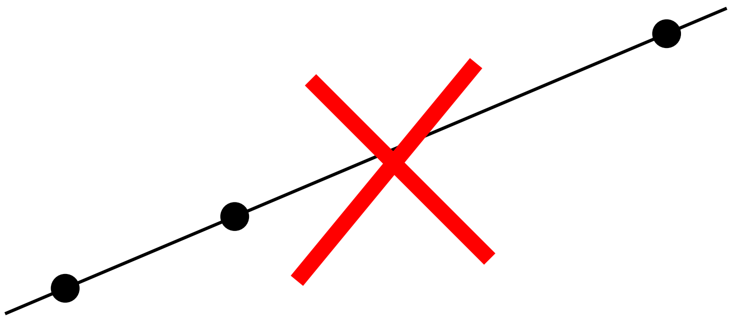
On Disjoint Holes in Point Sets

Manfred Scheucher

k-Gons

a finite point set P in the plane is
in *general position* if $\nexists$ collinear points in P



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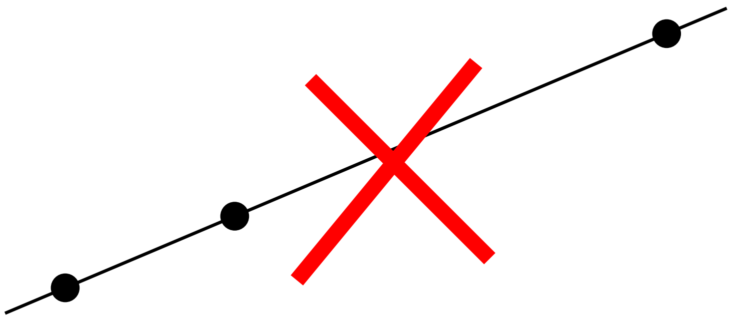

throughout this presentation, every set is in general position

k-Gons

a finite point set P in the plane is
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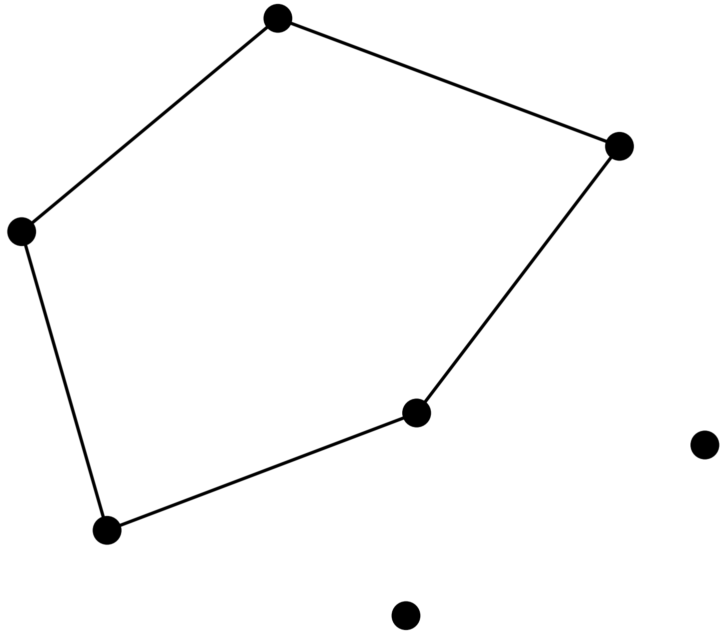
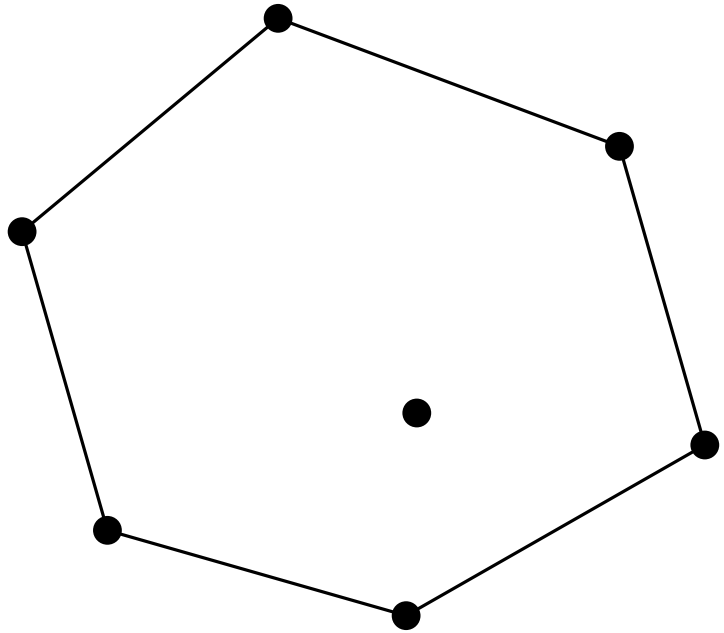
a *k-gon (in P)* is the vertex set of a convex k-gon



5-gon



6-gon

k-Gons

a finite point set P in the plane is
in *general position* if $\nexists$ collinear points in P

a *k-gon (in P)* is the vertex set of a convex k-gon

Theorem (Erdős and Szekeres '35).
$\forall\ k \geq 3$, $\exists$ a smallest integer $g(k)$ such that
every set of $g(k)$ points contains a k-gon.

k-Gons

Theorem. $2^{k-2}+1 \leq g(k) \leq \binom{2k-4}{k-2}$. [Erdős–Szekeres '35]

equality conjectured by Szekeres, Erdős offered 500$ for a proof

k-Gons

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$\vdots$ several improvements of order $4^{k-o(k)}$

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Known: $g(4) = 5$, $g(5) = 9$, $g(6) = 17$

computer assisted proof, 1500 CPU hours [Szekeres–Peters '06]

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NEW: 1 hour using SAT solvers

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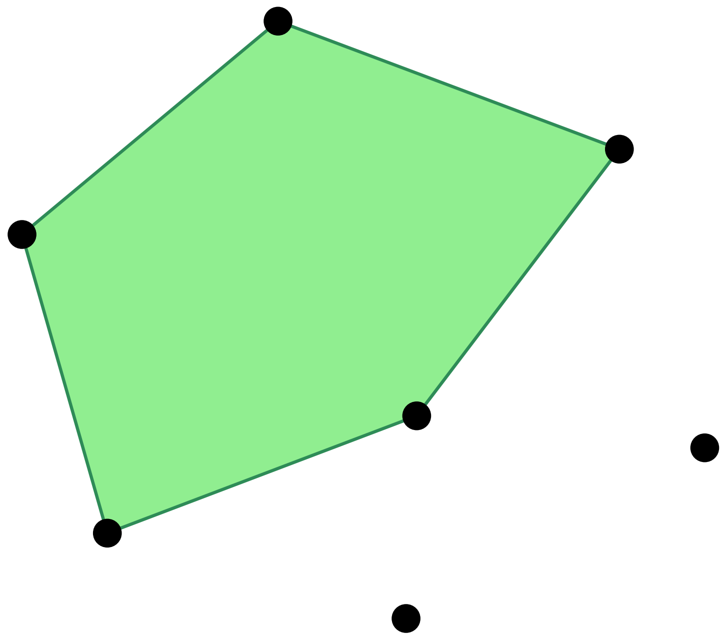
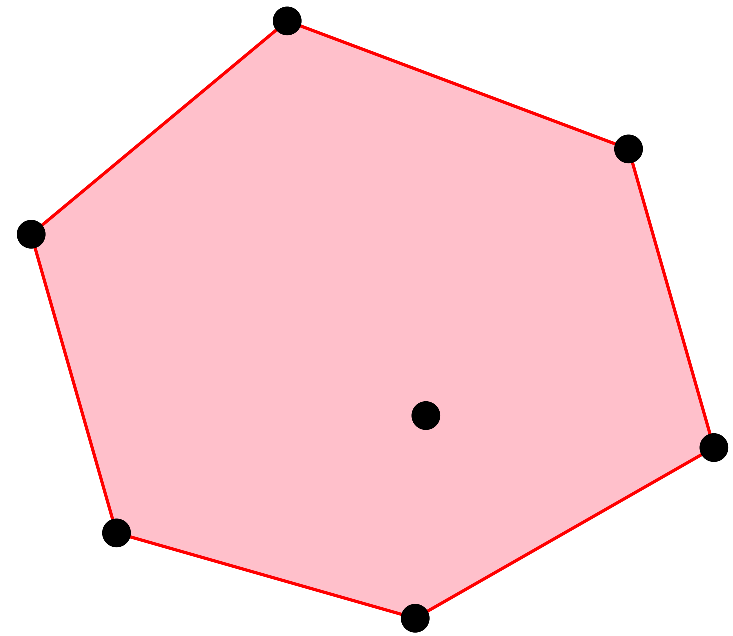
k-Holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P



5-hole



not a 6-hole

k-Holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P

$h(k)$ minimal s.t. any set of $h(k)$ points contains k-hole

Erdős, 1970': Is $h(k)$ finite for every k?

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- 3 points $\Rightarrow \exists$ 3-hole
- 5 points $\Rightarrow \exists$ 4-hole

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- 10 points $\Rightarrow \exists$ 5-hole [Harborth '78]

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- $\exists$ arbitrarily large point sets with no 7-hole [Horton '83]

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- $\exists$ arbitrarily large point sets with no 7-hole [Horton '83]
- Sufficiently large point sets $\Rightarrow \exists$ 6-hole
  [Gerken '08 and Nicolás '07, independently]

