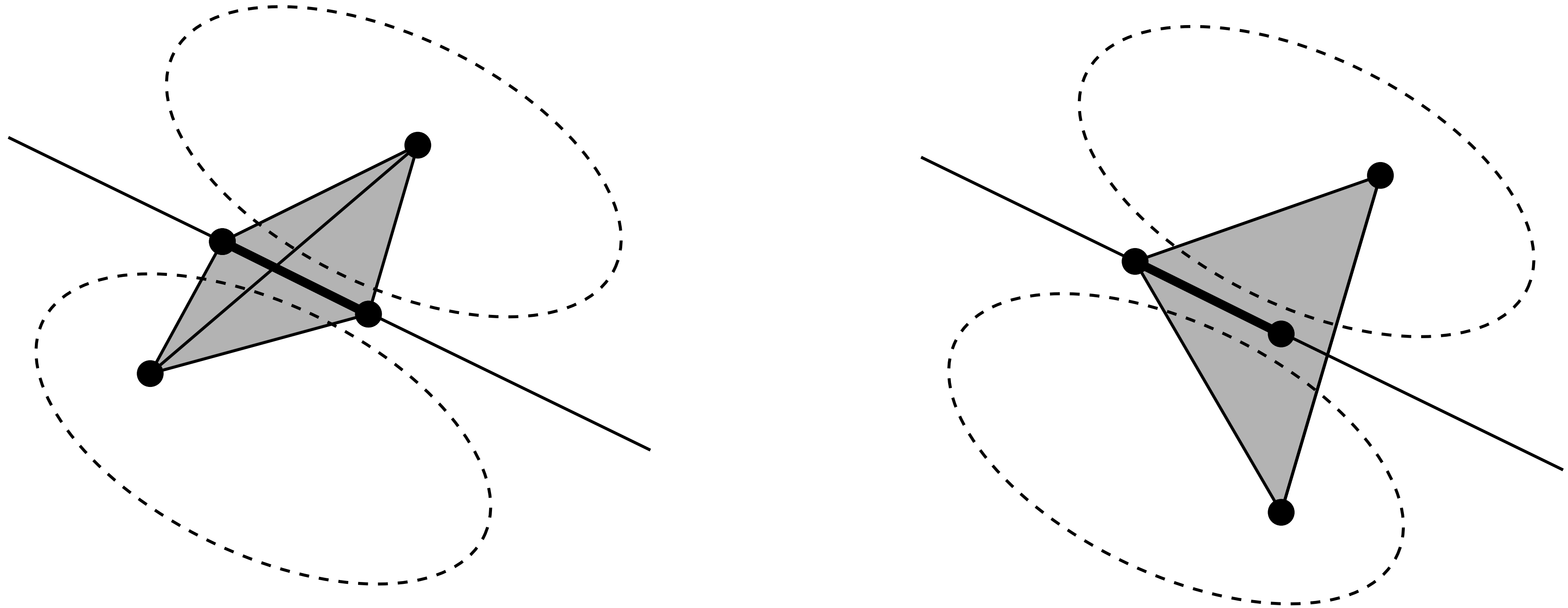

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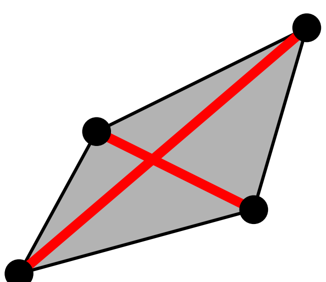
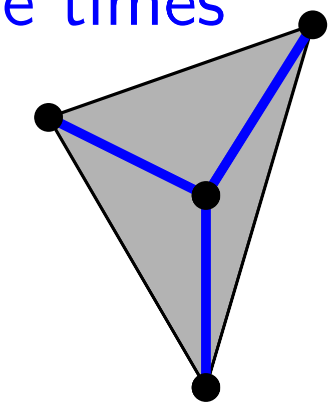


counted twice

counted three times



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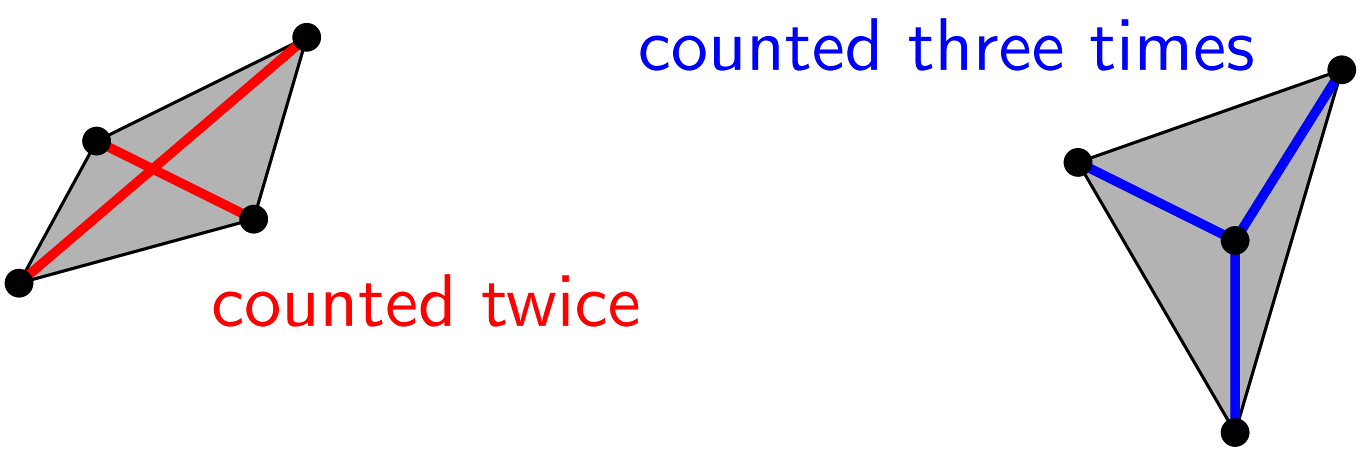

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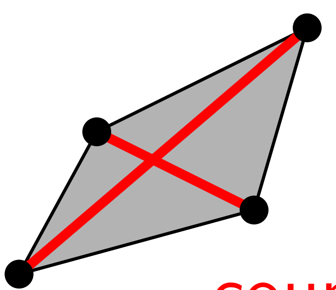
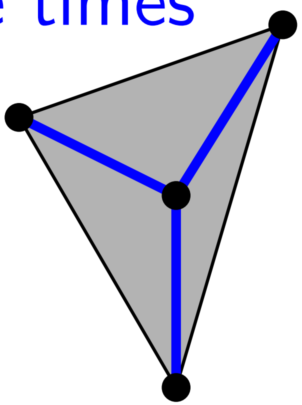
QED





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Remark: this crossings – k-edges relation generalizes to simple topological drawings of K_n
[Ábrego, Fernández-Merchant, Ramos, Salazar '11]

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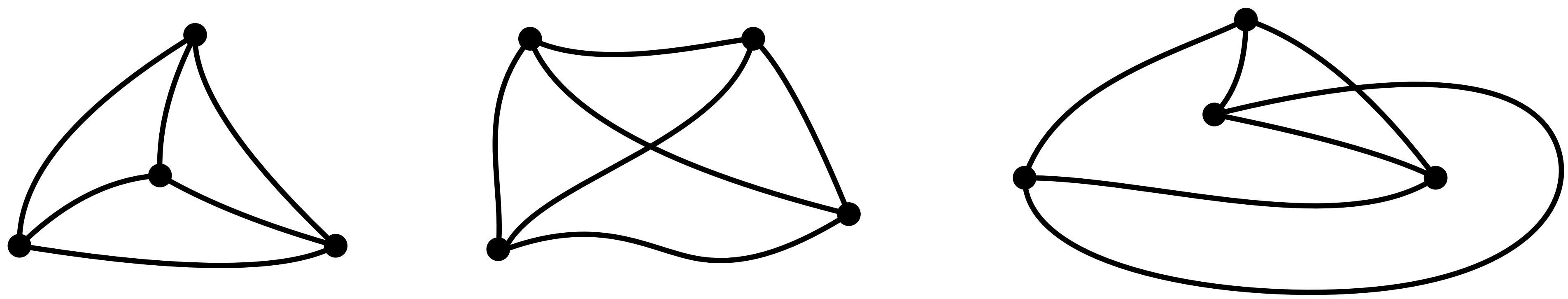
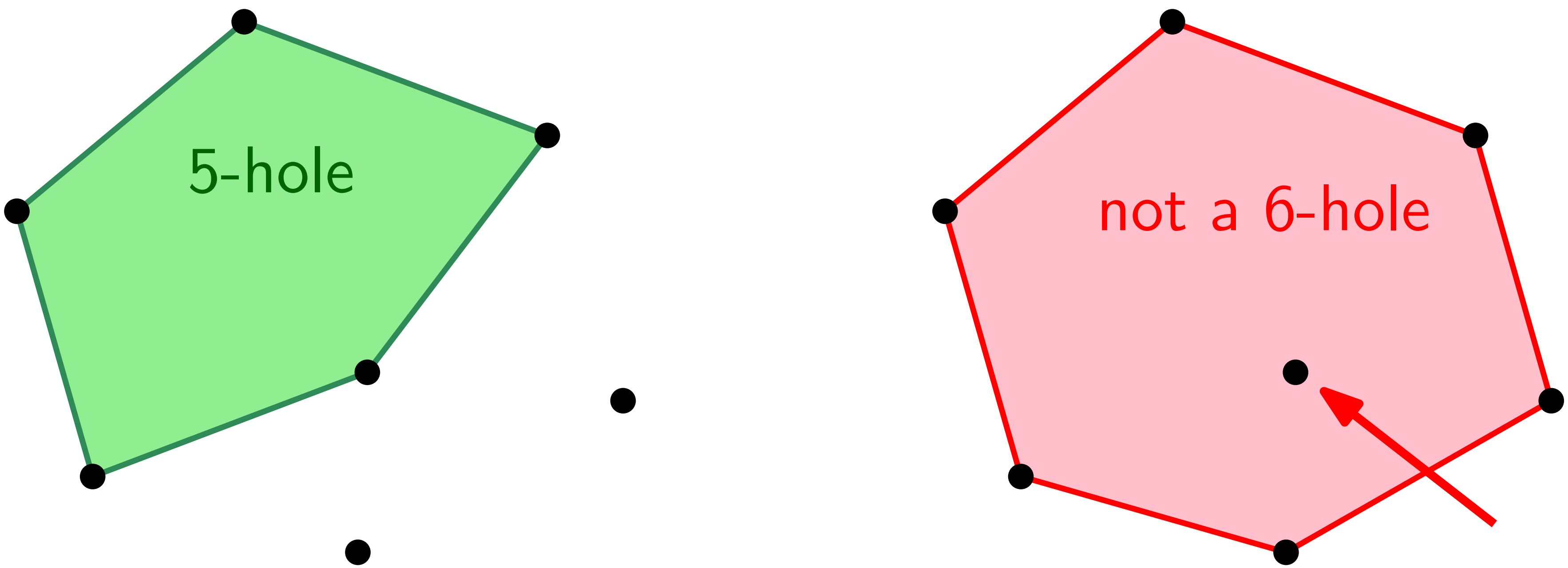


k-Holes

a k-hole (in S) is the vertex set of a convex k-gon containing no other points of S



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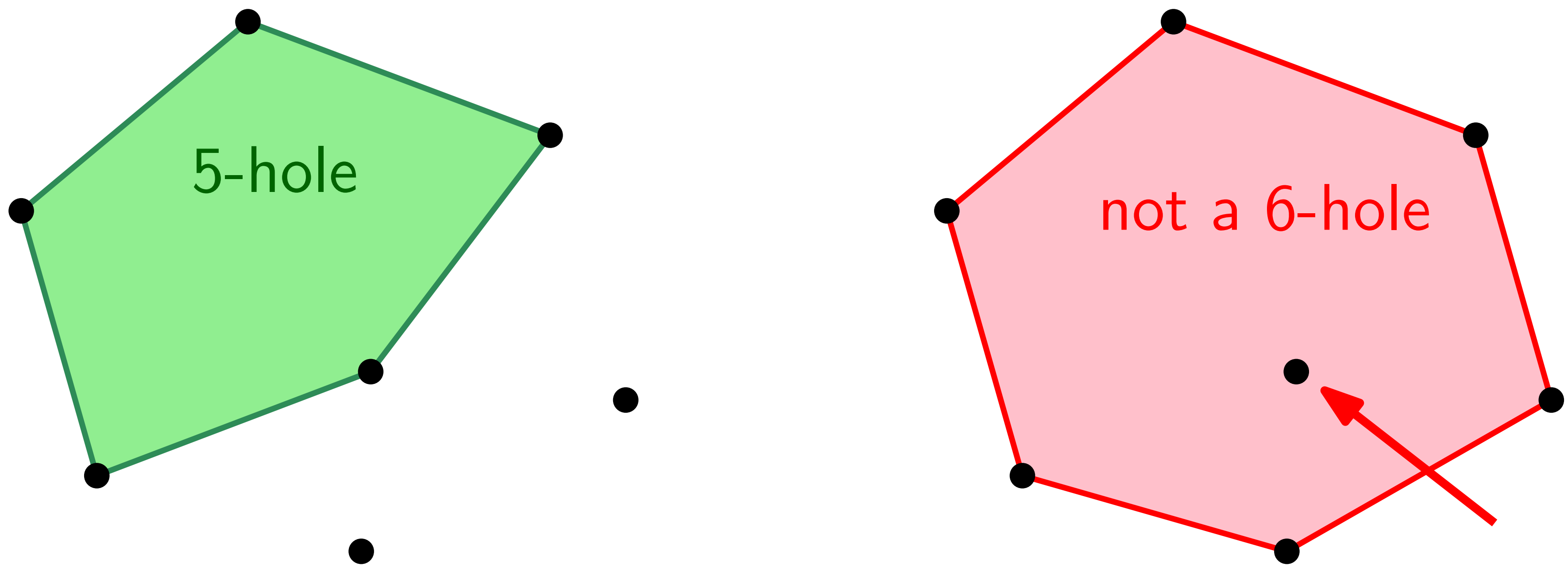

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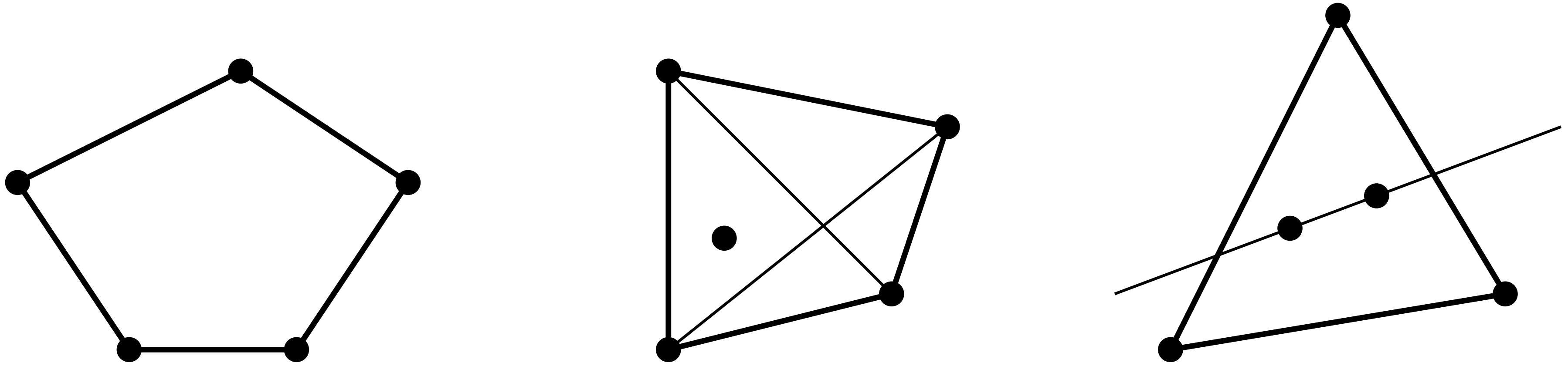

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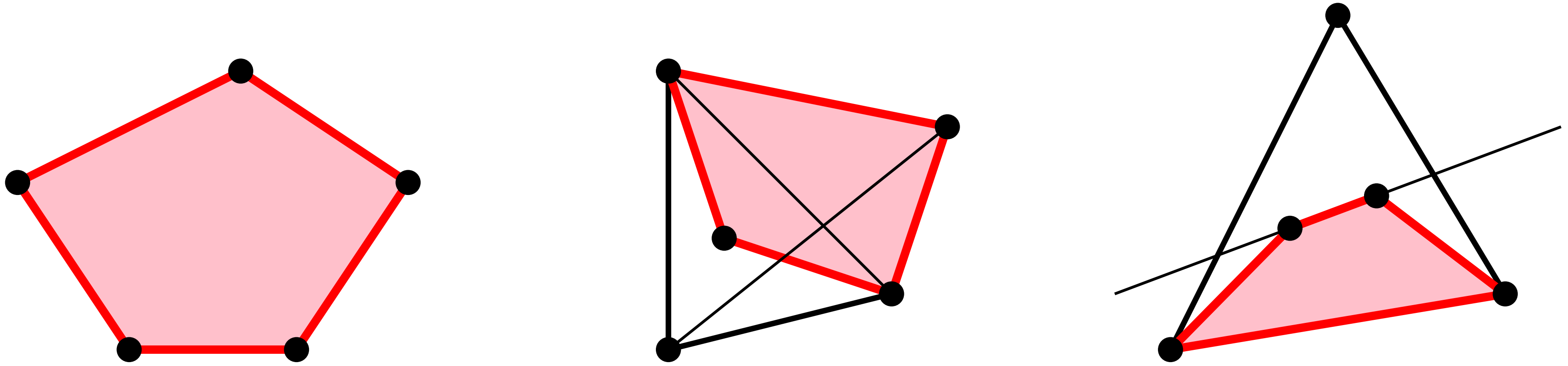

