Points, Lines, and Circles

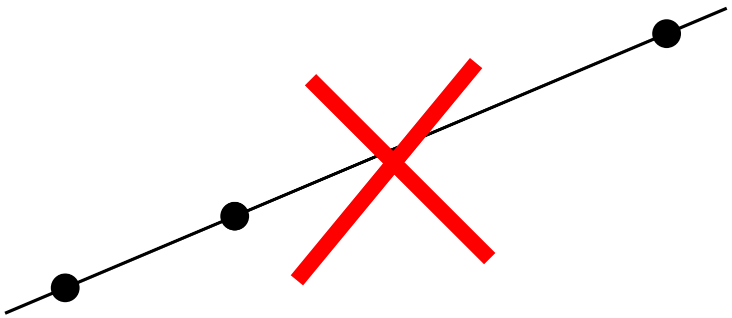
Some Contributions to Combinatorial Geometry

Manfred Scheucher

Part I:
Erdős–Szekeres-Type Problems

k-Gons

a finite point set P in the plane is
in *general position* if $\nexists$ collinear points in P



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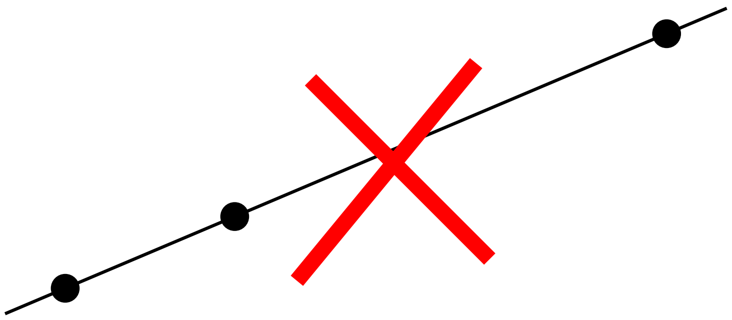
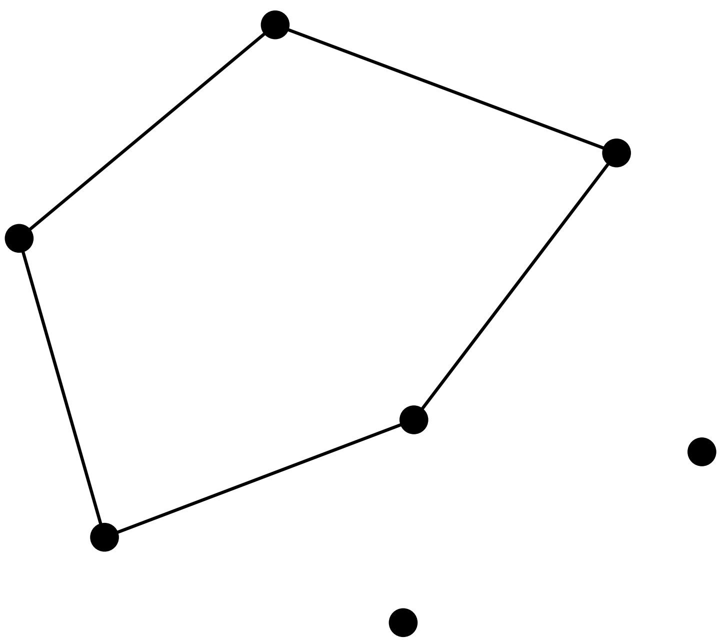


throughout this presentation, every set is in general position

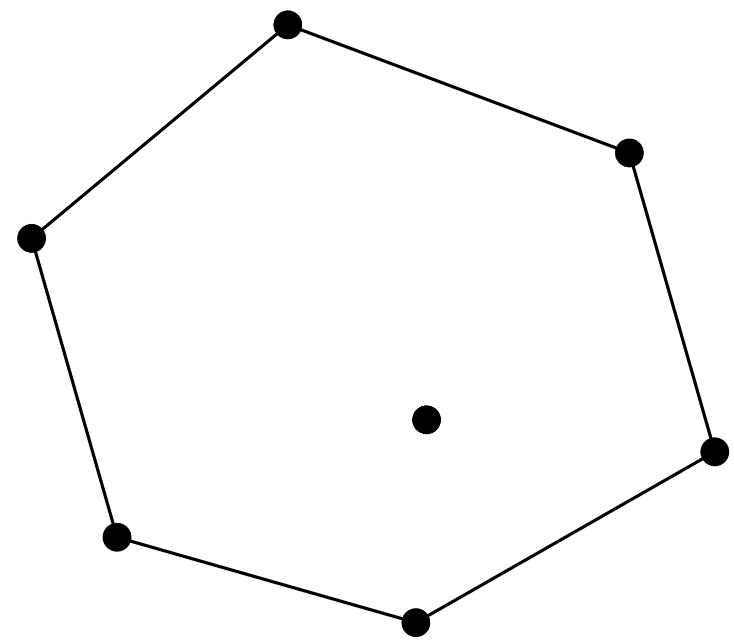
k-Gons

a finite point set P in the plane is in *general position* if $\nexists$ collinear points in P

a *k-gon (in P)* is the vertex set of a convex k-gon



5-gon



6-gon

k-Gons

a finite point set P in the plane is
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a *k-gon (in P)* is the vertex set of a convex k-gon

Theorem (Erdős and Szekeres '35).
$\forall\ k \geq 3$, $\exists$ a smallest integer $g(k)$ such that
every set of $g(k)$ points contains a k-gon.

k-Gons

Theorem. $2^{k-2} + 1 \leq g(k) \leq \binom{2k-4}{k-2}$. [Erdős–Szekeres '35]

↑

equality conjectured by Szekeres, Erdős offered 500$ for a proof

k-Gons

Theorem. $2^{k-2}+1 \leq g(k) \leq \binom{2k-4}{k-2}$. [Erdős–Szekeres '35]

$\vdots$ several improvements of order $4^{k-o(k)}$

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Known: $g(4) = 5$, $g(5) = 9$, $g(6) = 17$

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computer assisted proof, 1500 CPU hours [Szekeres–Peters '06]

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NEW: 1 hour using SAT solvers

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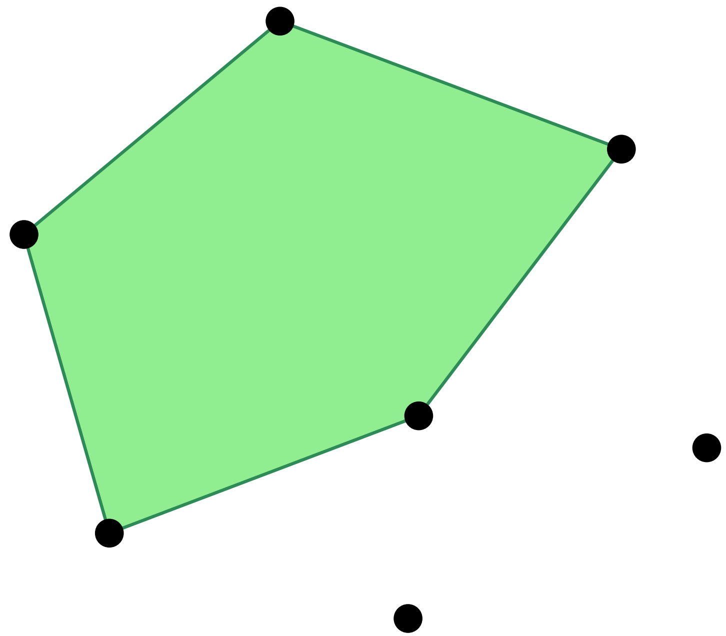
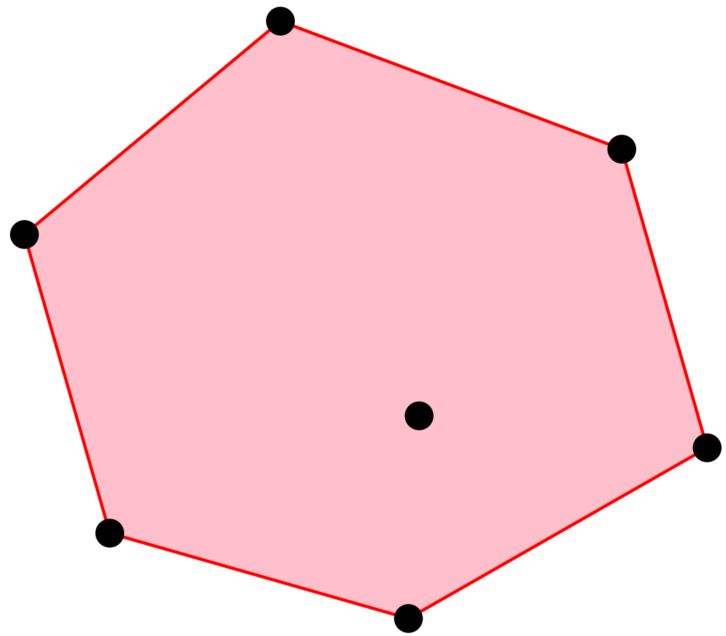
k-Holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P



5-hole



not a 6-hole

k-Holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P

Erdős, 1970's: For k fixed, does every suff. large point set contain k-holes?

k-Holes

$h(k)$ minimal s.t. any set of $h(k)$ points contains k-hole

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq 463$, $h(7) = \infty$

Harborth '78

Horton'83

Overmars '02

Gerken '08, Nicolas '07, Koshelev '09

k-Holes

$h(k)$ minimal s.t. any set of $h(k)$ points contains k-hole

exact value still unknown

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq 463$, $h(7) = \infty$

Harborth '78

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Quantity of k-Holes

$h_k(n)$ = minimum # of k-holes among all sets of n points

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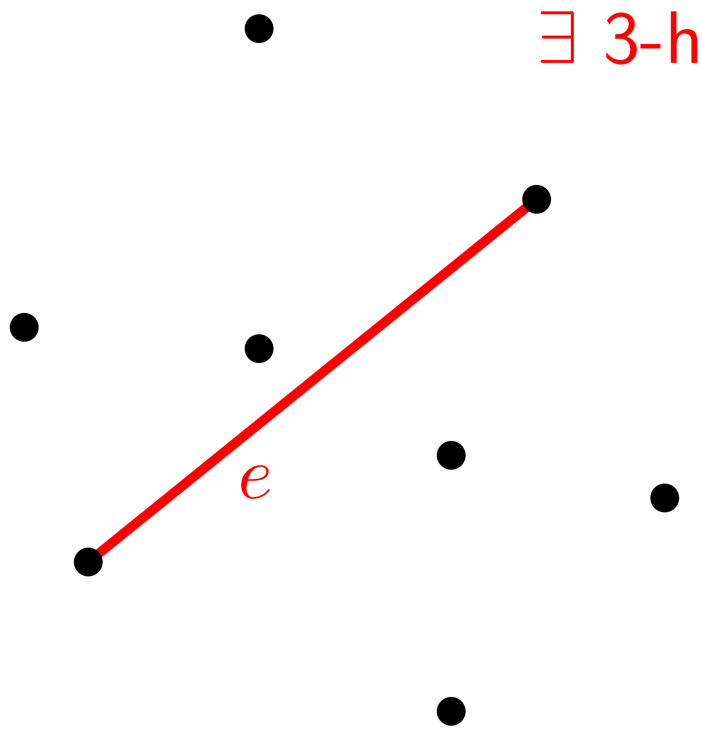
- $h_3(n) \geq \lfloor \frac{1}{3}\binom{n}{2} \rfloor = \Omega(n^2)$

$\forall$ edge e
$\exists$ 3-hole with closest point



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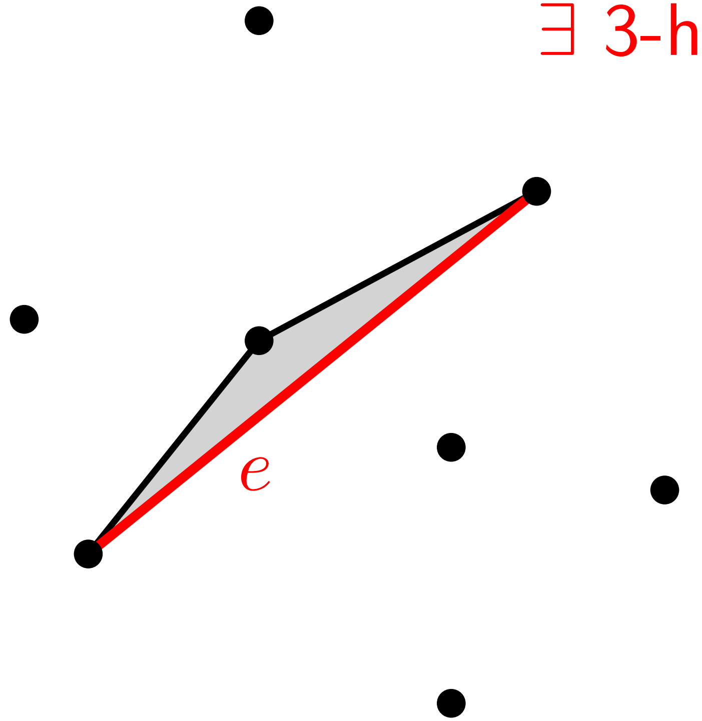

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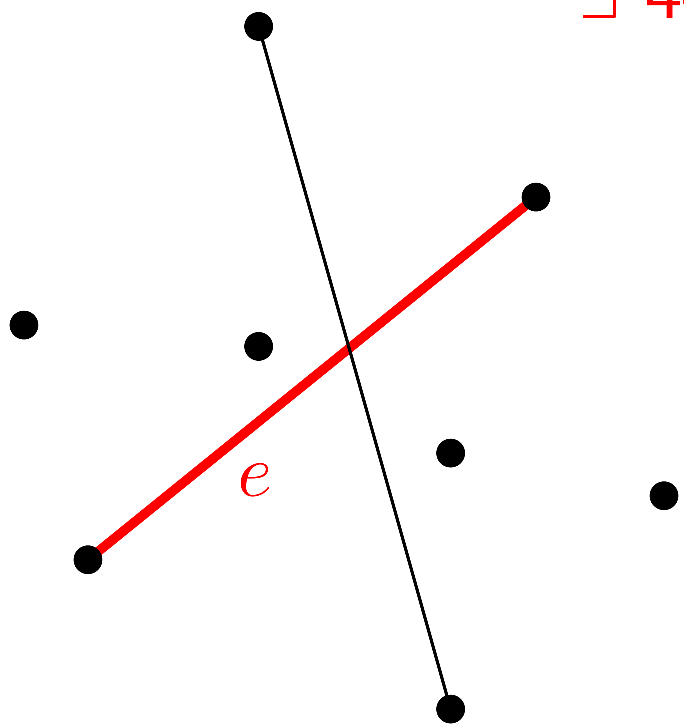

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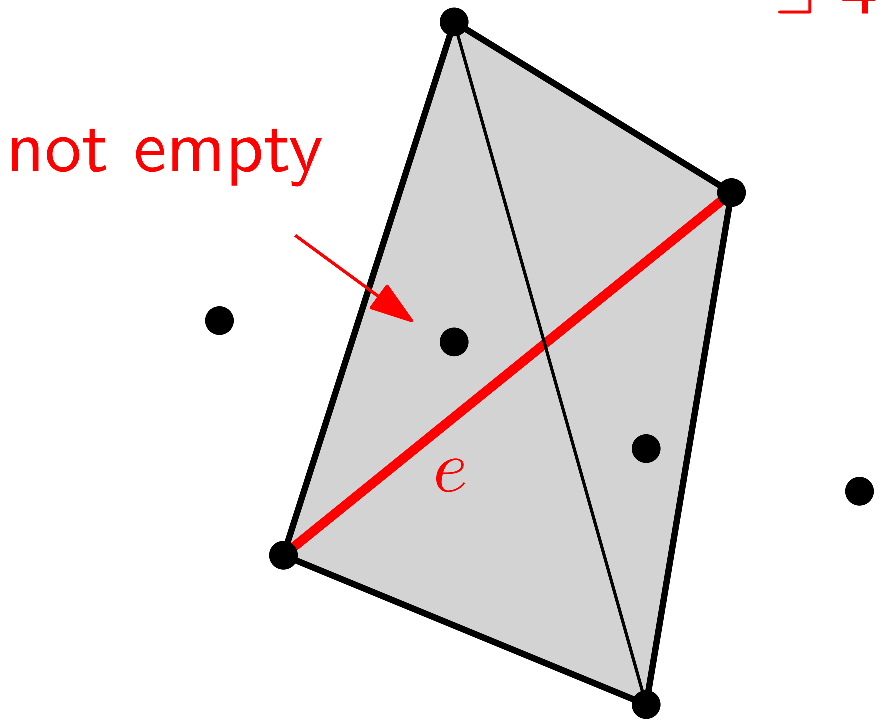

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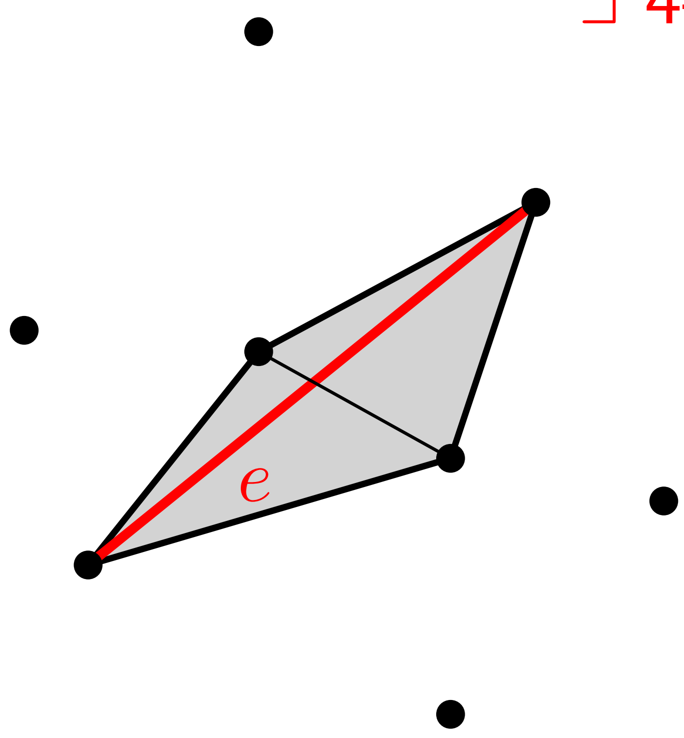

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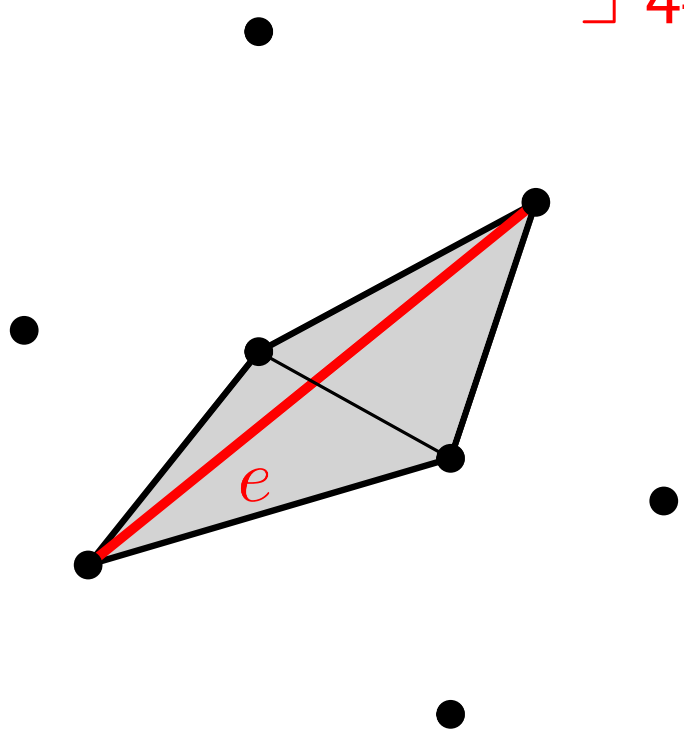

$O(n)$ uncrossed edges
(planar graph)

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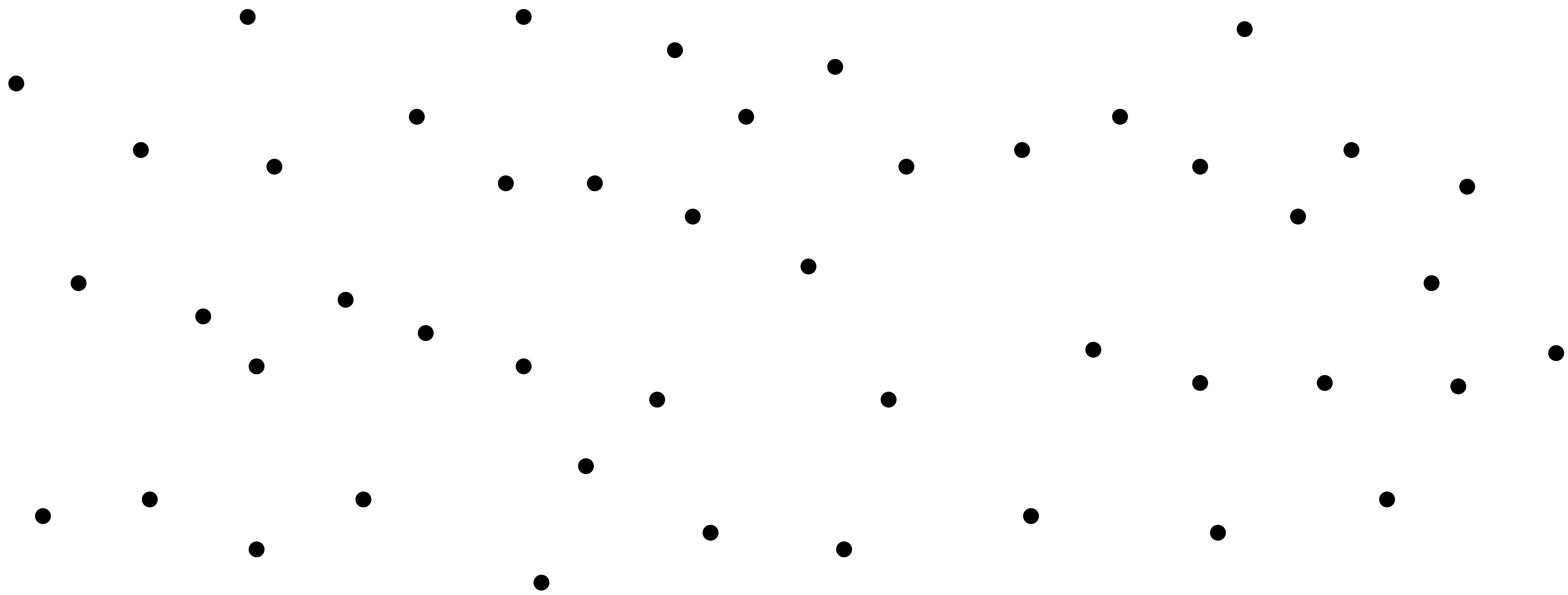
- $h_5(n) \geq \lfloor \frac{1}{10} n \rfloor = \Omega(n)$