k-Holes

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq 463$, $h(7) = \infty$

Harborth '78

Horton'83

Overmars '02

Gerken '08, Nicolas '07, Koshelev '09

k-Holes

exact value still unknown

- $h(4) = 5$, $h(5) = 10$, $30 \le h(6) \le 463$, $h(7) = \infty$

Harborth '78

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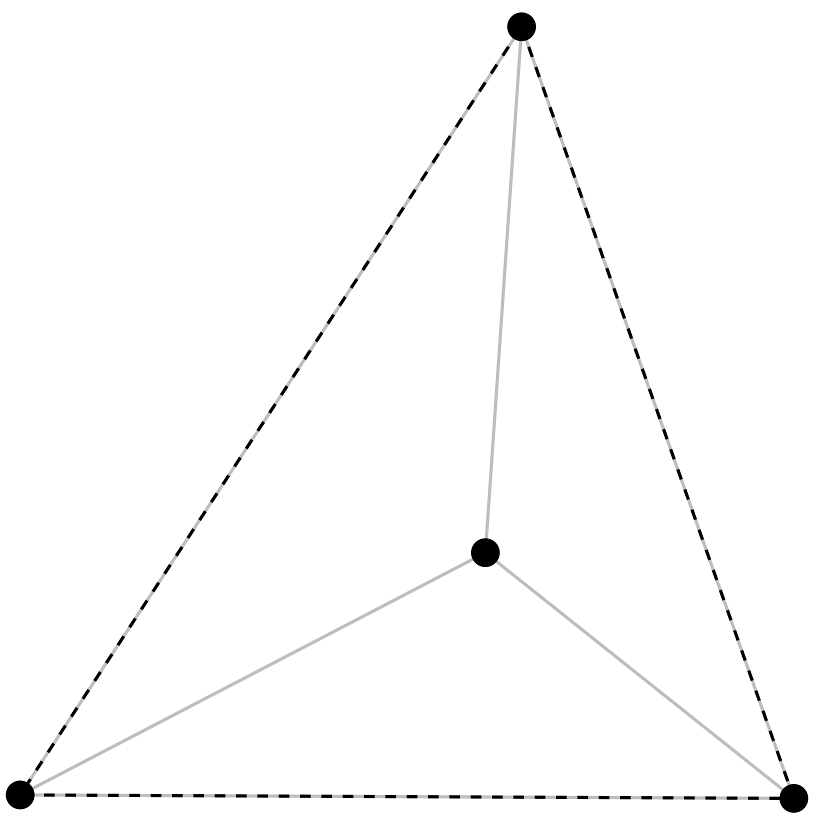
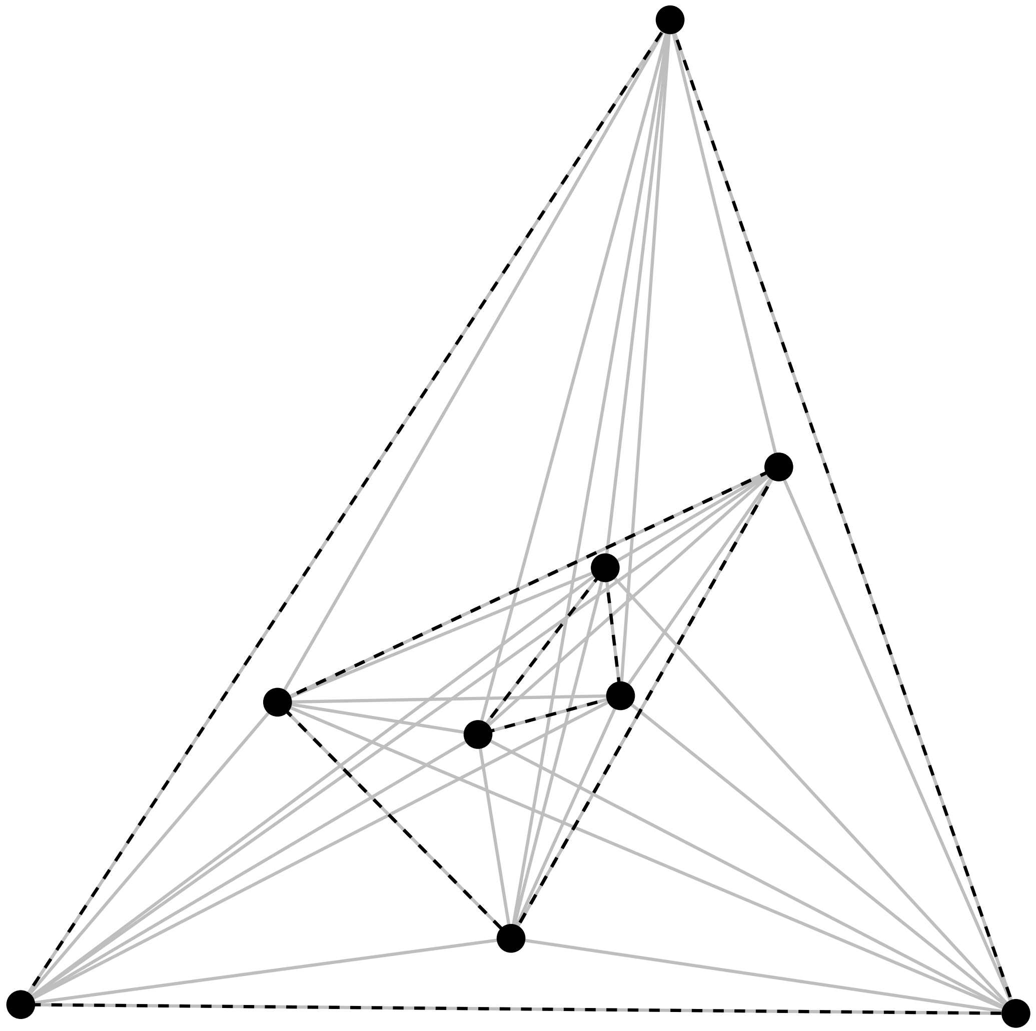
k-Holes



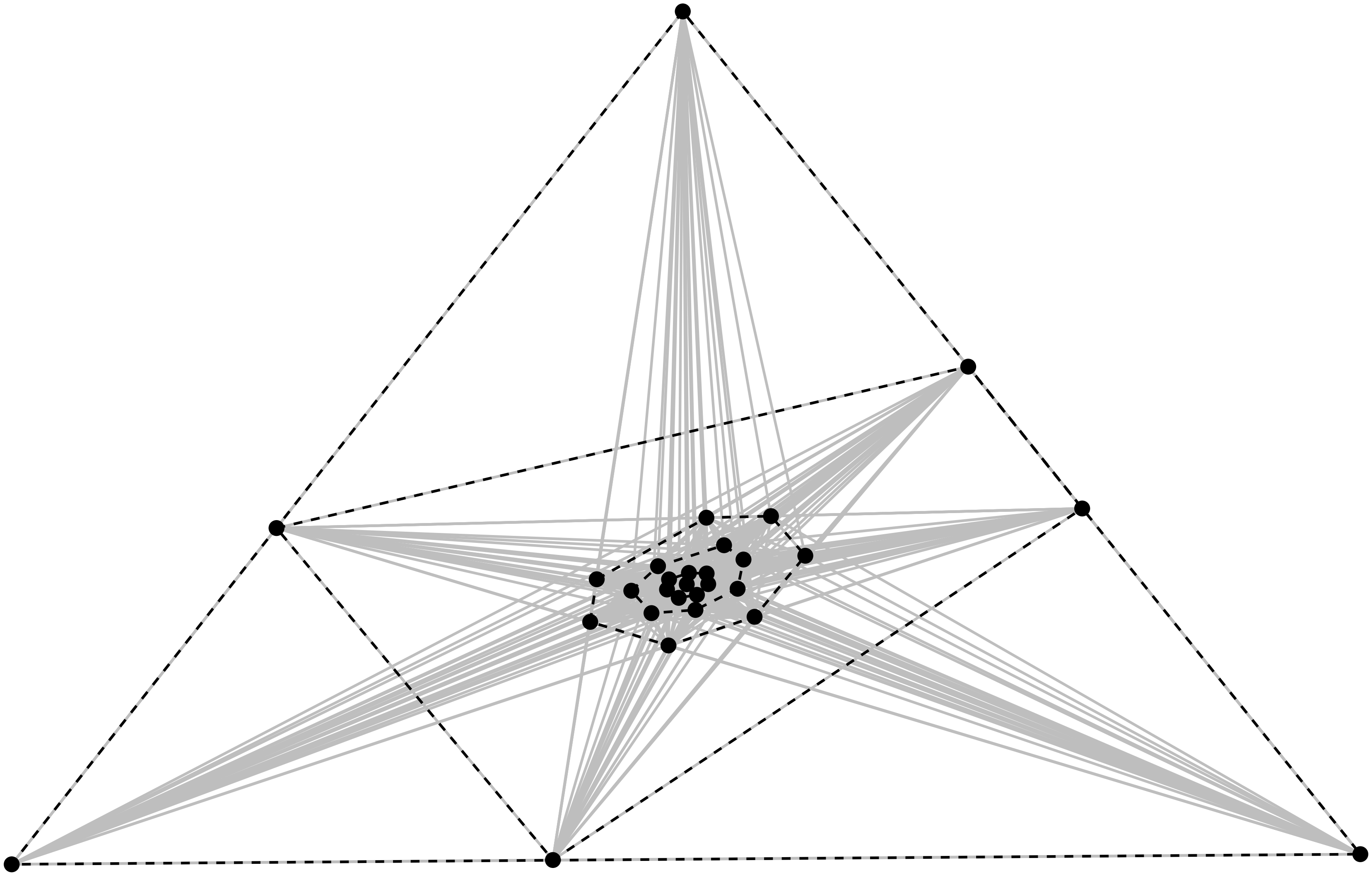
4 points, no 4-hole ($h(4) = 5$)

k-Holes



9 points, no 5-hole ($h(5) = 10$)