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- Sufficiently large point sets $\Rightarrow$ $\exists$ 6-hole [Gerken '08 and Nicolás '07, independently]

k-Holes

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq 463$, $h(7) = \infty$

Harborth '78

Overmars '02

Gerken '08, Nicolas '07, Koshelev '09

Horton'83

k-Holes

exact value remains unknown

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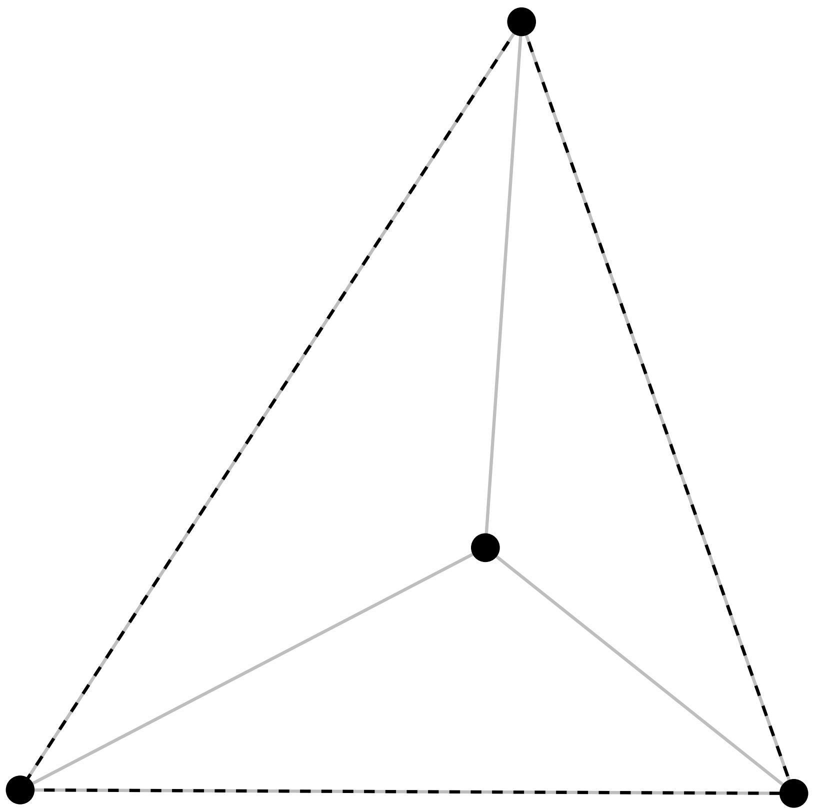
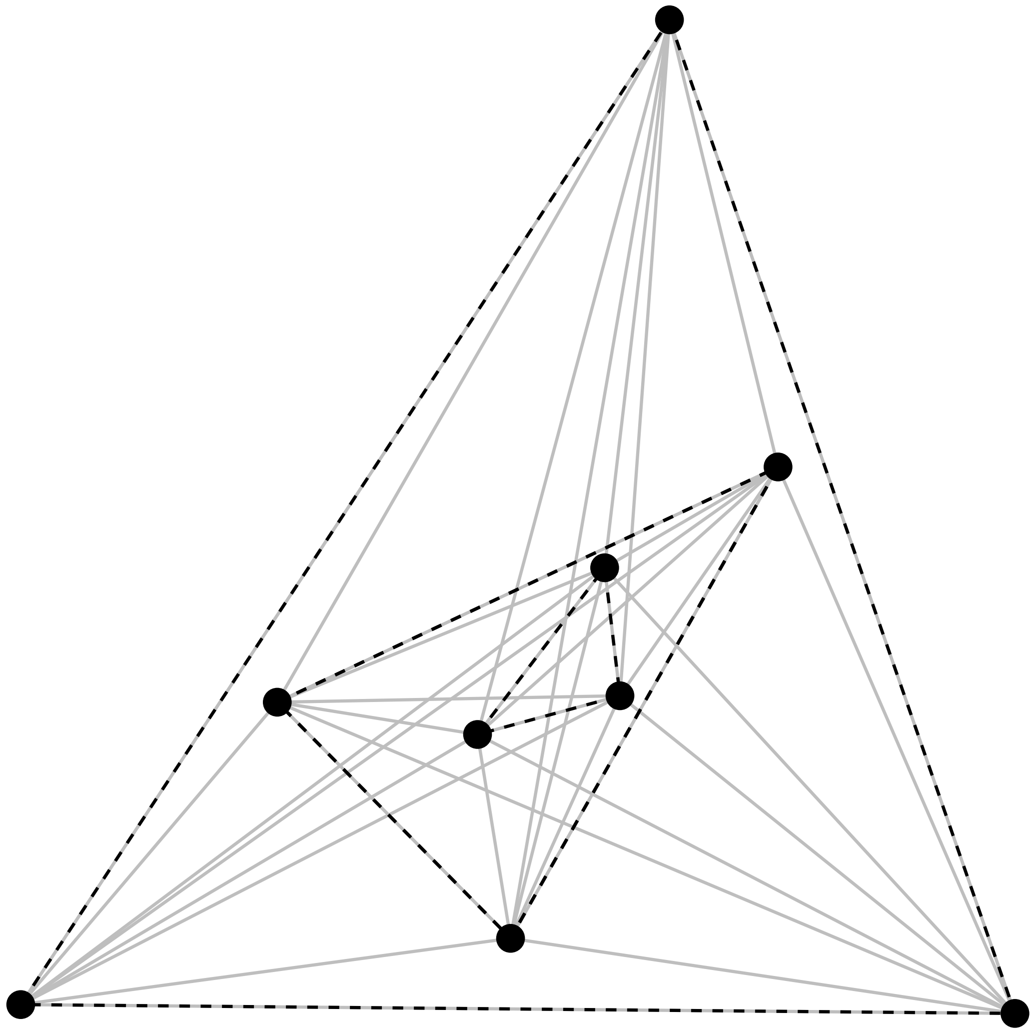


4 points, no 4-hole ($h(4) = 5$)

k-Holes



9 points, no 5-hole ($h(5) = 10$)

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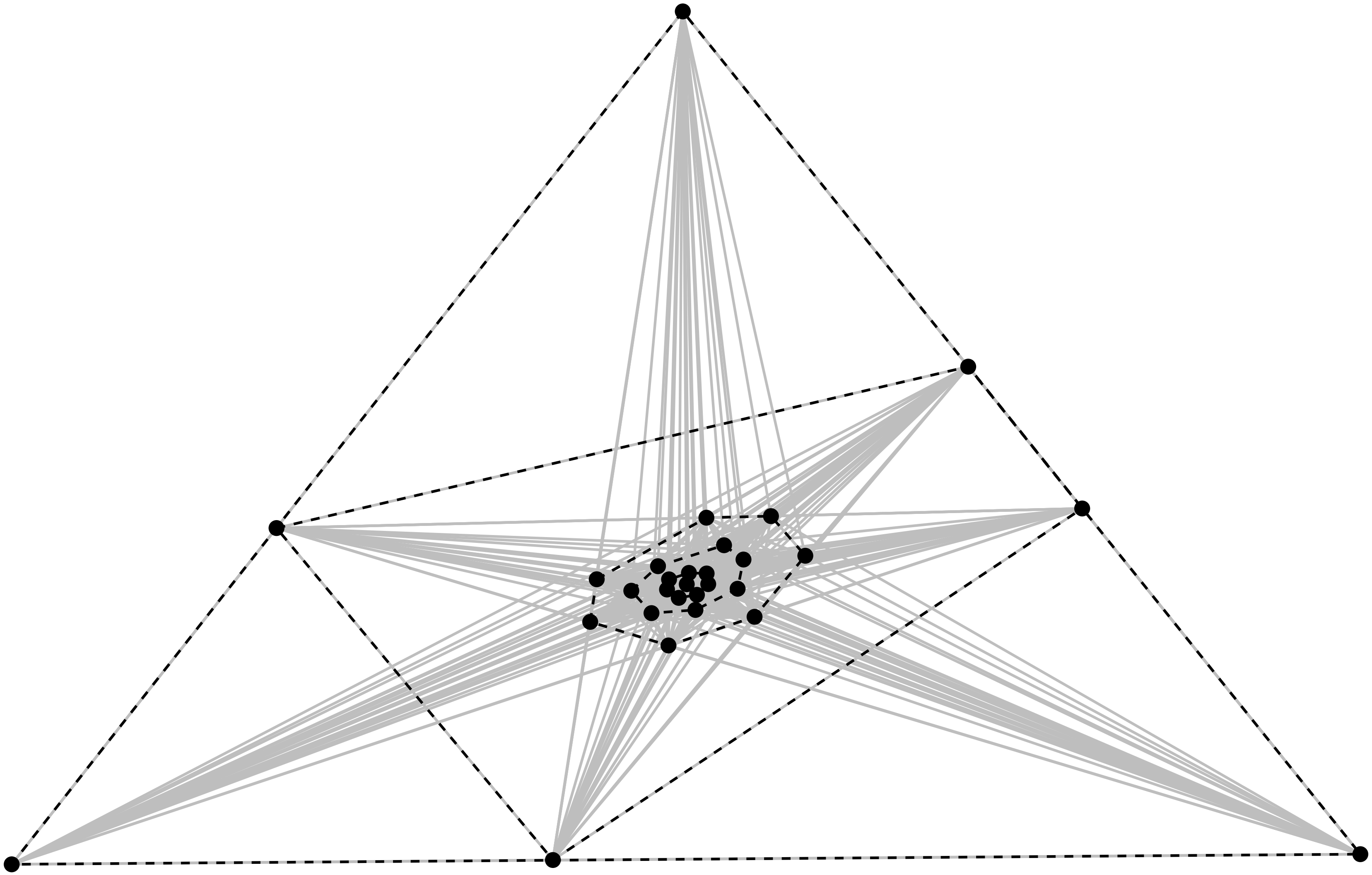
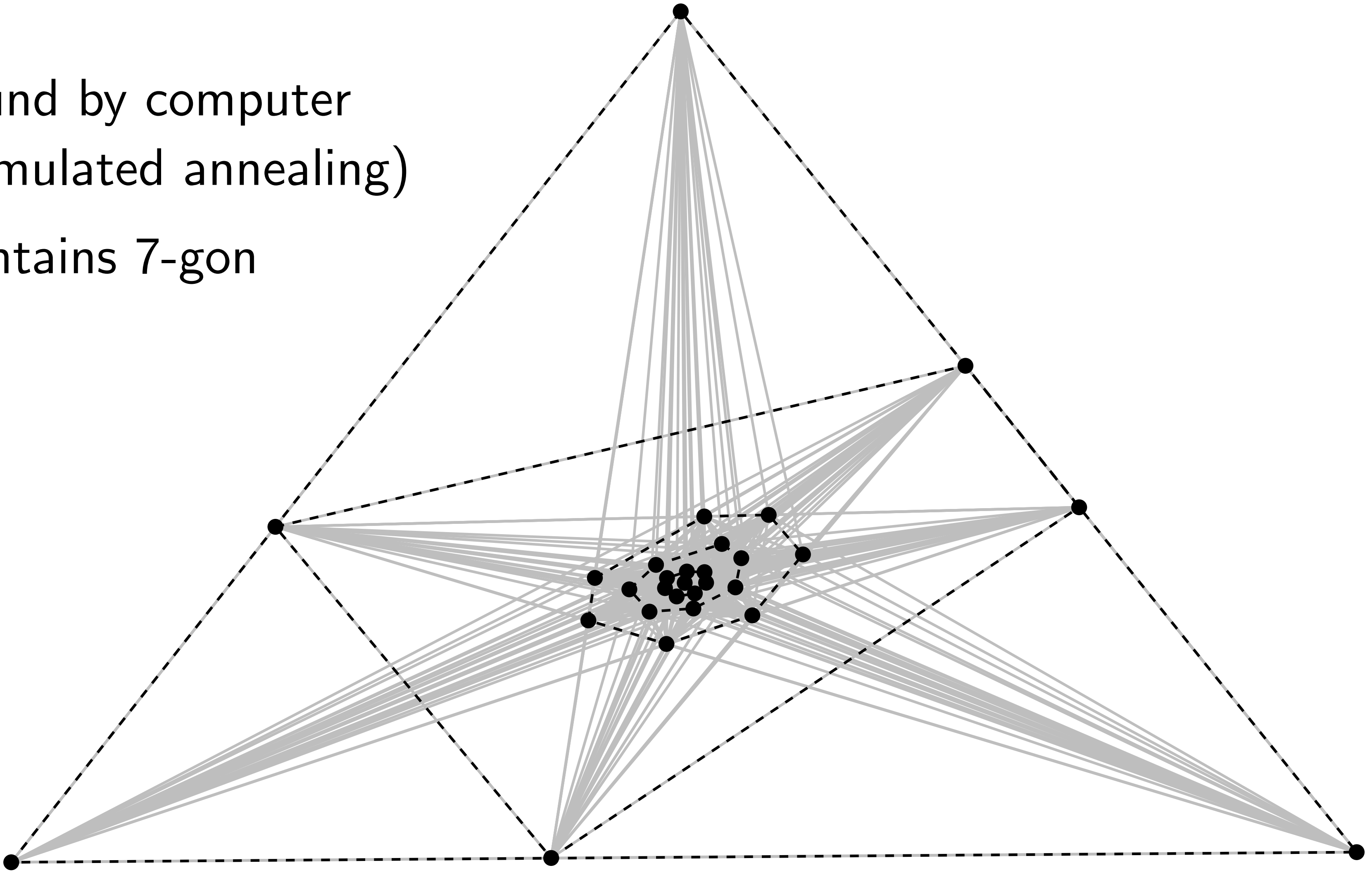


29 points, no 6-hole [Overmars '02] $(30 \leq h(6) \leq 463)$

k-Holes

- found by computer (simulated annealing)
- contains 7-gon



29 points, no 6-hole [Overmars '02] $(30 \leq h(6) \leq 463)$

k-Holes

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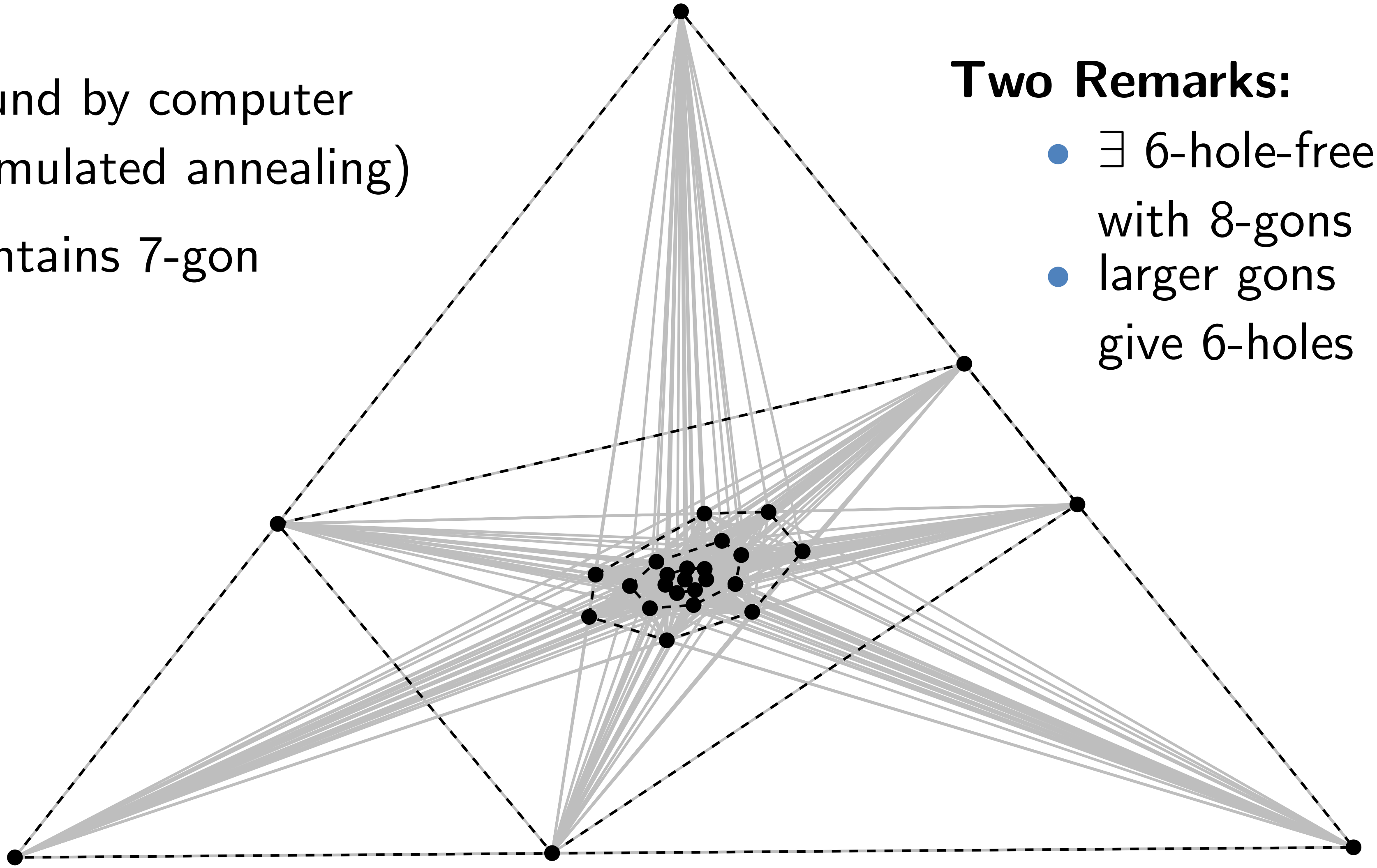


