On 4-Crossing-Families in Point Sets and an Asymptotic Upper Bound

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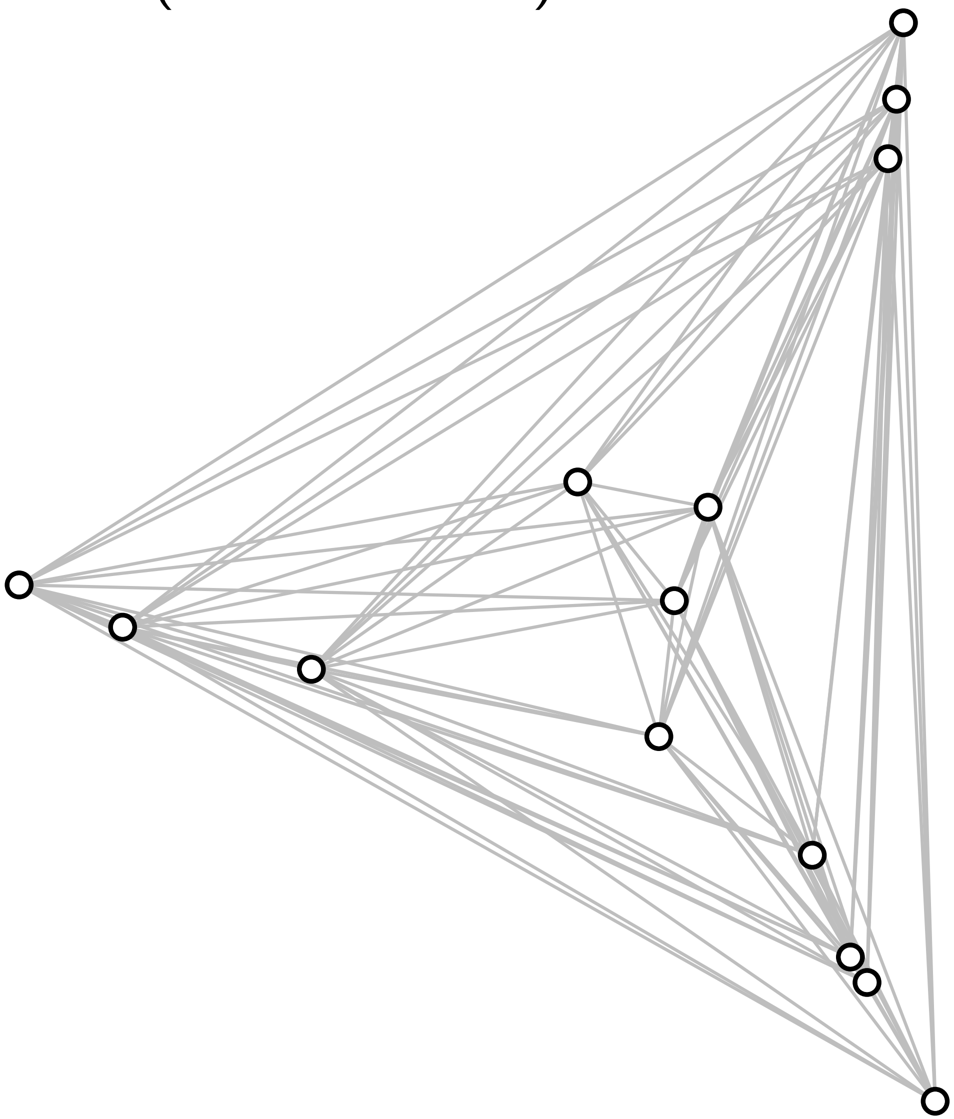
Crossing Families

- S set of n points in general position (no 3 collinear)