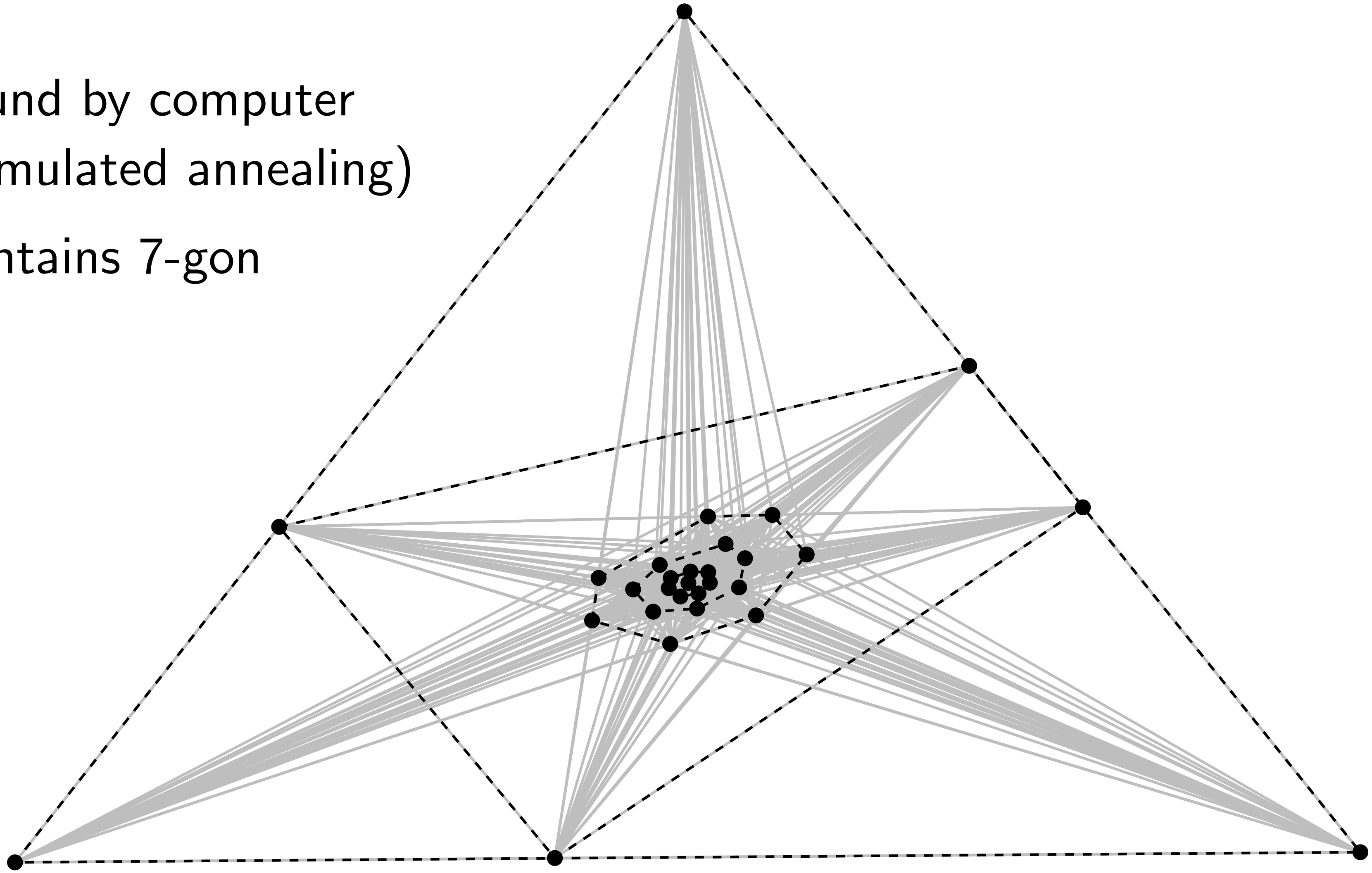
k-Holes



29 points, no 6-hole [Overmars '02] $(30 \leq h(6) \leq 463)$

k-Holes

- found by computer (simulated annealing)
- contains 7-gon

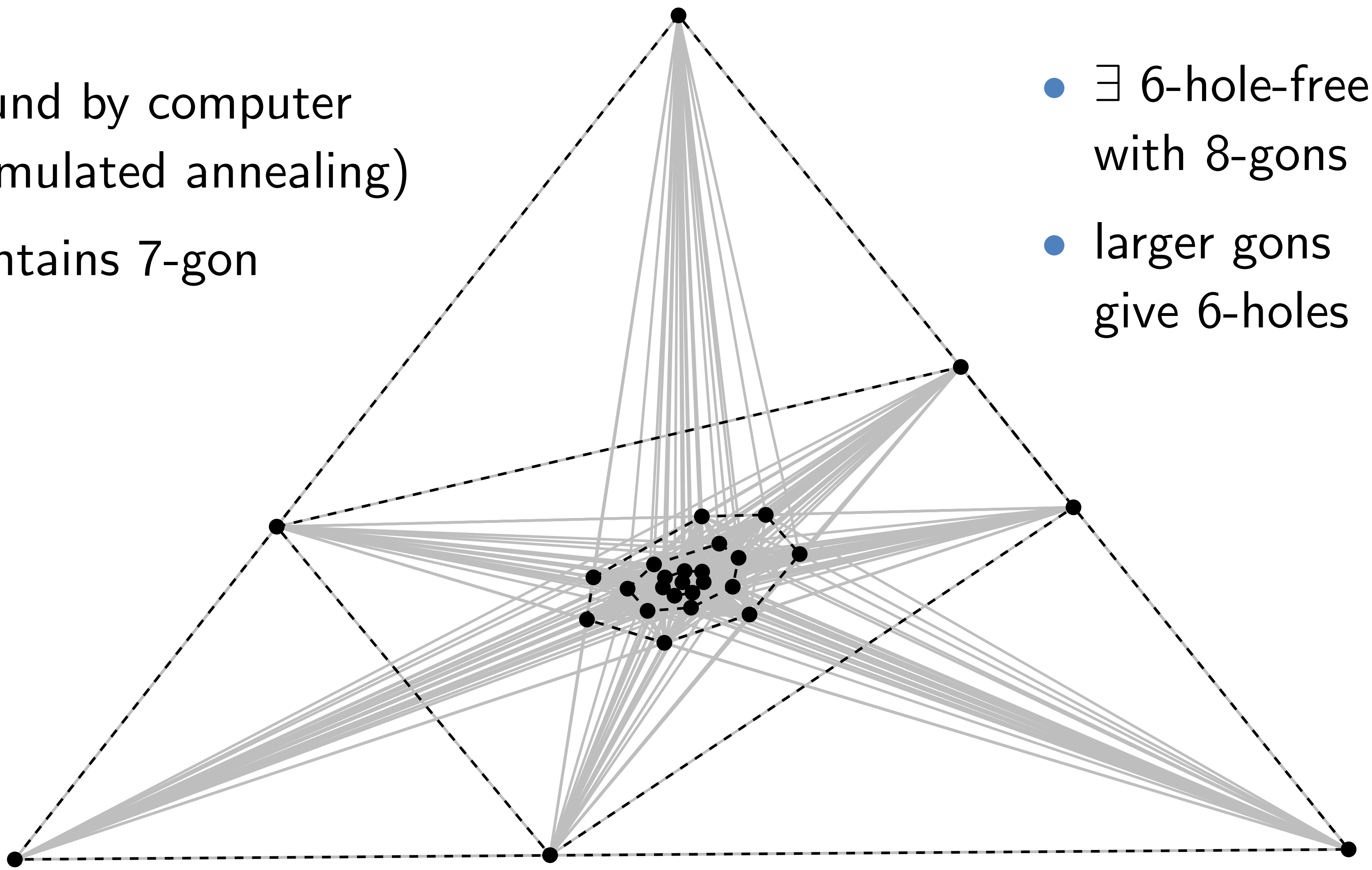


29 points, no 6-hole [Overmars '02] $(30 \leq h(6) \leq 463)$

k-Holes

- found by computer (simulated annealing)
- contains 7-gon

- $\exists$ 6-hole-free sets with 8-gons
- larger gons give 6-holes



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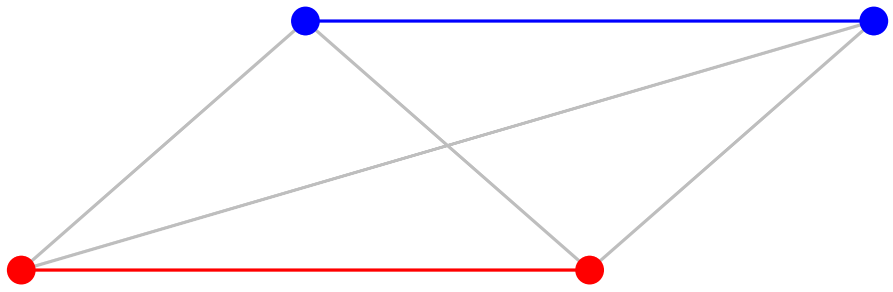
k-Holes

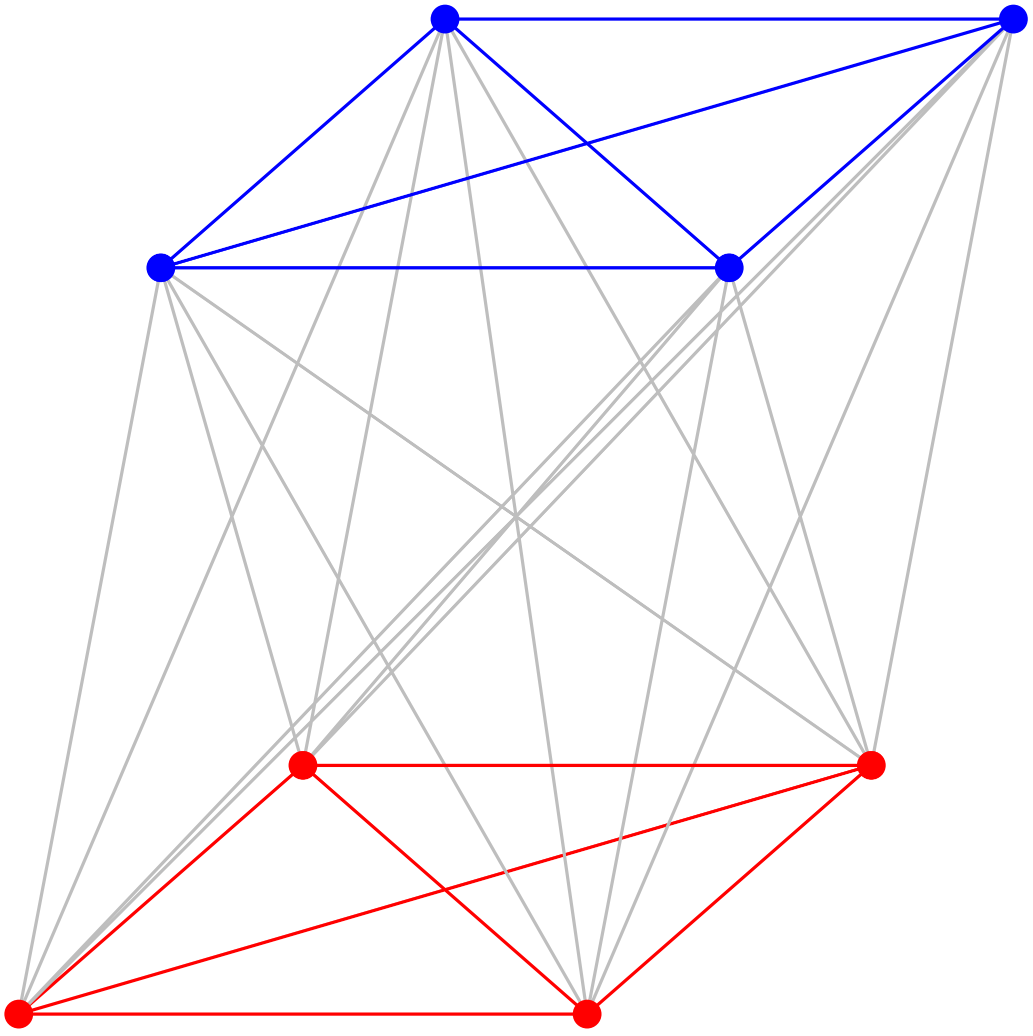
Horton's construction for $n = 2^1$ points, no 7-holes