Two Remarks:

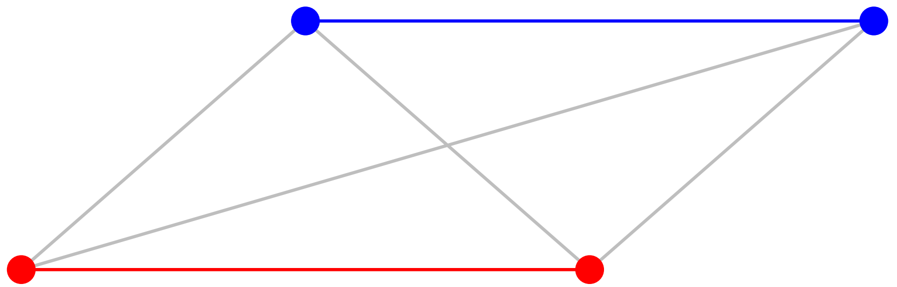
- $\exists$ 6-hole-free sets with 8-gons
- larger gons give 6-holes

29 points, no 6-hole [Overmars '02] $(30 \leq h(6) \leq 463)$

k-Holes

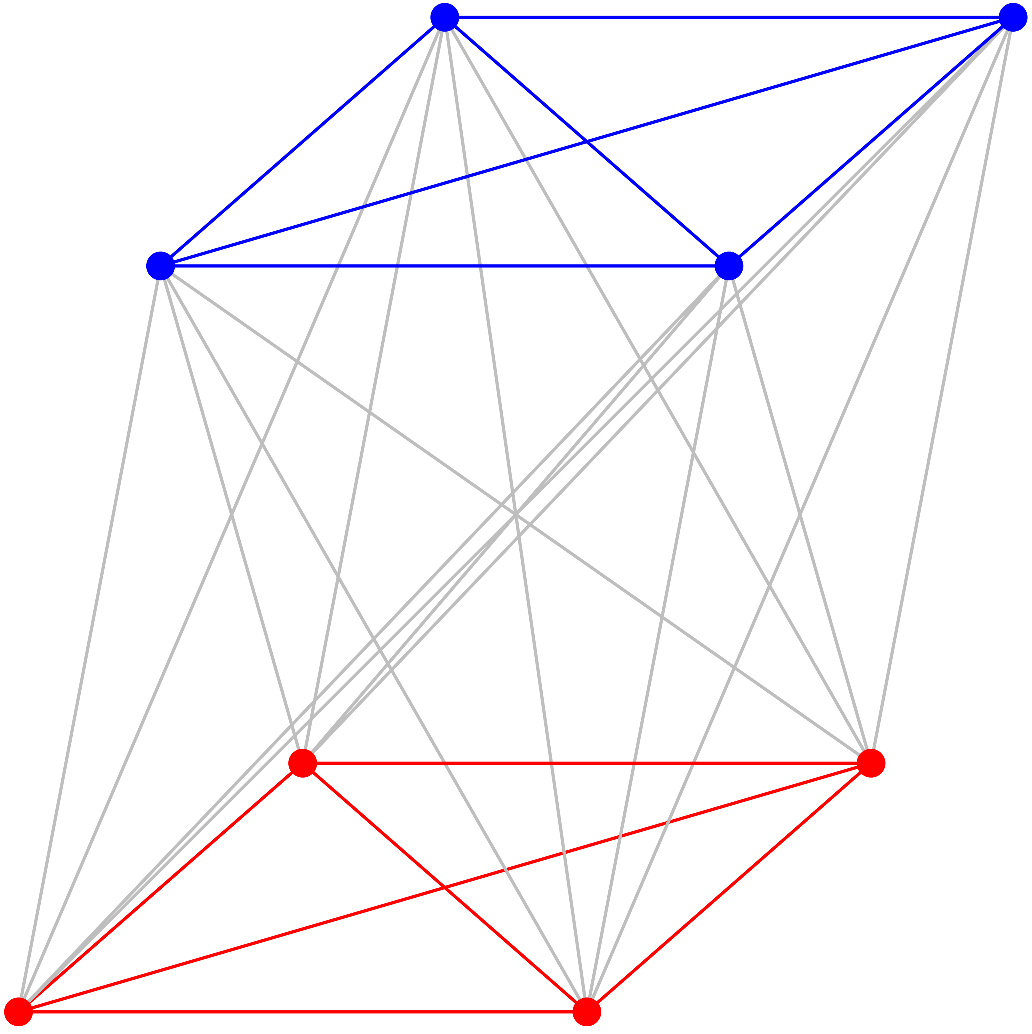
Horton's construction for $n = 2^1$ points, no 7-holes

k-Holes



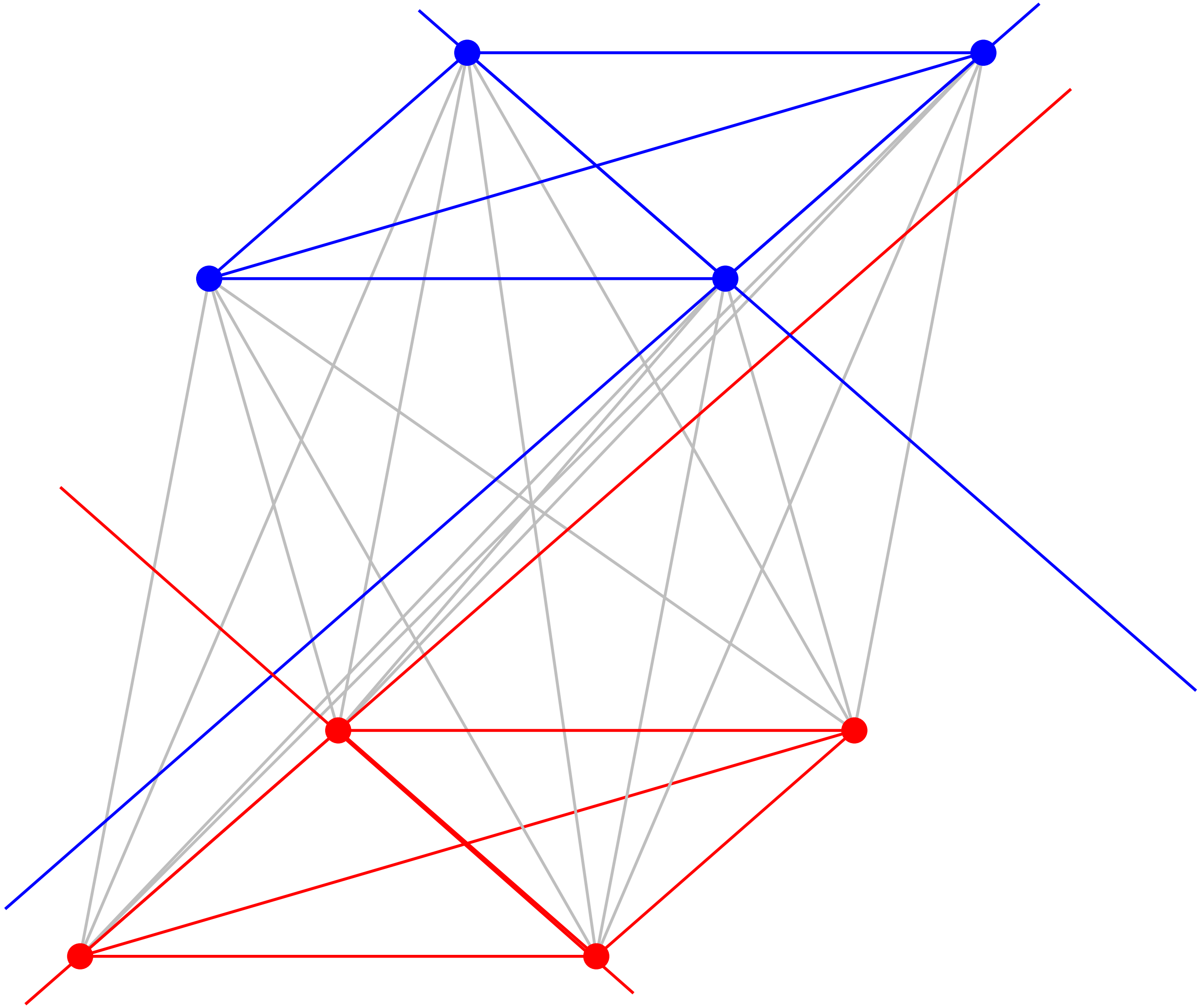
Horton's construction for $n = 2^2$ points, no 7-holes

k-Holes



Horton's construction for $n = 2^3$ points, no 7-holes

k-Holes



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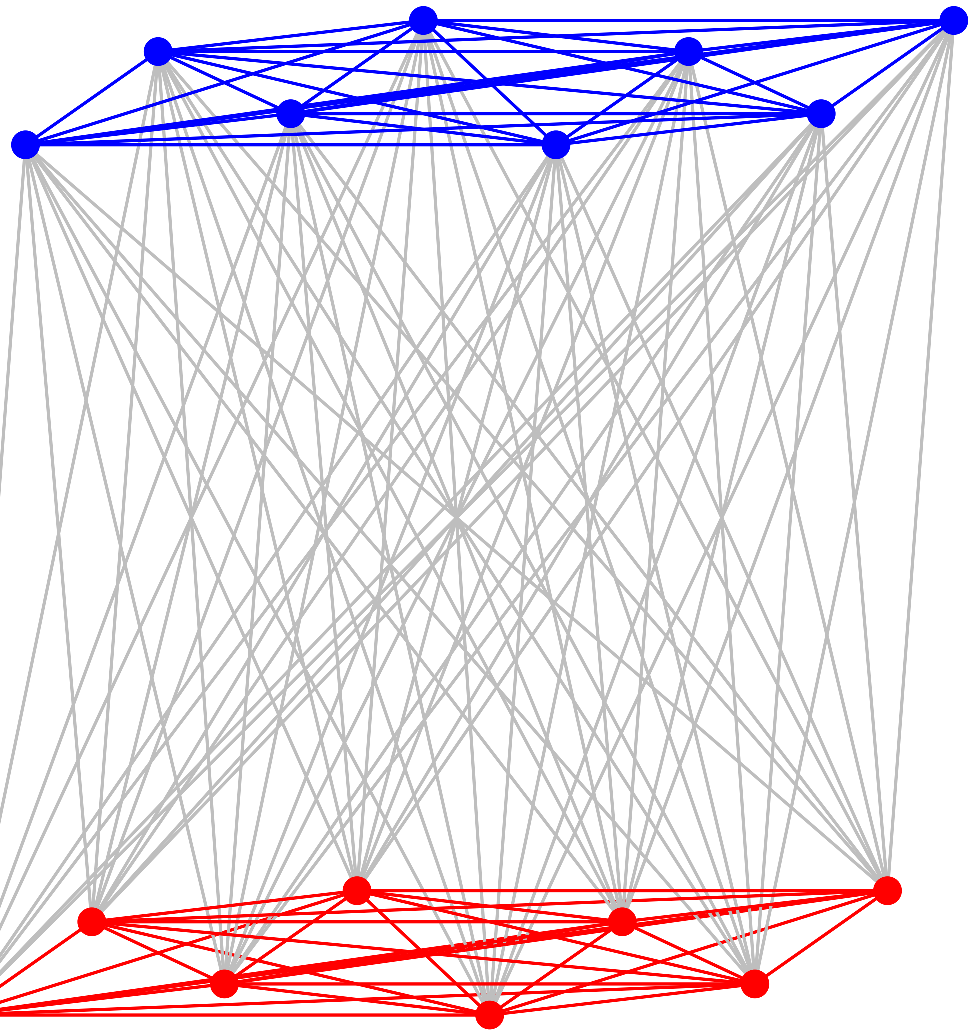
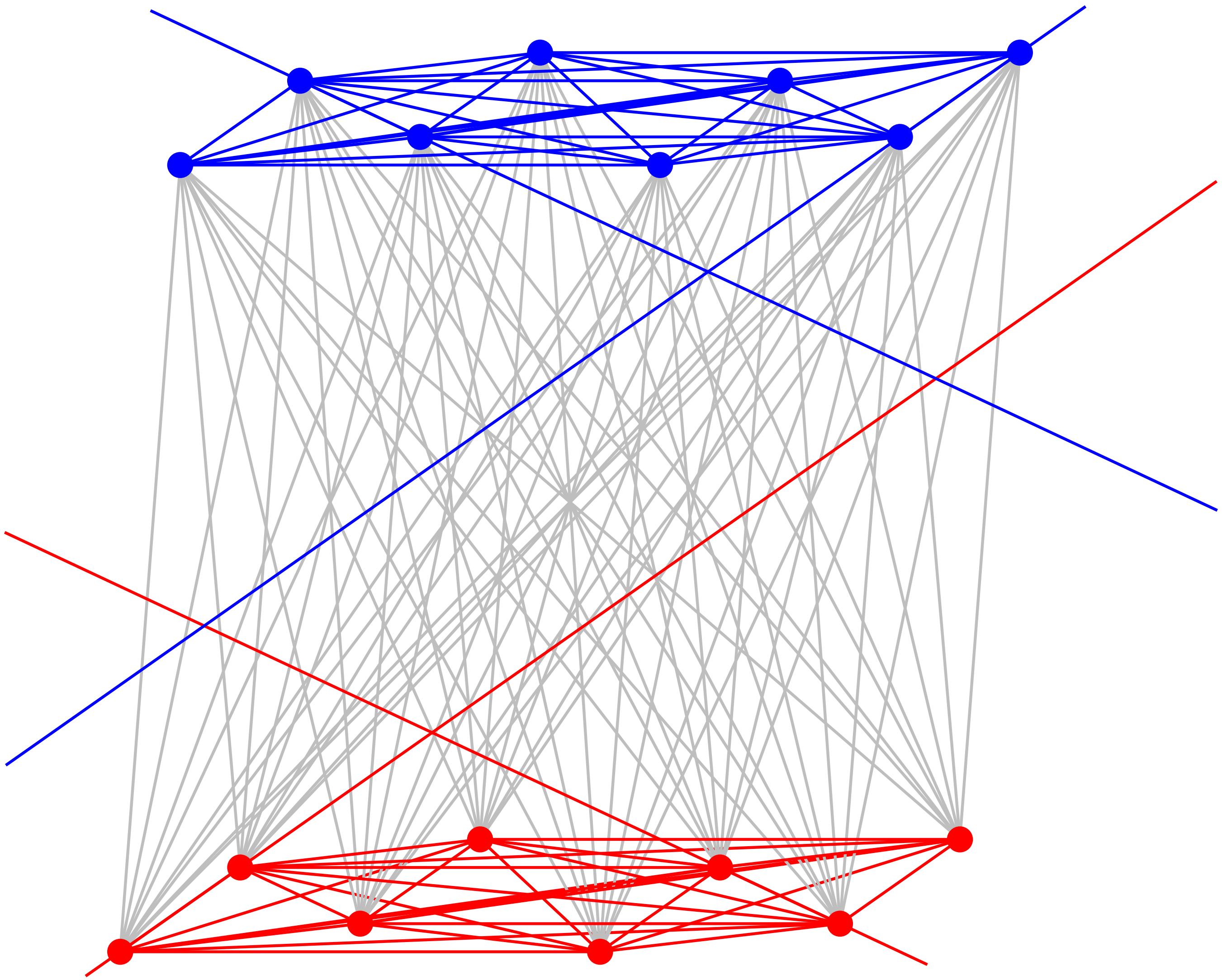