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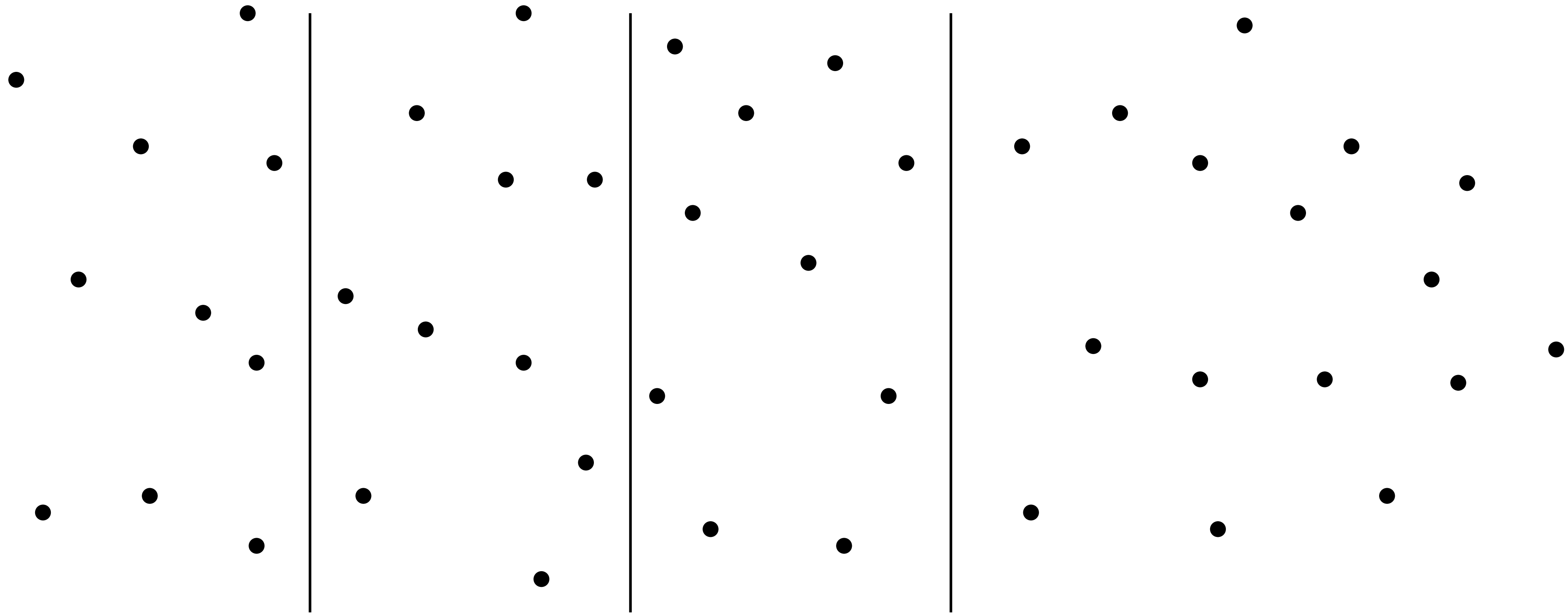

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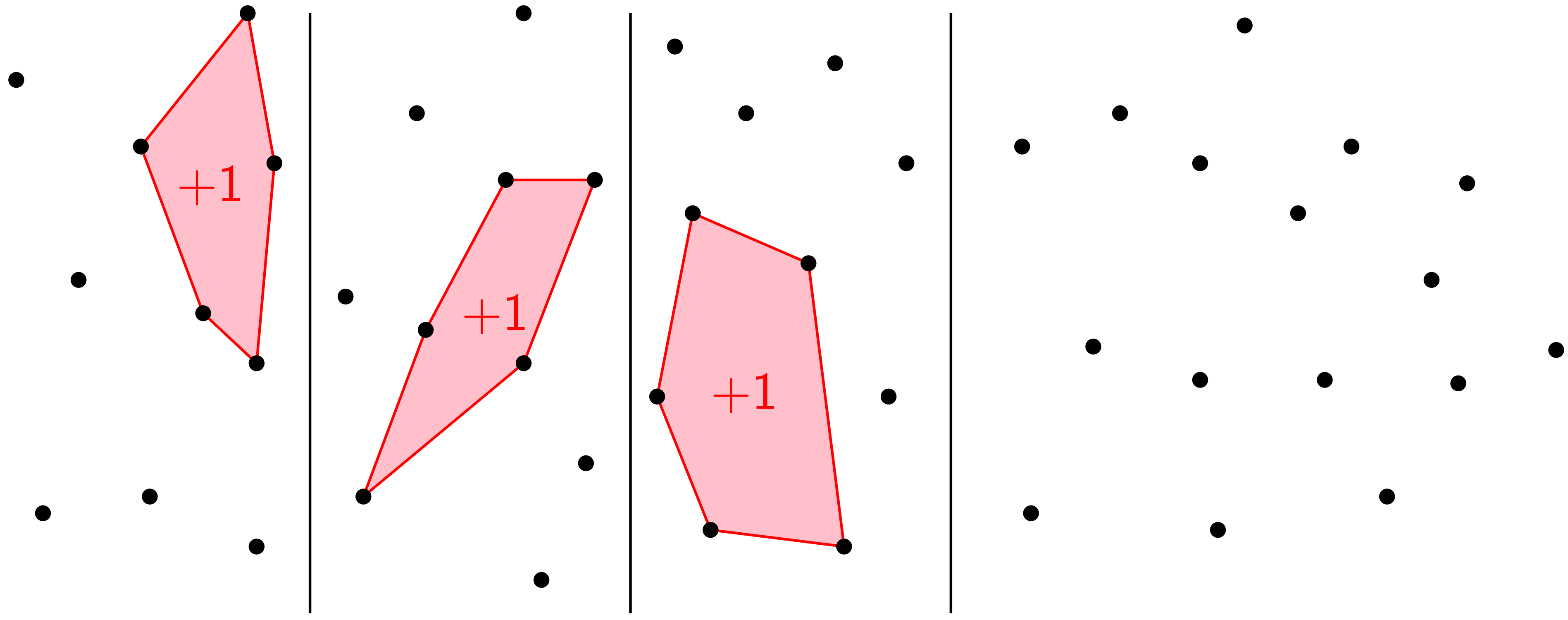

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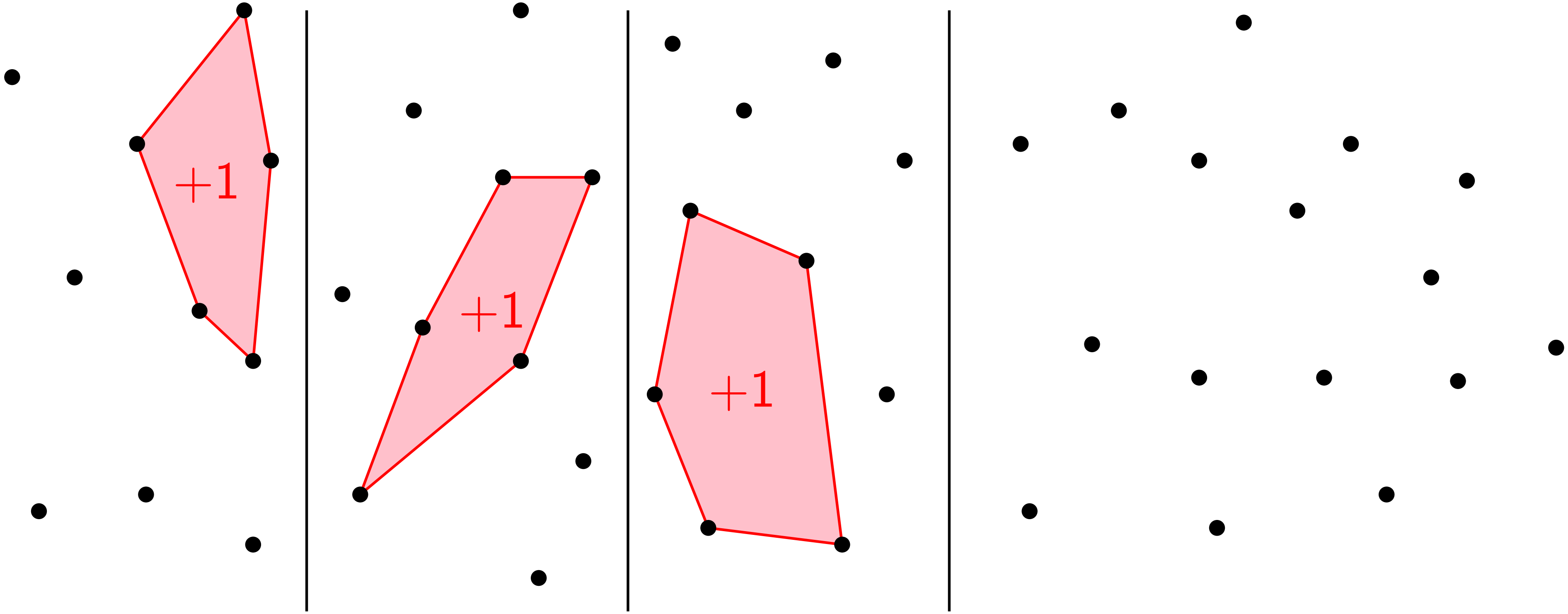
- same idea: $h_6(n) \geq \Omega(n)$



Quantity of k-Holes

$h_k(n)$ = minimum # of k-holes among all sets of n points

[Bárány and Füredi '87, Bárány and Valtr '04]

- $h_3(n)$ and $h_4(n)$ both $\Theta(n^2)$
- $h_k(n) = 0$ for $k \geq 7$ [Horton '83]
- $h_5(n)$ and $h_6(n)$ both $\Omega(n)$ and $O(n^2)$

[Harborth '78] [Gerken '08, Nicolás '07]

Quantity of k-Holes

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Conjecture 1: $h_5(n)$ is quadratic in n.

Conjecture 2: $h_5(n)$ is superlinear in n.

P. Brass, W. Moser, and J. Pach.
Research Problems in Discrete Geometry. 2005.

A Superlinear Lower Bound on the Number of 5-Holes

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Martin Balko
Thomas Hackl
Jan Kynčl

Irene Parada
Manfred Scheucher
Pavel Valtr
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A Superlinear Lower Bound

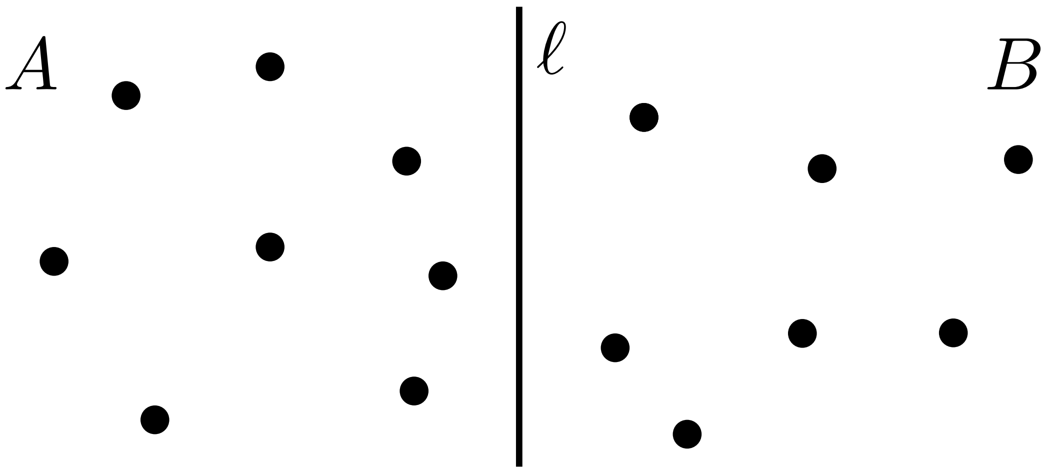
Theorem 1: $h_5(n) \geq \Omega(n \log^{4/5} n)$.

This solves Conjecture 2.

Conjecture 1 is still open.

A Superlinear Lower Bound

$P = A \cup B$ is *ℓ-divided* if line ℓ partitions P into A and B



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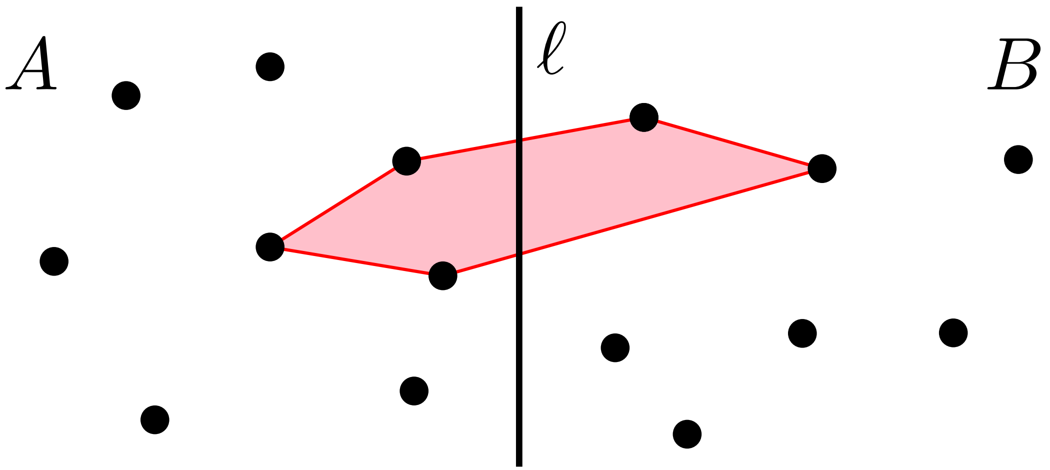
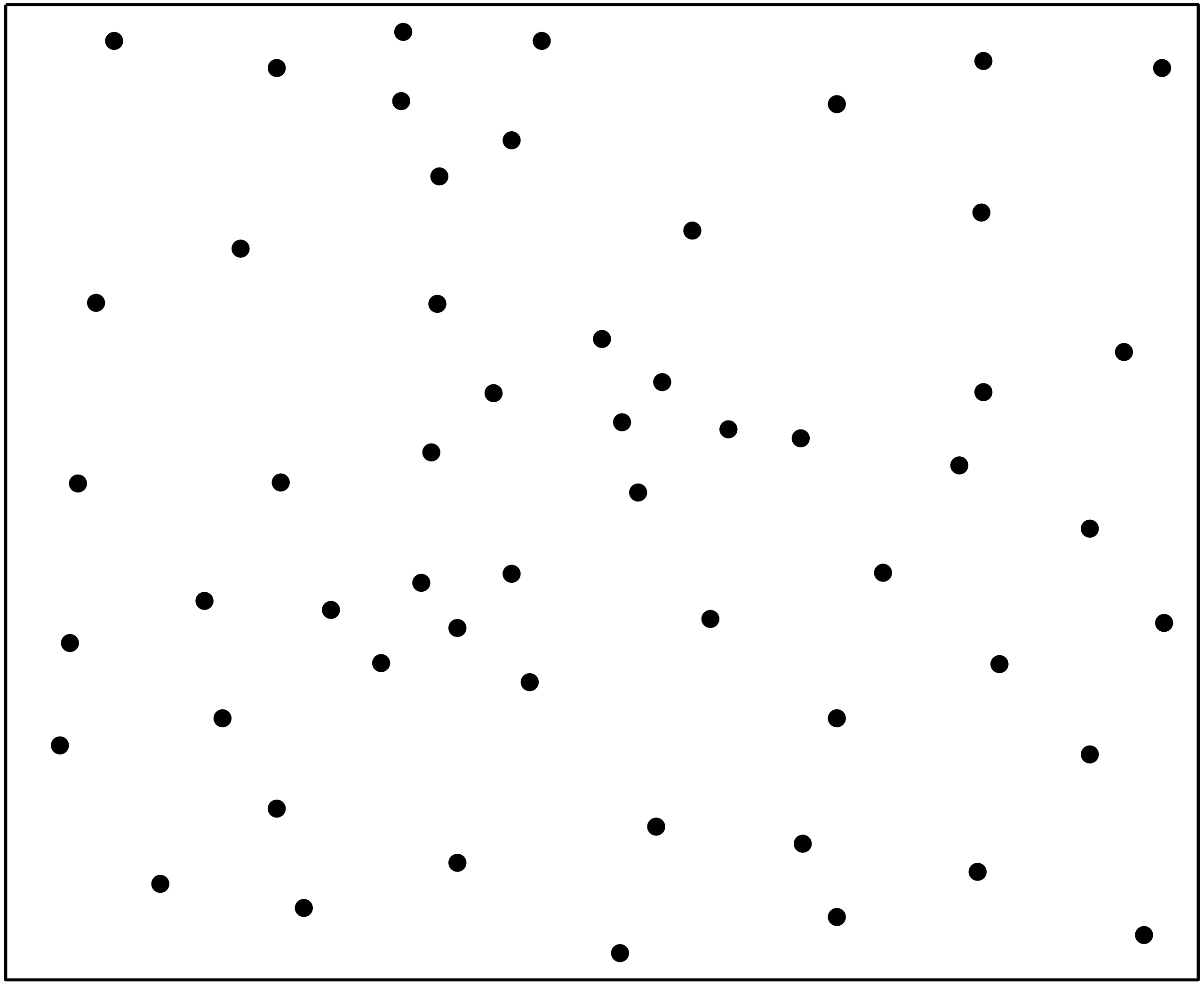


Theorem 3: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$,
A and B *not* in convex position $\Rightarrow \exists$ ℓ-divided 5-hole

Using Theorem 3 to prove Theorem 1



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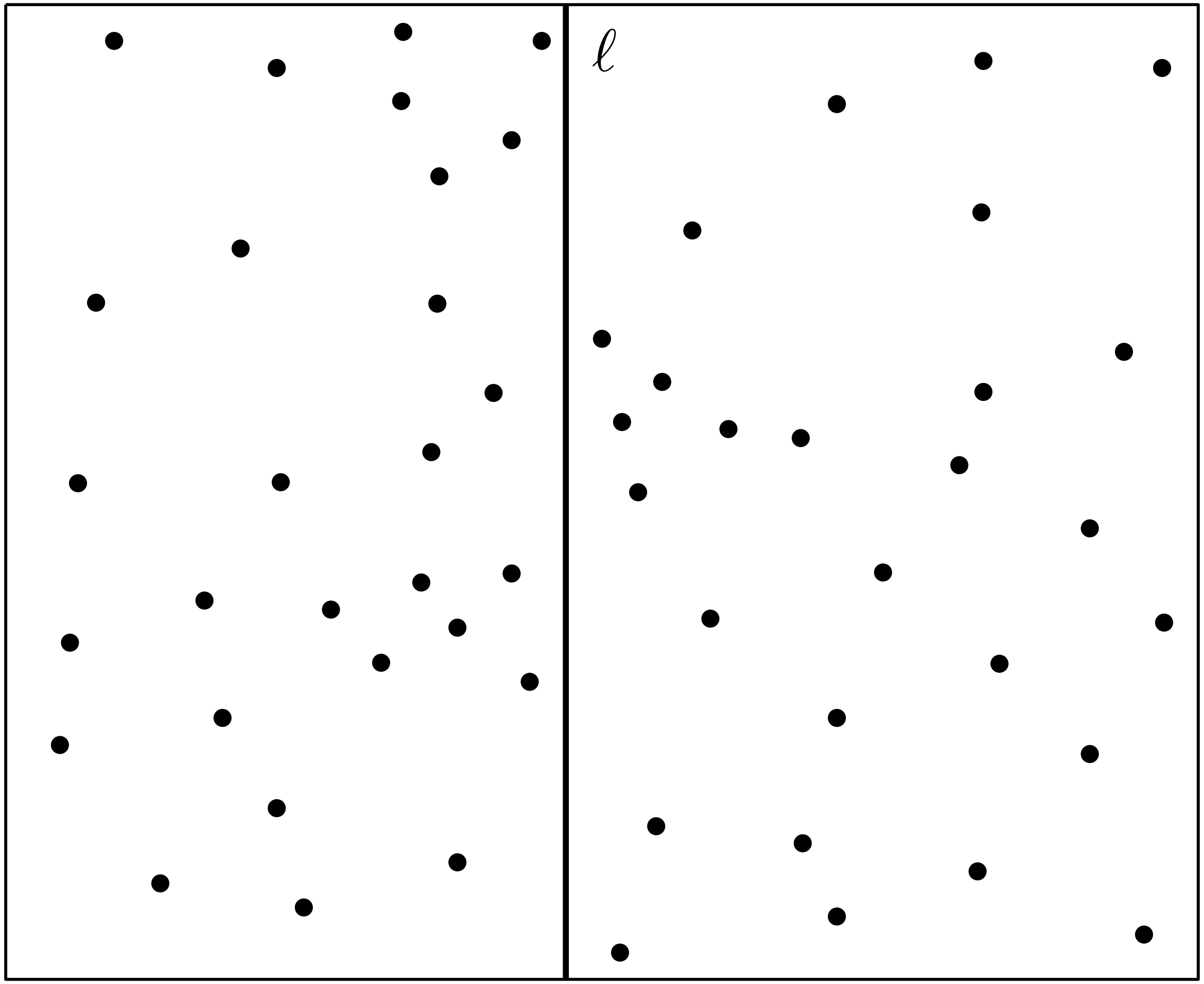

We partition

$$P = A \cup B$$

such that

$$|A| = \frac{n}{2} = |B|$$

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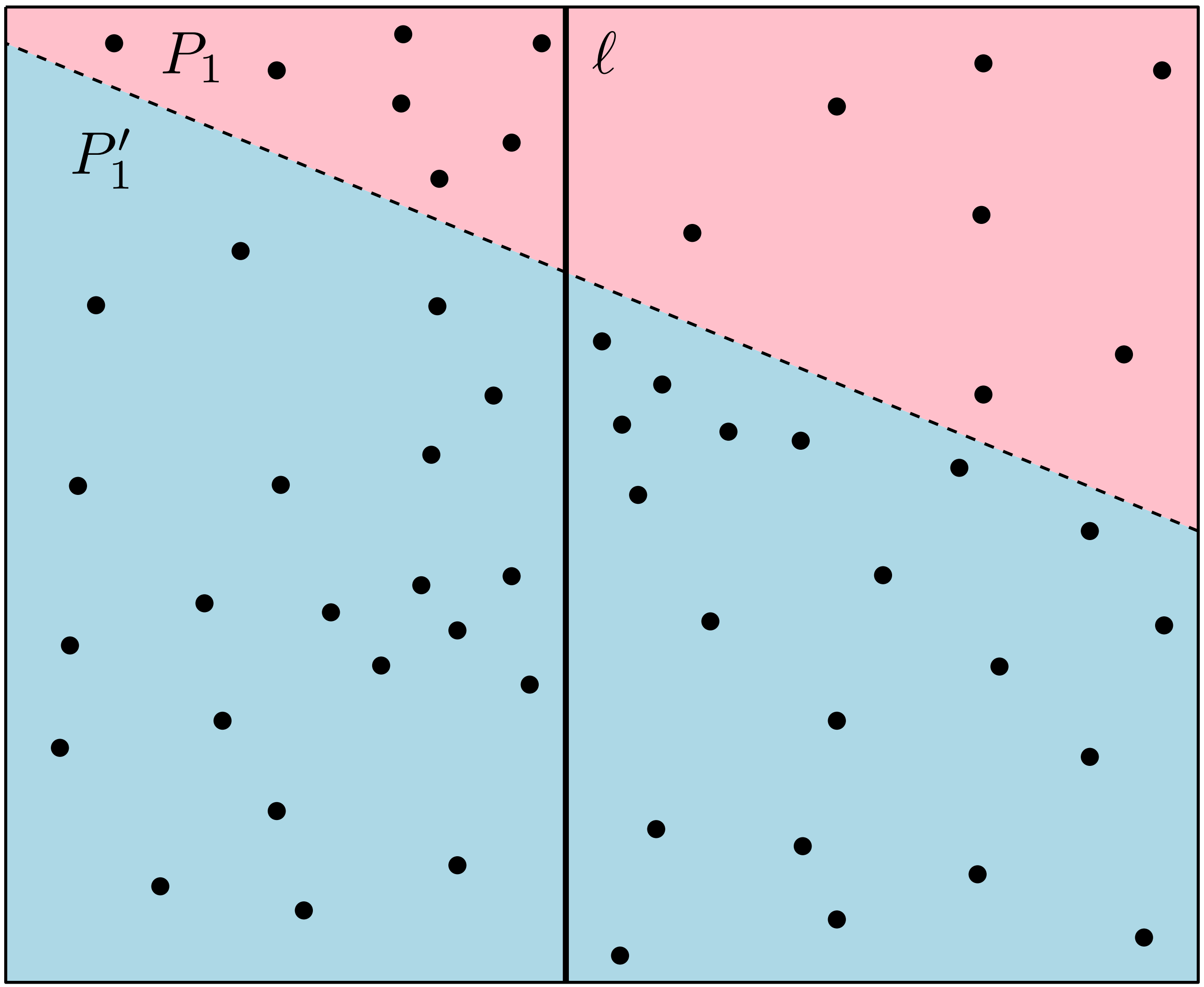