k-Holes



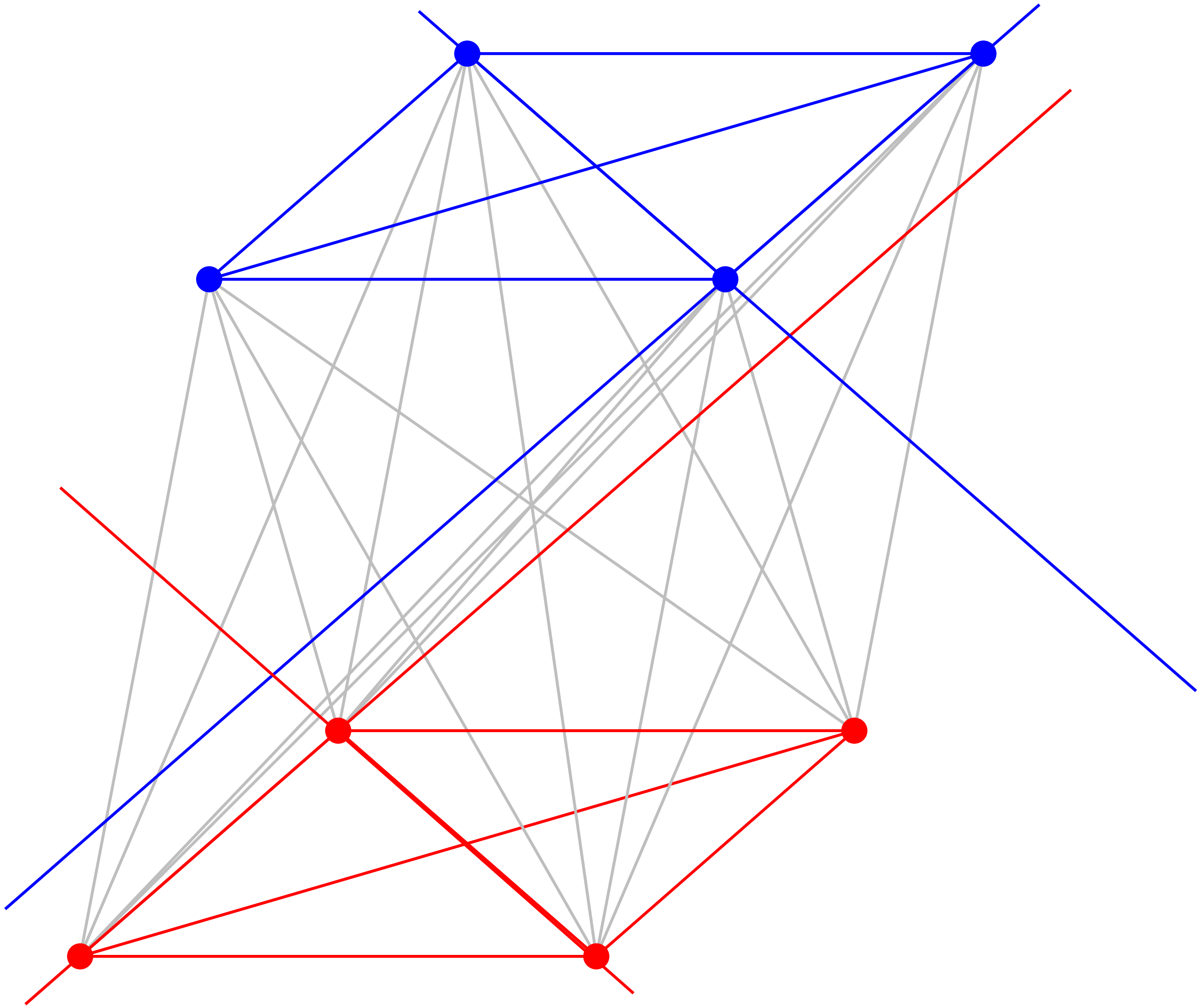
Horton's construction for $n = 2^2$ points, no 7-holes

k-Holes



Horton's construction for $n = 2^3$ points, no 7-holes

k-Holes



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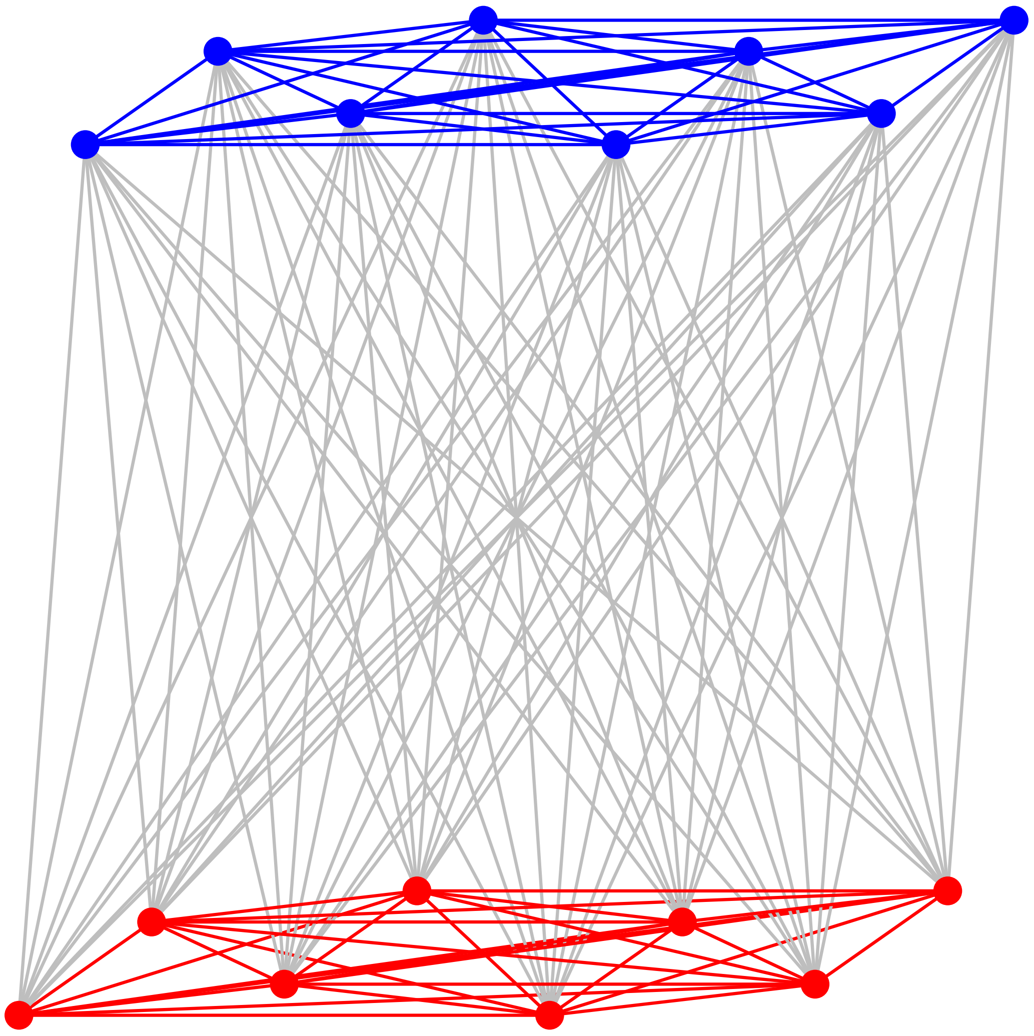
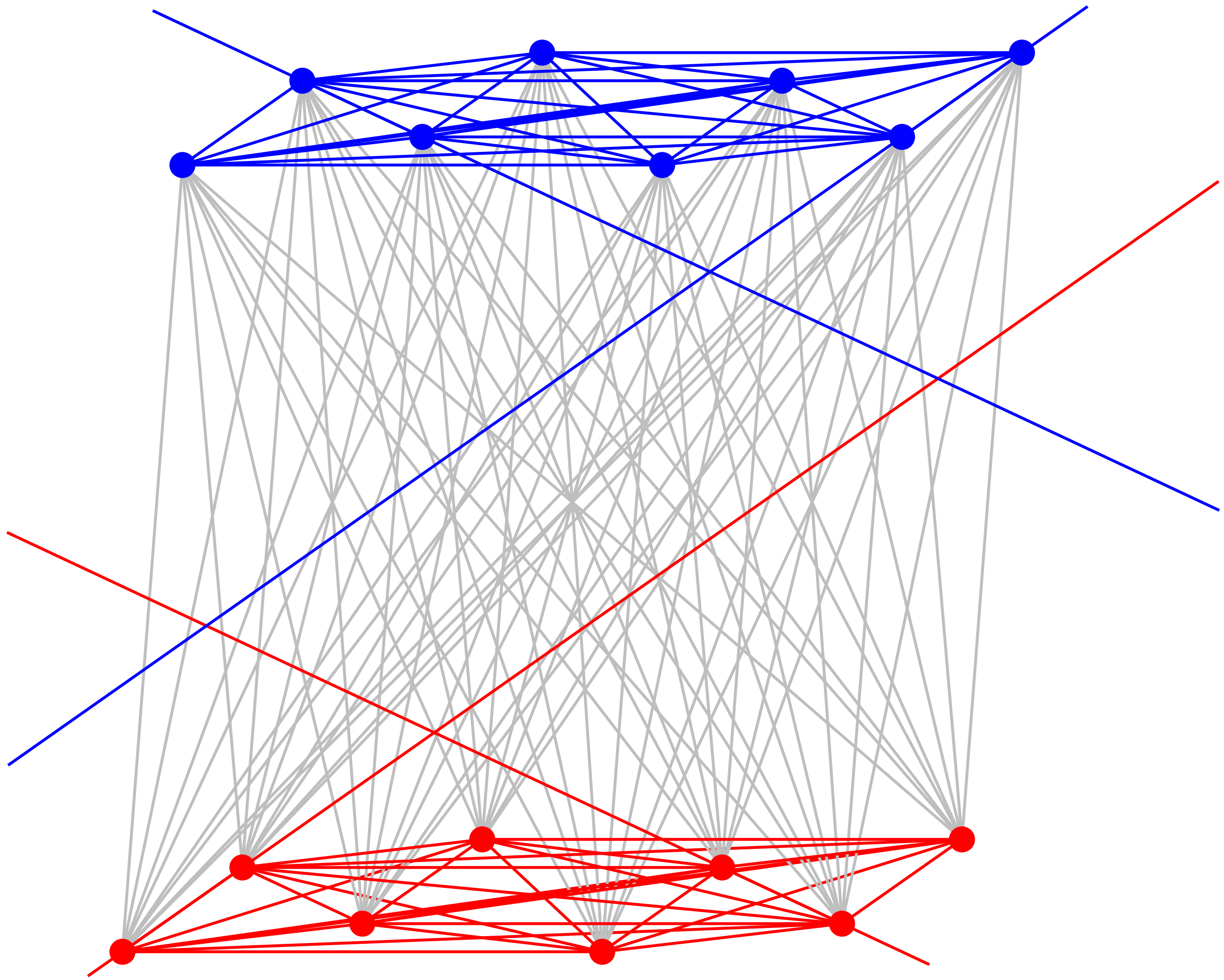
k-Holes