Horton's construction for $n = 2^4$ points, no 7-holes

k-Holes



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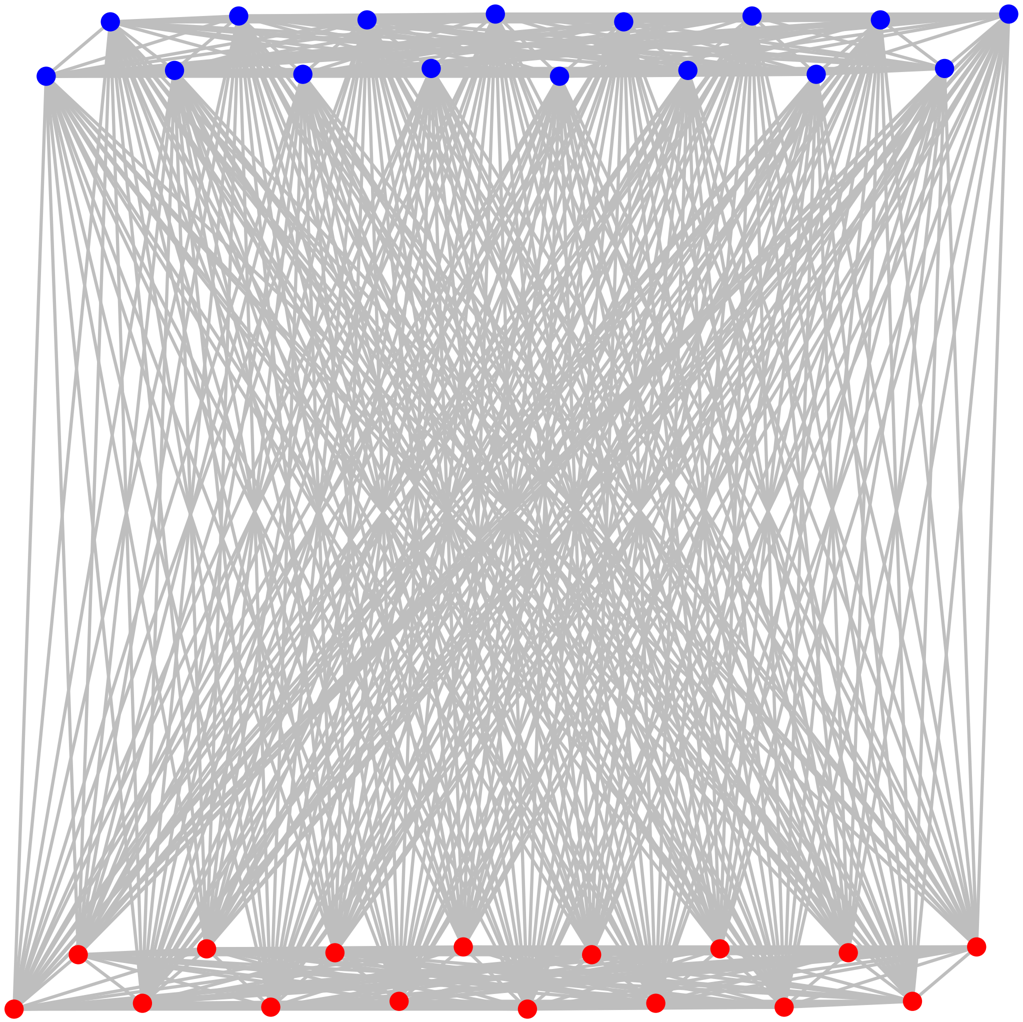
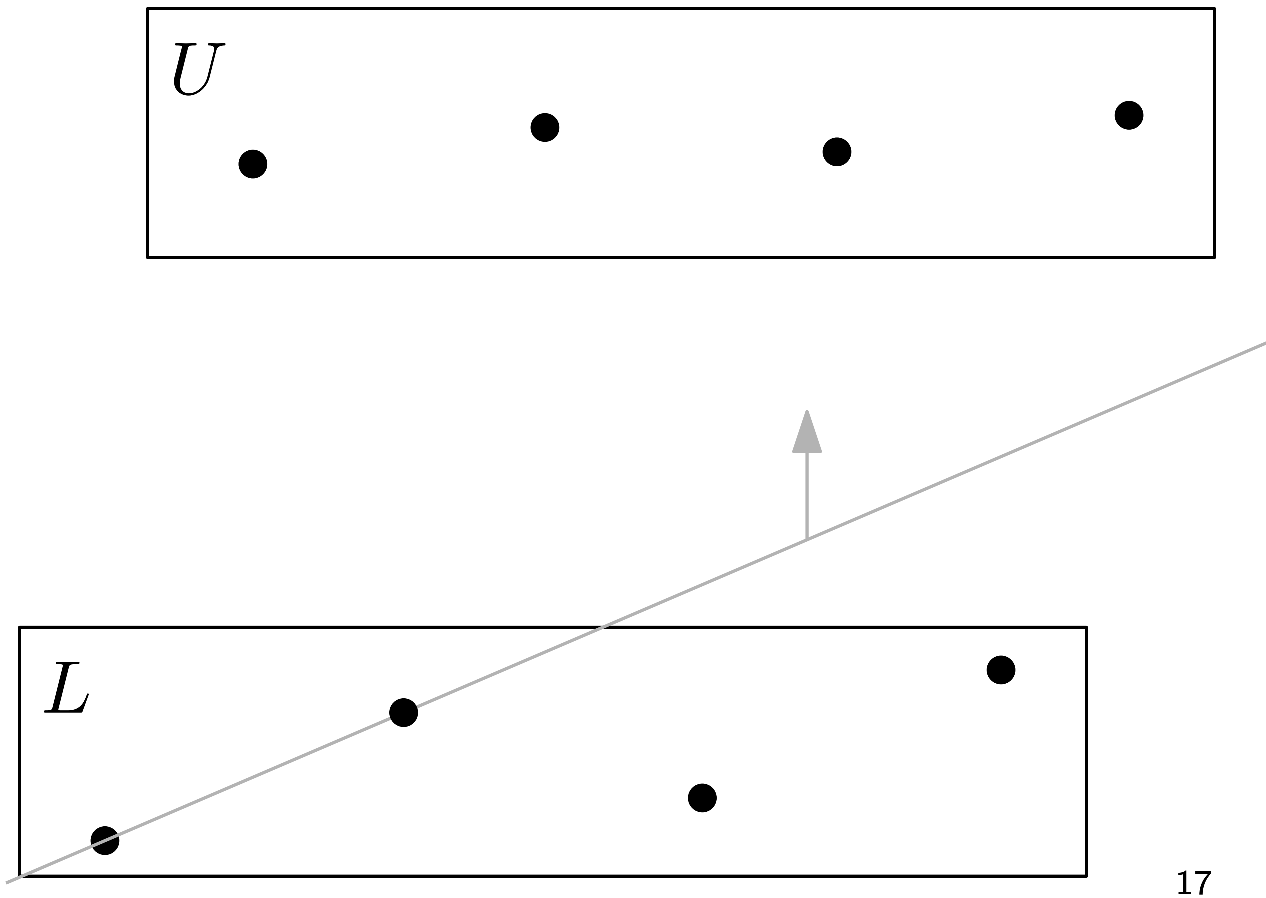


Horton's construction: $n = 2^k$ points, no 7-holes ($h(7) = \infty$)

Horton Sets

S_1 = single point, and S_n recursively: two copies L, U of $S_{\frac{n}{2}}$

- from left to right: points alternatingly in L and U
- U high above L



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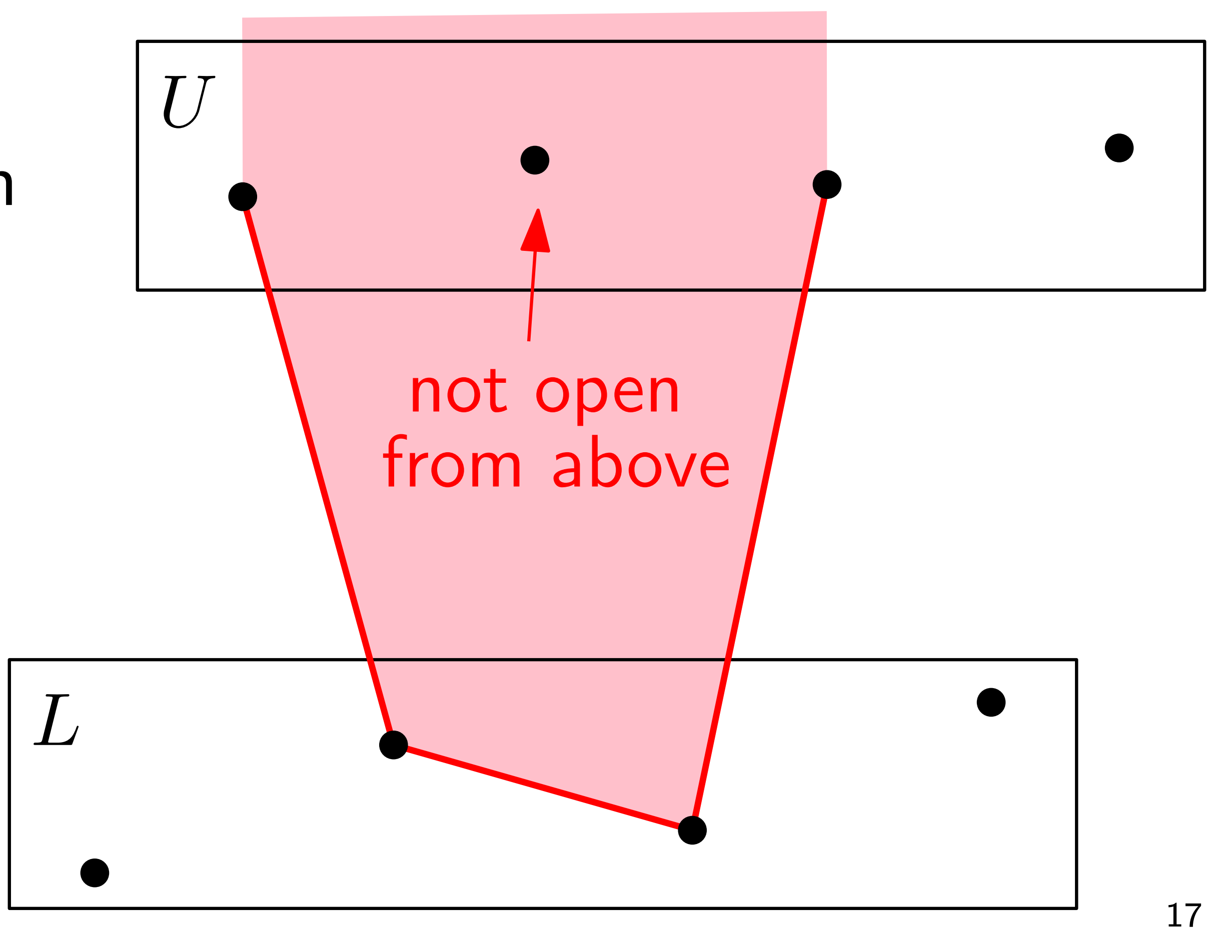

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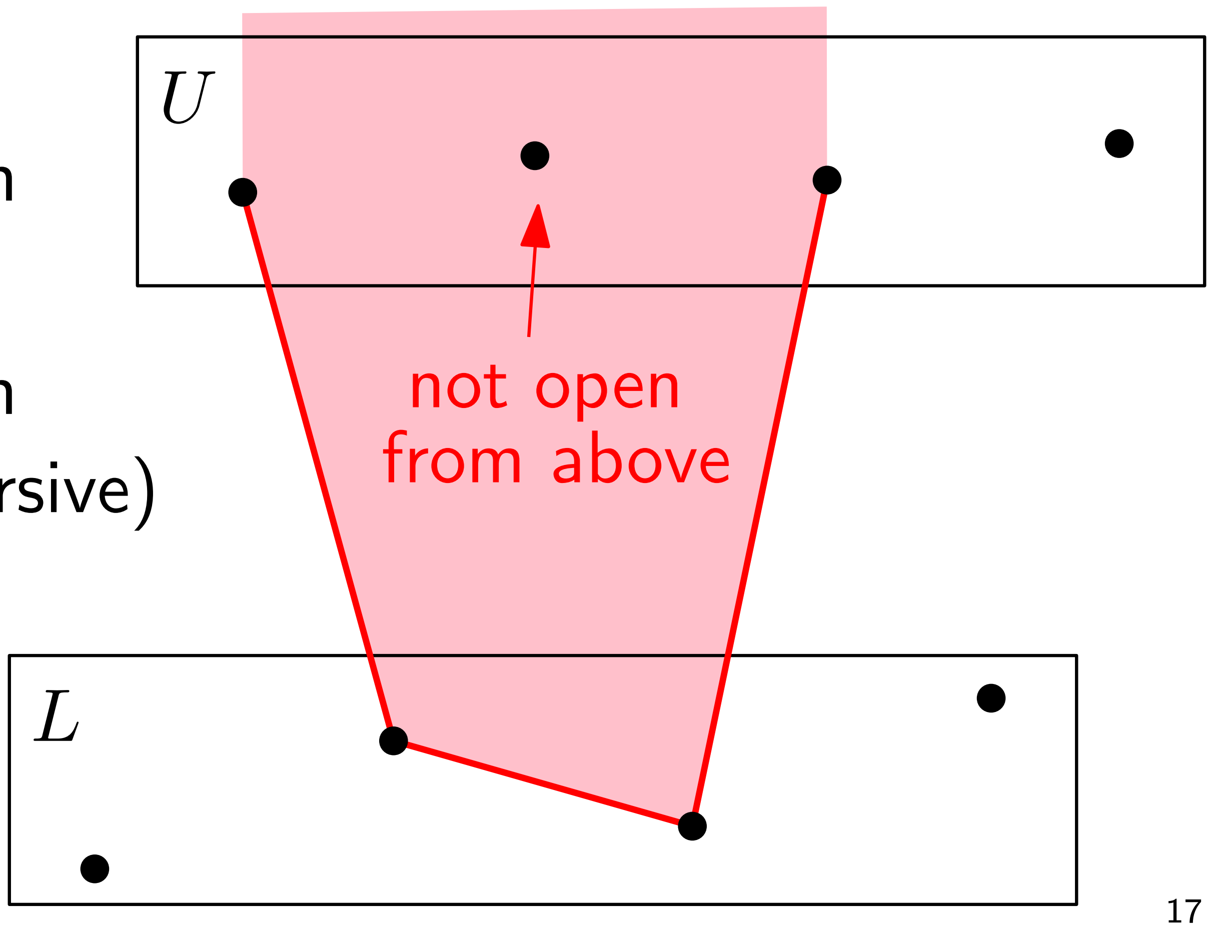

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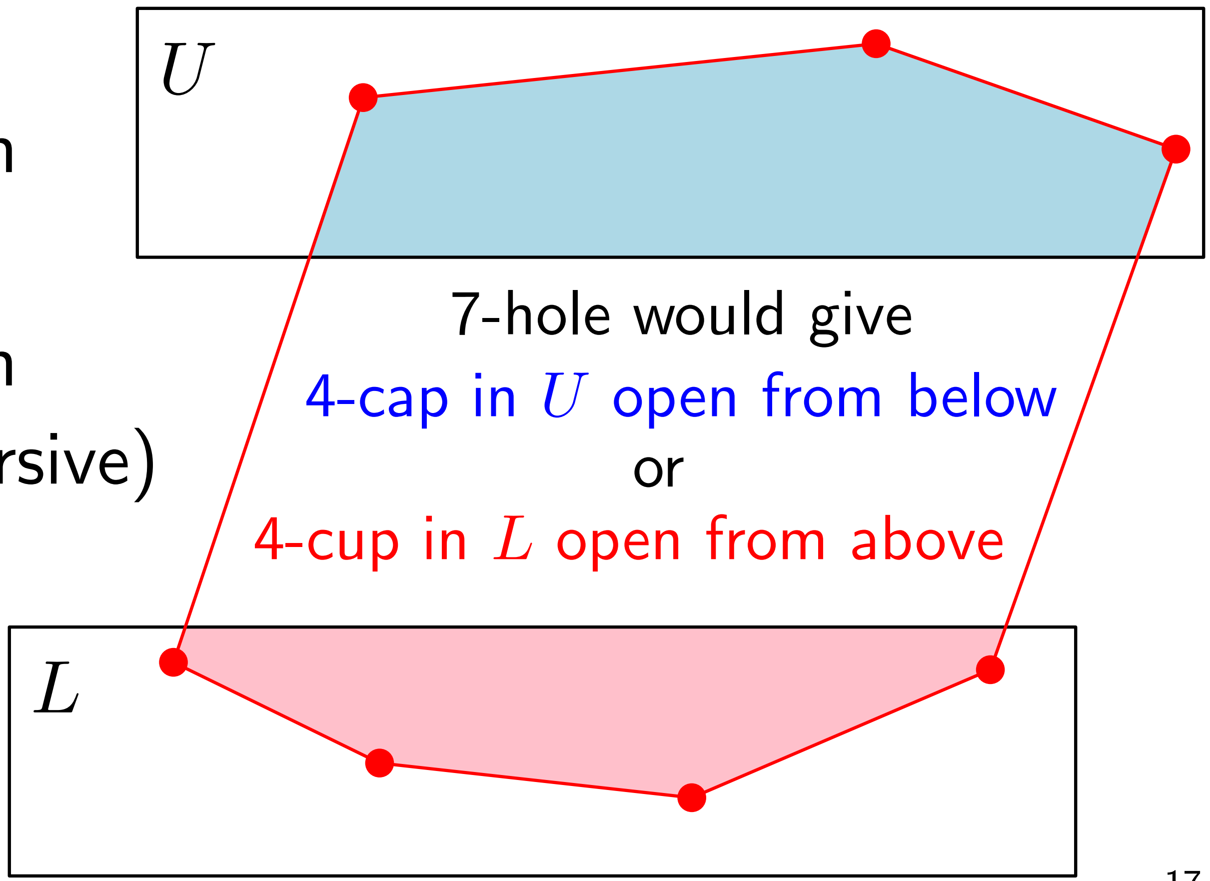