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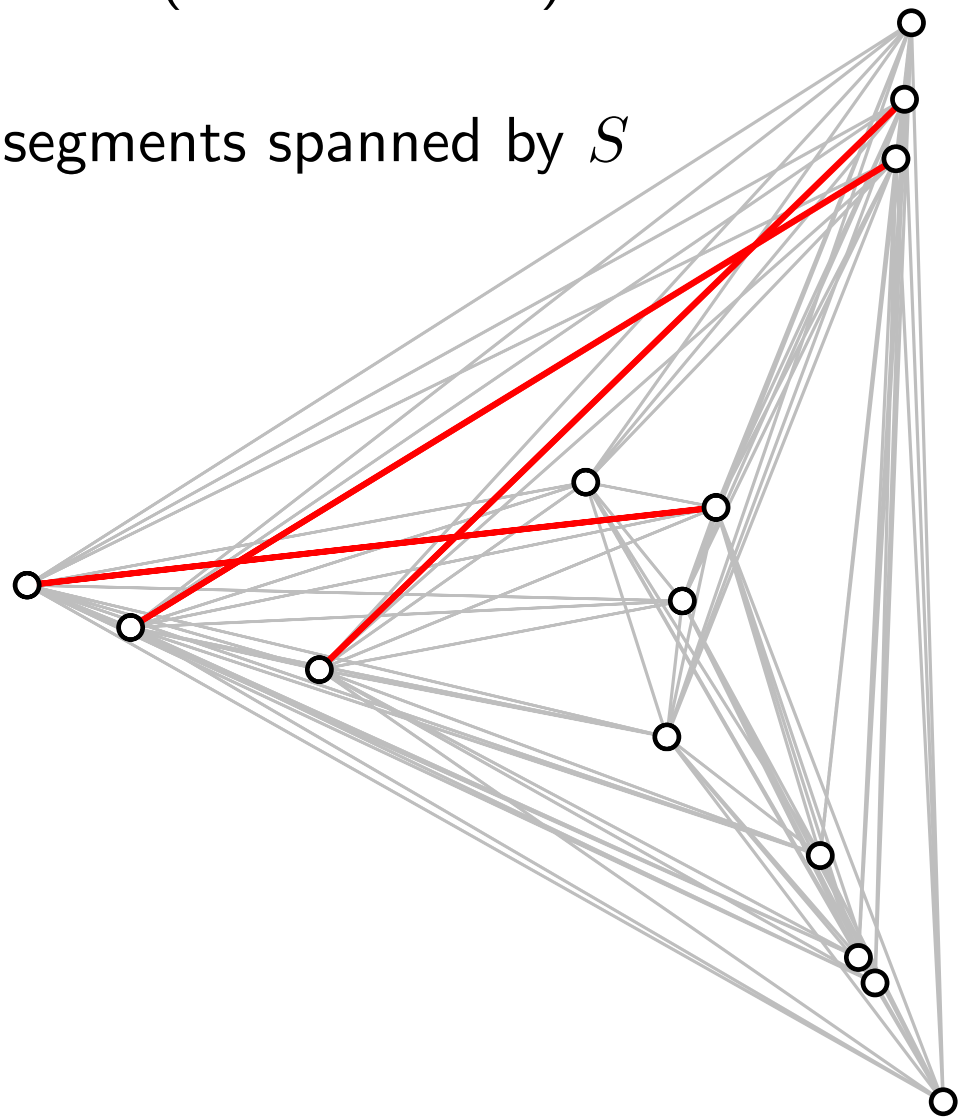