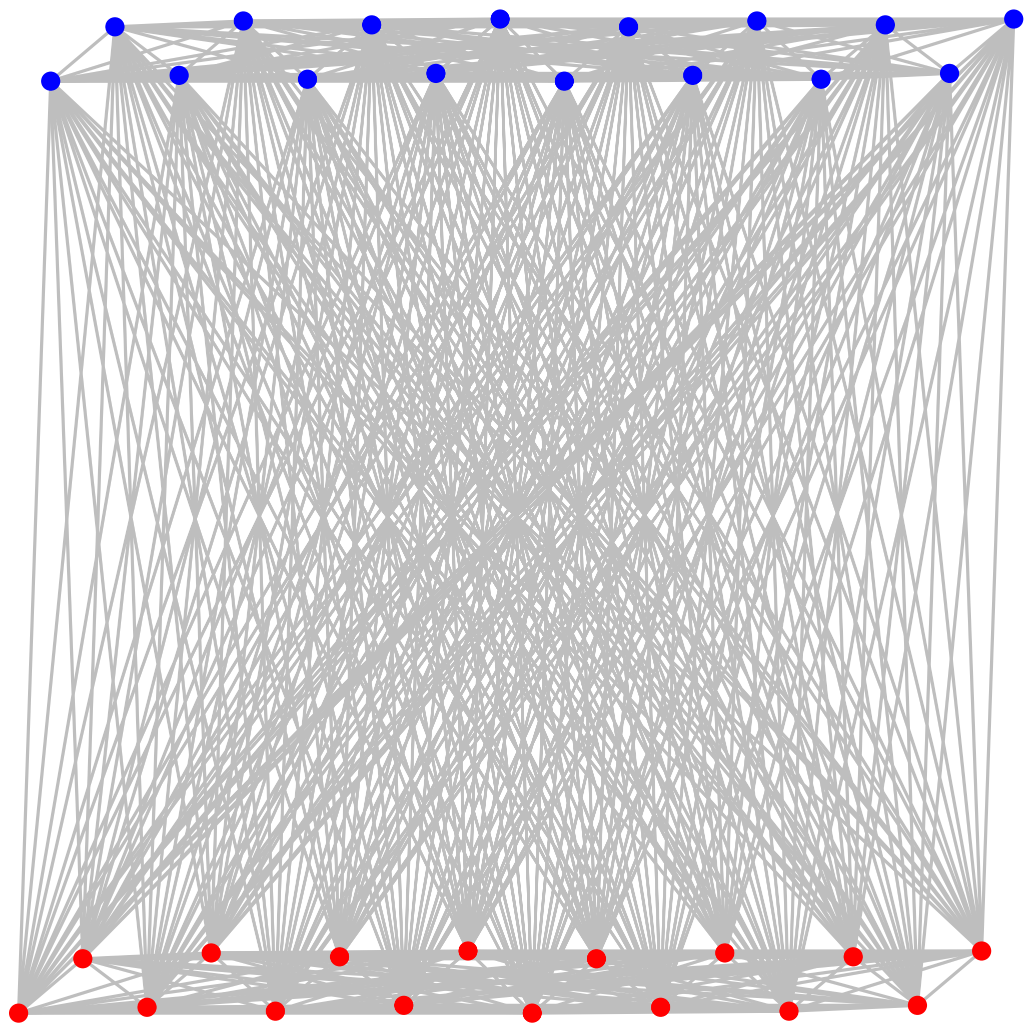
Horton's construction for $n = 2^4$ points, no 7-holes

k-Holes



Horton's construction for $n = 2^4$ points, no 7-holes

k-Holes



Horton's construction: $n = 2^k$ points, no 7-holes ($h(7) = \infty$)

Quantity of k-Holes

$h_k(n)$ = minimum # of k-holes among all sets of n points

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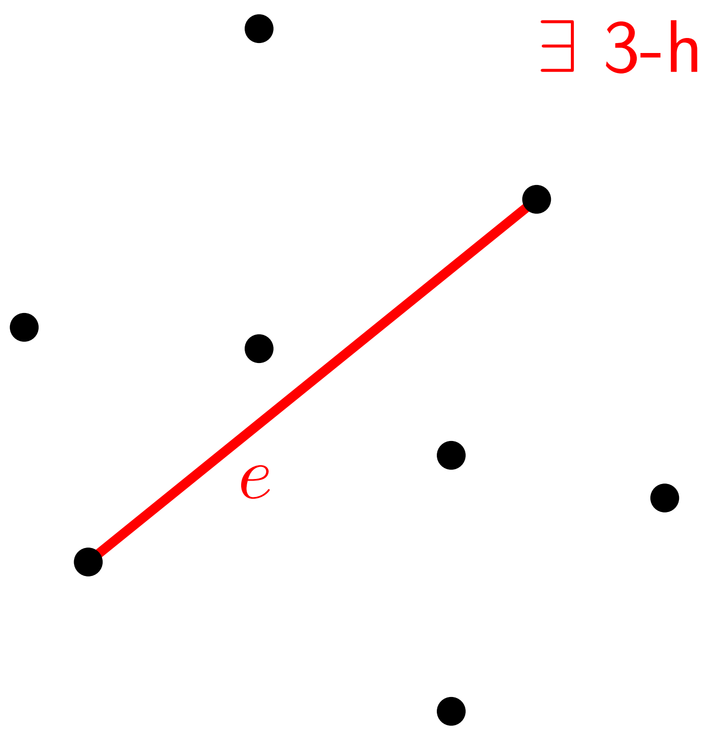
- $h_3(n) \geq \lfloor \frac{1}{3}\binom{n}{2} \rfloor = \Omega(n^2)$

$\forall$ edge e

$\exists$ 3-hole with closest point



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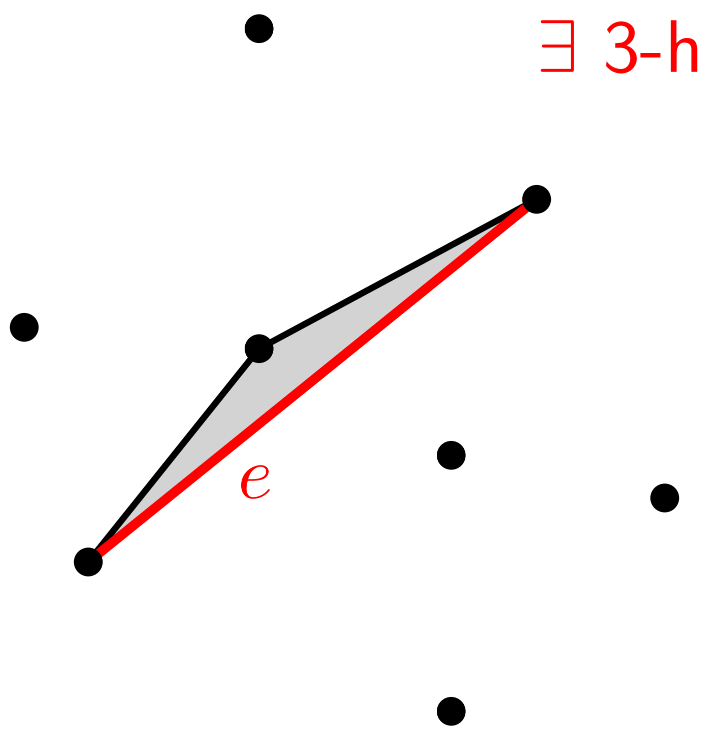

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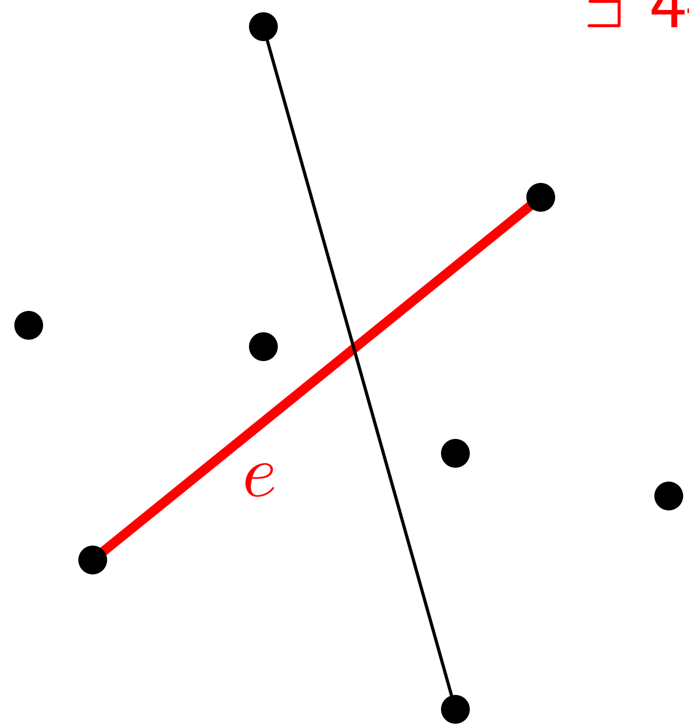

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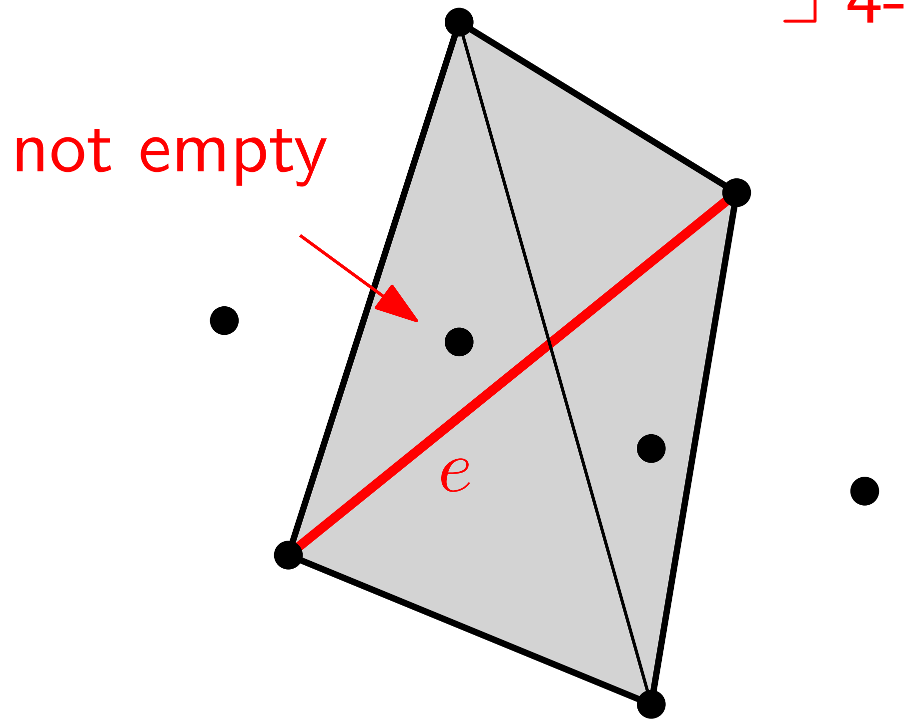

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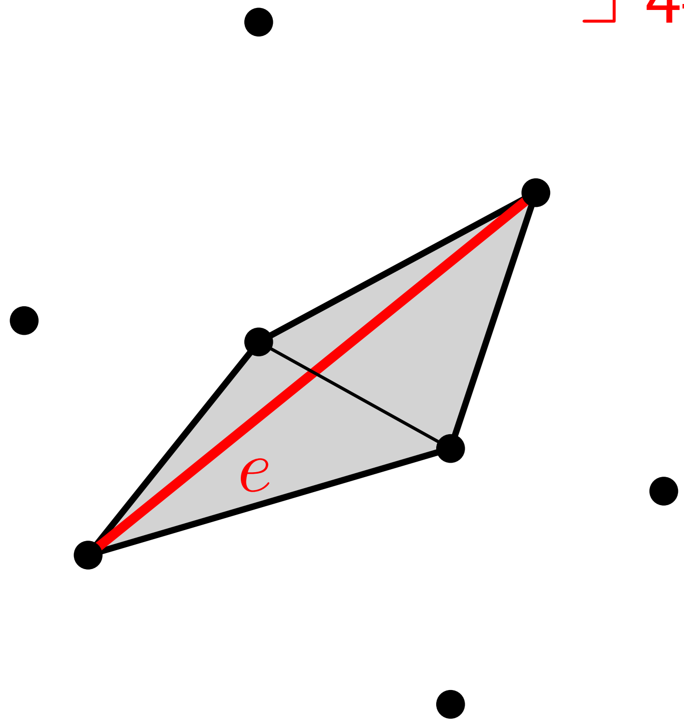