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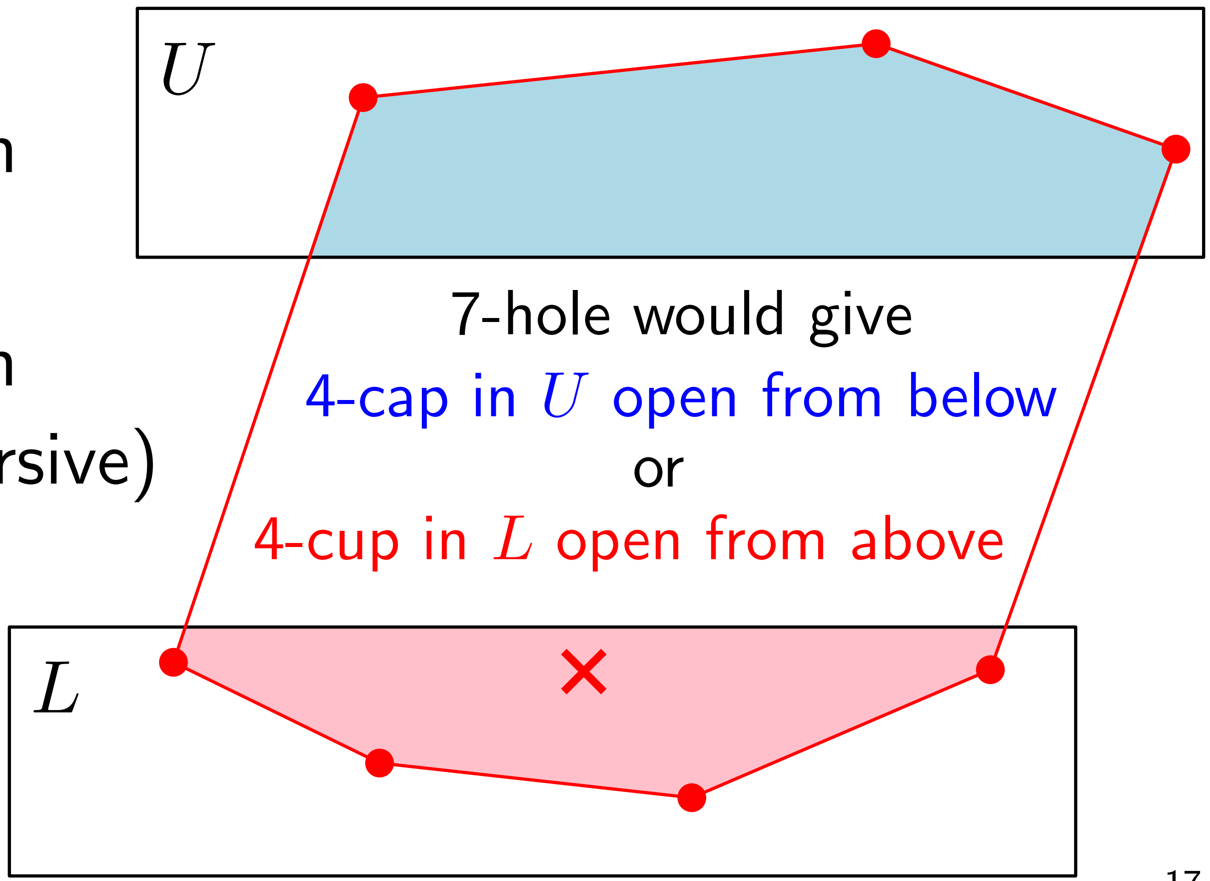

Existence of Holes

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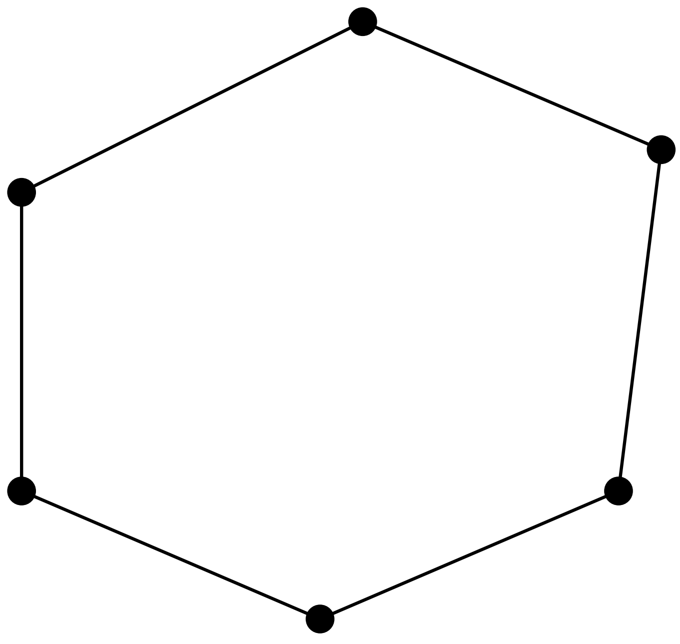
- we show $h(5) \leq g(6)$:

consider 6-gon G with minimum number of interior points I



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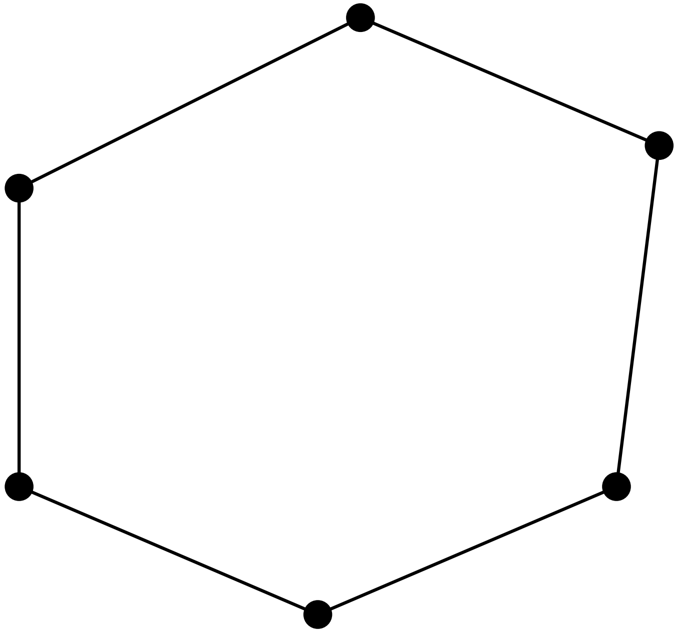

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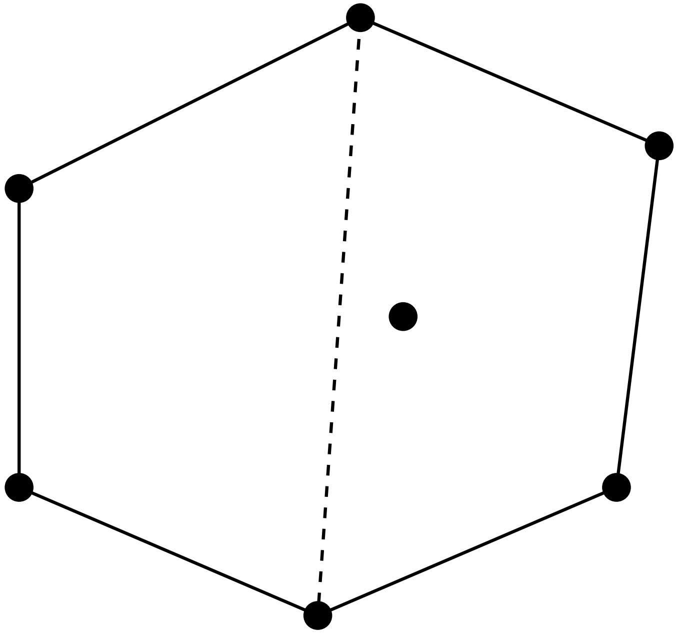

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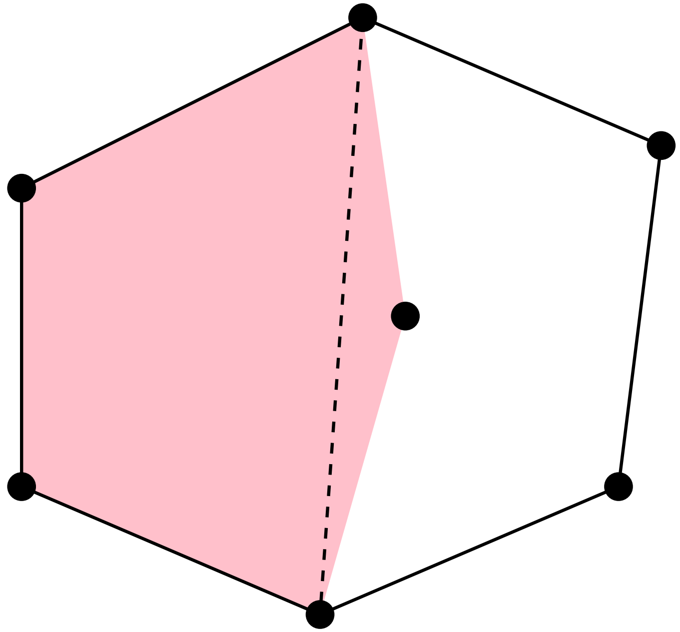

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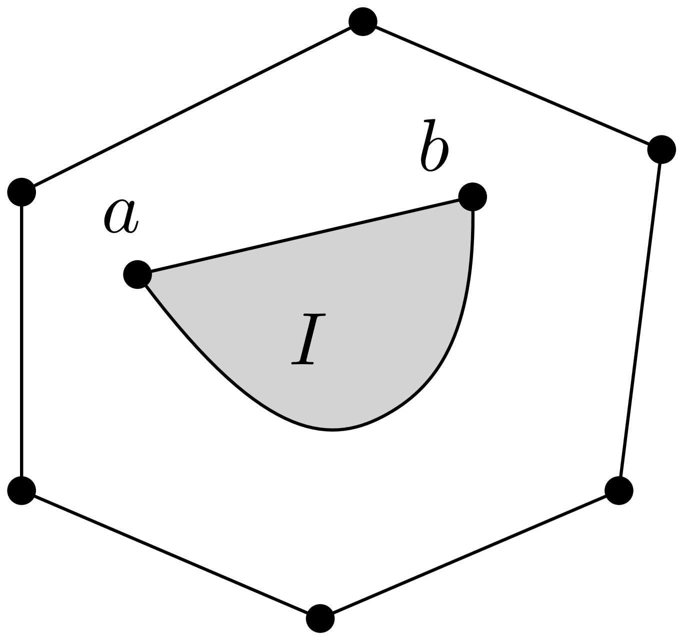

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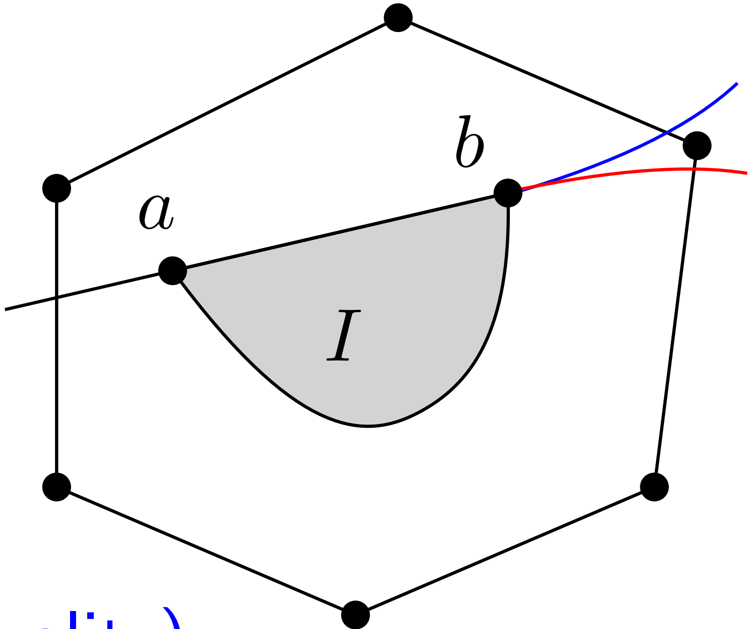
if I is empty, G is a 6-hole

if I contains 1 point, we find a 5-hole

if I contains ≥ 2 points,
we choose ab as bounding edge of $\mathrm{conv}(I)$,
and find a 5-hole or smaller 6-gon G'
(contradiction to minimality)



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  using the estimate $g(k) \le \binom{2k-5}{k-2} + 1$ [Tóth & Valtr '05]

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Q1: $\exists$ shorter (computed assisted?) proof for existence of 6-holes?

Q2: is $h(6)$ bounded in terms of $\max\{const, g(7)\}$?

k-holes: First and Second Moment Invariant

Thm (affine 1st moment, Edelman and Jamison '85):

$$\underbrace{h_0(P)}_{1} - \underbrace{h_1(P)}_{n} + \underbrace{h_2(P)}_{\binom{n}{2}} - h_3(P) \pm \ldots = \sum_{k \geq 0} (-1)^k h_k(P) = 0$$

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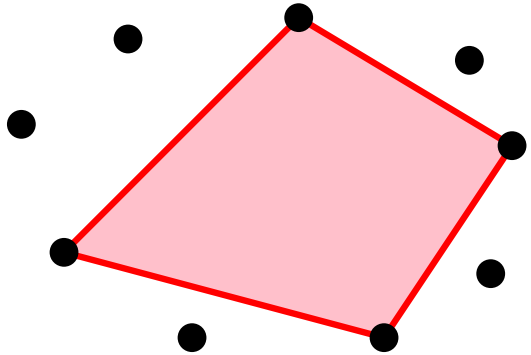
Proof Idea: holds for n points in convex position,



any k-subset is k-hole

$$\sum_{k \geq 0} (-1)^k \binom{n}{k} = (1 + (-1))^n = 0$$

Binomial theorem

k-holes: First and Second Moment Invariant

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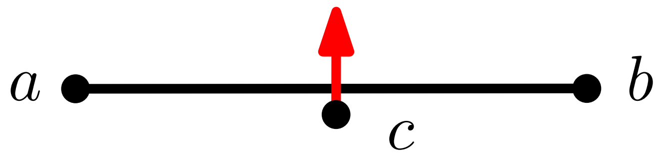
Proof Idea: holds for n points in convex position,

and invariant to mutations!