Generalized Ham-Sandwich Cut:

$$P = P_1 \cup P_1'$$

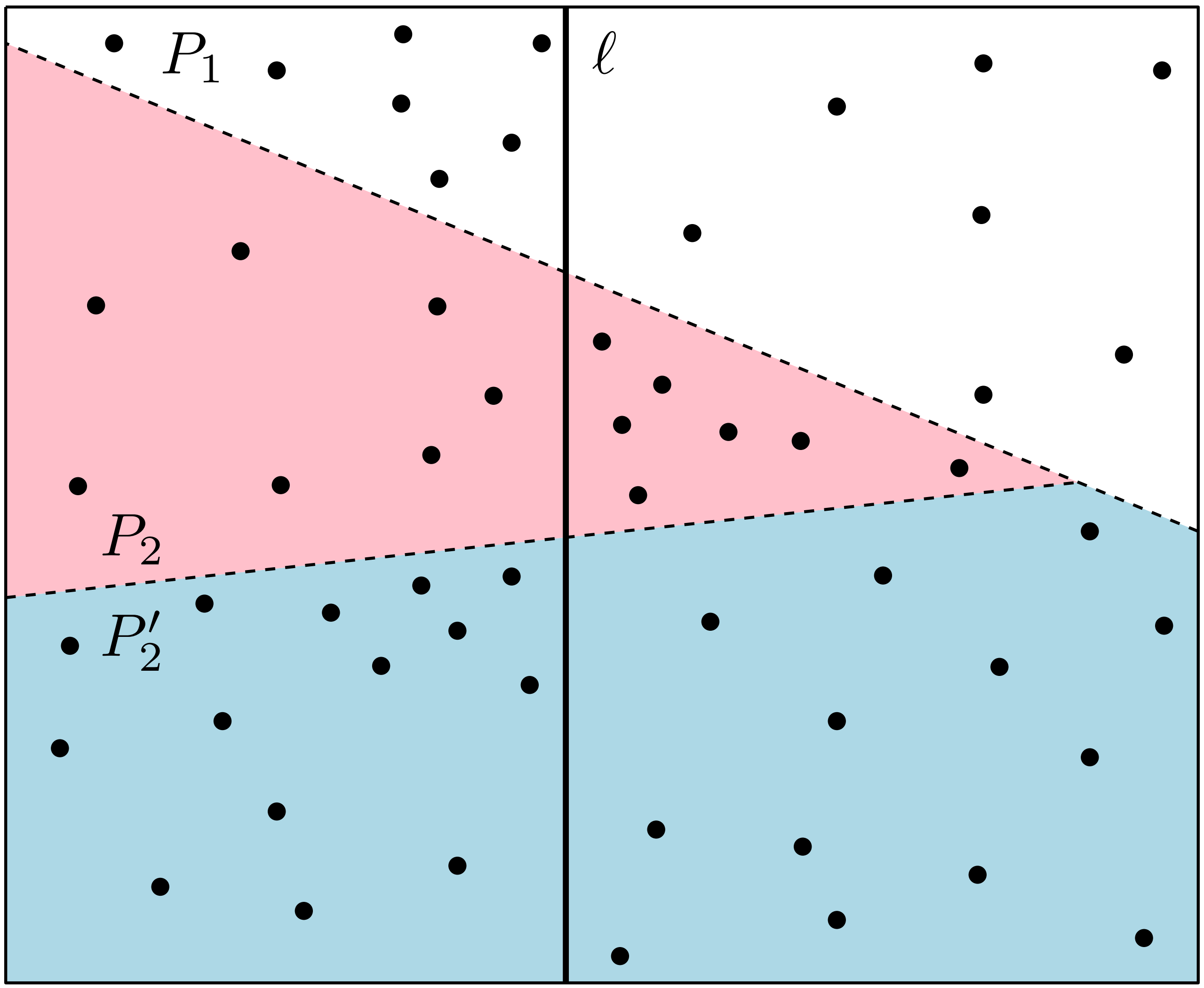
such that

$$|A \cap P_1| = r$$

$$|B \cap P_1| = r$$

(r=blocksize)

Using Theorem 3 to prove Theorem 1



Generalized Ham-Sandwich Cut:

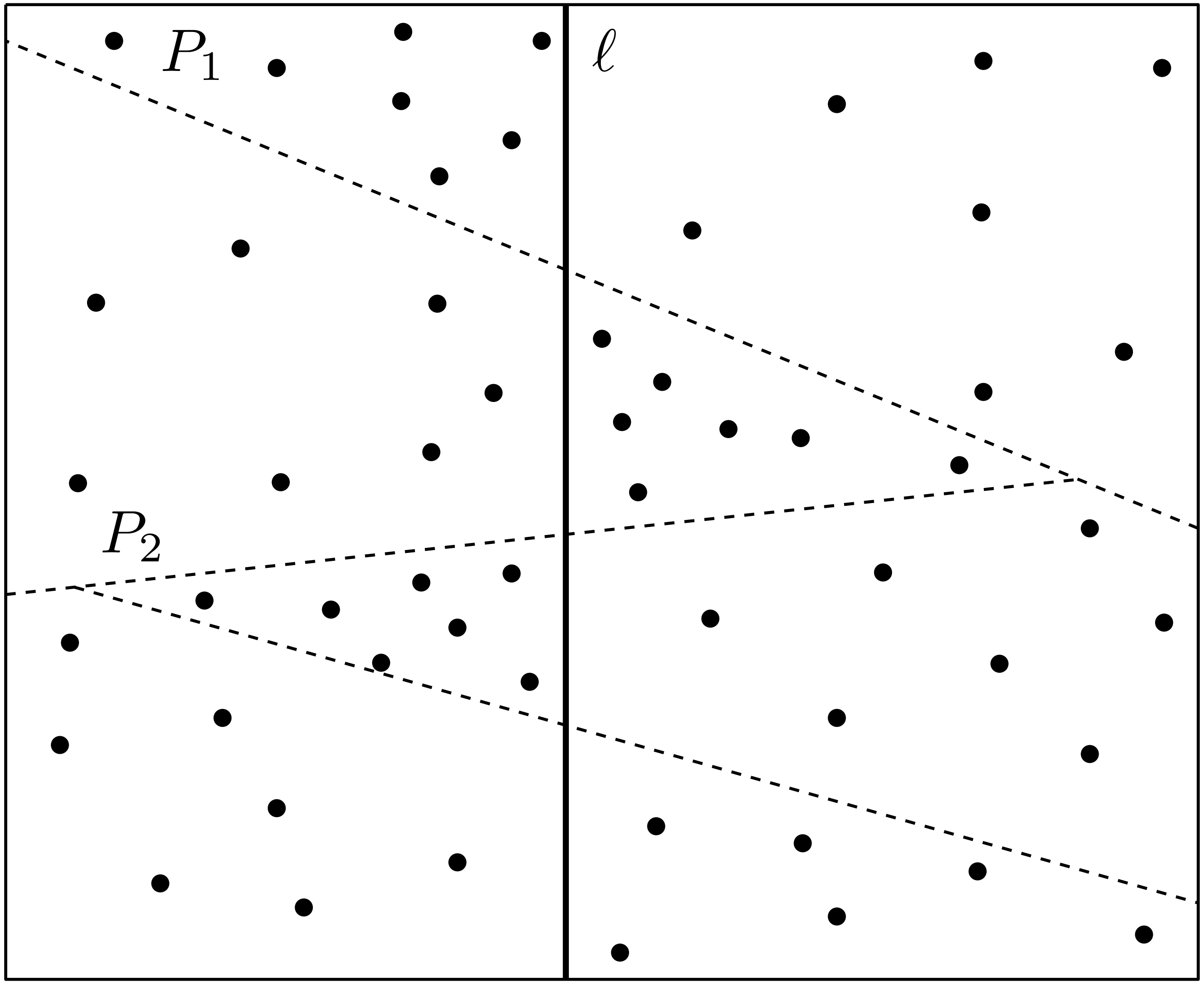
$$P_1 = P_2 \cup P_2'$$

such that

$$|A \cap P_2| = r$$

$$|B \cap P_2| = r$$

Using Theorem 3 to prove Theorem 1



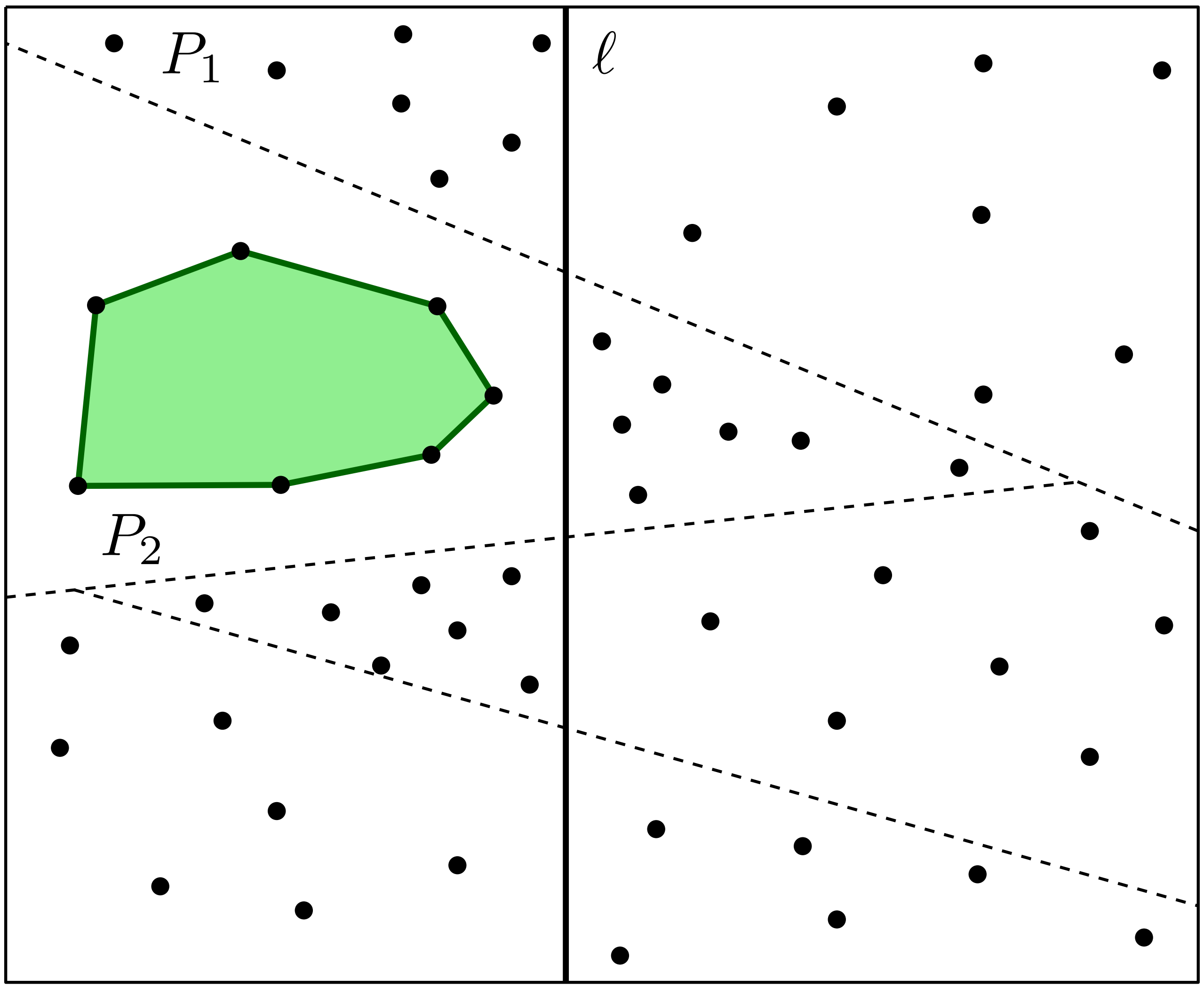
and so on...

$\Rightarrow s = \frac{n}{2r}$ blocks

$P_1, \ldots, P_s$

each of size $r + r$

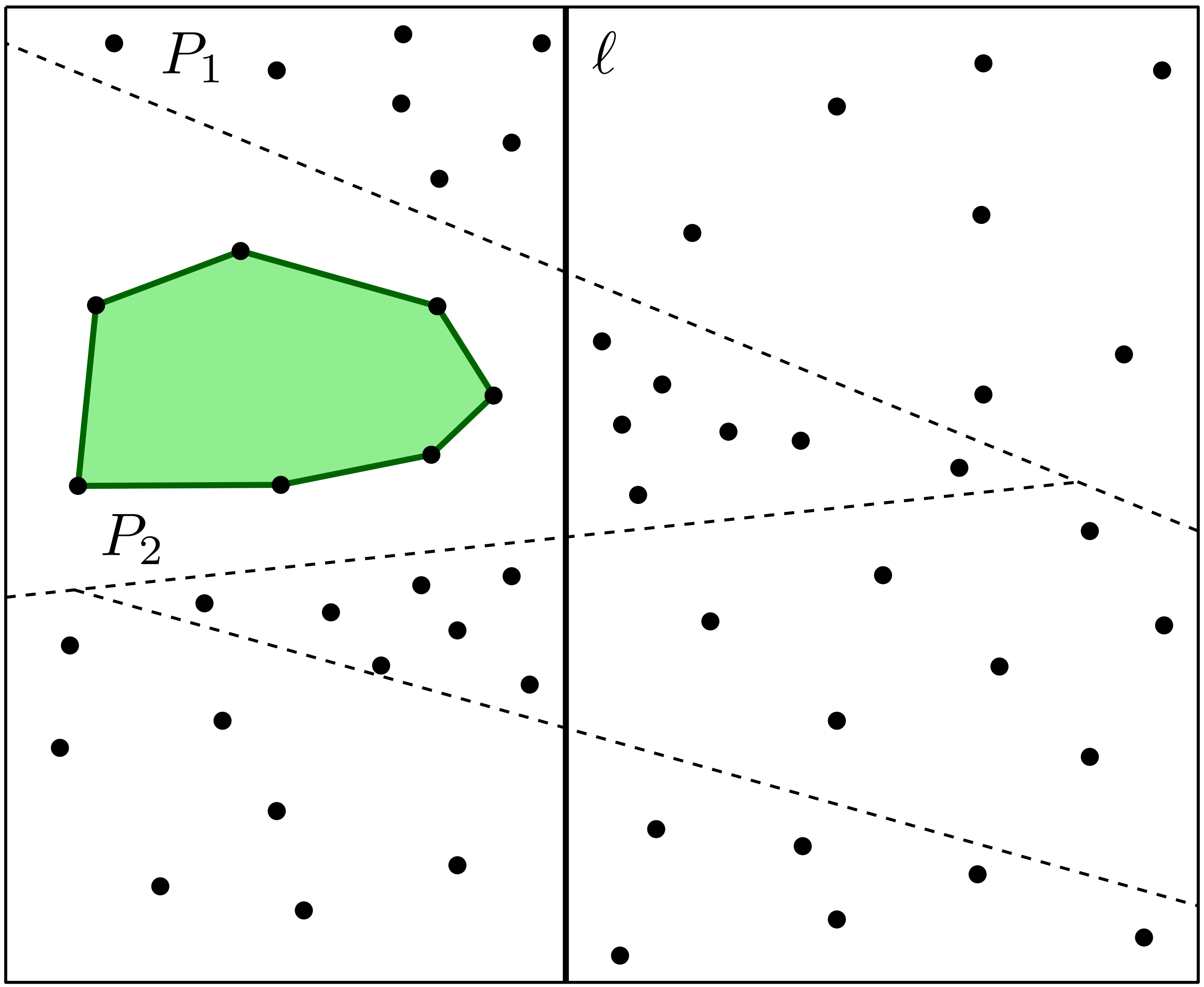
Using Theorem 3 to prove Theorem 1



If $P_i \cap A$ or $P_i \cap B$ in convex pos.

$\Rightarrow \binom{r}{5}$ 5-holes

Using Theorem 3 to prove Theorem 1

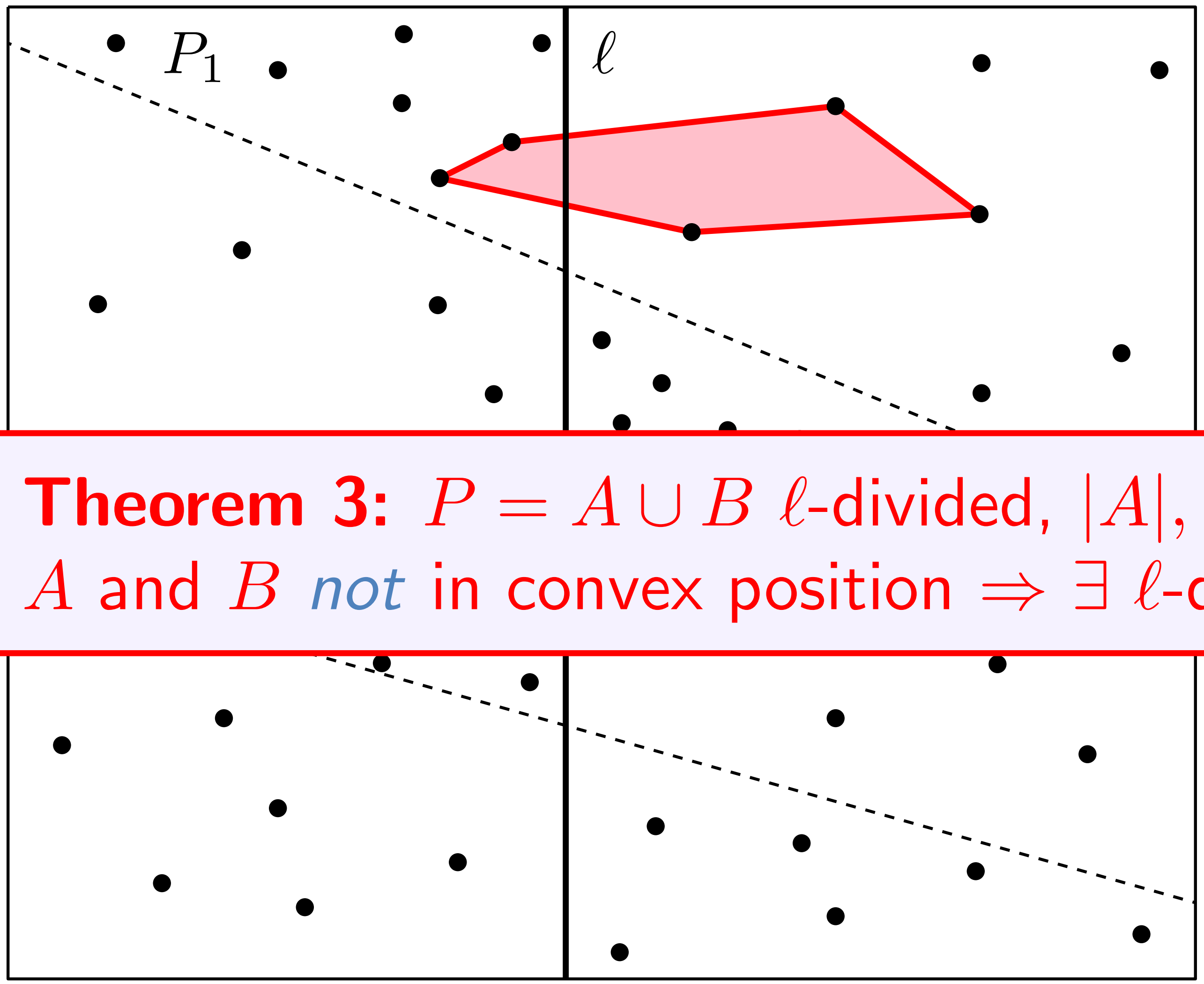


If $P_i \cap A$ or $P_i \cap B$ in convex pos.

$\Rightarrow \binom{r}{5}$ 5-holes

If this is the case for half of all blocks, we count at least $\frac{s}{2}\binom{r}{5}$ 5-holes

Using Theorem 3 to prove Theorem 1

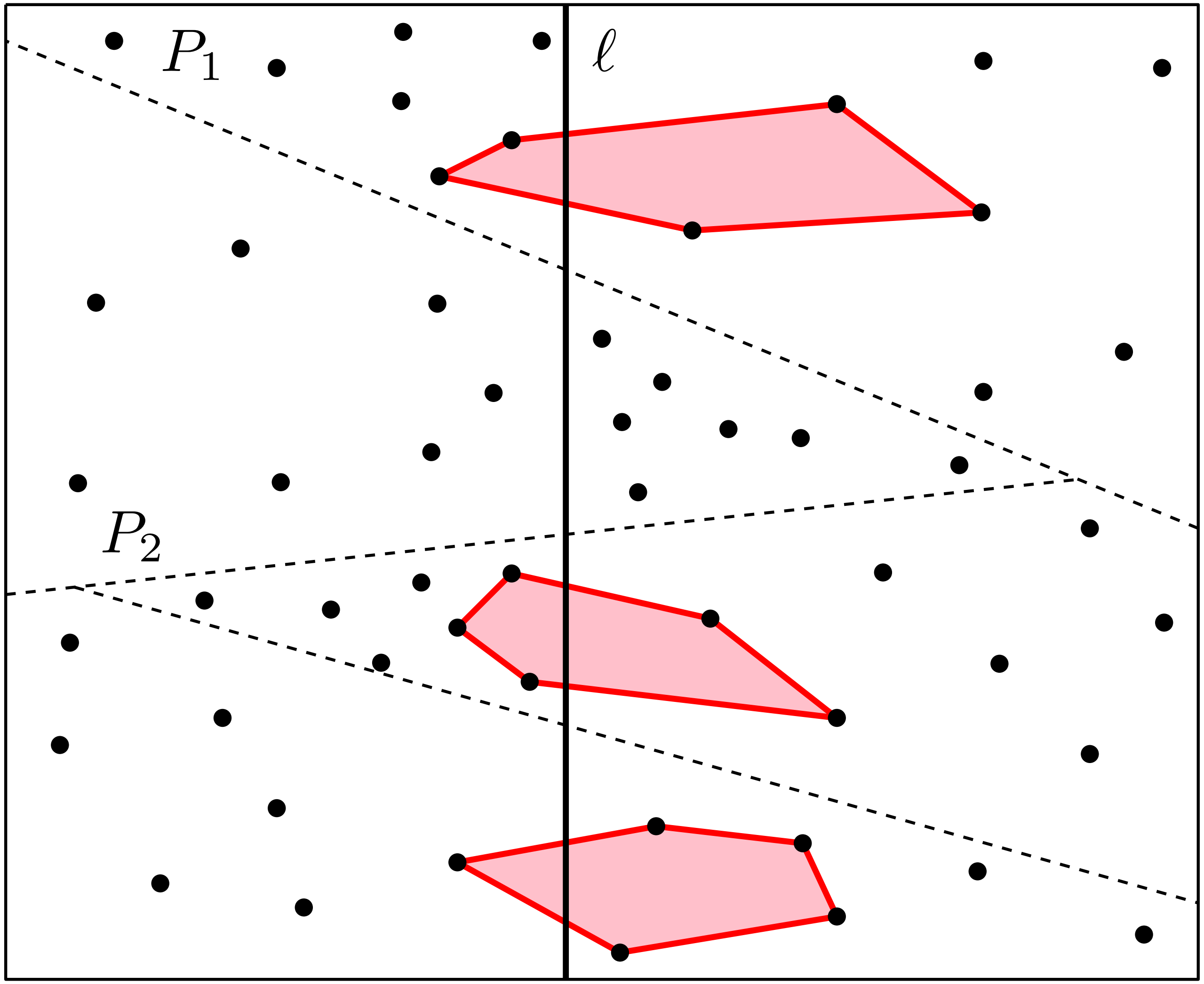


If $P_i \cap A$ and $P_i \cap B$ both not convex

$\overset{Th.3}{\Rightarrow}$ $\exists$ ℓ-divided 5-hole

Theorem 3: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$,
A and B *not* in convex position $\Rightarrow$ $\exists$ ℓ-divided 5-hole

Using Theorem 3 to prove Theorem 1



If $P_i \cap A$ and $P_i \cap B$ both not convex

$\overset{Th.3}{\Rightarrow} \exists\ \ell$-divided 5-hole

If this is the case for half of all blocks, we count at least $h_5(A) + h_5(B) + \frac{s}{2}$ 5-holes

Using Theorem 3 to prove Theorem 1

We set $r = \log^{1/5} n$, $s = \frac{n}{2r}$.

(block-size) (# of blocks)

In the first case

$$h_5(P) \geq \frac{s}{2}\binom{r}{5} \geq cn\log^{4/5} n$$

and in the second case

$$h_5(P) \geq h_5(A) + h_5(B) + \frac{s}{2} \overset{\text{Induction}}{\geq} cn\log^{4/5} n.$$

This completes the proof. $\square$

Proof of Theorem 3

Theorem 3: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$,
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A and B *not* in convex position $\Rightarrow \exists$ ℓ-divided 5-hole

Proof:

- Suppose $\nexists$ ℓ-divided 5-hole.
- If $|A| = 5 = |B|$, the statement follows from Harborth's result

P contains 5-hole ($h_5(10) = 1$),
not in A or B (both not in convex pos.)