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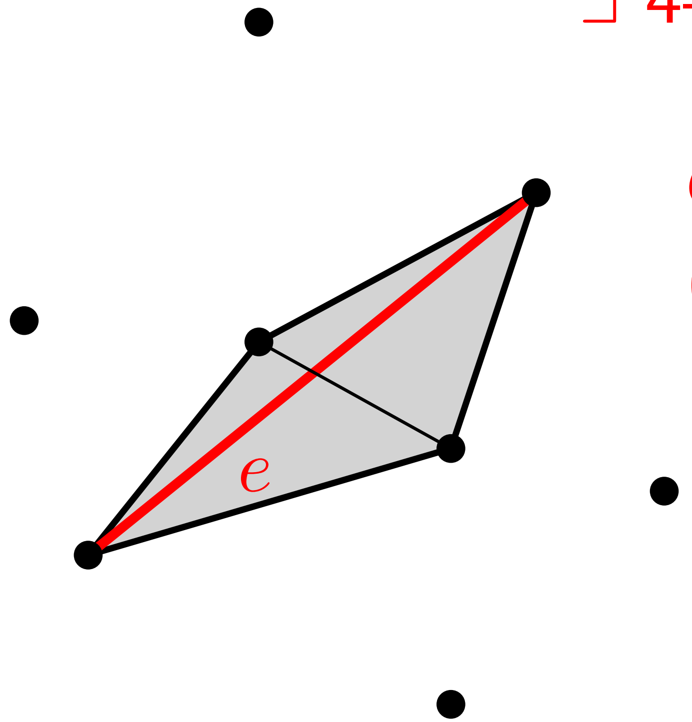


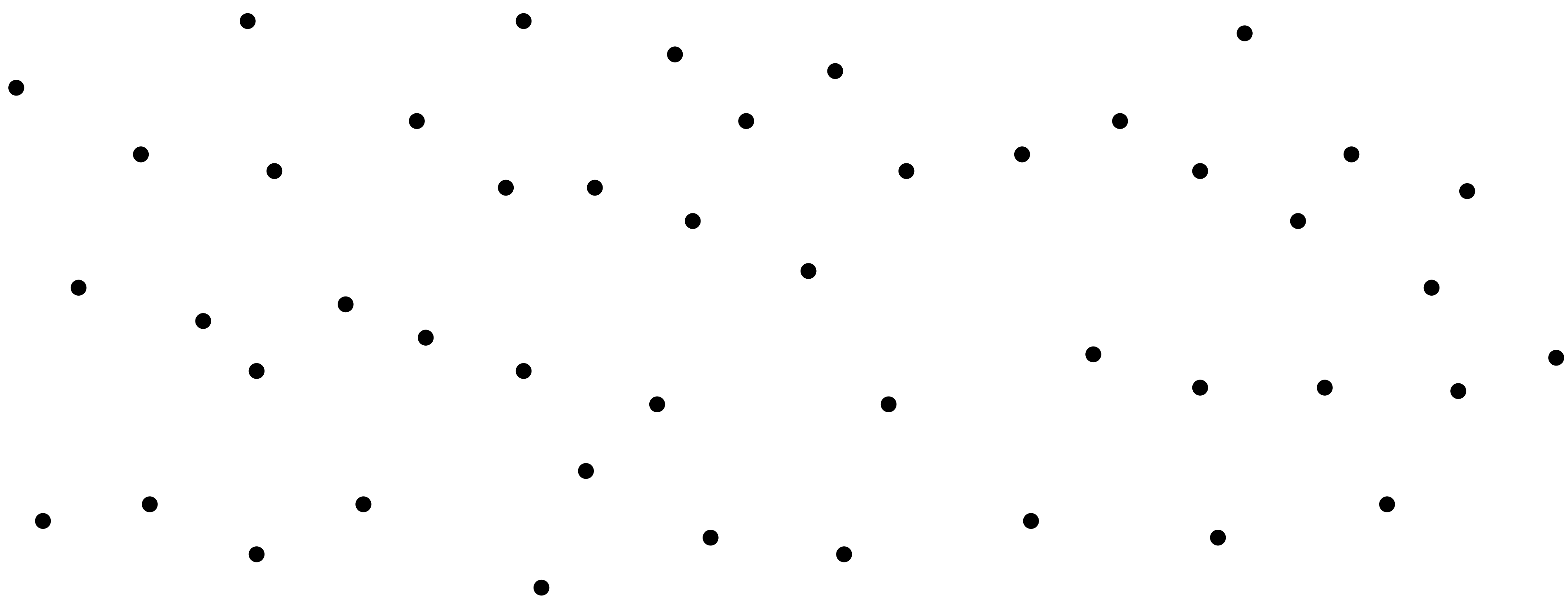
$\forall$ crossed edge e
$\exists$ 4-hole with diagonal e

$O(n)$ uncrossed edges
(planar graph)

Quantity of k-Holes

$h_k(n)$ = minimum # of k-holes among all sets of n points

- $h_5(n) \geq \lfloor \frac{1}{10} n \rfloor = \Omega(n)$



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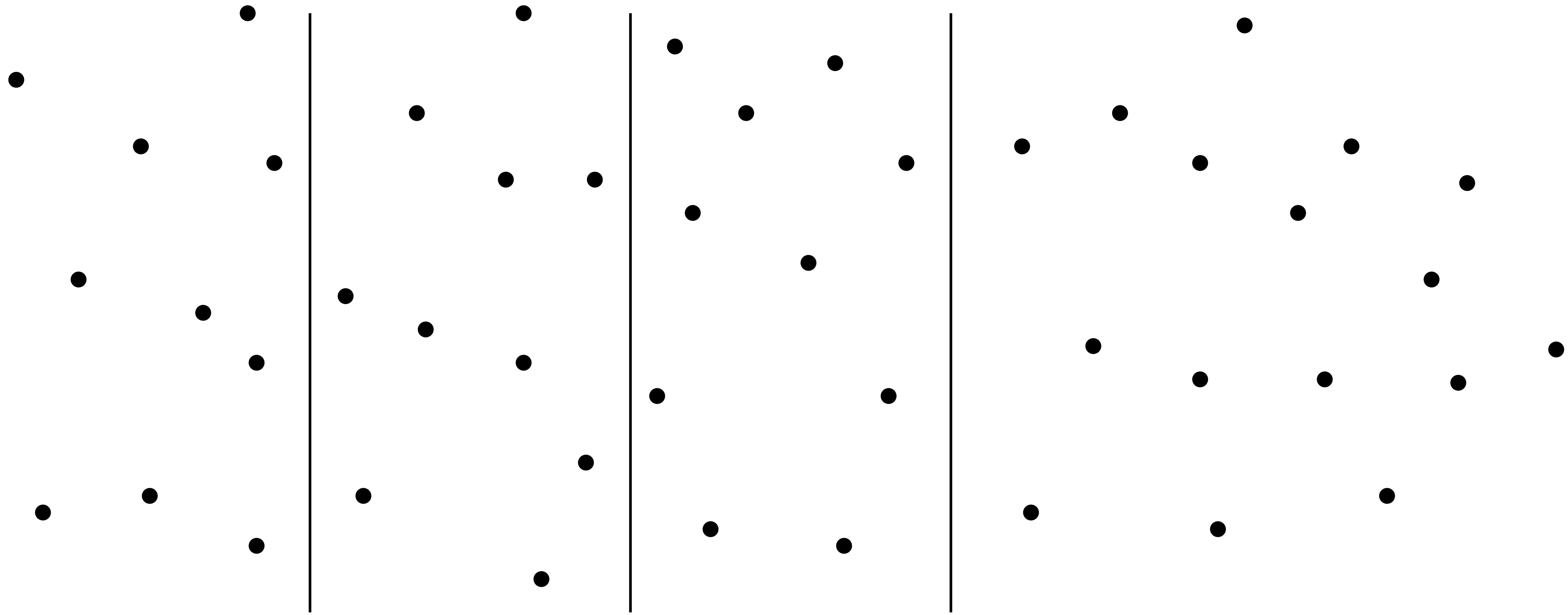

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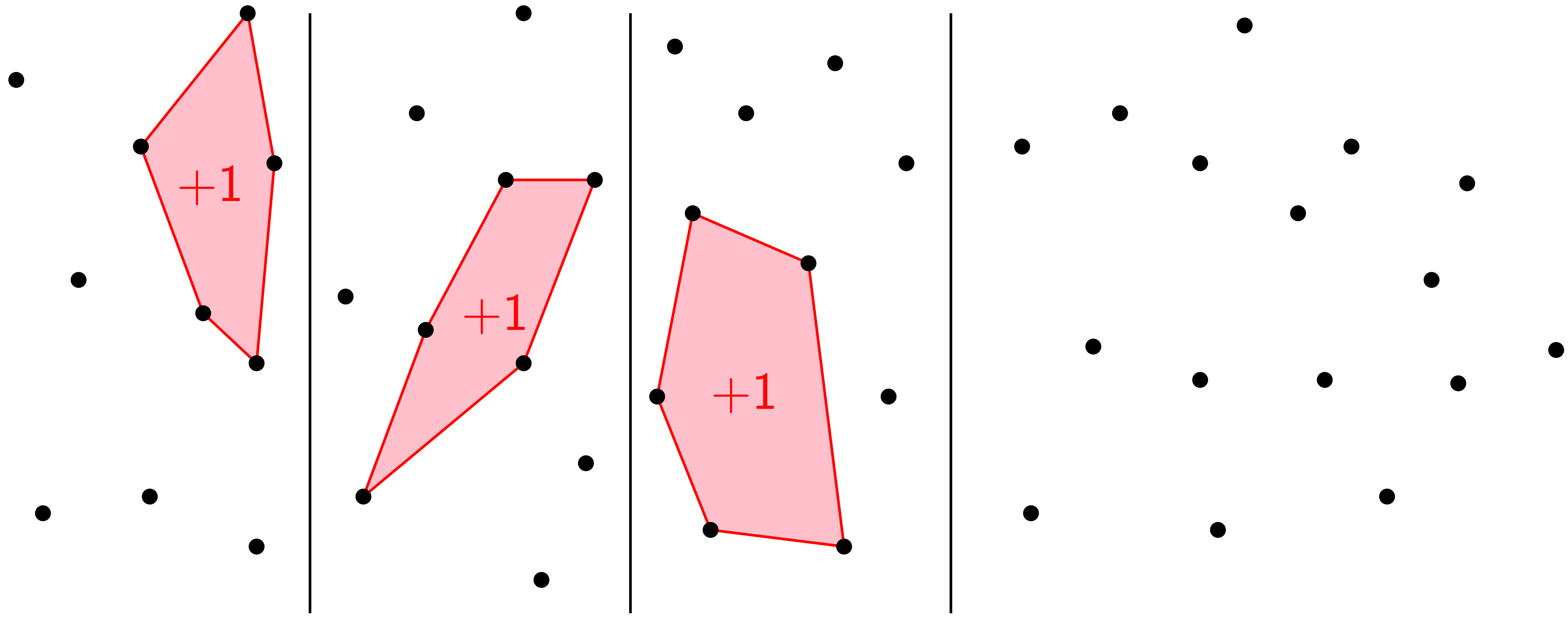