(i.e., when a point moves over a line)



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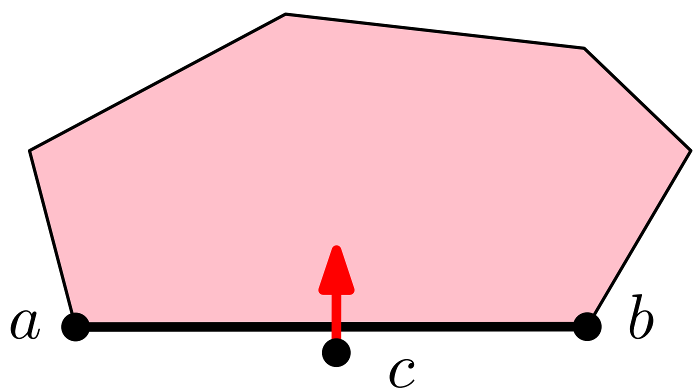

for each k-hole which we get/destroy with ab
we also get/destroy a $(k+1)$-hole with abc

k-holes: First and Second Moment Invariant

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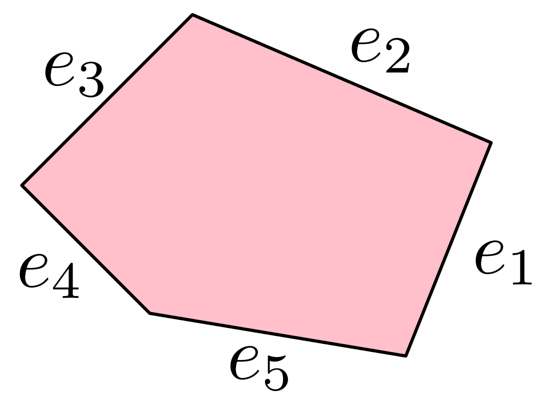
Idea: $k \cdot h_k(P) = \sum_e h_k(P; e)$



Quantity of k-Holes

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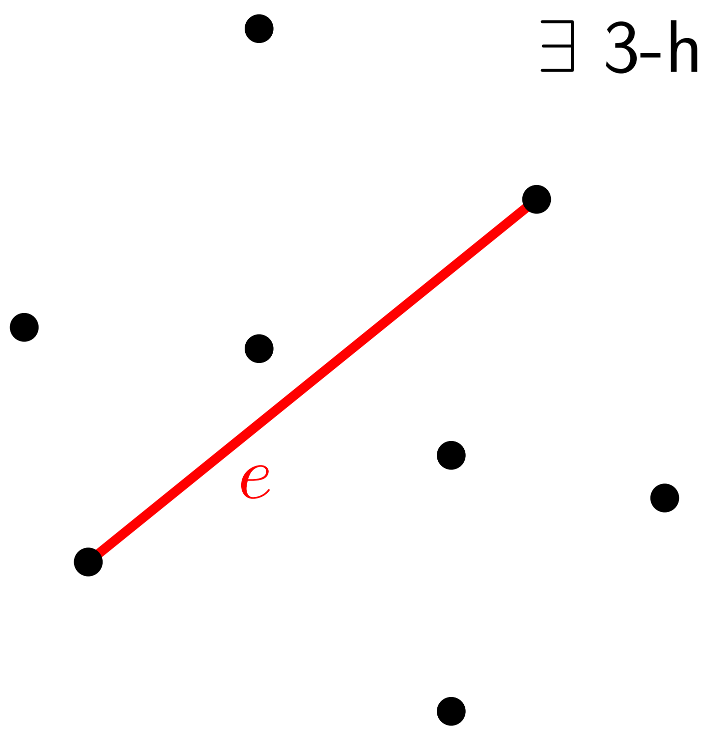
- $h_3(n) \geq \lfloor \frac{1}{3}\binom{n}{2} \rfloor = \Omega(n^2)$

$\forall$ edge e

$\exists$ 3-hole with closest point



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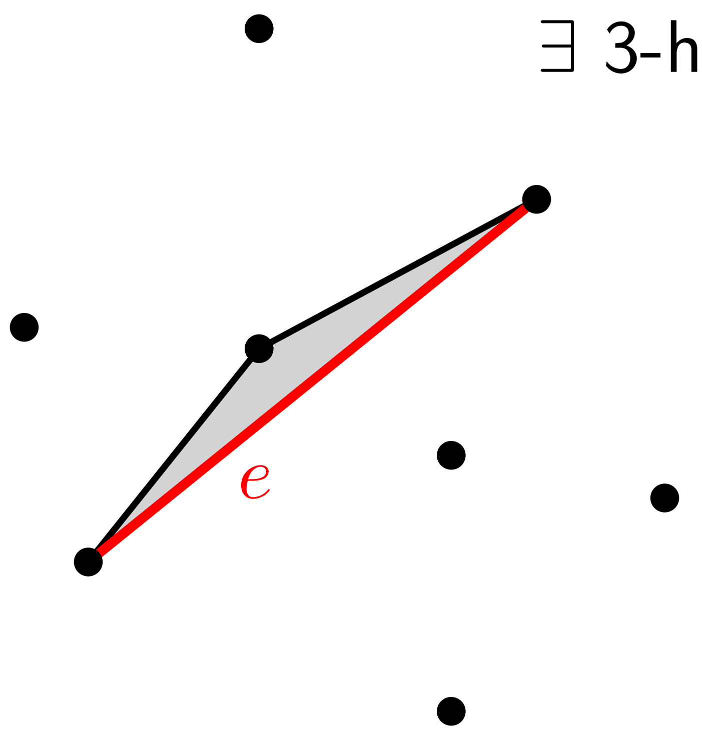

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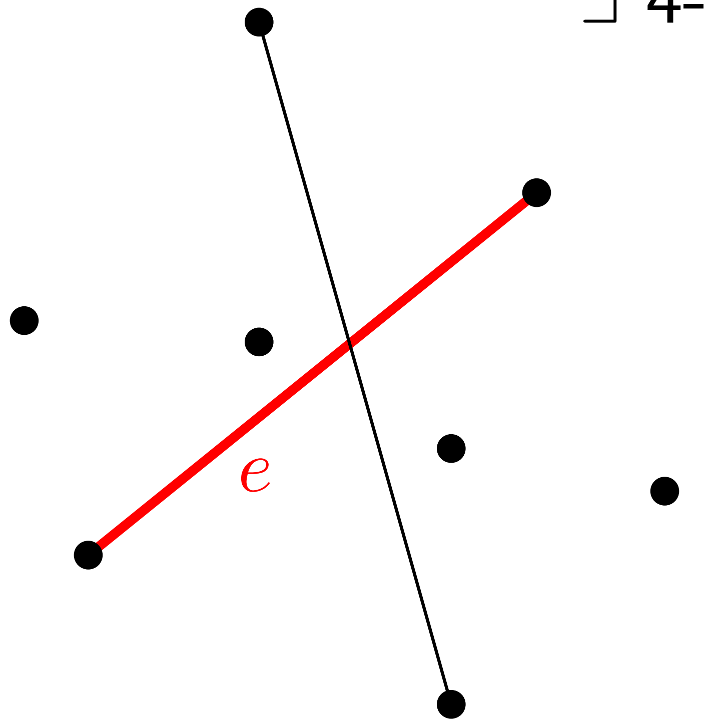
$\forall$ crossed edge e

$\exists$ 4-hole with diagonal e



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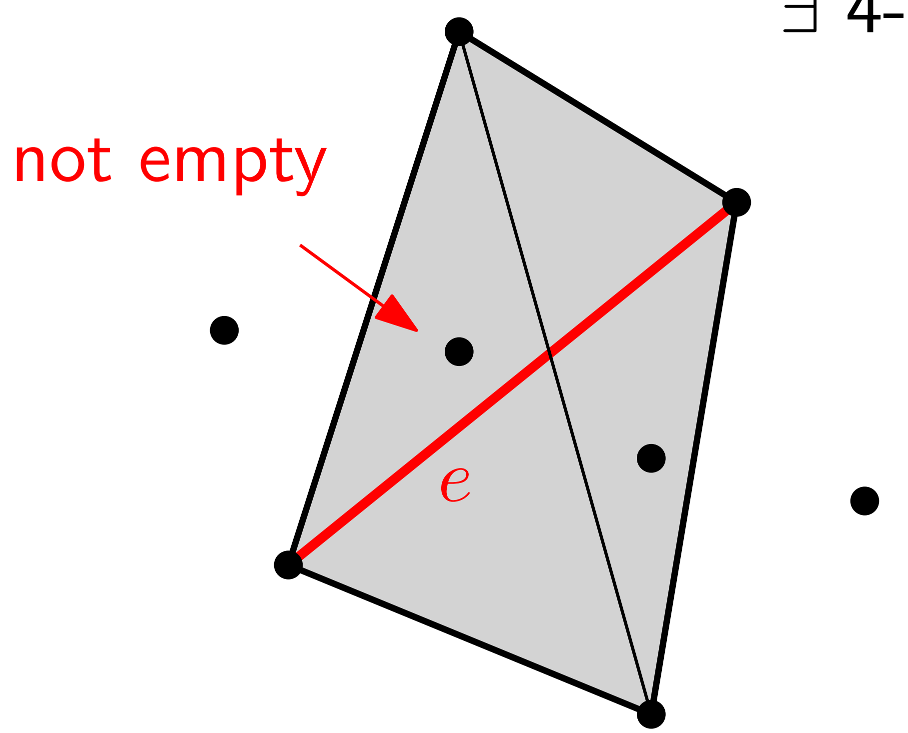

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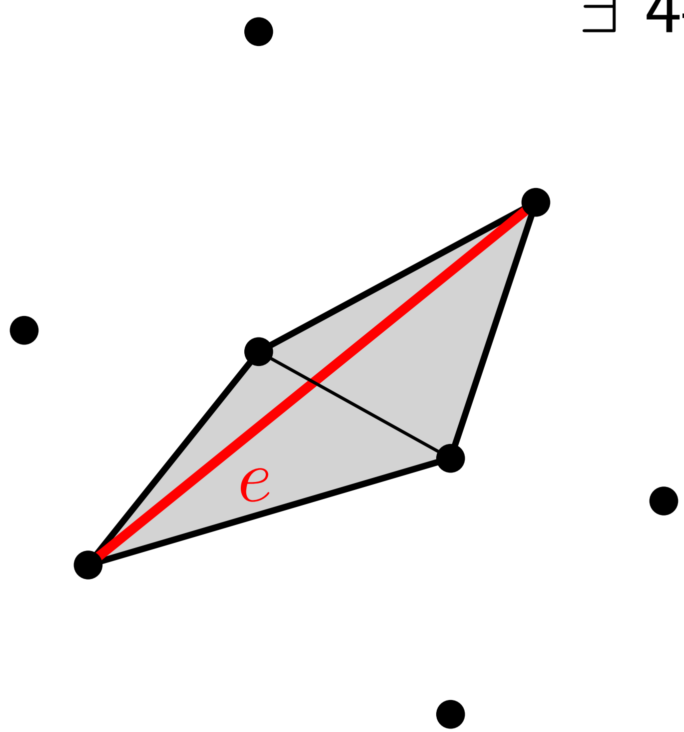

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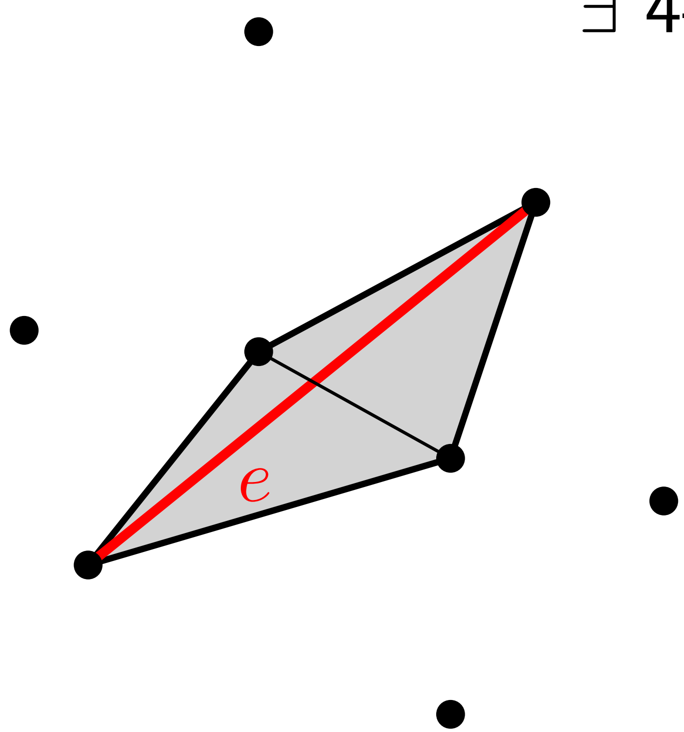

$O(n)$ uncrossed edges
(planar graph)

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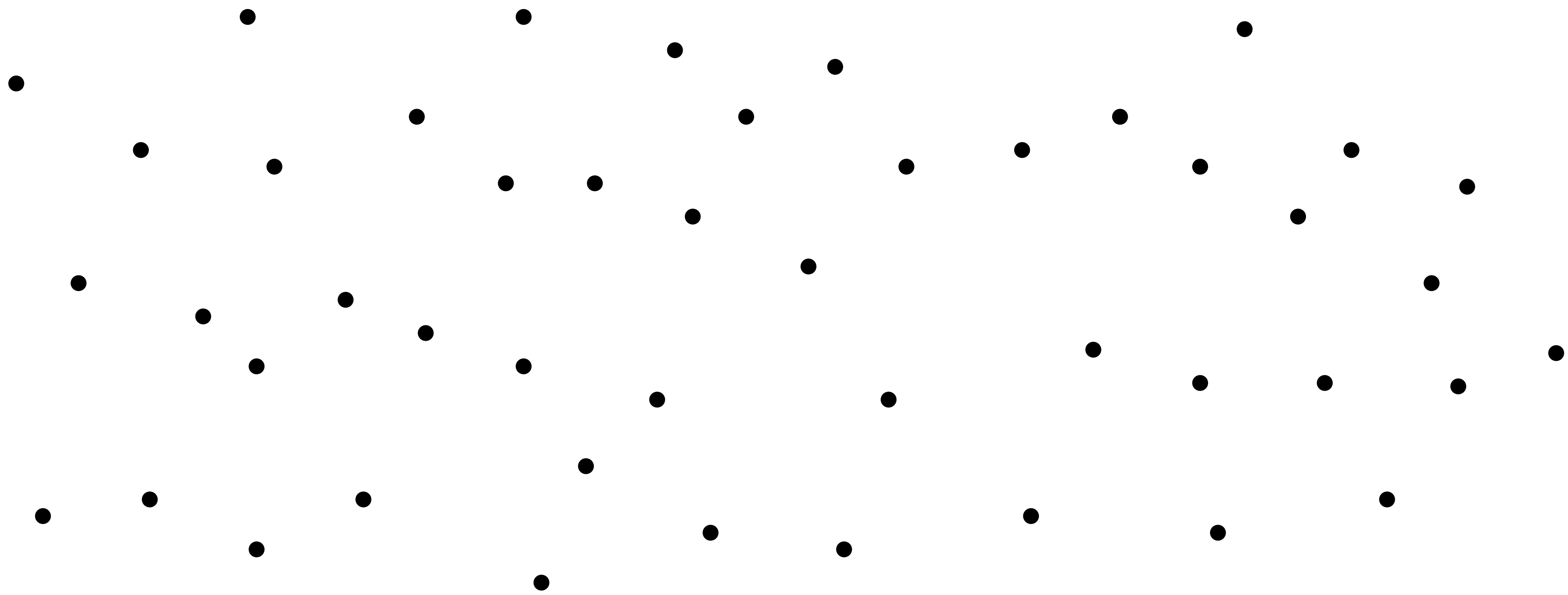

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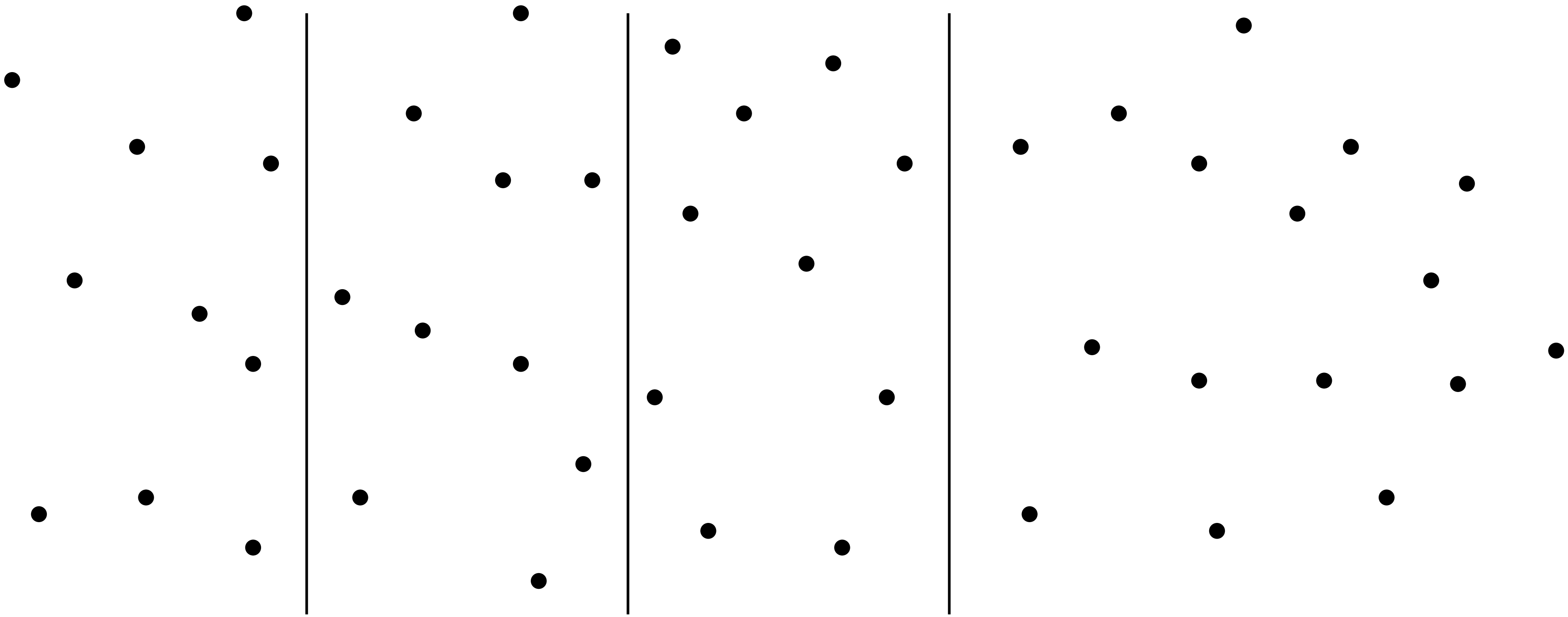