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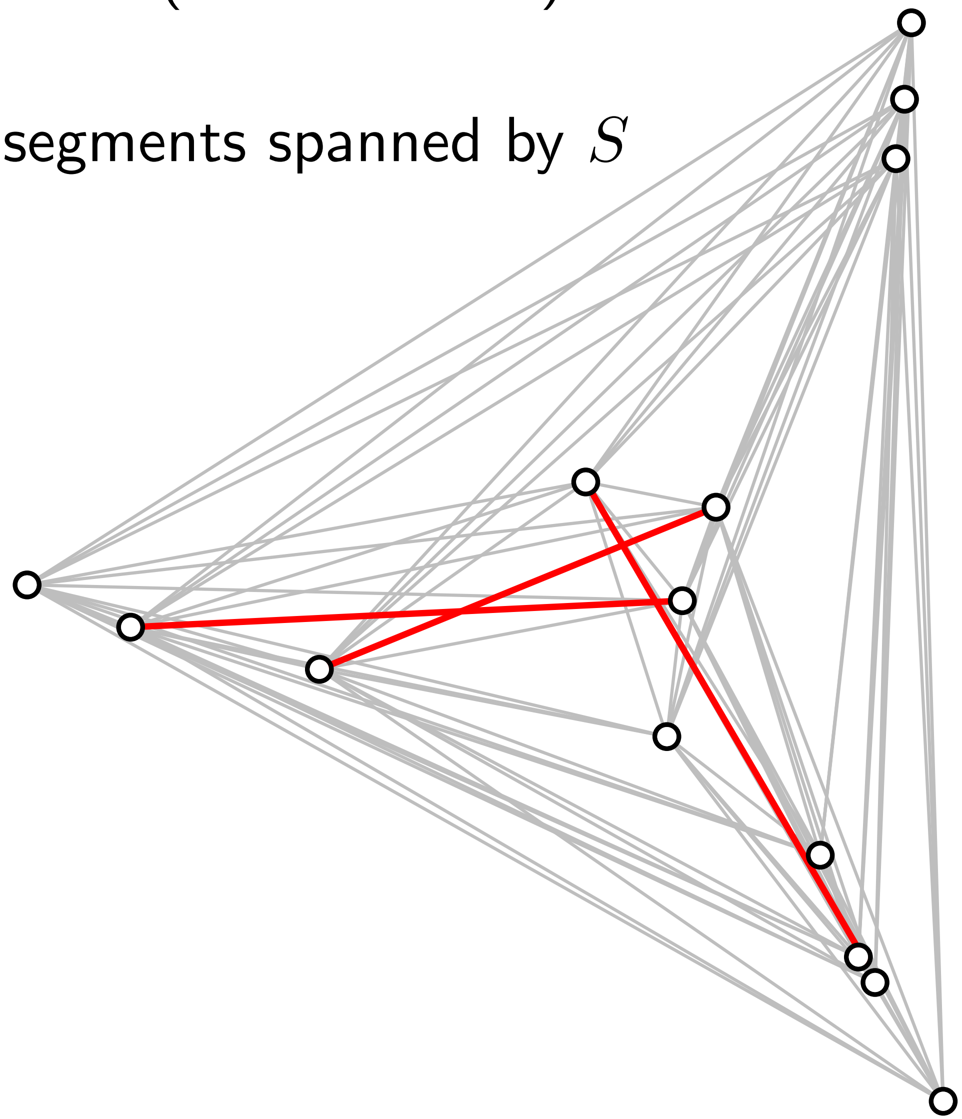

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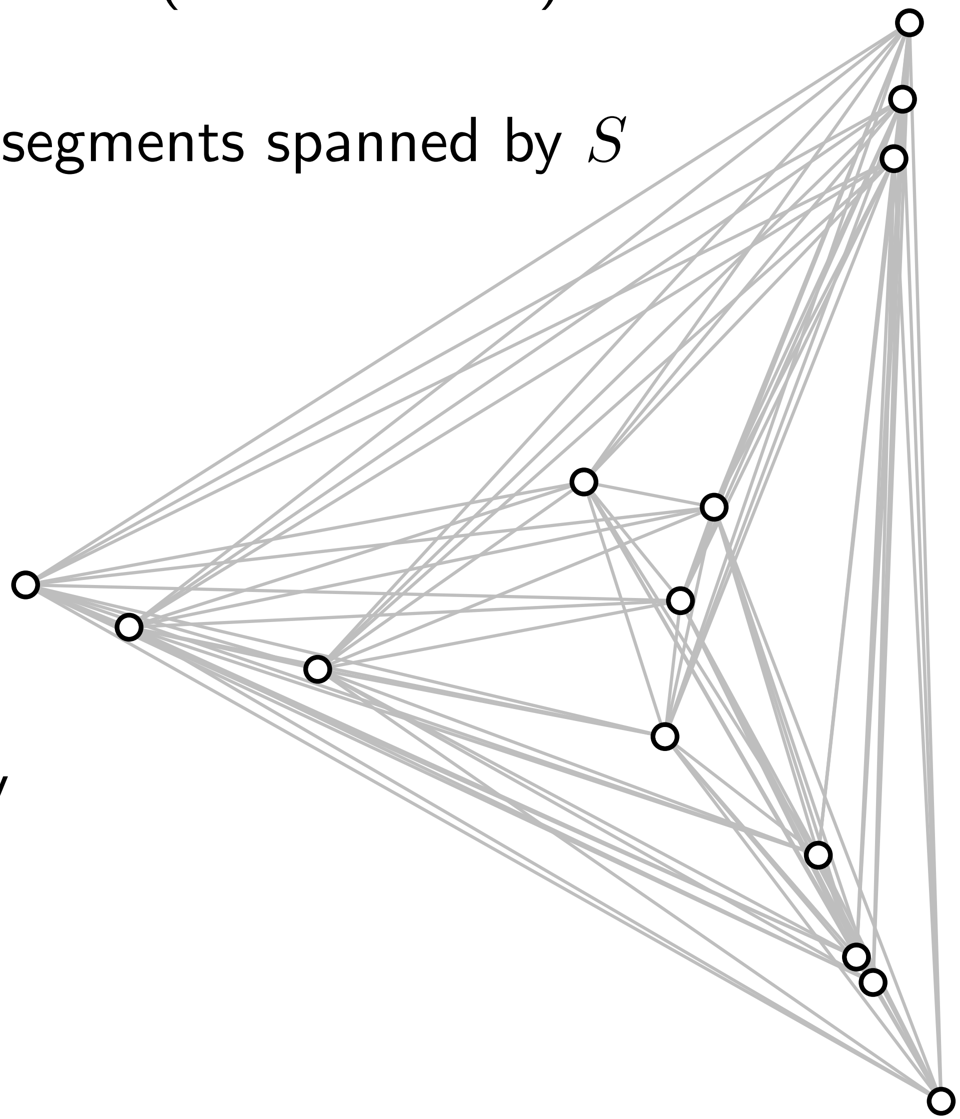