Quantity of k-Holes

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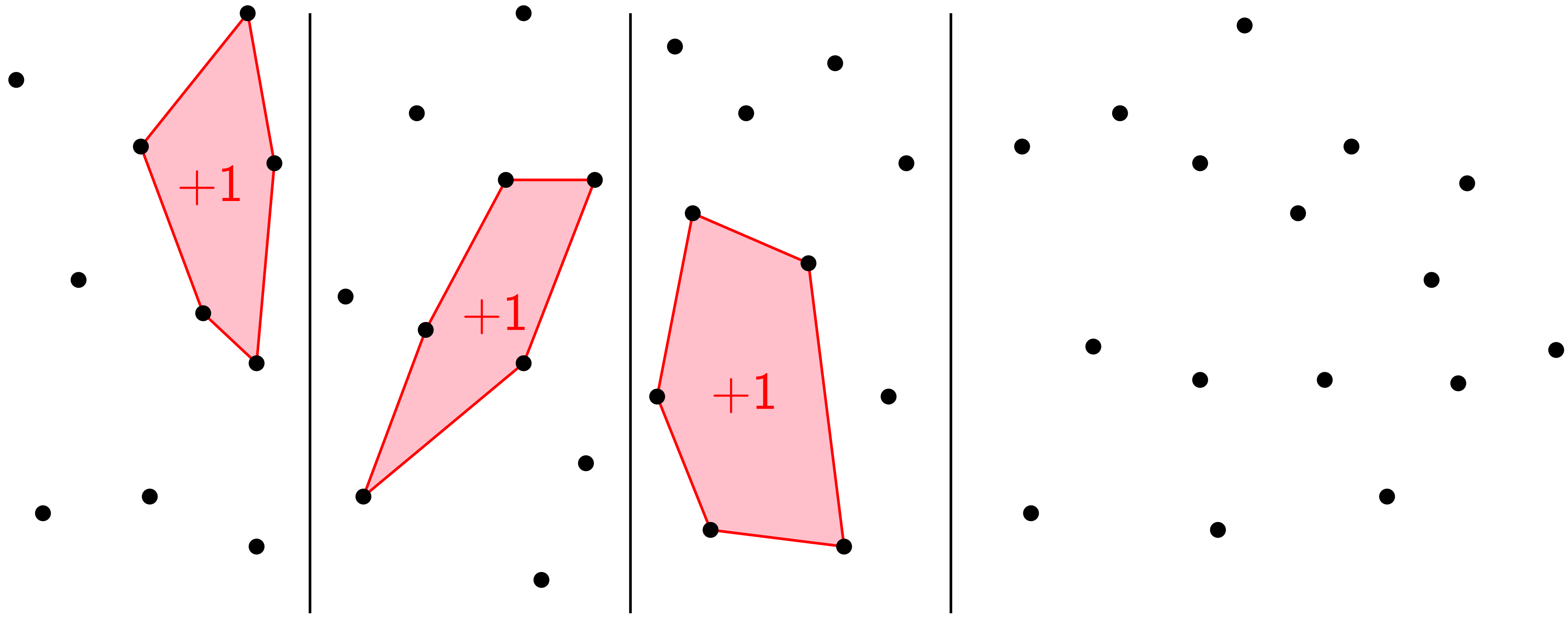
- $h_5(n) \geq \lfloor \frac{1}{10} n \rfloor = \Omega(n)$
- same idea: $h_6(n) \geq \Omega(n)$



Quantity of k-Holes

- $h_3(n)$ and $h_4(n)$ quadratic

Bárány and Füredi '87, Bárány and Valtr '04

Quantity of k-Holes

- $h_3(n)$ and $h_4(n)$ quadratic
- $h_5(n) = \Omega(n \log^{4/5} n)$ and $h_6(n) = \Omega(n)$

Gerken '08, Nicolás '07

Aichholzer, Balko, Hackl, Kynl, Parada, S., Valtr, and Vogtenhuber '17

Quantity of k-Holes

- $h_3(n)$ and $h_4(n)$ quadratic
- $h_5(n) = \Omega(n \log^{4/5} n)$ and $h_6(n) = \Omega(n)$
- $h_k(n) = 0$ for $k \geq 7$

Horton '83

Quantity of k-Holes

- $h_3(n)$ and $h_4(n)$ quadratic
- $h_5(n) = \Omega(n \log^{4/5} n)$ and $h_6(n) = \Omega(n)$
- $h_k(n) = 0$ for $k \geq 7$
- $h_k(n)$ determined for small values of n

n	9	10	11	12	13	14	15	16	17	18	19
$h_5(n)$	0	1	2	3	3	6	9	11	≤ 16	≤ 21	≤ 26

Harborth '78

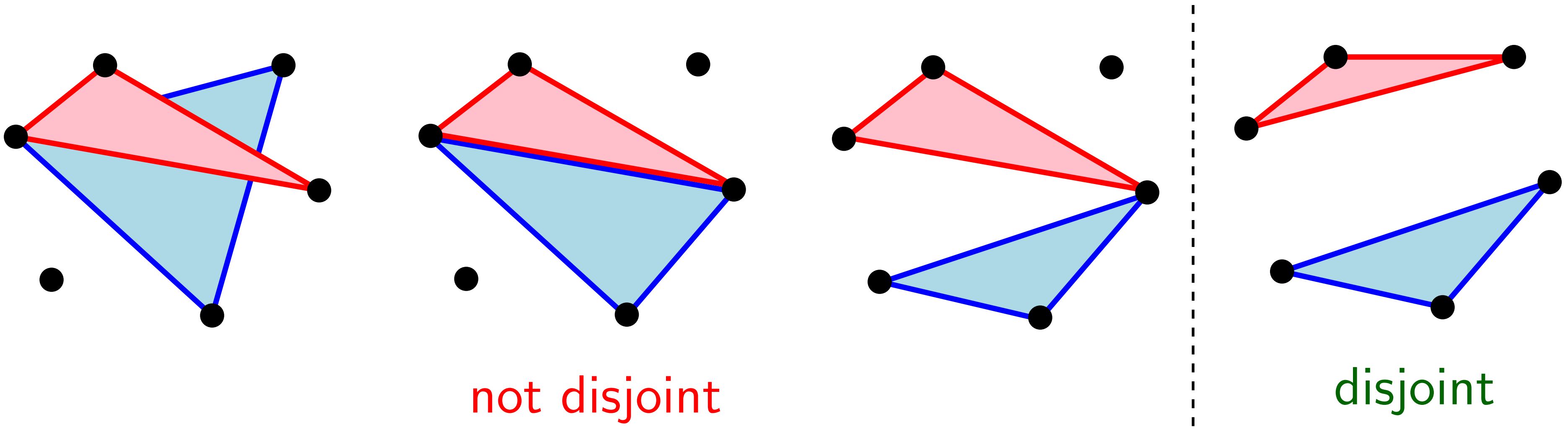
Dehnhart '87

Bachelor's thesis (S'13)

via SAT

Disjoint k-Holes

Hosono and Urabe '01:
What is the smallest number $h(k_1, k_2)$ such that every set of $h(k_1, k_2)$ points determines a k_1-hole and a k_2-hole, that are <u>disjoint</u>?



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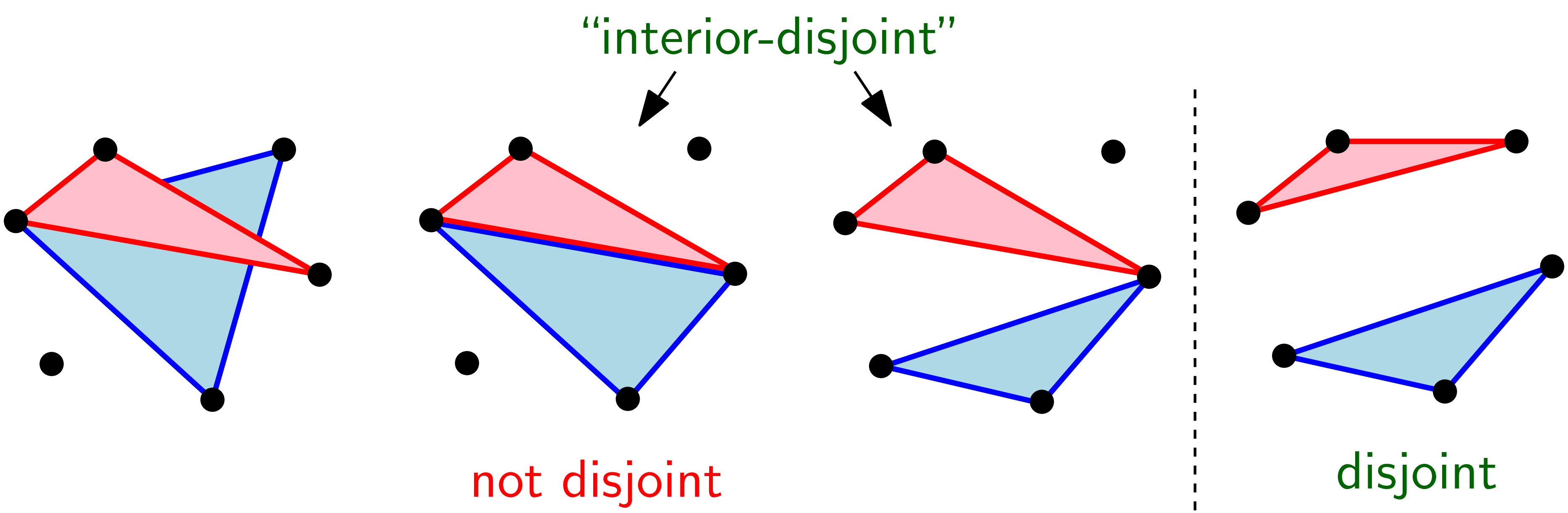

Disjoint k-Holes

Hosono–Urabe ('01, '05, '08)

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17..20

Minimum number $h(k_1, k_2)$ of points
such that disjoint k_1- and k_2-holes exist

Disjoint k-Holes

Hosono–Urabe ('01, '05, '08)

Bhattacharya–Das '11

	2	3	4	5
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Disjoint k-Holes

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Bhattacharya–Das '11

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17*

NEW

Minimum number $h(k_1, k_2)$ of points such that disjoint k_1- and k_2-holes exist

Theorem: $h(5,5) = 17.$

Disjoint k-Holes

Hosono–Urabe ('01, '05, '08)

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17*

NEW

2-parametric

3-parametric

You–Wei '15

	2	3	4
2	8	9	11
3		10	12
4			14

$h(k_1, k_2, 4)$

	2	3	4	5
2	10	11	11..14	17*
3		12	13..14	17..19*
4			15..17	17..23*
5				22*..27*

NEW

$h(k_1, k_2, 5)$

Disjoint k-Holes

Bárány–Károlyi '01 and Hosono–Urabe ('01, '08)