Proof of Theorem 3

Theorem 3: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$, A and B *not* in convex position $\Rightarrow \exists$ ℓ-divided 5-hole

Proof:

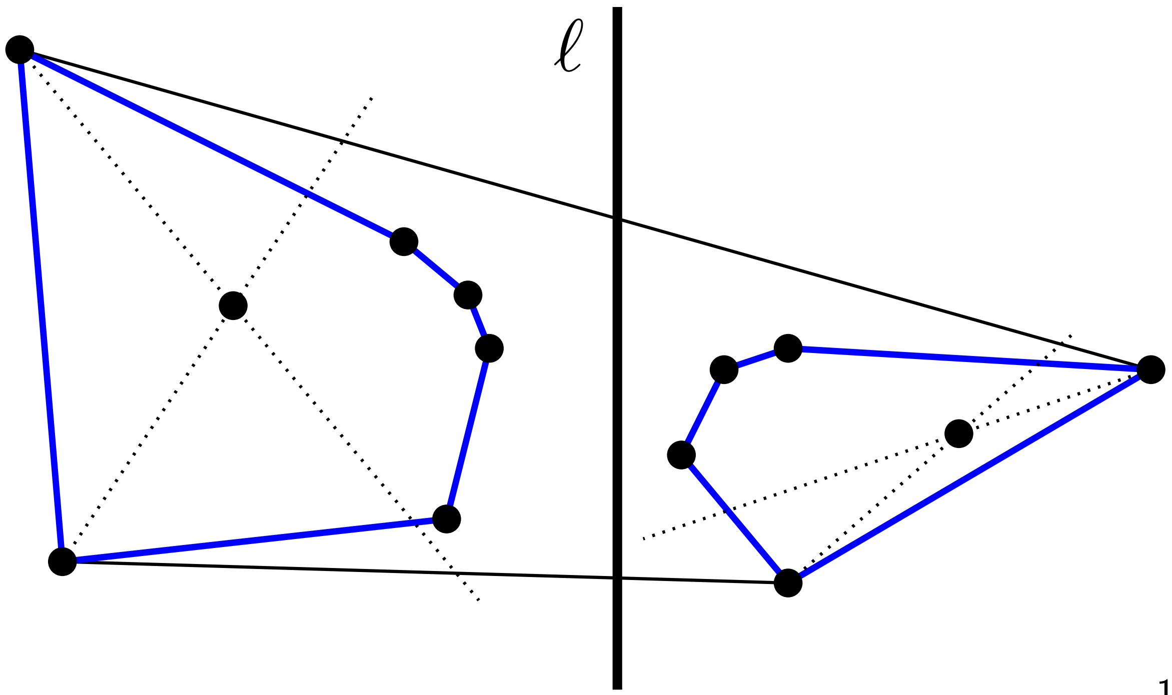
- Suppose $\nexists$ ℓ-divided 5-hole.
- If $|A| = 5 = |B|$, the statement follows from Harborth's result
- Hence we assume $|A| \geq 6$ or $|B| \geq 6$.

Proof of Theorem 3

By iteratively removing extremal points from P,
we find Q with $Q \cap A$, $Q \cap B$ *not* in convex position and

- $Q \cap A$ or $Q \cap B$ has 5 points, or
- Q is *ℓ-critical* with $|Q \cap A|, |Q \cap B| \geq 6$.

Q is minimal!



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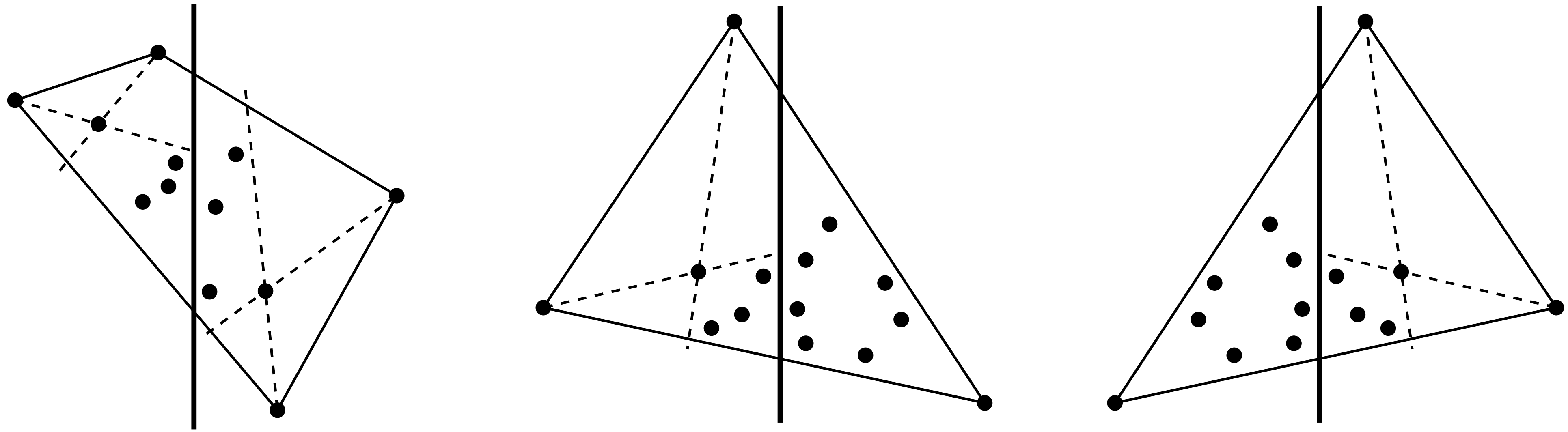
Case 1: If $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$ (or vice versa), consider $Q \cap A$ with the 6 leftmost points of $Q \cap B$ and apply

Lemma 14 (Computer-assisted): $|A| = 5$, $|B| = 6$, A not in convex position $\Rightarrow \exists$ ℓ-divided 5-hole.

Proof of Theorem 3

Case 2: If Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$, then, by Lemma 17,

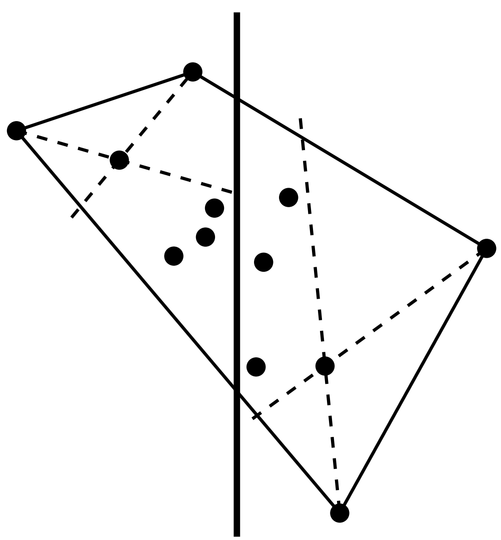
"2+2" "2+1" "1+2" (sym.)



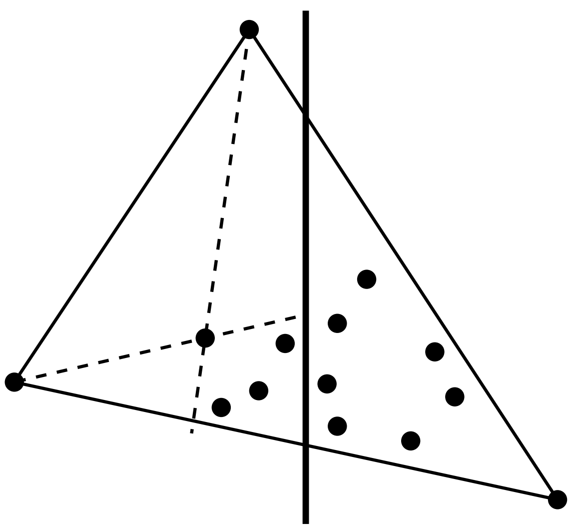
Proof of Theorem 3

Case 2: If Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$, then, by Lemma 17, $|A \cap \partial\,\mathrm{conv}(Q)| = 2$ (w.l.o.g.)

"2+2"



"2+1"



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We obtain $|Q \cap B| < |Q \cap A|$ from

Proposition 21+22: $C = A \cup B$ ℓ-critical, $|A|, |B| \geq 6$, no ℓ-divided 5-hole,

(i) $|A \cap \partial\,\mathrm{conv}(C)| = 2 \Rightarrow |B| \leq |A| - 1$.

Proof of Theorem 3

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(ii) $|A \cap \partial\,\mathrm{conv}(C)| = 1 \Rightarrow |B| \leq |A|.$

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By exchanging the roles of A and B, we obtain $|Q \cap A| \leq |Q \cap B|$ – a contradiction $\square$

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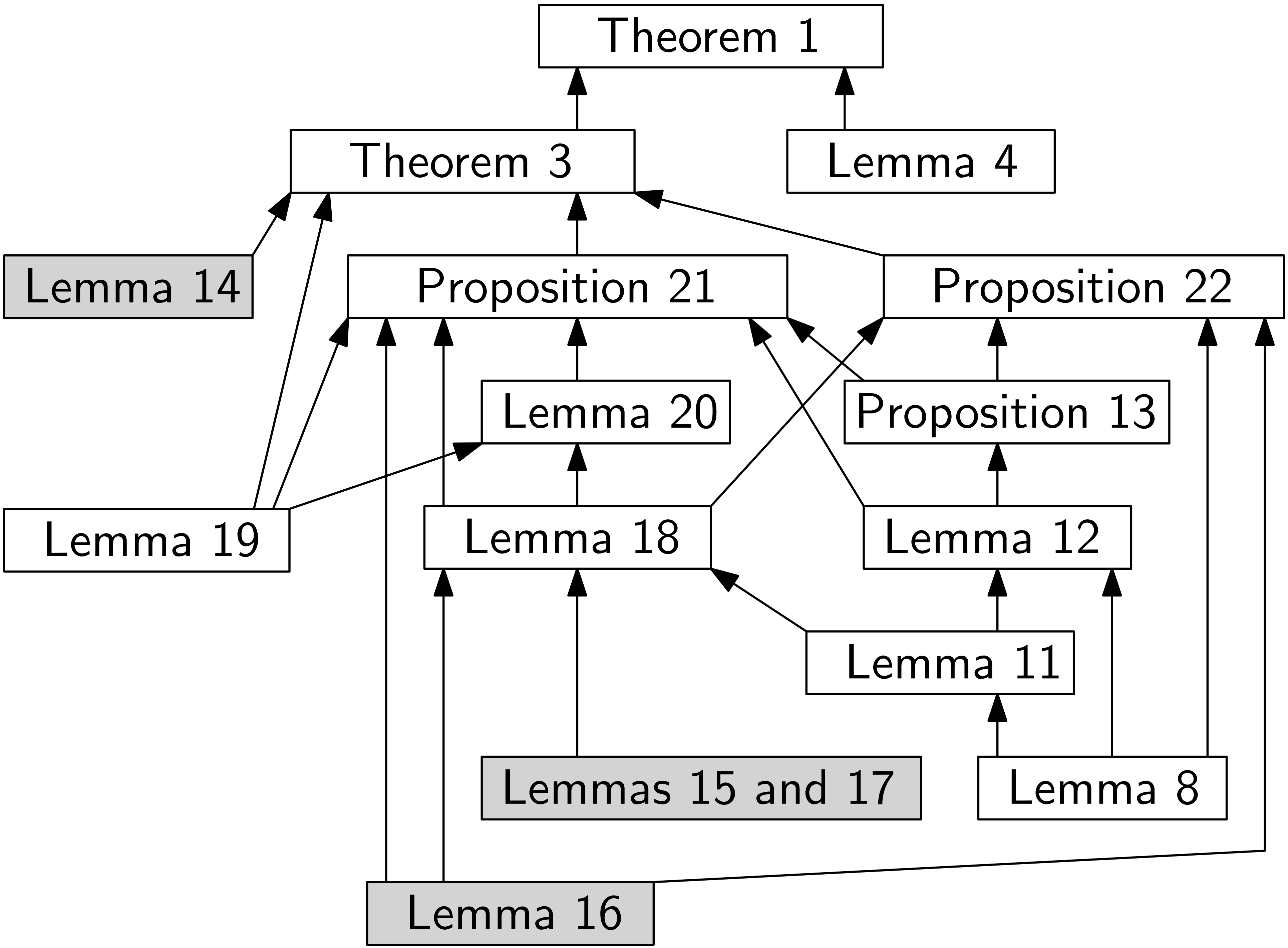
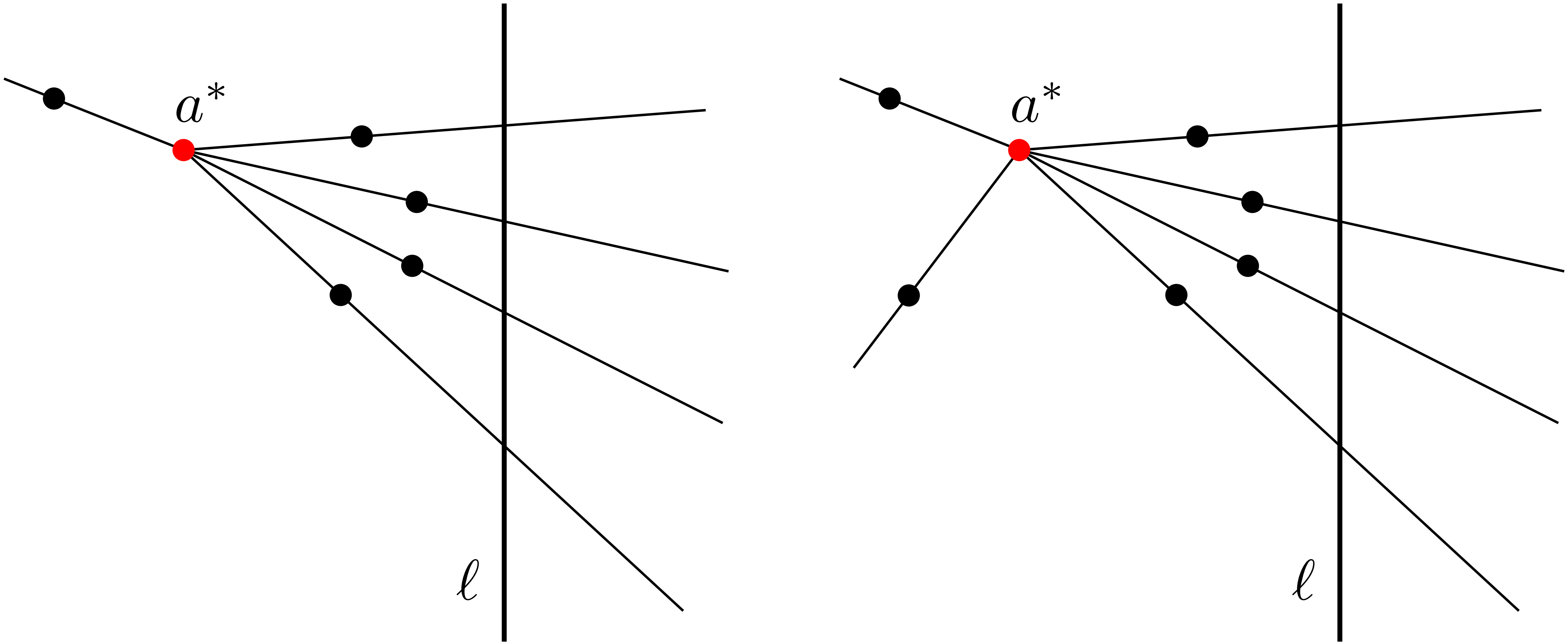

Theorem 1
Theorem 3
Lemma 4
Lemma 14
Proposition 21
Proposition 22
Lemma 20
Proposition 13
Lemma 19
Lemma 18
Lemma 12
Lemma 11
Lemmas 15 and 17
Lemma 8
Lemma 16

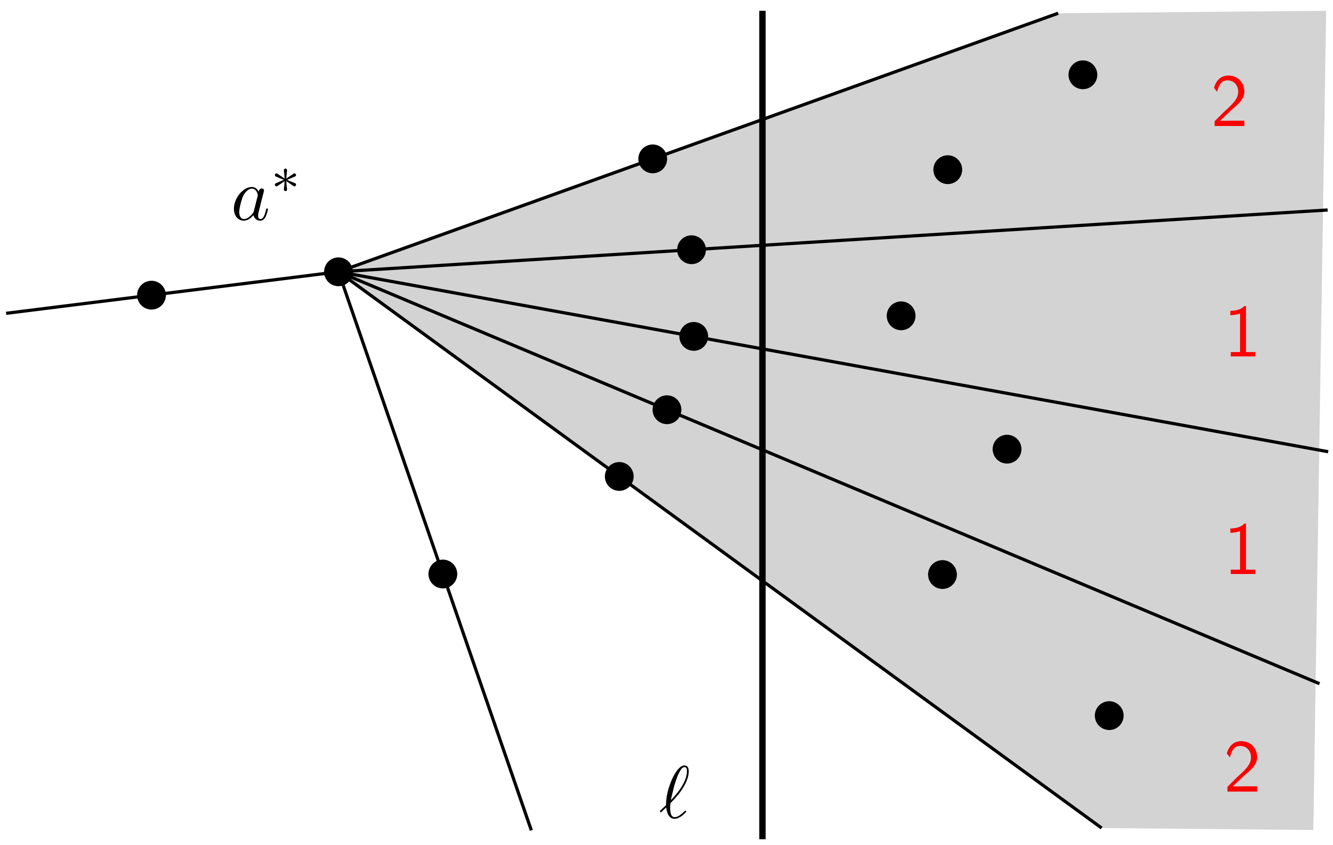
a^*-wedges

- a^* ... rightmost inner point of A



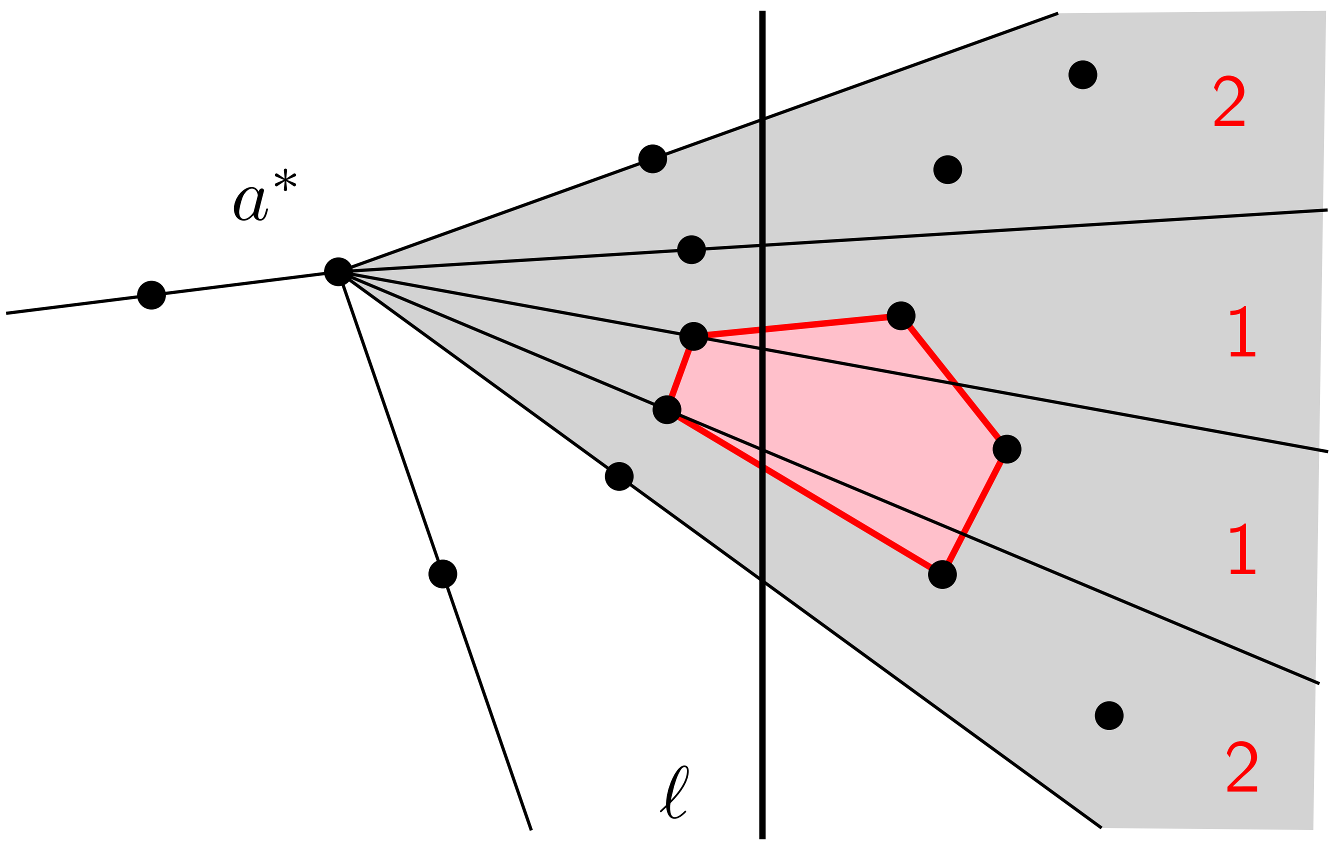
Sequences of a^*-wedges

Lemma 12: $(2, 1, \ldots, 1, 2)$ is forbidden pattern



Sequences of a^*-wedges

Lemma 12: $(2, 1, \ldots, 1, 2)$ is forbidden pattern

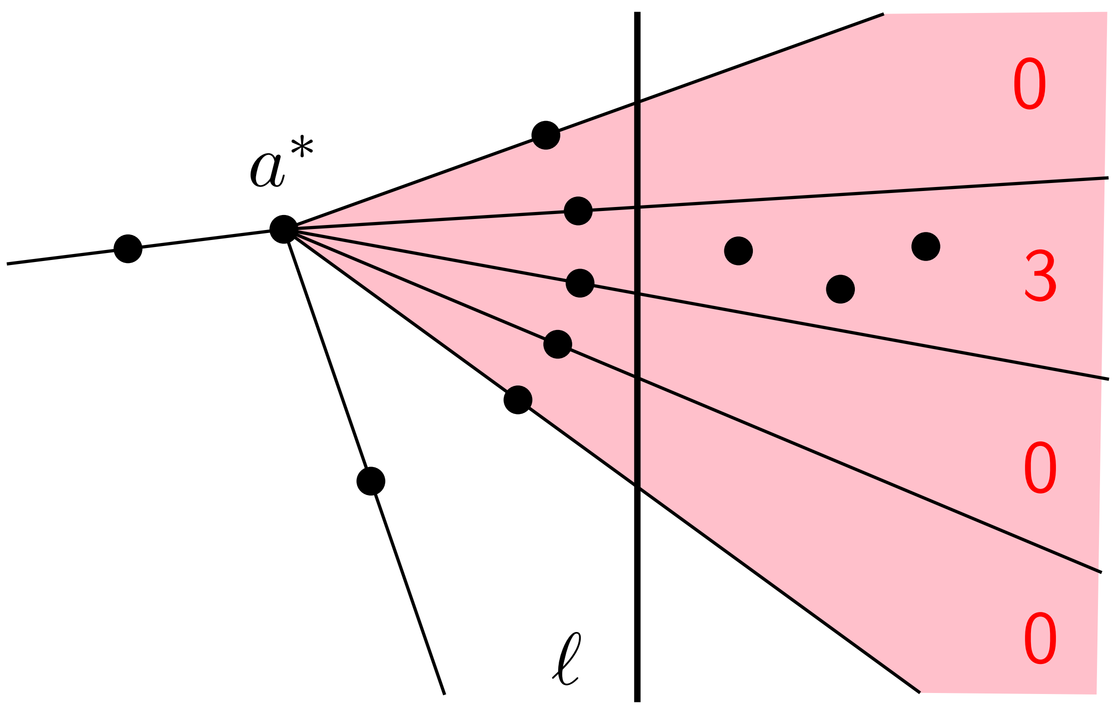


Sequences of a^*-wedges

Lemma 18:

(i) $w_i + w_{i+1} + w_{i+2} \geq 4 \Rightarrow w_i, w_{i+1}, w_{i+2} \leq 2.$

(ii) $w_i + \ldots + w_{i+3} \geq 4 \Rightarrow w_i, \ldots, w_{i+3} \leq 2.$



Sequences of a^*-wedges

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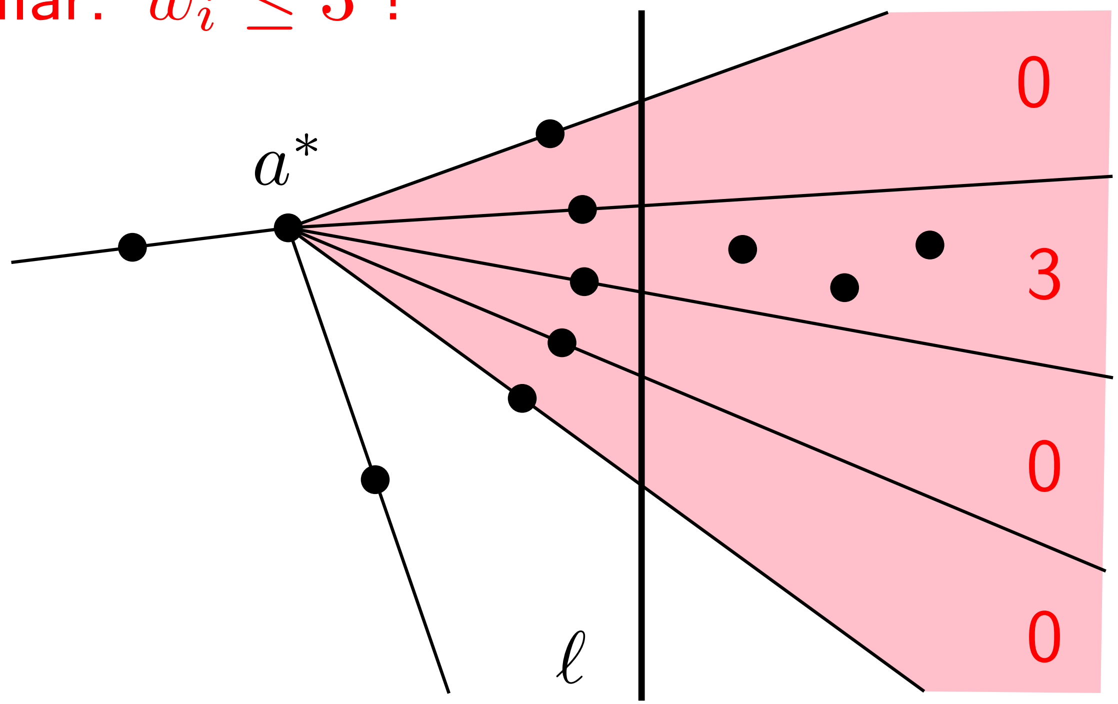
(ii) $w_i + \ldots + w_{i+3} \geq 4 \Rightarrow w_i, \ldots, w_{i+3} \leq 2.$

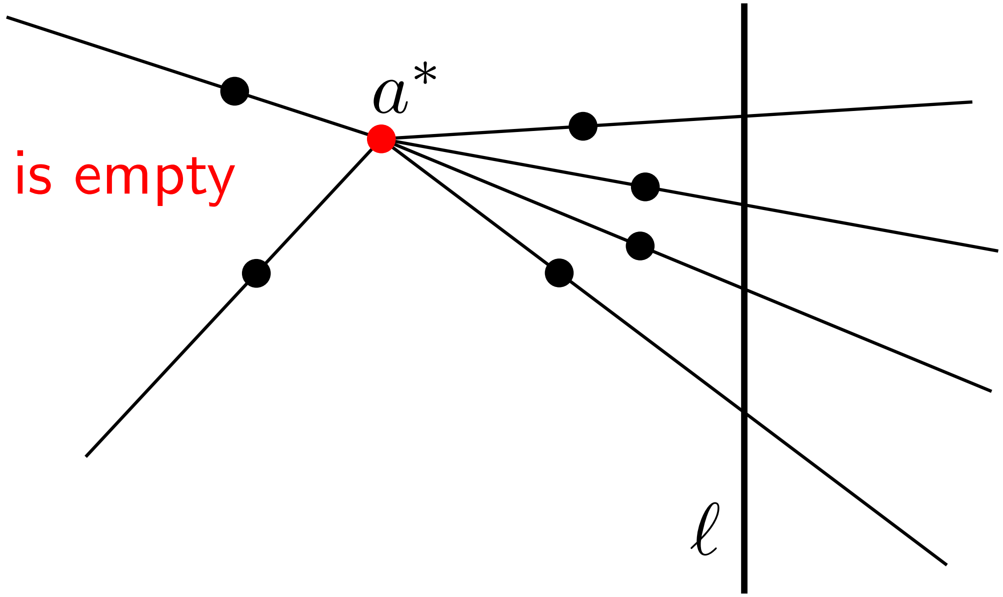
In particular: $w_i \leq 3$!



Sequences of a^*-wedges

Proposition 21: $|A \cap \partial\, \mathrm{conv}(C)| = 2 \Rightarrow |B| \leq |A| - 1.$

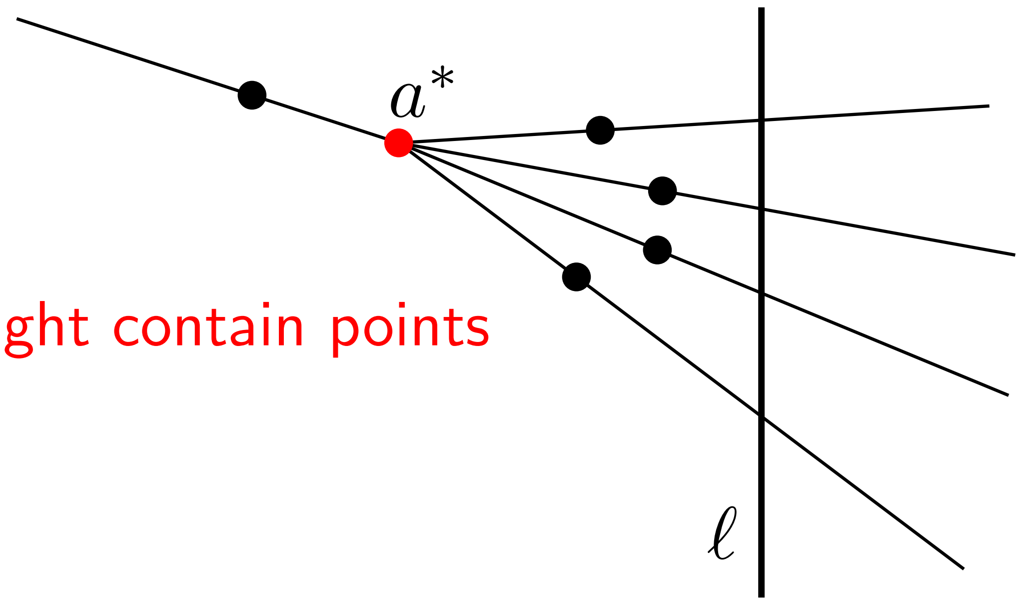
one wedge is empty



Sequences of a^*-wedges

Proposition 21: $|A \cap \partial\,\mathrm{conv}(C)| = 2 \Rightarrow |B| \leq |A| - 1.$

Proposition 22: $|A \cap \partial\,\mathrm{conv}(C)| = 1 \Rightarrow |B| \leq |A|.$



all wedges might contain points

Sequences of a^*-wedges

Proposition 21: $|A \cap \partial\,\mathrm{conv}(C)| = 2 \Rightarrow |B| \leq |A| - 1$.

Proposition 22: $|A \cap \partial\,\mathrm{conv}(C)| = 1 \Rightarrow |B| \leq |A|$.

Theorem 3: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$,
A and B *not* in convex position $\Rightarrow \exists$ ℓ-divided 5-hole

Theorem 1: $h_5(n) \geq \Omega(n \log^{4/5} n)$.

Results

Theorem 1: $h_5(n) \geq \Omega(n \log^{4/5} n)$.

Theorem 2: $h_3(n) \geq n^2 + \Omega(n \log^{2/3} n)$ and

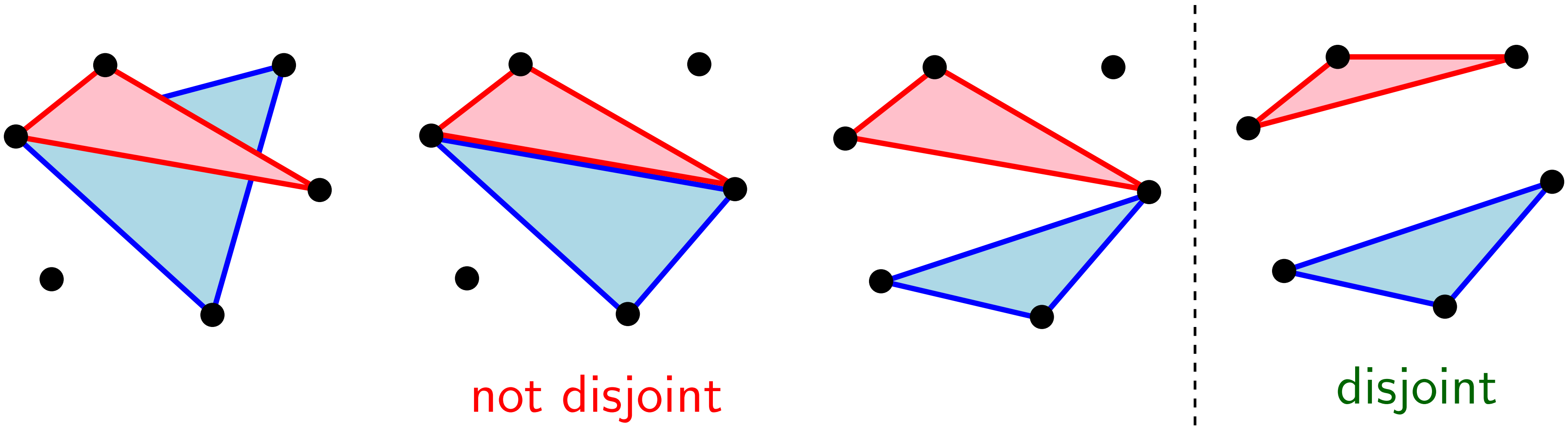
$$h_4(n) \geq \frac{n^2}{2} + \Omega(n \log^{3/4} n).$$

On Disjoint Holes in Point Sets

Manfred Scheucher

Disjoint k-Holes

Hosono and Urabe '01:
What is the smallest number $h(k_1, k_2)$ such that every set of $h(k_1, k_2)$ points determines a k_1-hole and a k_2-hole with *disjoint* convex hulls?



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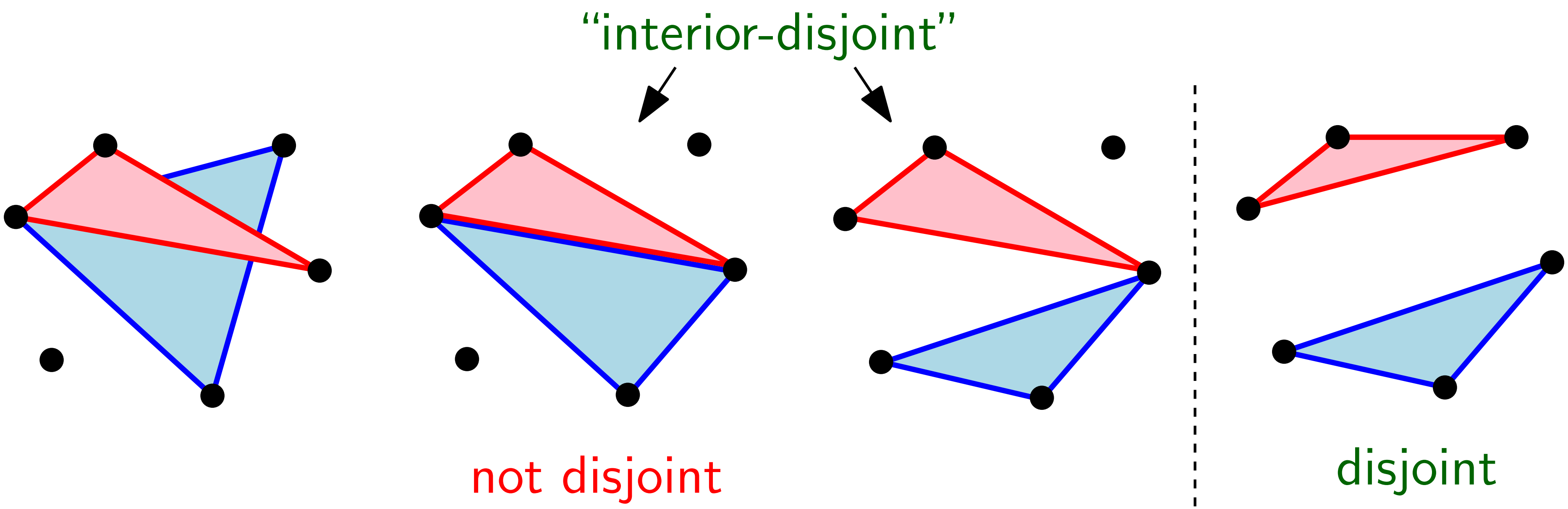

Disjoint k-Holes

Hosono–Urabe ('01, '05, '08)

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17..20

Minimum number $h(k_1, k_2)$ of points such that disjoint k_1- and k_2-holes exist

Disjoint k-Holes

Hosono–Urabe ('01, '05, '08)

Bhattacharya–Das '11

	2	3	4	5
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3		6	7	10
4			9	12
5				17..19

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Disjoint k-Holes

Hosono–Urabe ('01, '05, '08)

Bhattacharya–Das '11

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17*

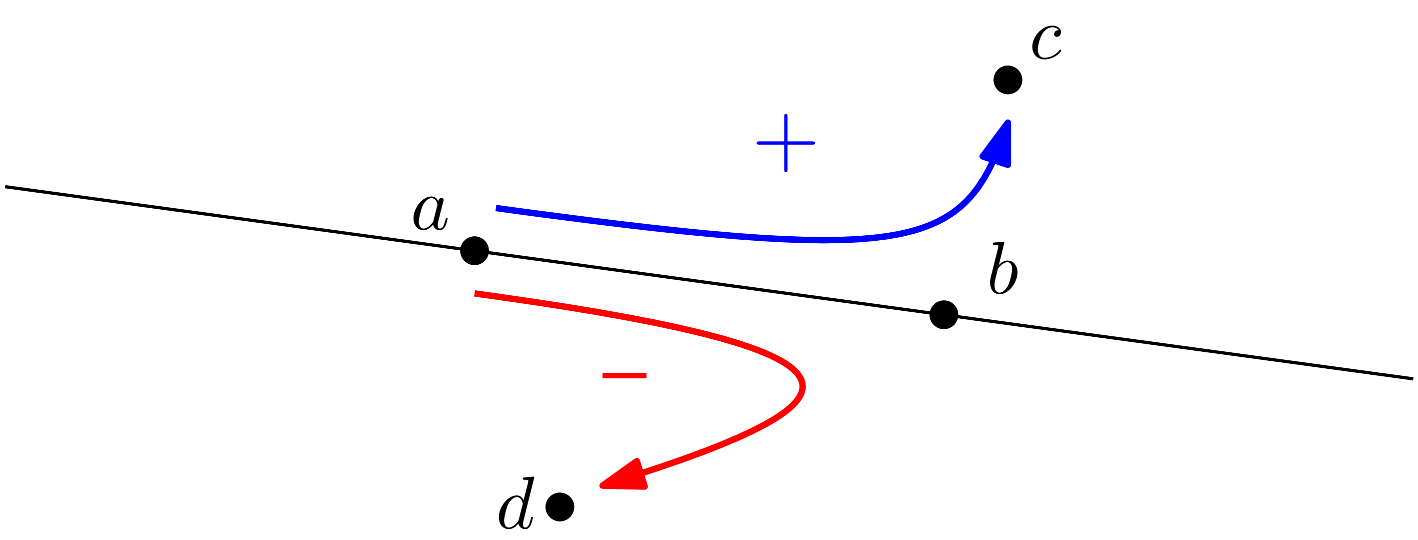
NEW

Minimum number $h(k_1, k_2)$ of points such that disjoint k_1- and k_2-holes exist

Theorem: $h(5,5) = 17$.

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+,-\}$



SAT Model

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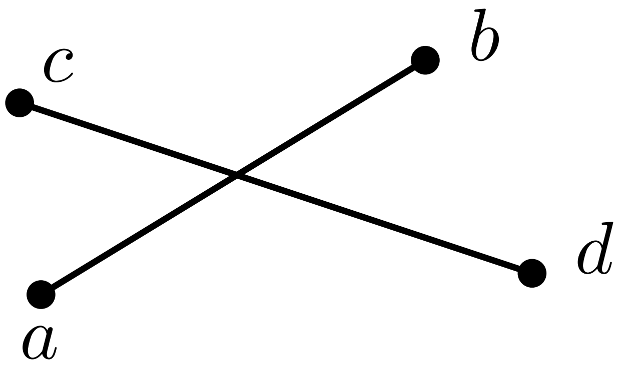
- *crossings* (4-gons),
  otherwise *containment* (point-in-triangle)

$\chi_{abc} \neq \chi_{abd}$ and $\chi_{cda} \neq \chi_{cdb}$



SAT Model

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- axiomatize "point set": *chirotope/signotope* axioms
- *crossings* (4-gons),
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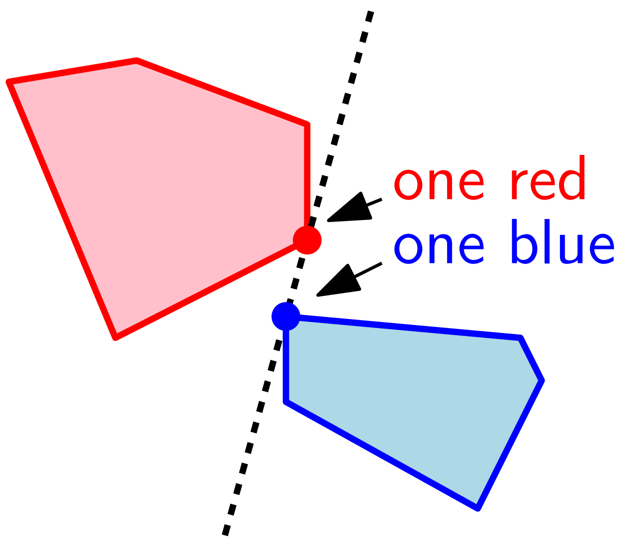
- *k-gons* and *k-holes*

  Carathéodory: every 4-tuple in k-gon is in convex positon

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+, -\}$
- axiomatize "point set": *chirotope/signotope* axioms
- *crossings* (4-gons),
  otherwise *containment* (point-in-triangle)
- *k-gons* and *k-holes*
- *disjointness* also via triple-orientations



(Un)Satisfiablity and SAT-Solvers

- Given Boolean formula, is there an assignment such that the formula is true?
- NP-complete, but good heuristics

Further Results

- $h(k_1, k_2, k_3)$ and more parameters
- Variant: interior-disjoint holes
- Classical Erdős–Szekeres: $g(6) = 17$ (with 1 CPU hour)

Part II:
Arrangements of Pseudocircles

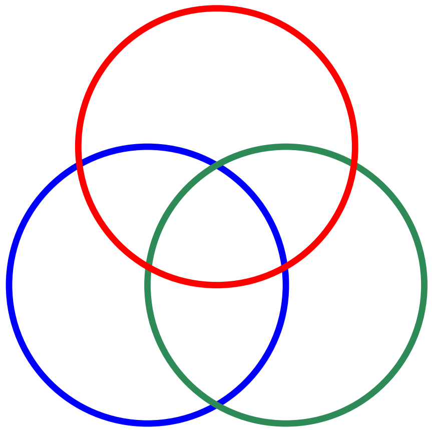
Arrangements of Pseudocircles

Stefan Felsner and Manfred Scheucher

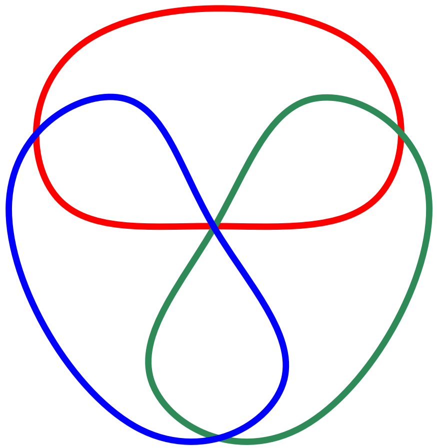
Definitions

pseudocircle . . . simple closed curve

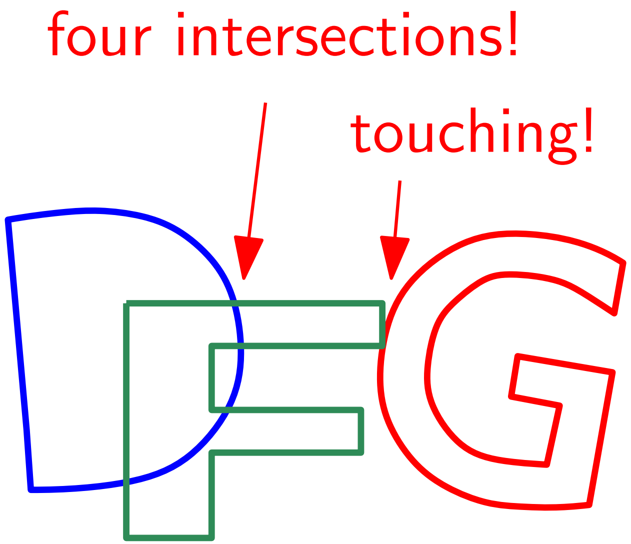
arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross



arrangement



arrangement



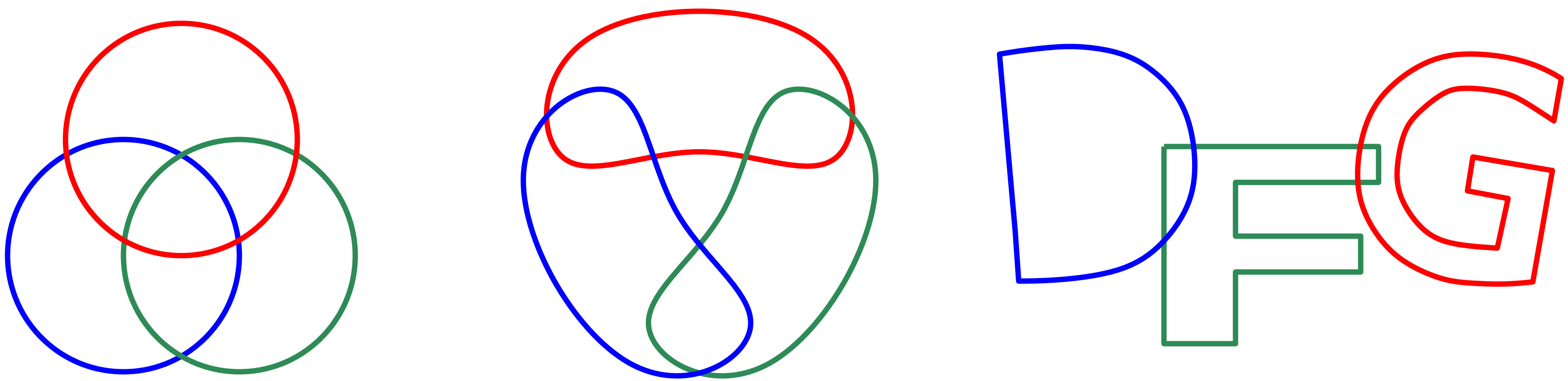
no arrangement

Definitions

simple . . . no 3 pcs. intersect in common point

connected . . . intersection graph is connected

assumptions throughout presentation



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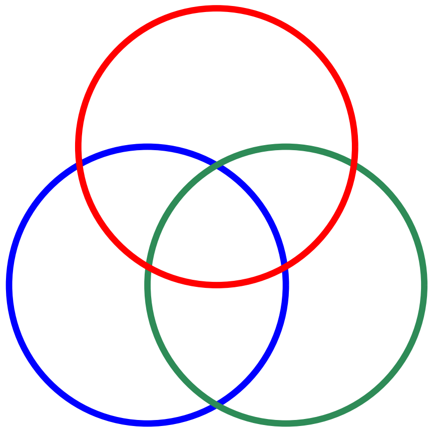
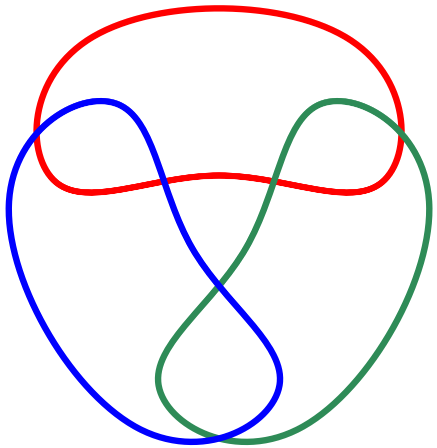
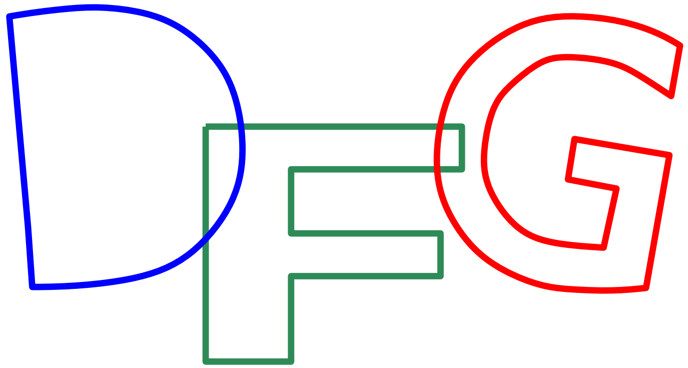