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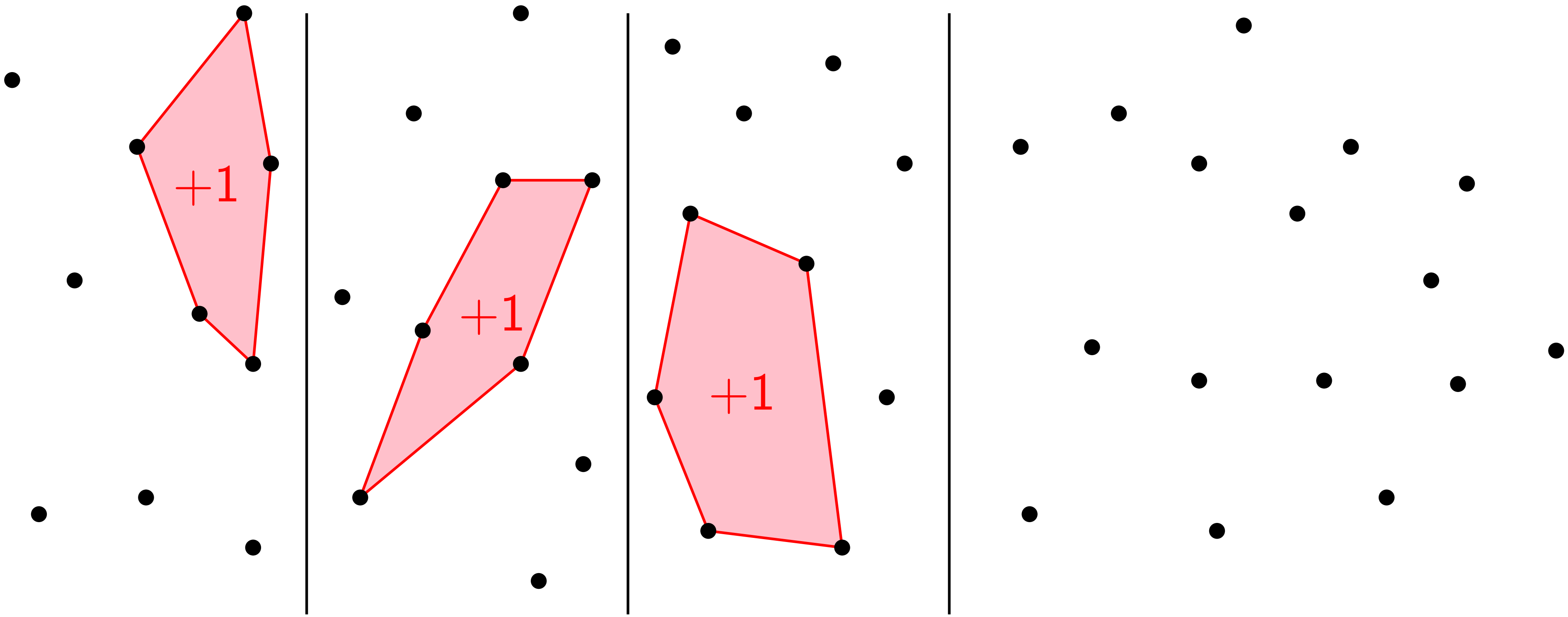

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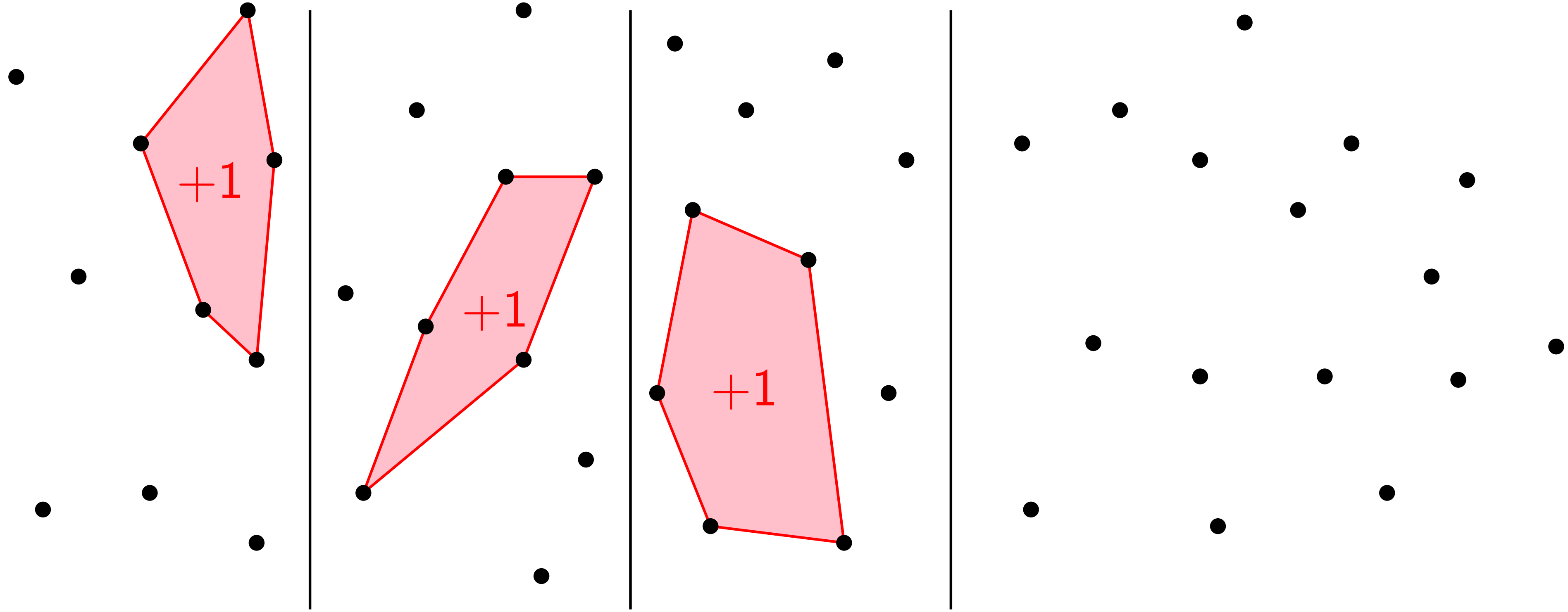
- same idea: $h_6(n) \geq \Omega(n)$



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[Bárány and Füredi '87]

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[Bárány and Füredi '87]

- h_3, h_4 both in $\Theta(n^2)$
- h_5 in $\Omega(n \log^{4/5} n)$ and $O(n^2)$

  [Aichholzer, Balko, Hackl, Kynčl, Parada, S., Valtr, and Vogtenhuber '17]
  (computer assisted proof, 20 pages)

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- $h_6(n)$ in $\Omega(n)$ and $O(n^2)$
  [Gerken '08, Nicolás '07]

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- $h_6(n)$ in $\Omega(n)$ and $O(n^2)$
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Conjecture: $h_5(n)$ and $h_6(n)$ are both in $\Theta(n^2)$

Horton Sets II

Horton set S_n defined recursively: two copies L, U of $S_{\frac{n}{2}}$

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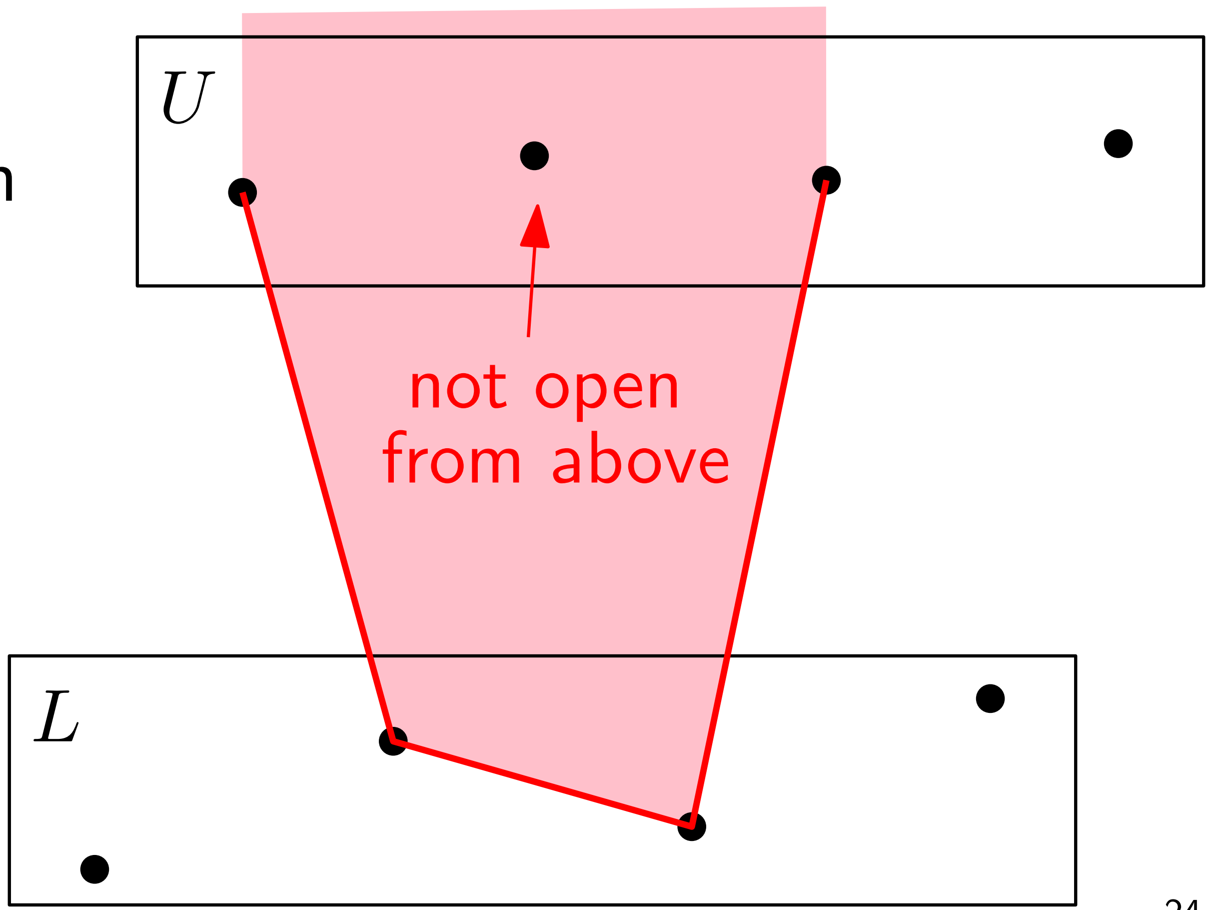

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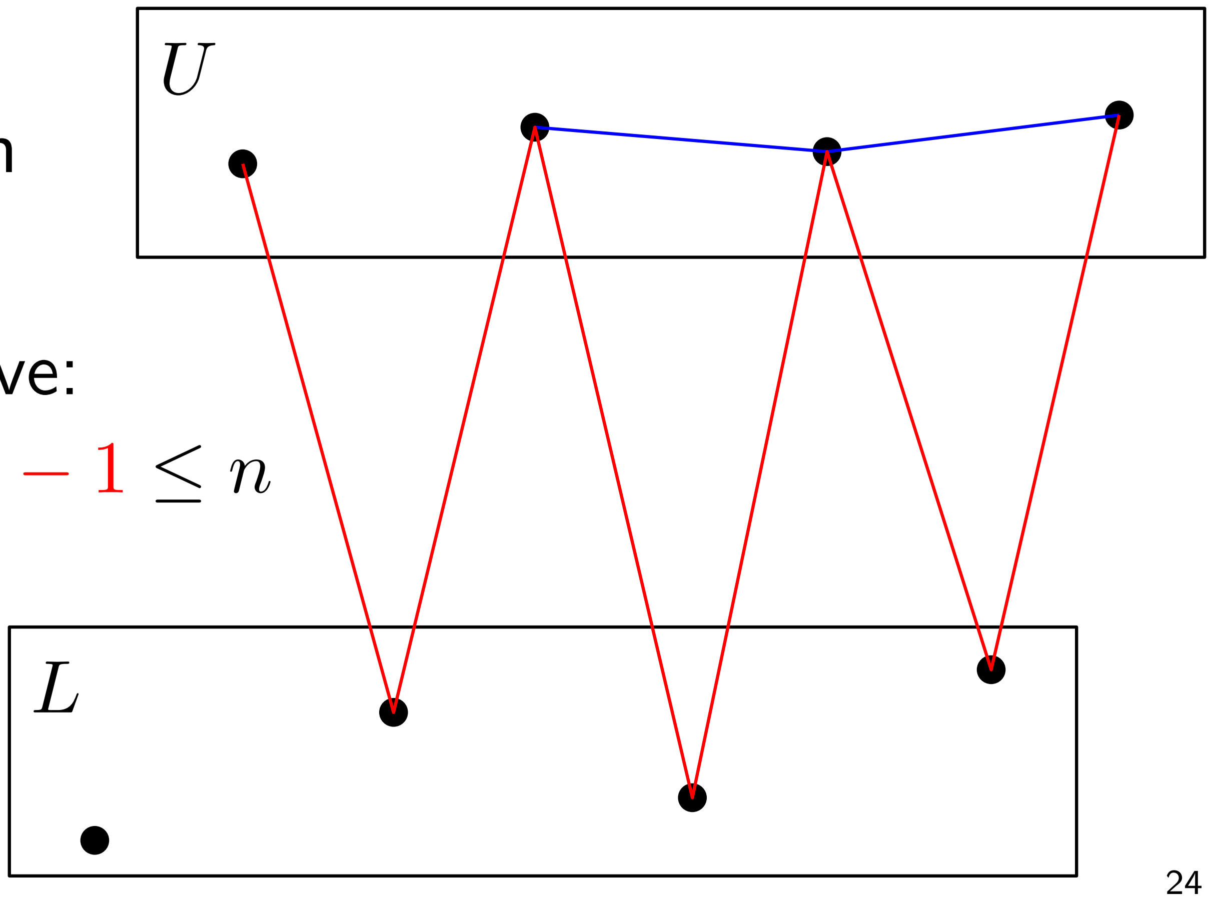
- cups open from above: $U_3(n) = U_3(\frac{n}{2}) + \frac{n}{2} - 1 \leq n$



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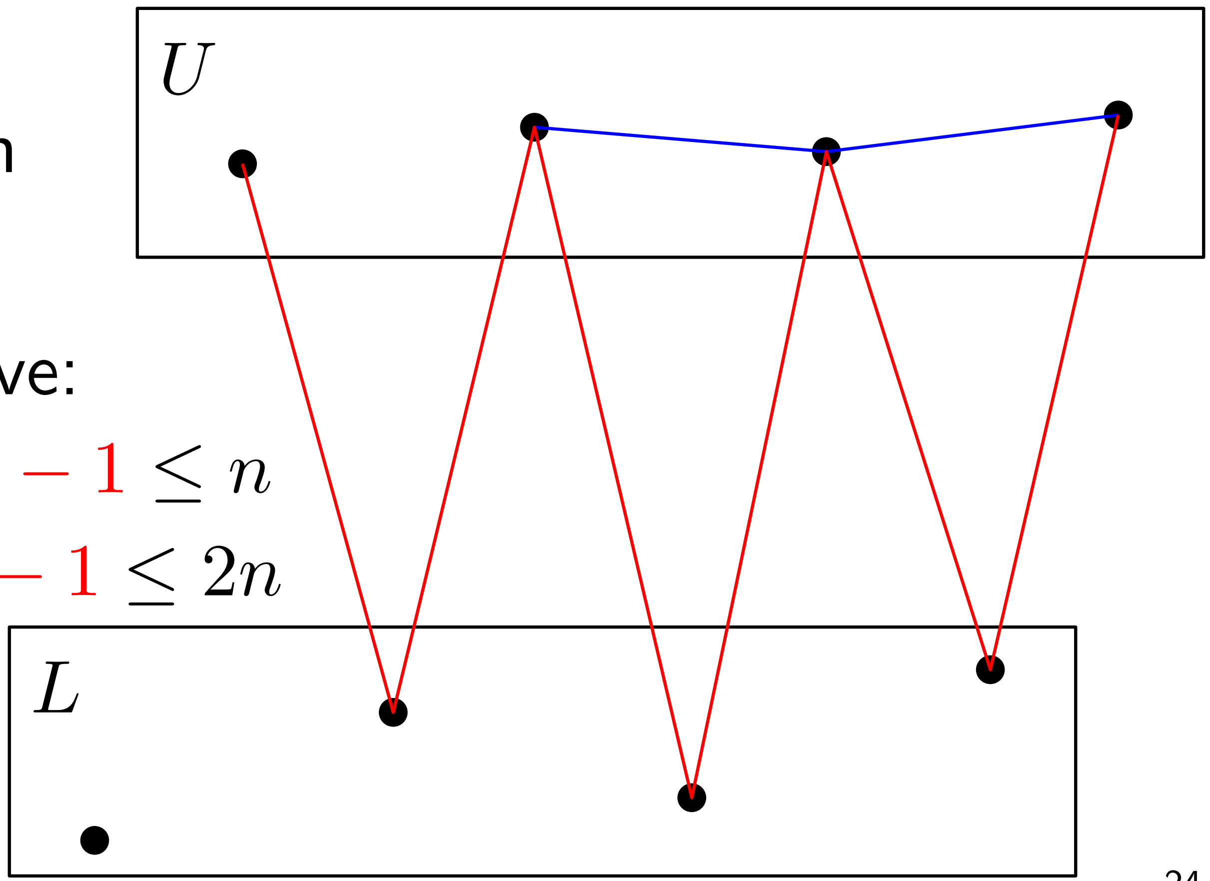
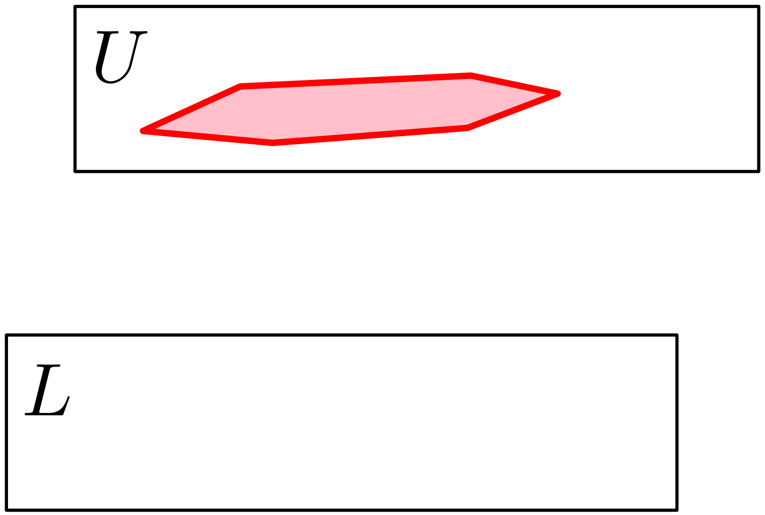
- cups open from above:
  $U_3(n) = U_3(\frac{n}{2}) + \frac{n}{2} - 1 \leq n$
  $U_2(n) = U_2(\frac{n}{2}) + n - 1 \leq 2n$