$F_k(n)$ ... min. # of disjoint k-holes in n points

$$
\begin{aligned}
F_k(n) &= \lfloor n/k \rfloor && \text{for } k = 1, 2, 3 \\
3n/13 + o(n) \leq F_4(n) &< \quad n/4 \\
\lfloor 2n/17 \rfloor \leq F_5(n) &< \quad n/6 \\
\lfloor n/h(6) \rfloor \leq F_6(n) &< \; n/12 \\
F_k(n) &= \qquad 0 && \text{for } k \geq 7.
\end{aligned}
$$

NEW → $\lfloor 2n/17 \rfloor$

Disjoint k-Holes

Variant: also *interior-disjoint* holes have been studied

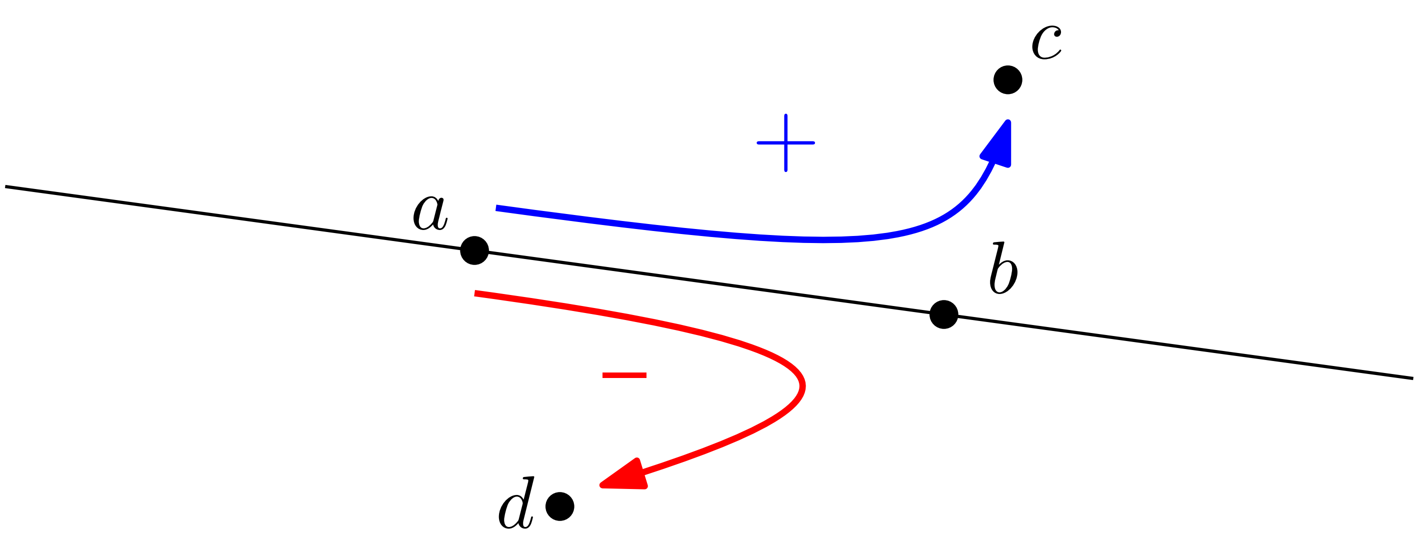
Devillers et al. 2003
Sakai–Urrutia 2007
Cano et al. 2015
Biniaz–Maheshwari–Smid 2017
Hosono–Urabe 2018
. . .

	3	4	5
3	4	5	10
4		7	10
5			15*

NEW

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+, -\}$



$$\chi_{abc} = \operatorname{sgn}\det\begin{pmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{pmatrix}$$

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+,-\}$
- axiomatize "point set": *chiritope/signotope* axioms

Grassmann-Plücker relations for r-dim. vectors (we have $r=3$):

$$\det(x_1,\dots,x_r)\cdot\det(y_1,\dots,y_r) =$$
$$\sum_{i=1}^{r}\det(y_i,x_2,\dots,x_r)\cdot\det(y_1,\dots,y_{i-1},x_1,y_{i+1},\dots,y_r)$$

SAT Model

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  Grassmann-Plücker relations:

  $$\det(x_1,\ldots,x_r)\cdot\det(y_1,\ldots,y_r) = \\ \sum_{i=1}^{r}\det(y_i,x_2,\ldots,x_r)\cdot\det(y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_r)$$

  - exchange axioms:
    if $\chi_{y_i,x_2,\ldots,x_r}\cdot\chi_{y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_r} \geq 0$ for every i,
    then $\chi_{x_1,\ldots,x_r}\cdot\chi_{y_1,\ldots,y_r} \geq 0$

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+,-\}$
- axiomatize "point set": *chiritope/signotope* axioms
  - alternating axioms: $\Theta(n^3)$ many
    $\chi_{x_{\pi(1)},x_{\pi(2)},x_{\pi(3)}} = \operatorname{sgn}(\pi) \cdot \chi_{x_1,x_2,x_3}$
  - exchange axioms: $\Theta(n^6)$ many
    if $\chi_{y_i,x_2,\ldots,x_r} \cdot \chi_{y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_r} \geq 0$ for every i,
    then $\chi_{x_1,\ldots,x_r} \cdot \chi_{y_1,\ldots,y_r} \geq 0$

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+, -\}$

- axiomatize "point set": *chiritope/signotope* axioms

  - alternating axioms: $\Theta(n^3)$ many

    $\chi_{x_{\pi(1)},x_{\pi(2)},x_{\pi(3)}} = \mathrm{sgn}(\pi) \cdot \chi_{x_1,x_2,x_3}$

necessary conditions but not sufficient (stretchability!)

  - exchange axioms:

    if $\chi_{y_i,x_2,\ldots,x_r} \cdot \chi_{y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_r} \geq 0$ for every i,
    then $\chi_{x_1,\ldots,x_r} \cdot \chi_{y_1,\ldots,y_r} \geq 0$

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+,-\}$
- axiomatize "point set": *chiritope/signotope* axioms
  - alternating axioms: $\Theta(n^3)$ many
    $\chi_{x_{\pi(1)},x_{\pi(2)},x_{\pi(3)}} = \operatorname{sgn}(\pi) \cdot \chi_{x_1,x_2,x_3}$

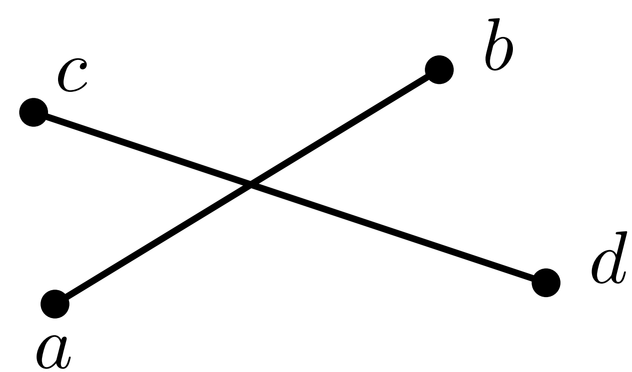
Felsner–Weil '01, Balko–Fulek–Kynčl '15:

  - signotope axioms: for $i<j<k<l$, $\Theta(n^4)$ many
    the sequence $\chi_{ijk}, \chi_{ijl}, \chi_{ikl}, \chi_{jkl}$ (lex. order) changes sign at most once

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+,-\}$
- axiomatize "point set": *chiritope/signotope* axioms
- *crossings* (two crossing edges = 4-gon),
  otherwise *containment* (point-in-triangle)

$\chi_{abc} \neq \chi_{abd}$ and $\chi_{cda} \neq \chi_{cdb}$



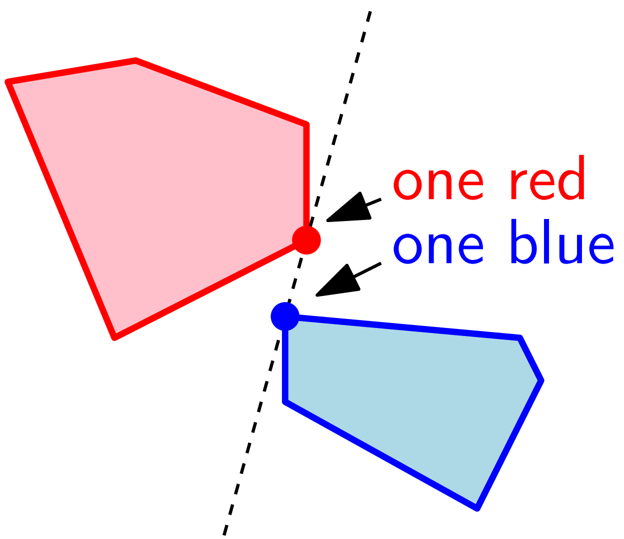
SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+, -\}$
- axiomatize "point set": *chiritope/signotope* axioms
- *crossings* (two crossing edges $=$ 4-gon),
  otherwise *containment* (point-in-triangle)
- k*-gons* and k*-holes*

  (Carathéodory: every 4-tuple in k-gon is in convex positon)

SAT Model

- variables for *triple-orientations*: $\chi_{abc} \in \{+,-\}$
- axiomatize "point set": *chiritope/signotope* axioms
- *crossings* (two crossing edges = 4-gon), otherwise *containment* (point-in-triangle)
- *k-gons* and *k-holes*
- *disjointness* also via triple-orientations



Crucial Elements

- **Lemma:** Let $S = \{s_1, \ldots, s_n\}$, s_1 extremal, and $s_2, \ldots, s_n$ sorted around s_1.

  There exists $\tilde{S}$ of the same order type as S
  (in particular, sorted around first point)
  with increasing x-coordinates.

  w.l.o.g.

- Harborth's result: Any 10 consecutive points give 5-hole

SAT Model Modifications

- **Interior-disjoint Holes**

- **Classical Erdős–Szekeres:**
  $g(6) = 17$ in about 1 hour

- **Counting 5-holes:**
  variables $X_{abcde;k}$ indicates whether $a < \ldots < e$ form the k-th 5-hole in lexicographic order

(Un)Satisfiablity and SAT-Solvers

- Given Boolean formula, is there an assignment such that the formula is true?
- NP-complete, but quite good heuristics
- we used the SAT-solvers *glucose* and *picosat*
- Satisfiability efficiently verifiable (check solution)
- UNSAT certificates (e.g. DRAT, tool *DRAT-trim*)

A Python/Pycosat Example

```
$ ipython
Python 2.7.15
...
In [1]: import pycosat

In [2]: CNF = [[1,2,3],[-1,-2,-3]]

In [3]: for sol in pycosat.itersolve(CNF): print sol
[-1, -2, 3]
[-1, 2, -3]
[-1, 2, 3]
[1, 2, -3]
[1, -2, 3]
[1, -2, -3]
```

$(x_1 \vee x_2 \vee x_3)$

$(\neg x_1 \vee \neg x_2 \vee \neg x_3)$

THANK

YOU