Krupp



NonKrupp



3-Chain

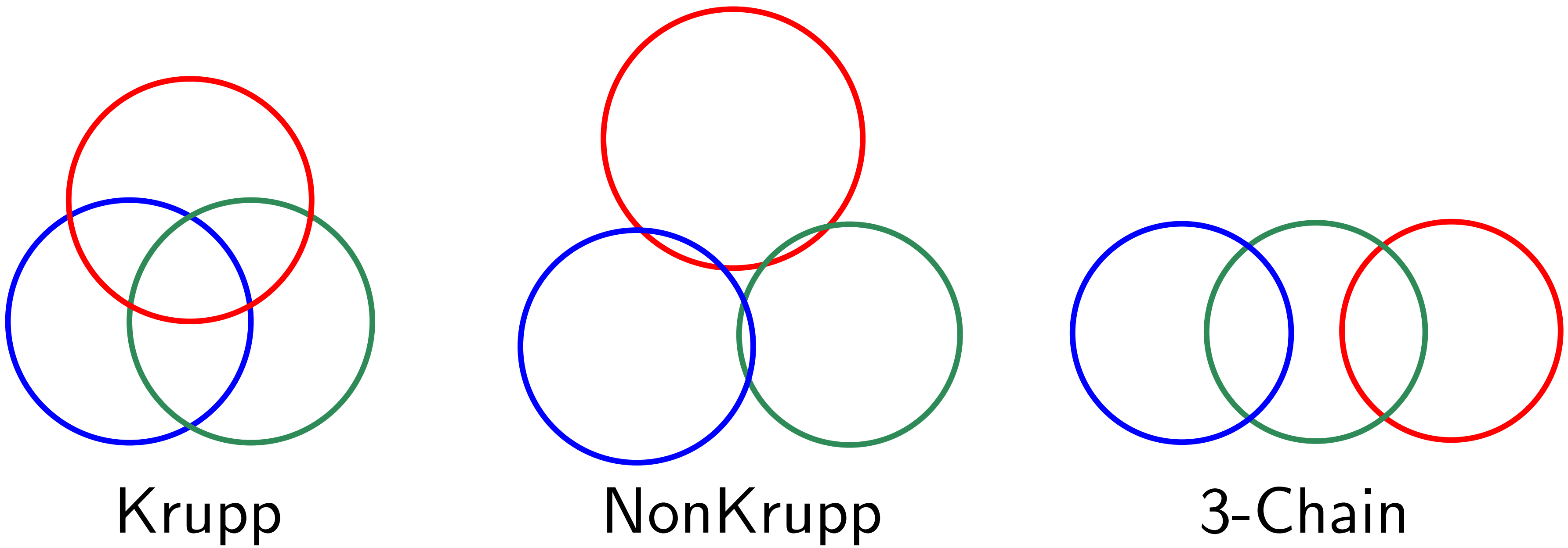
Definitions

simple . . . no 3 pcs. intersect in common point

connected . . . intersection graph is connected

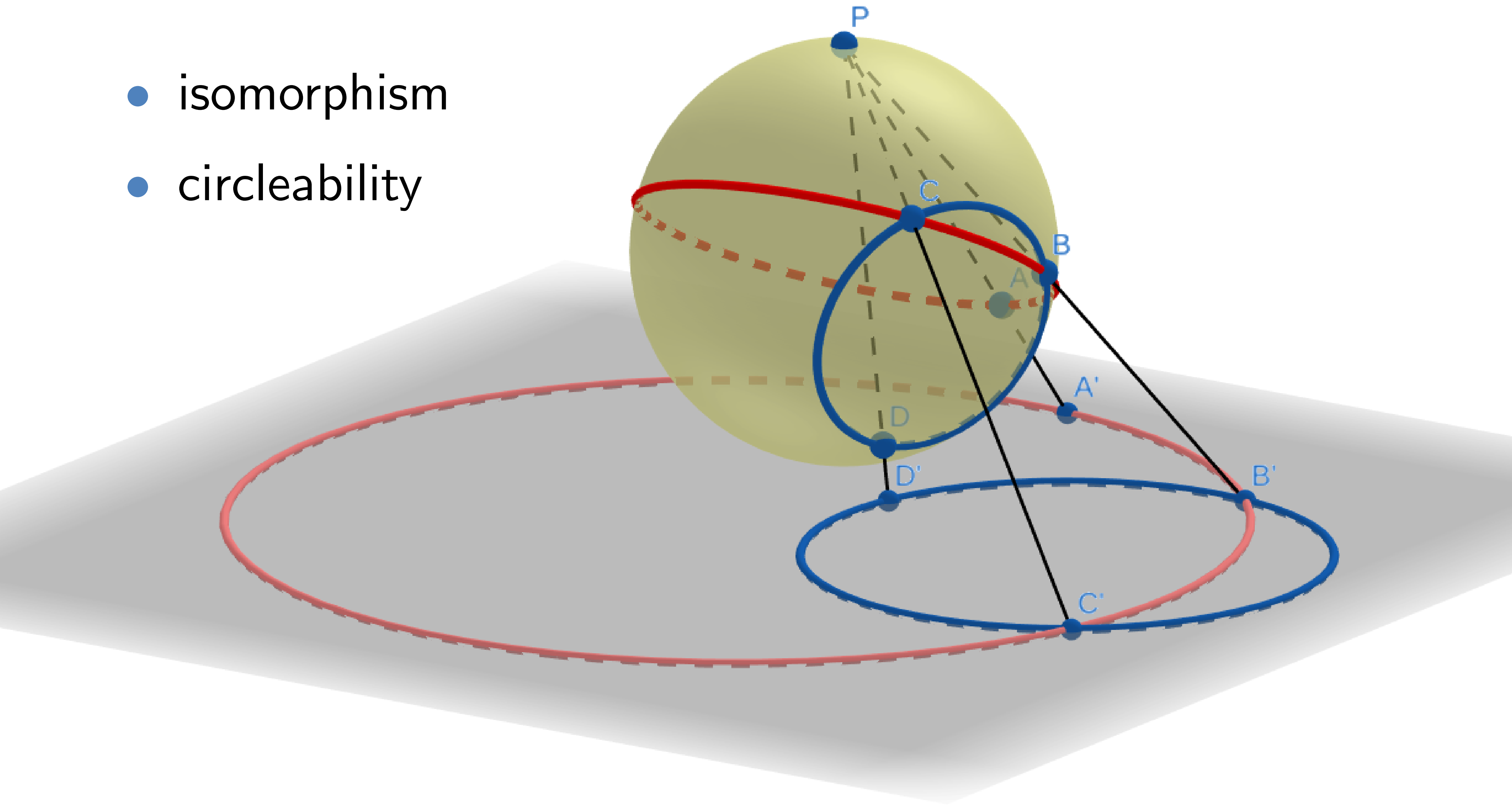
assumptions throughout presentation

circleable . . . ∃ isomorphic arrangement of circles



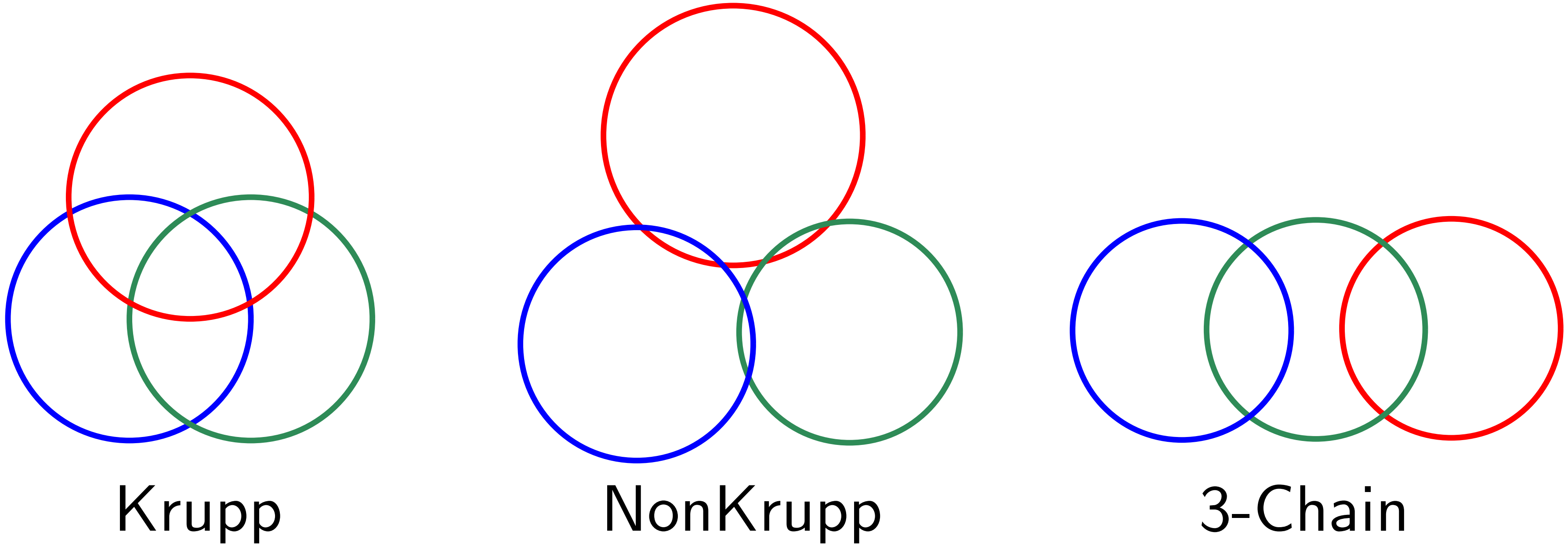
Plane VS Sphere

- isomorphism
- circleability



Hierarchy of Arrangements

connected . . . graph of arrangement is connected



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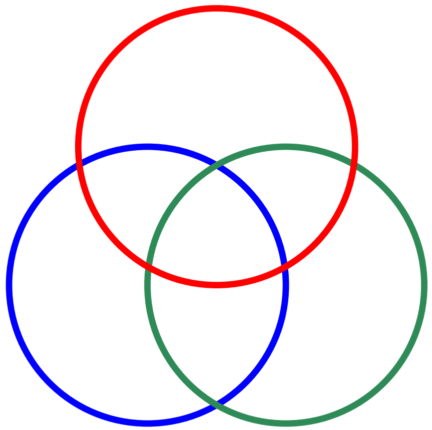
connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice



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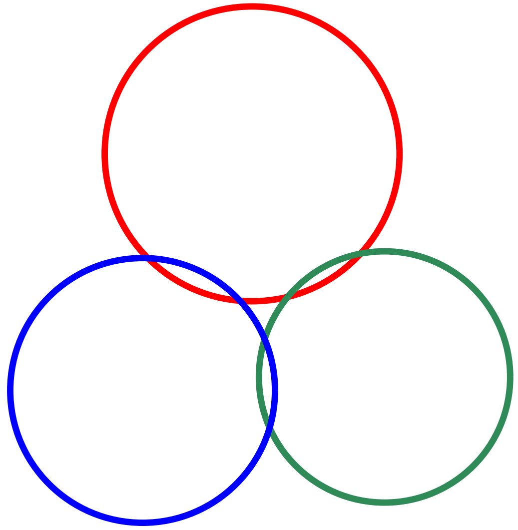

NonKrupp

Hierarchy of Arrangements

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↑

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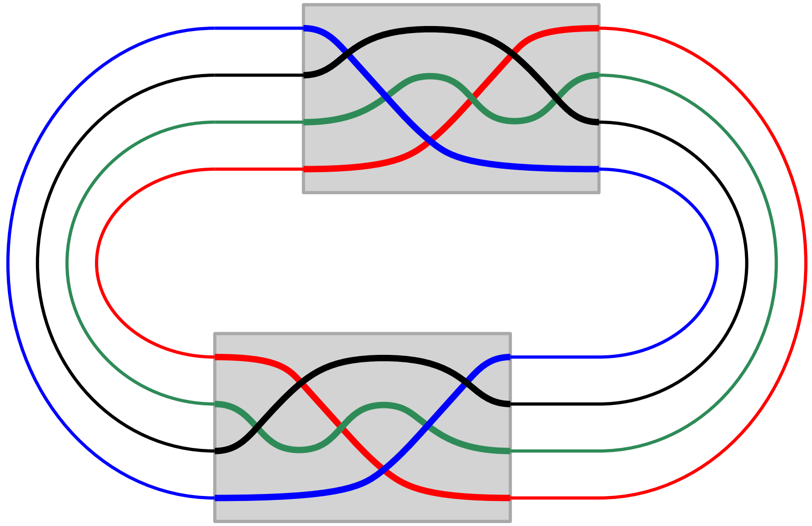
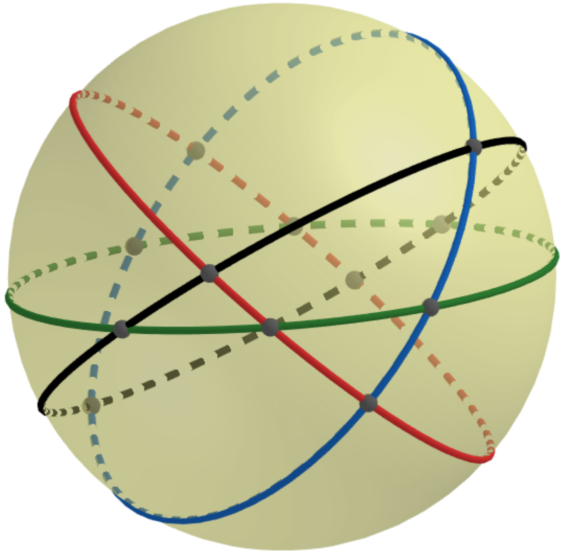
↑

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp





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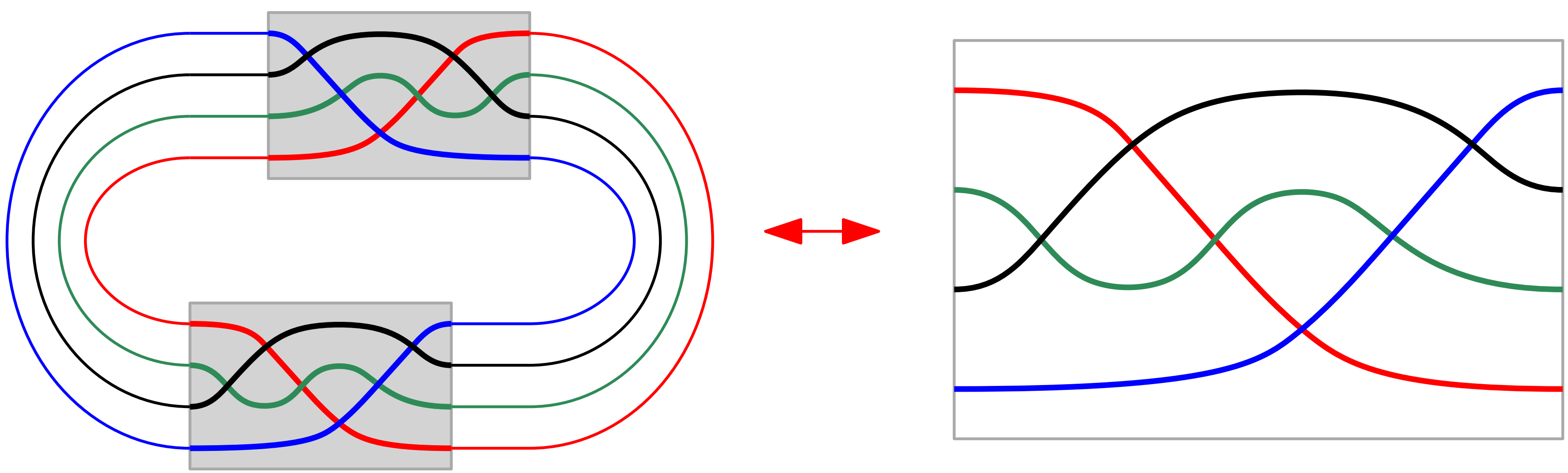

Hierarchy of Arrangements

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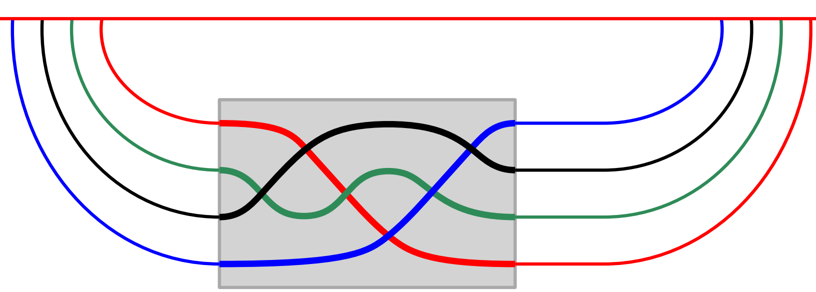
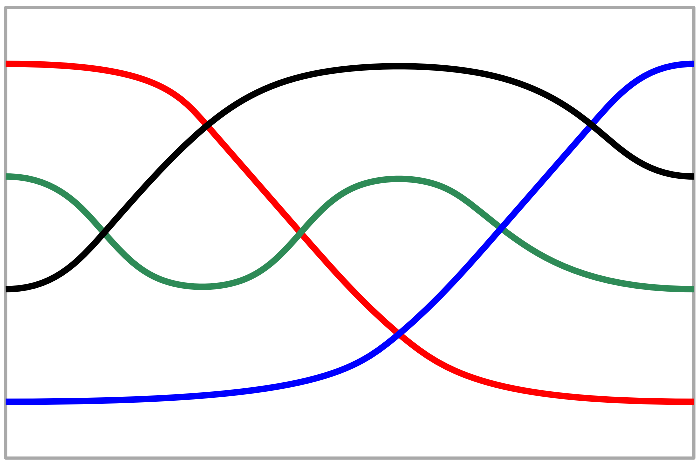
↑

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp

one-to-one correspondence
to pseudoline arrangements



↔



Number of Arrangements

n	3	4	5	6	7
connected	3	21	984	609 423	?
+digon-free	1	3	30	4 509	?
intersecting	2	8	278	145 058	447 905 202
+digon-free	1	2	14	2 131	3 012 972
great-p.c.s	1	1	1	4	11

Table: # of combinatorially different arragements of n pcs.

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Table: # of combinatorially different arragements of n pcs.