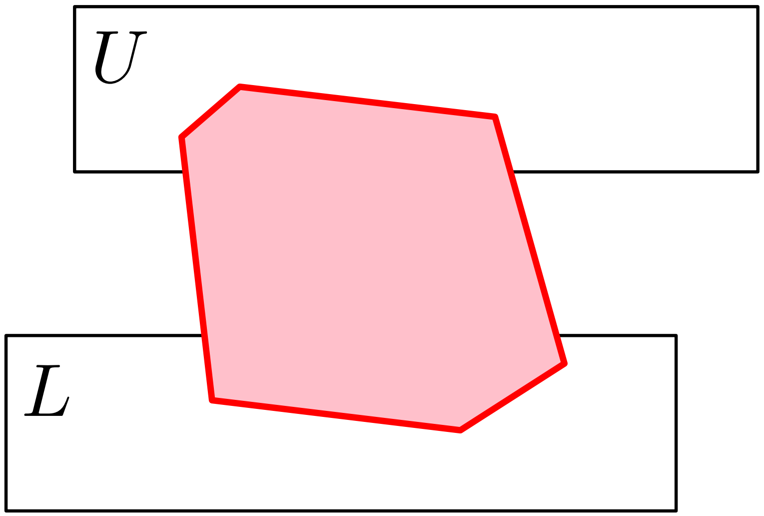
Horton Sets II

- each 6-hole is one of two types:

entirely from U or L



points from both, U and L



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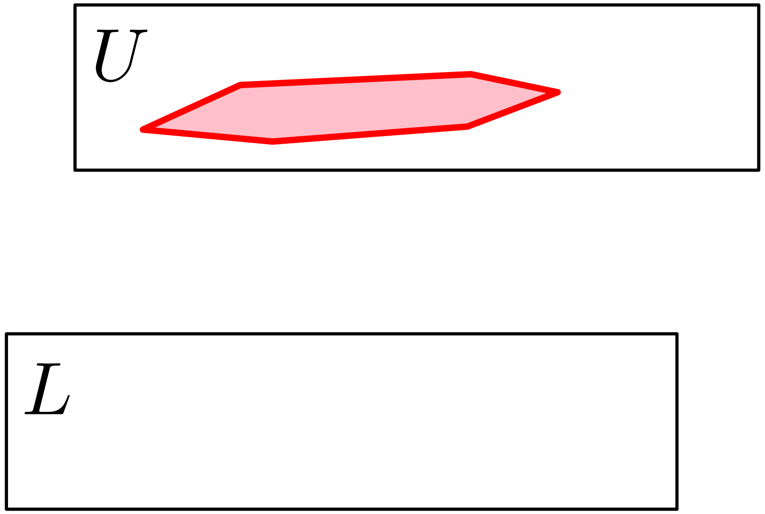

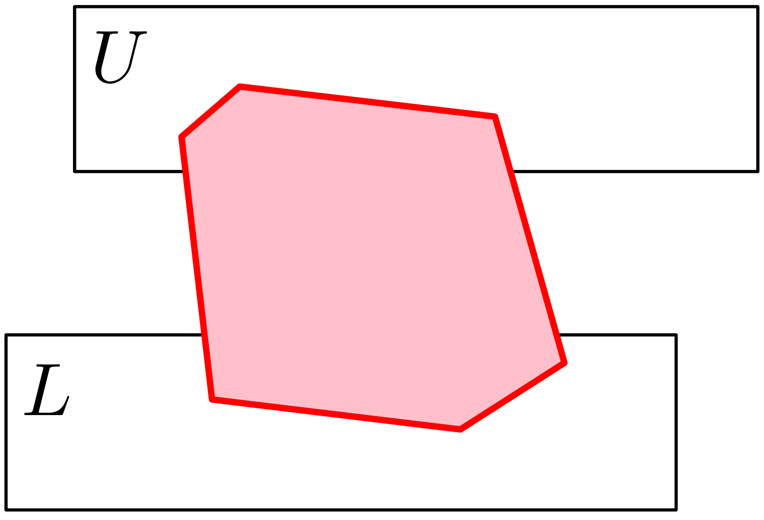

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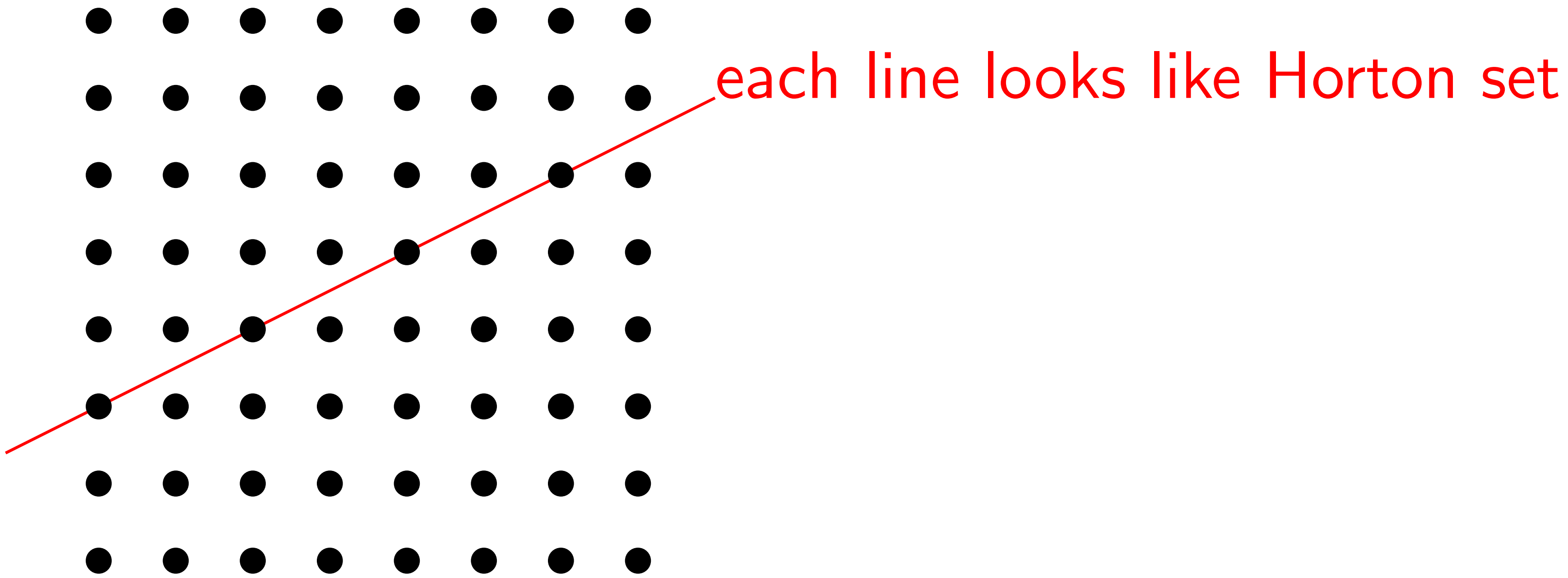
- similar recurrences for 3-, 4-, and 5-holes

Horton Lattice

- Horton lattice: perturbation of $\sqrt{n} \times \sqrt{n}$ grid



gives currently best bounds for h_3, h_4, h_5, h_6 and no 7-holes [Bárány & Valtr '04]

What about Higher Dimensions??

Higher Dimensions

a finite point set P in $\mathbb{R}^d$ is in general position
if no $d + 1$ points lie in a common hyperplane

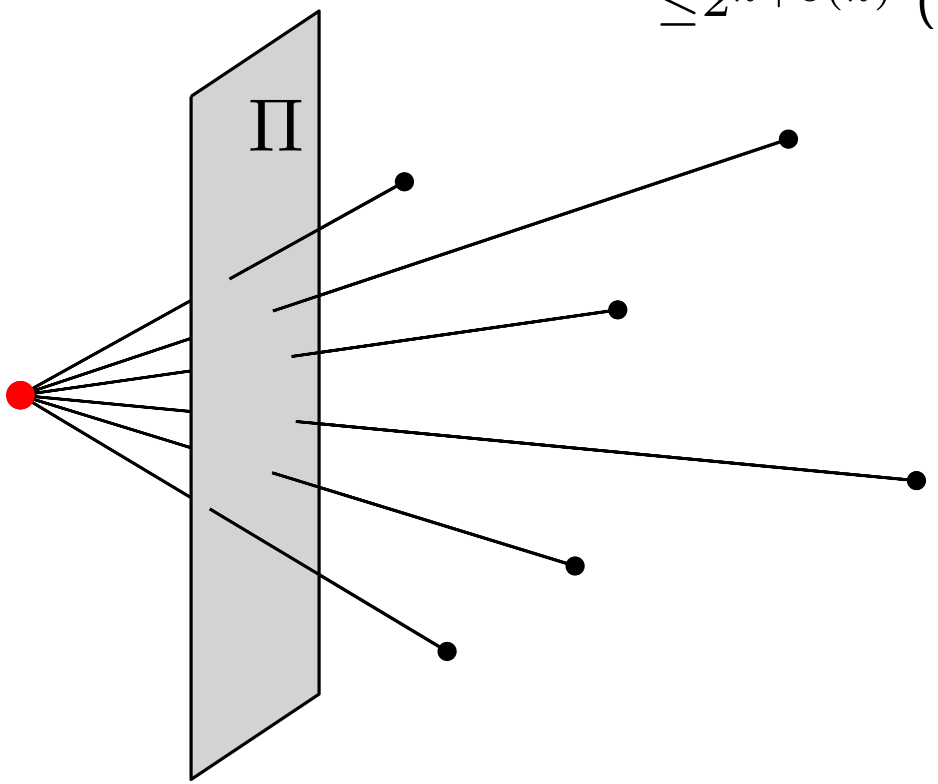
k-gon $= k$ points in convex position

k-hole $= k$-gon with no other points of P in its convex hull

Higher Dimensional k-Gons

dimension reduction (Károlyi '01):

$$g^{(d)}(k) \leq g^{(d-1)}(k-1)+1 \leq \ldots \leq \underbrace{g^{(2)}(k-d+1)+d-2}_{\leq 2^{k+o(k)} \text{ (Suk'17)}}$$



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asymptotic behavior remains unknown for $d \geq 3$

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  [Gerken '07, Nicolás '07] [Horton '87]

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    - $\exists$ d-dimensional Horton sets without $d^{d+o(d)}$-holes

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  - in particular, $7 \leq H(3) \leq 22$ remains open

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- do exponentially large holes exist?

Number of Holes in Higher Dimensions

- random sets give $O(n^d)$ bounds for k-holes

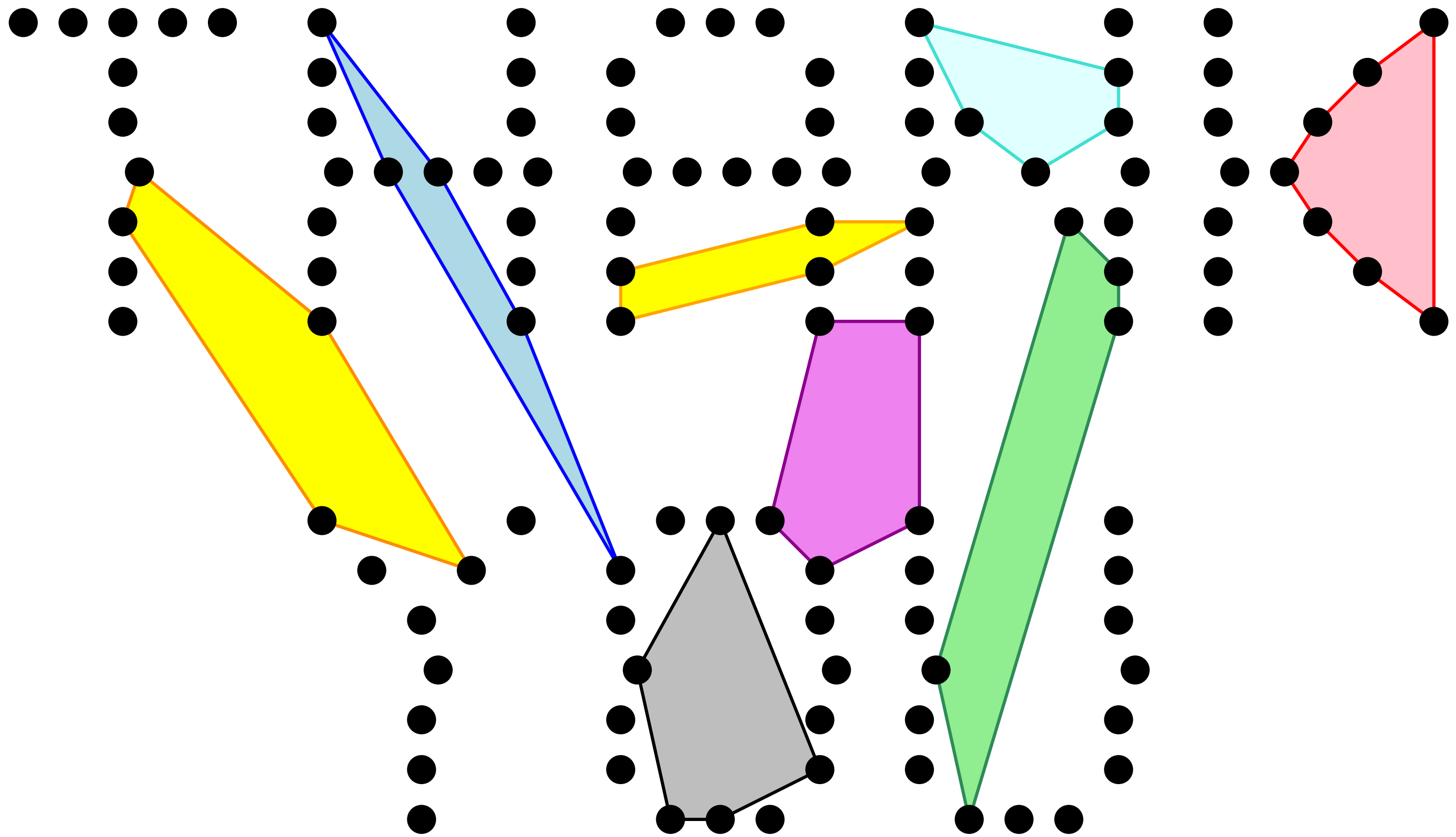
Theorem (Balko S. Valtr '20 + '21). Let $d \geq 2$ and $k \geq d+1$, and let K be a convex body in $\mathbb{R}^d$.
If S is a set of n points chosen uniformly and independently at random from K,
then the expected number of k-holes in S is $\Theta(n^d)$.

In particular:
$\exists$ sets of n points in $\mathbb{R}^d$ with $O(n^d)$ many k-holes

Number of Holes in Higher Dimensions

- random sets give $O(n^d)$ bounds for k-holes
- d-dimensional Horton sets with $d > 2$ contain $\Omega(n^{\min\{k,2^{d-1}\}})$ many k-holes [Balko S. Valtr '20]
- no explicit construction known with $O(n^d)$ k-holes

THANK
YOU


THANK
YOU