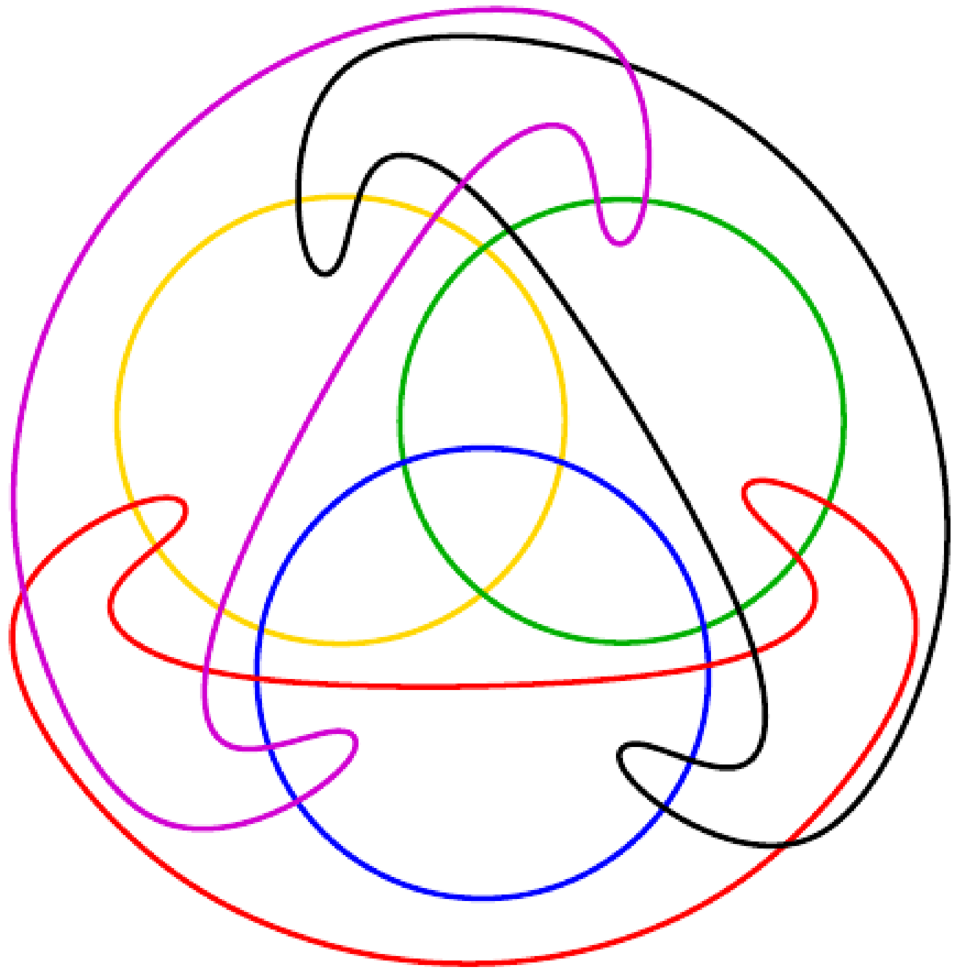
Proposition: $2^{\Theta(n^2)}$ arrangements of pcs

Proposition: $2^{\Theta(n \log n)}$ arrangements of circles

Circleability

Circleability Results

- non-circleability of intersecting $n = 6$ arrangement [Edelsbrunner and Ramos '97]



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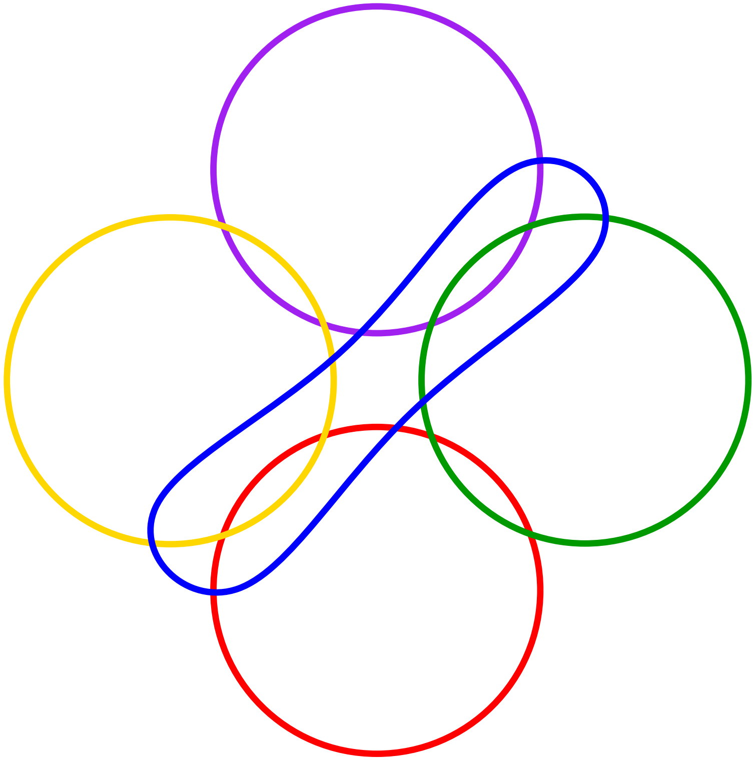
- non-circleability of $n = 5$ arrangement [Linhart and Ortner '05]



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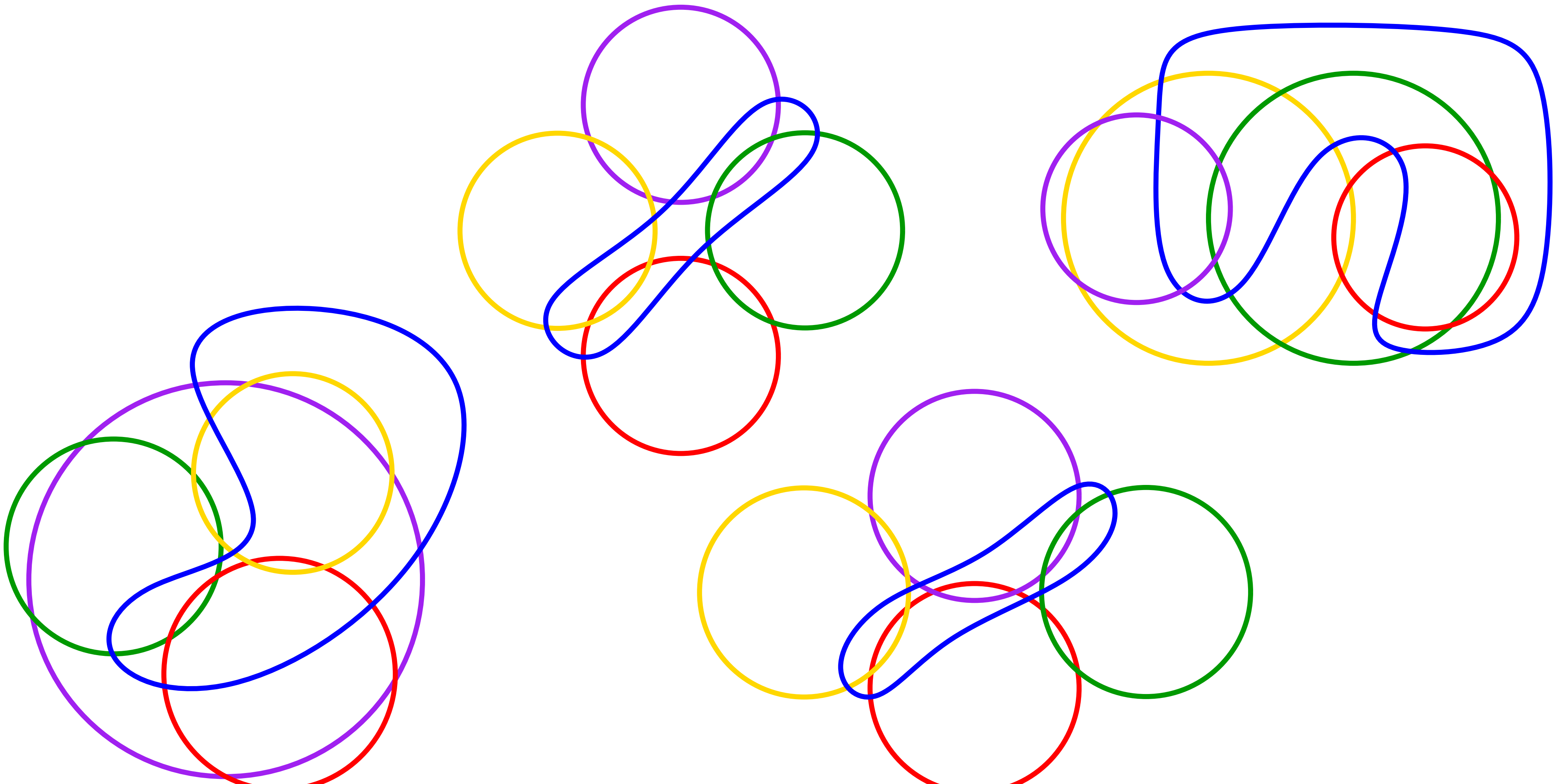
- NP-hardness of circleability [Kang and Müller '14]

Circleability Results

Theorem. There are exactly 4 non-circleable $n = 5$ arrangements (984 classes).



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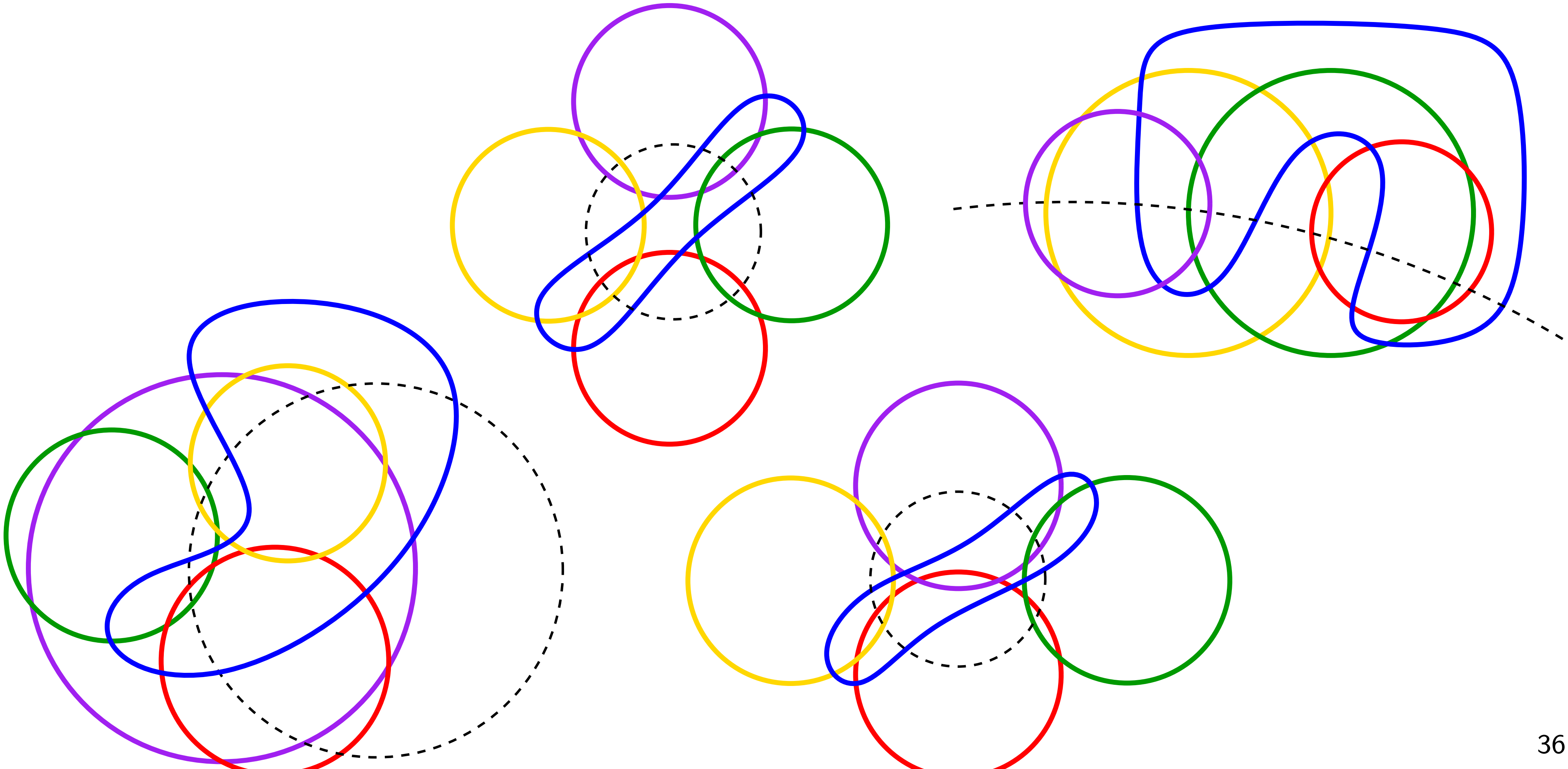
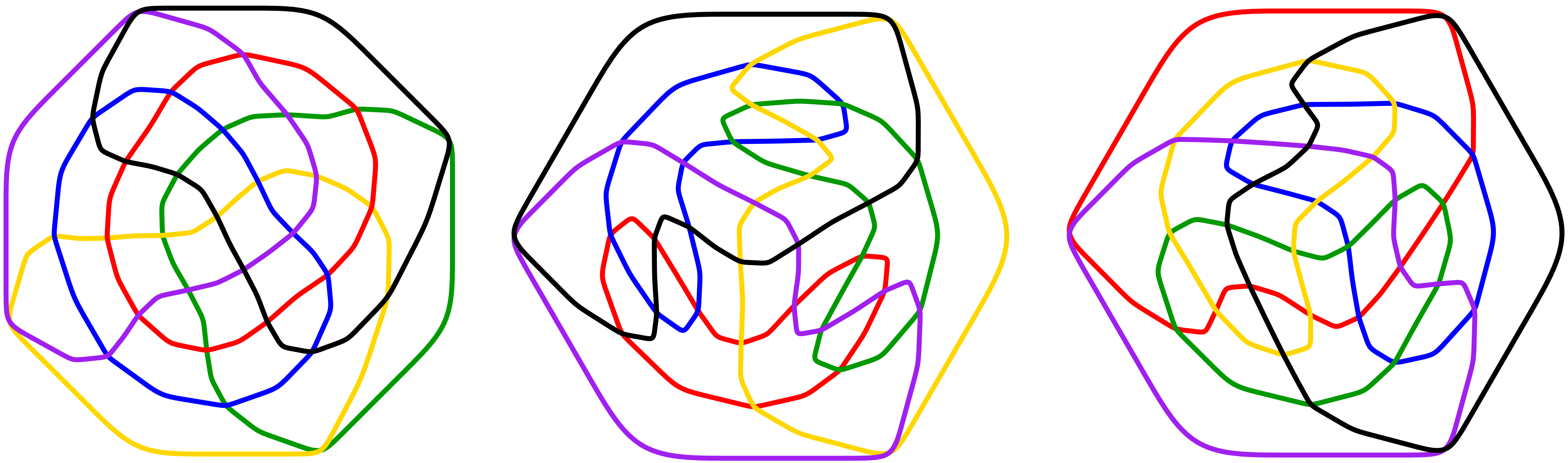


Circleability Results

Theorem. There are exactly 3 non-circleable digon-free intersecting $n = 6$ arrangements (2131 classes).



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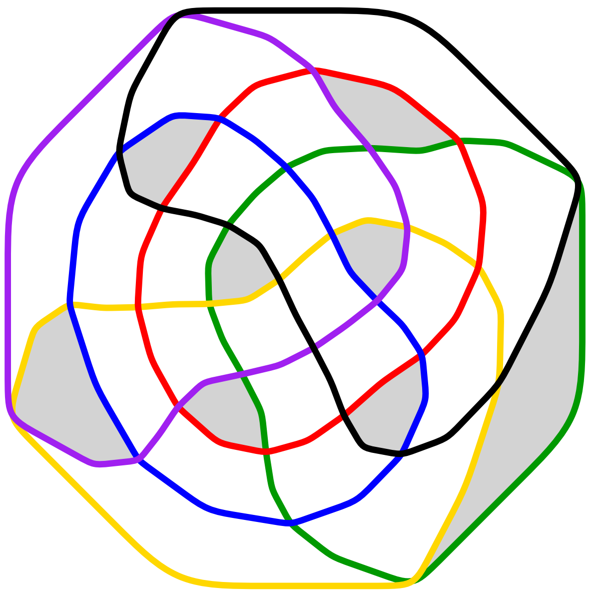
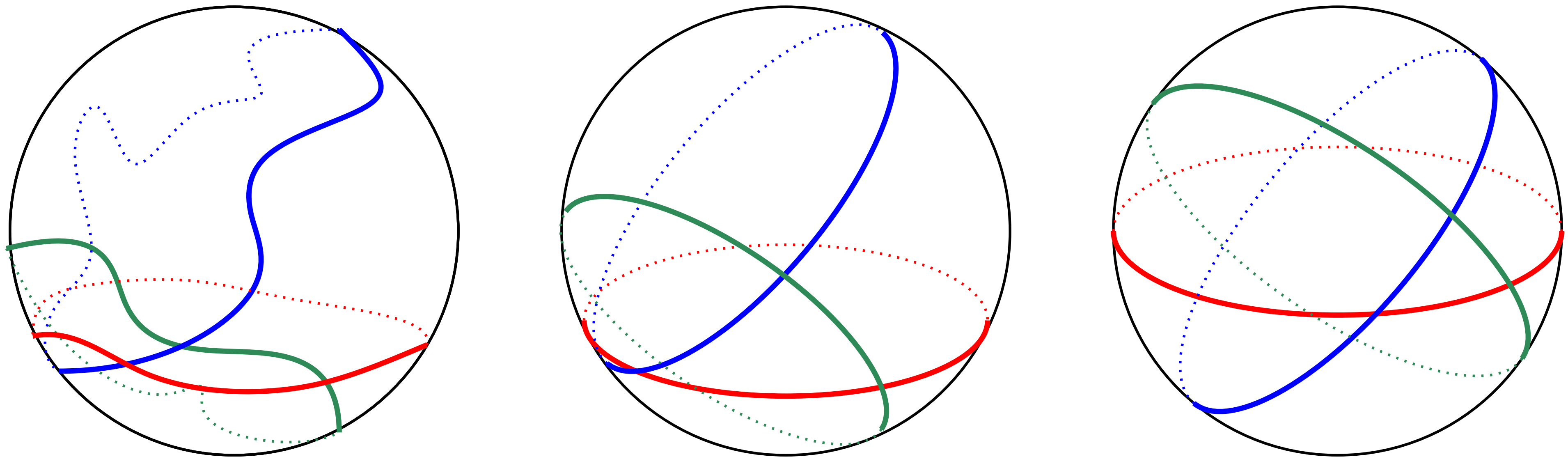


$\mathcal{N}_6^{\triangle}$ is unique digon-free intersecting with 8 triangular cells

Grünbaum Conjecture: $p_3 \geq 2n - 4$

Great-(Pseudo)Circles

Great-Circle Theorem:
An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.



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Proof:

- $C_1, \ldots, C_n$ circles on $\mathbb{S}^2$, E_i plane through C_i

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Proof:

- $C_1, \ldots, C_n$ circles on $\mathbb{S}^2$, E_i plane through C_i
- move planes towards the origin
- all triples Krupp
  $\Rightarrow$ all intersections remain inside
  $\Rightarrow$ no events
- we obtain a great-circle arrangement

$\square$

Great-(Pseudo)Circles

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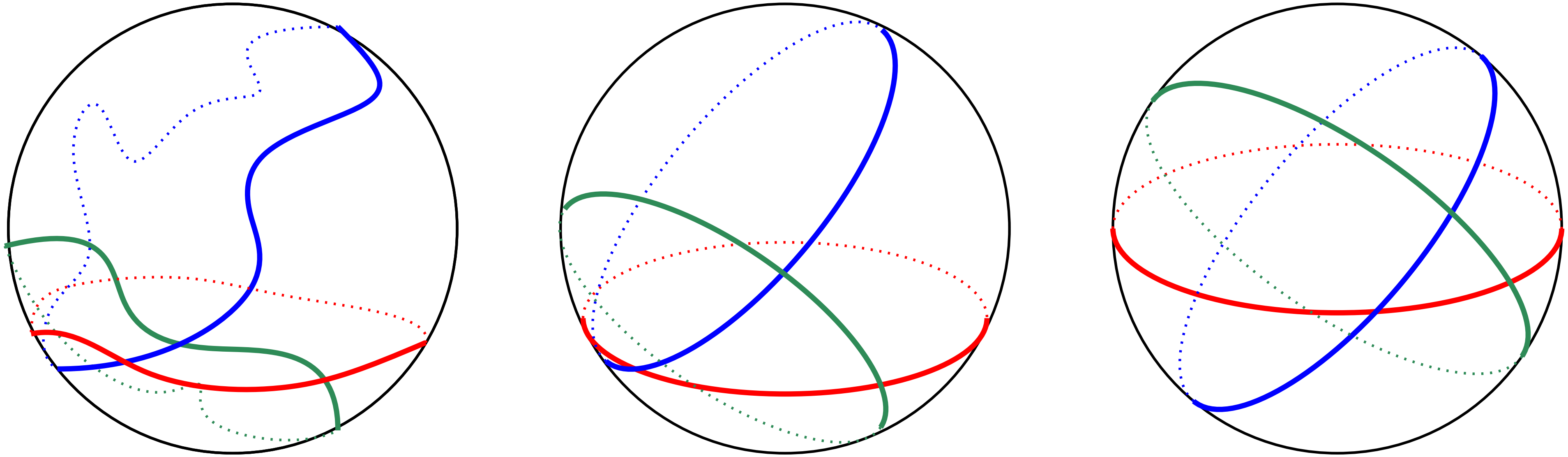
Proof:



Great-(Pseudo)Circles

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Corollaries:

- $\forall$ non-stretchable arr. of pseudolines
  $\exists$ corresponding non-circleable arr. of pseudocircles

Great-(Pseudo)Circles

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- deciding circleability is $\exists\mathbb{R}$-complete

(NP $\subseteq \exists\mathbb{R} \subseteq$ PSPACE)

Great-(Pseudo)Circles

Great-Circle Theorem:

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Corollaries:

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  $\exists$ corresponding non-circleable arr. of pseudocircles
- deciding circleability is $\exists\mathbb{R}$-complete
- $\exists$ infinite families of minimal non-circ. arrangements
- $\exists$ arr with a disconnected realization space
- ...

Triangles in Arrangements

Triangles in Arrangements

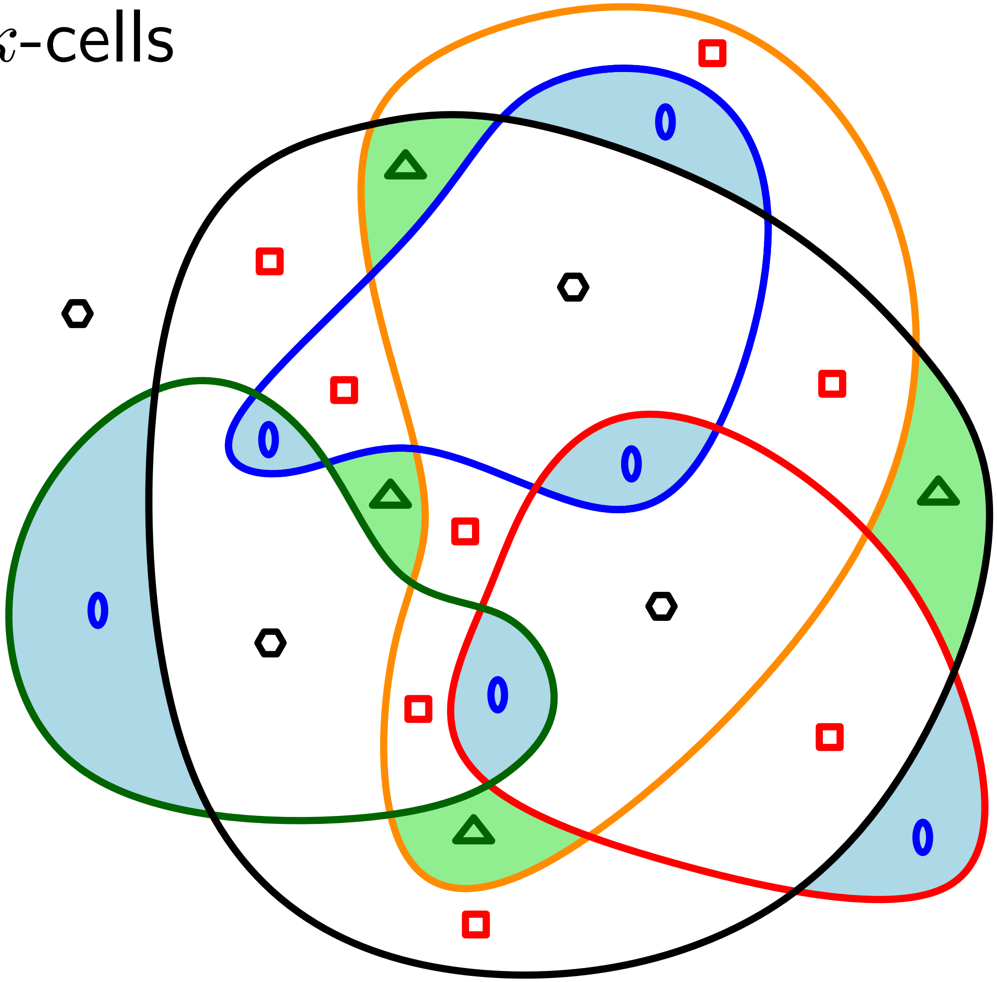
assumption throughout this part:

intersecting . . . any 2 pseudocircles cross twice

Cells in Arrangements

digon, *triangle*, *quadrangle*, *pentagon*, . . . , k*-cell*

p_k . . . # of k-cells



$p_2 = 6$

$p_3 = 4$

$p_4 = 8$

$p_5 = 0$

$p_6 = 4$

Triangles in **Digon-free** Arrangements

Grünbaum's Conjecture ('72):

$$p_2 = 0 \Rightarrow p_3 \geq 2n - 4 \text{ ?}$$

Triangles in **Digon-free** Arrangements

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Known:

- $p_3 \geq 4n/3$ [Snoeyink and Hershberger '91]
- $p_3 \geq 4n/3$ for non-simple arrangements,
  tight for infinite family [Felsner and Kriegel '98]

Triangles in **Digon-free** Arrangements

Grünbaum's Conjecture ('72):

$$p_2 = 0 \Rightarrow p_3 \geq 2n - 4 \text{ ?}$$

Known:

- $p_3 \geq 4n/3$ [Snoeyink and Hershberger '91]
- $p_3 \geq 4n/3$ for non-simple arrangements, tight for infinite family [Felsner and Kriegel '98]

Our Contribution:

- disprove Grünbaum's Conjecture
- $p_3 < 1.\overline{45}n$
- **New Conjecture:** $4n/3$ is tight

Triangles in Digon-free Arrangements

Theorem. The minimum number of triangles in digon-free arrangements of n pseudocircles is

(i) 8 for $3 \leq n \leq 6$.

(ii) $\lceil \frac{4}{3}n \rceil$ for $6 \leq n \leq 14$.

(iii) $\frac{16k}{11k+1} < 1.\overline{45}n$ for all $n = 11k + 1$ with $k \in \mathbb{N}$.

Triangles in Digon-free Arrangements

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- Grünbaum's Conjecture first violated at $n = 8$

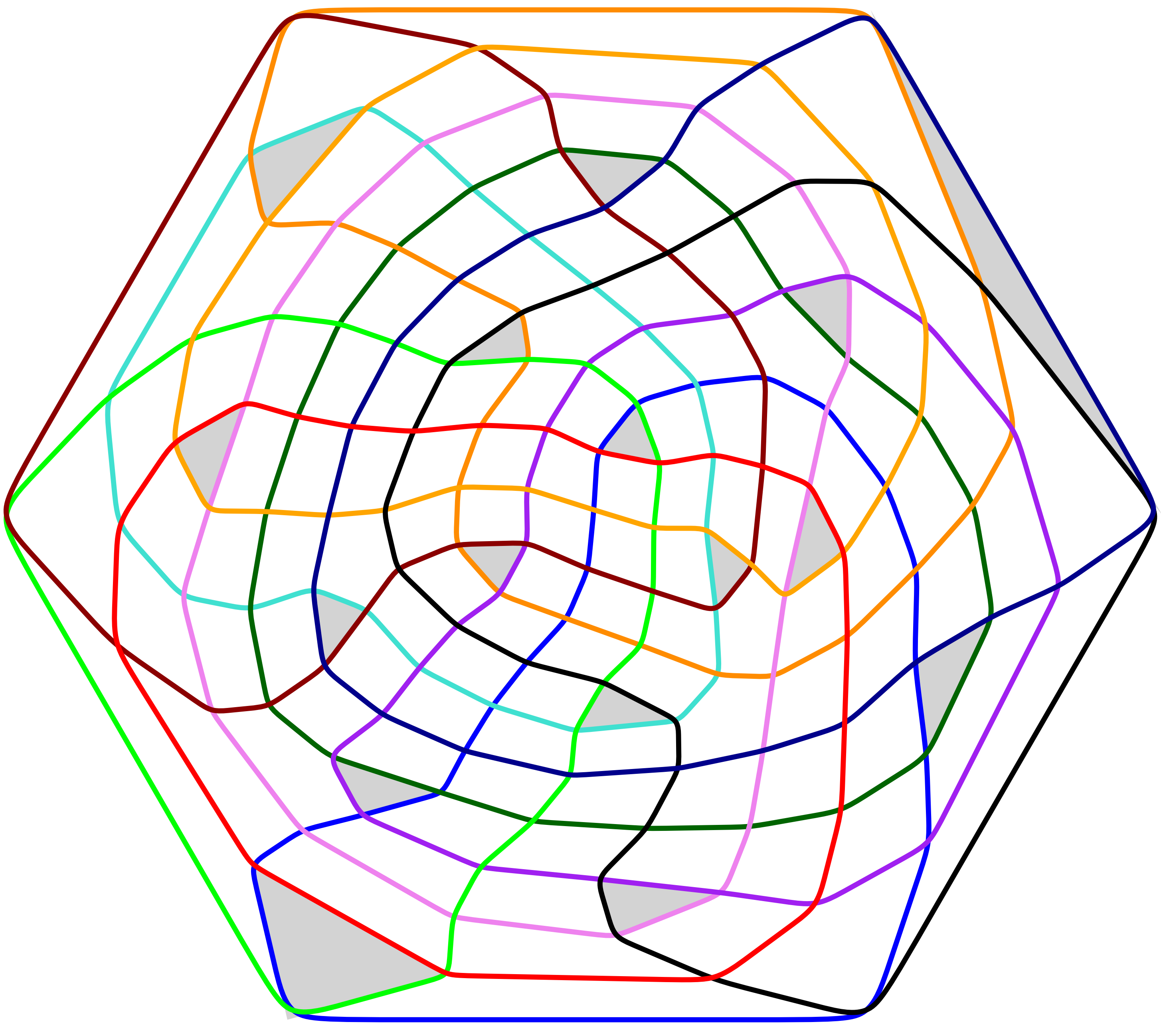
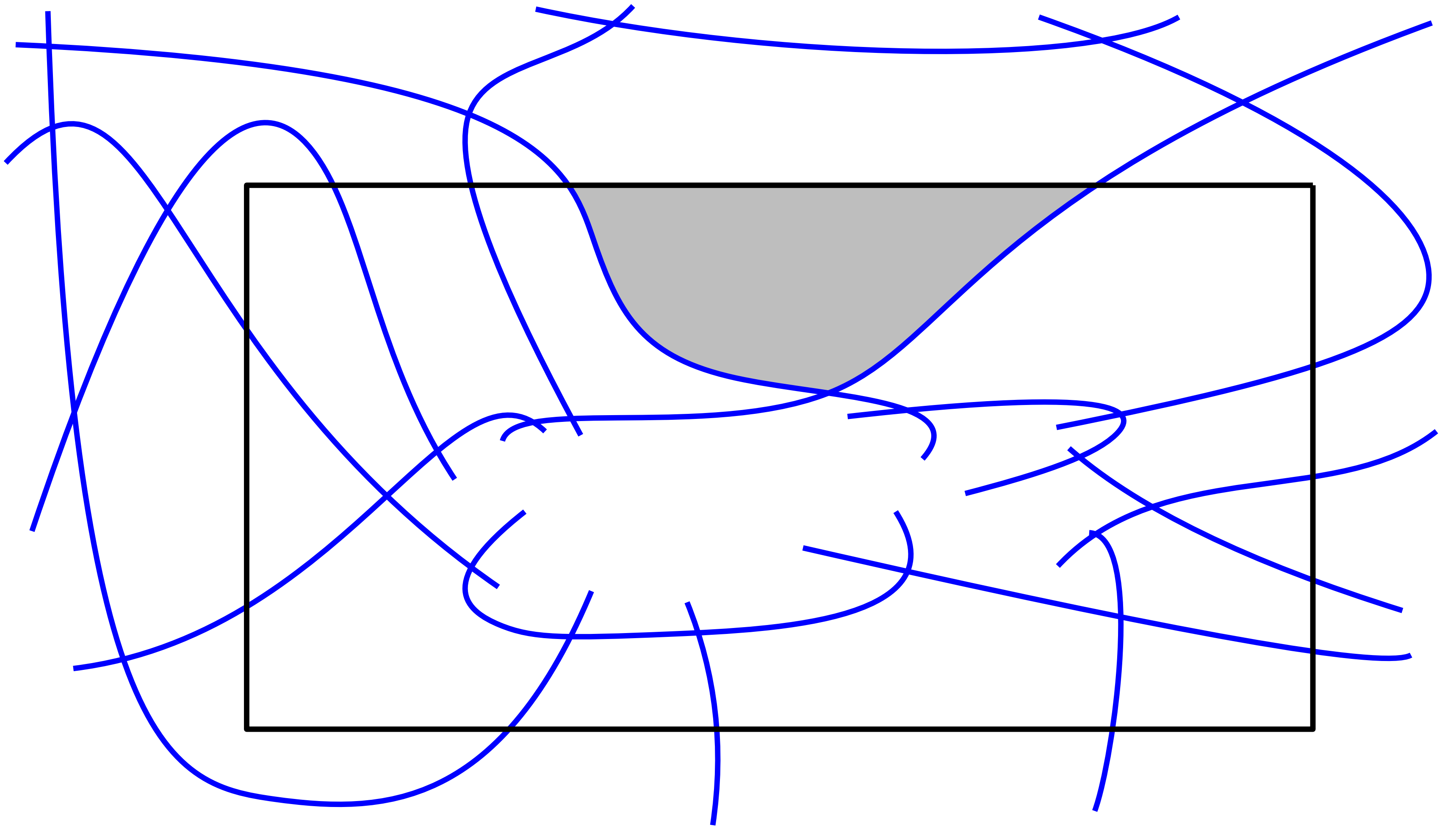


Figure: Arrangement of $n = 12$ pcs with $p_3 = 16$ triangles.

Recursive Construction



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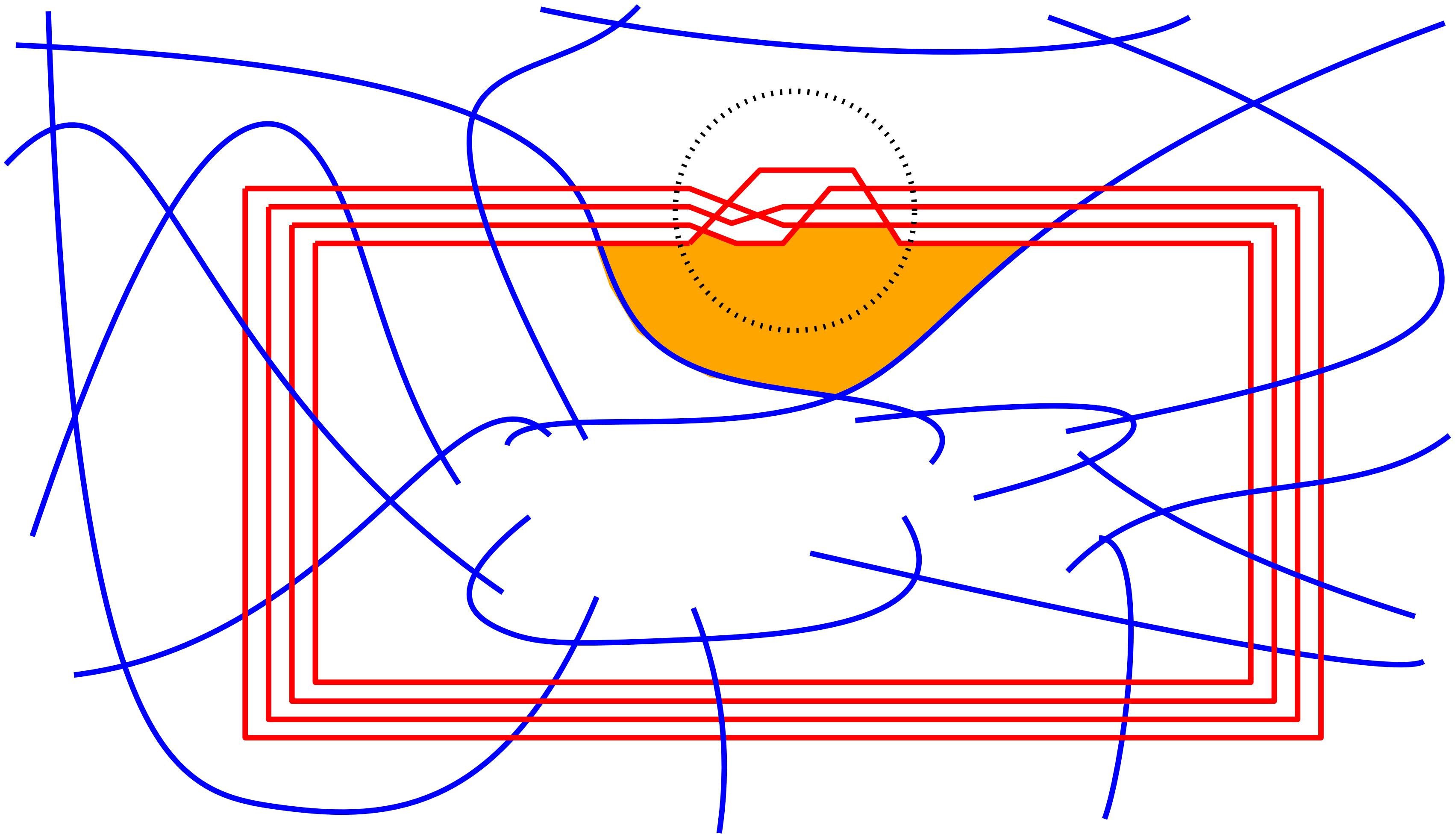

Triangles in Digon-free Arrangements

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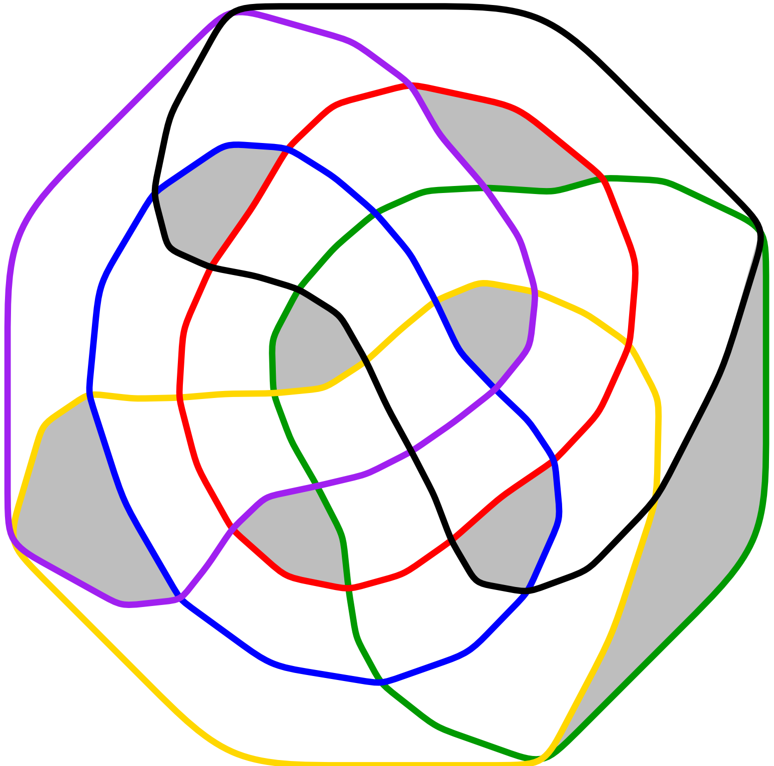
Conjecture. $\lceil 4n/3 \rceil$ is tight for infinitely many n.

Triangles in Digon-free Arrangements

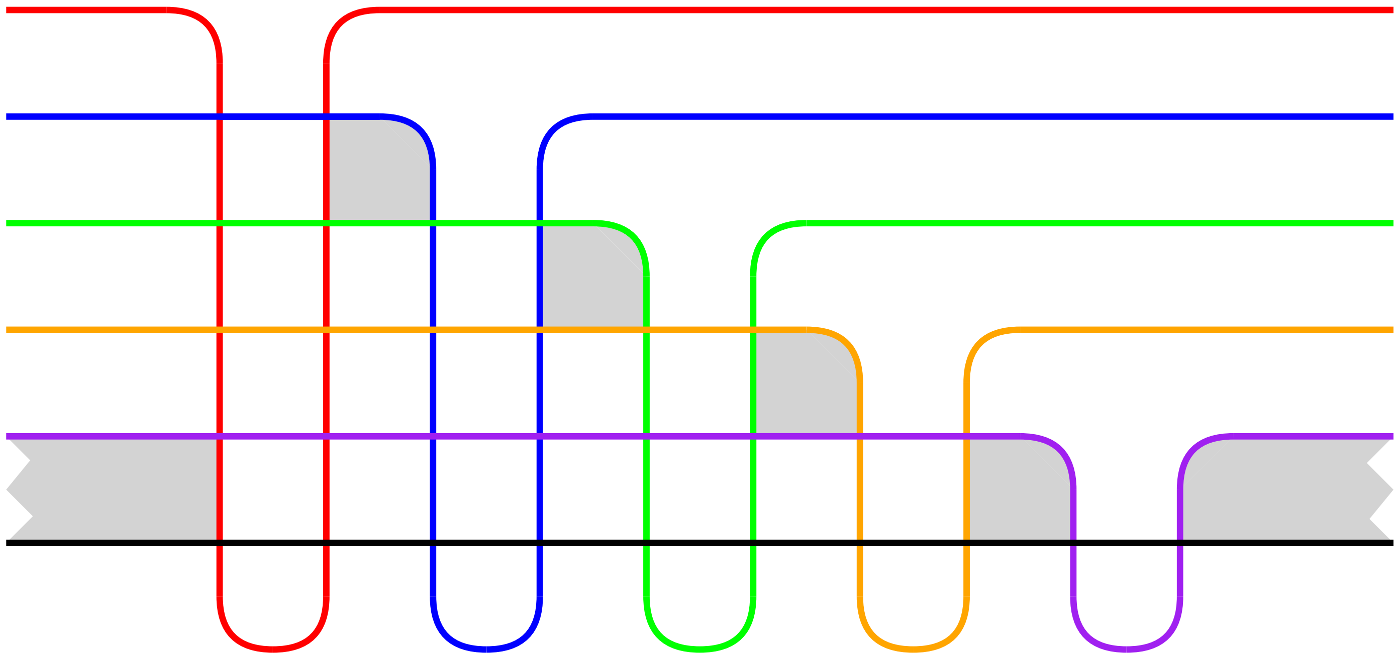
- $\exists$ unique arrangement $\mathcal{N}_6^\triangle$ with $n = 6, p_3 = 8$
- $\mathcal{N}_6^\triangle$ appears as a subarrangement of every arr. with $p_3 < 2n - 4$ for $n = 7, 8, 9$
- $\mathcal{N}_6^\triangle$ is non-circularizable
- $\Rightarrow$ Grünbaum's Conjecture might still be true for arrangements of circles!



Triangles in Arrangements **with Digons**

Theorem. $p_3 \geq 2n/3$

Conjecture. $p_3 \geq n - 1$



Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

(proof is based on a discharging argument)

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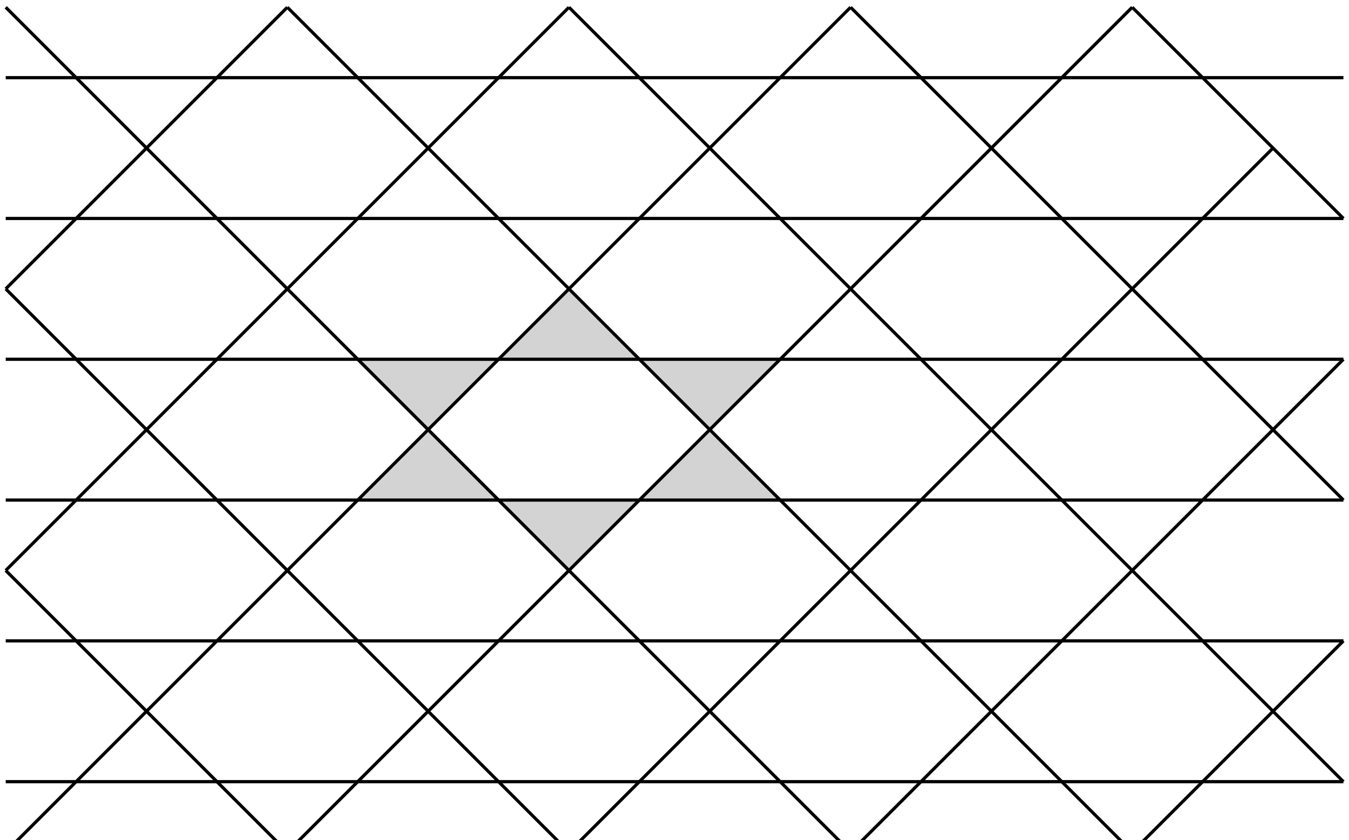
- $\frac{4}{3}\binom{n}{2}$ construction based on line arr. [Blanc '11]



Maximum Number of Triangles

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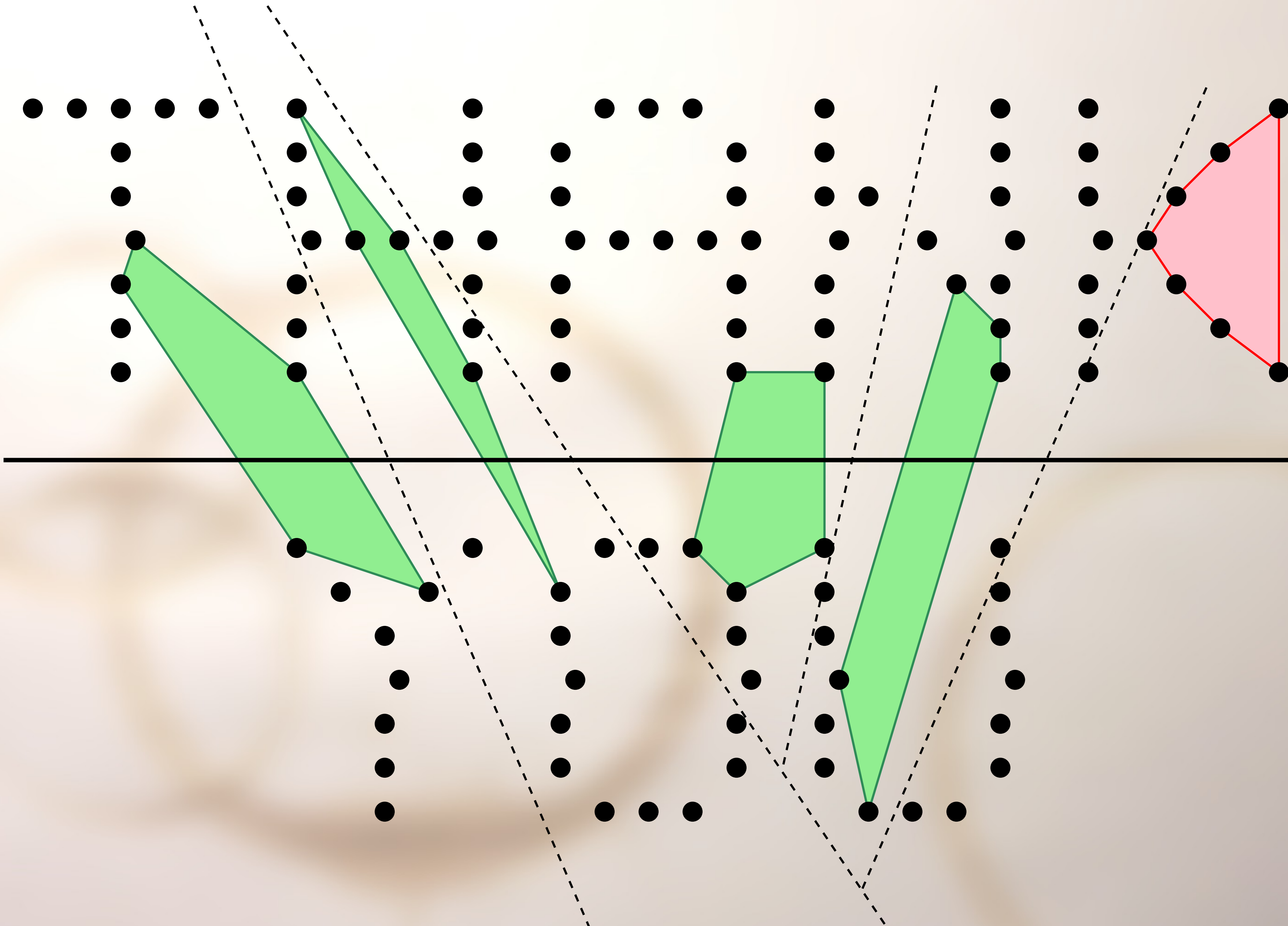
$\frac{4}{3}\binom{n}{2} + 1$

n	2	3	4	5	6	7	8	9	10
simple	0	8	8	13	20	29	≥ 37	≥ 48	≥ 60
+digon-free	-	8	8	12	20	29	≥ 37	≥ 48	≥ 60
$\lfloor \frac{4}{3}\binom{n}{2} \rfloor$	1	4	8	13	20	28	37	48	60

- Question: $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$?

THANK YOU


THANK
YOU