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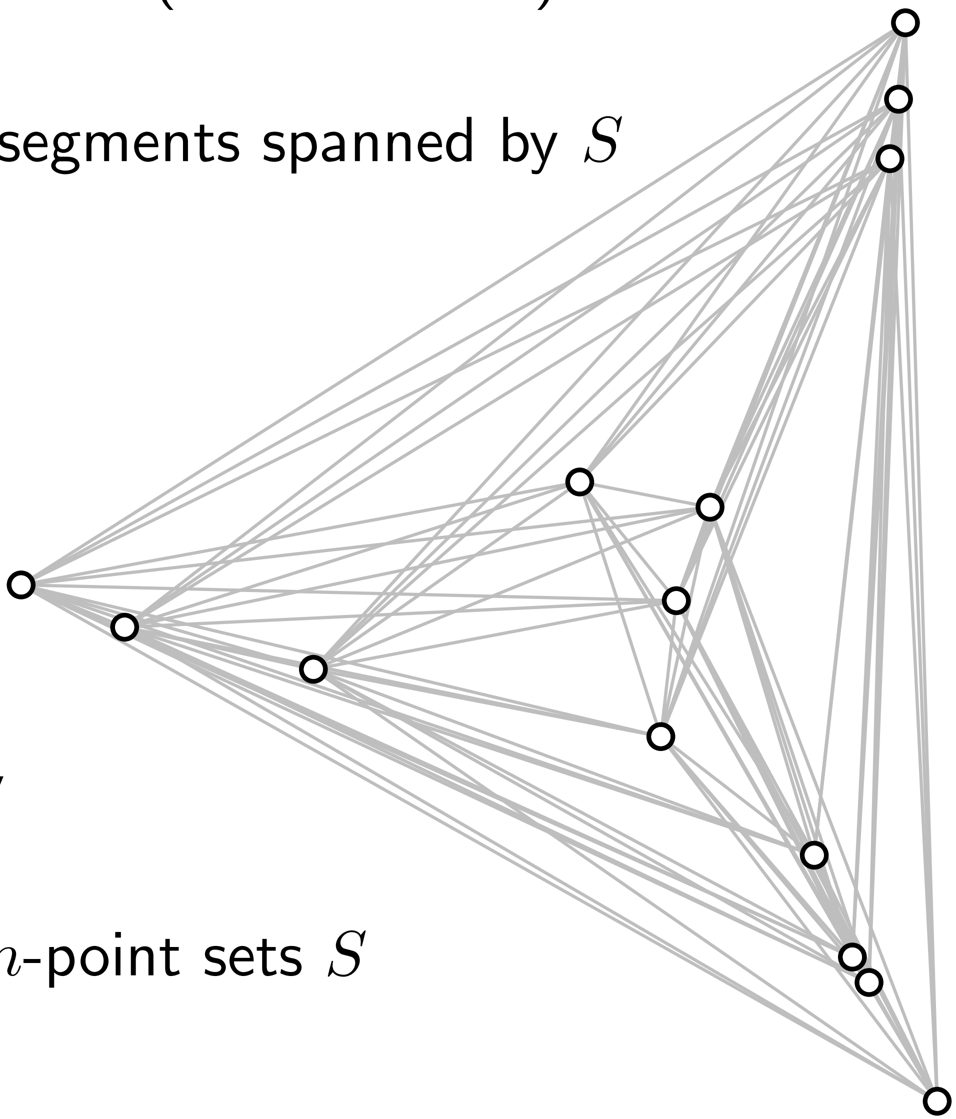

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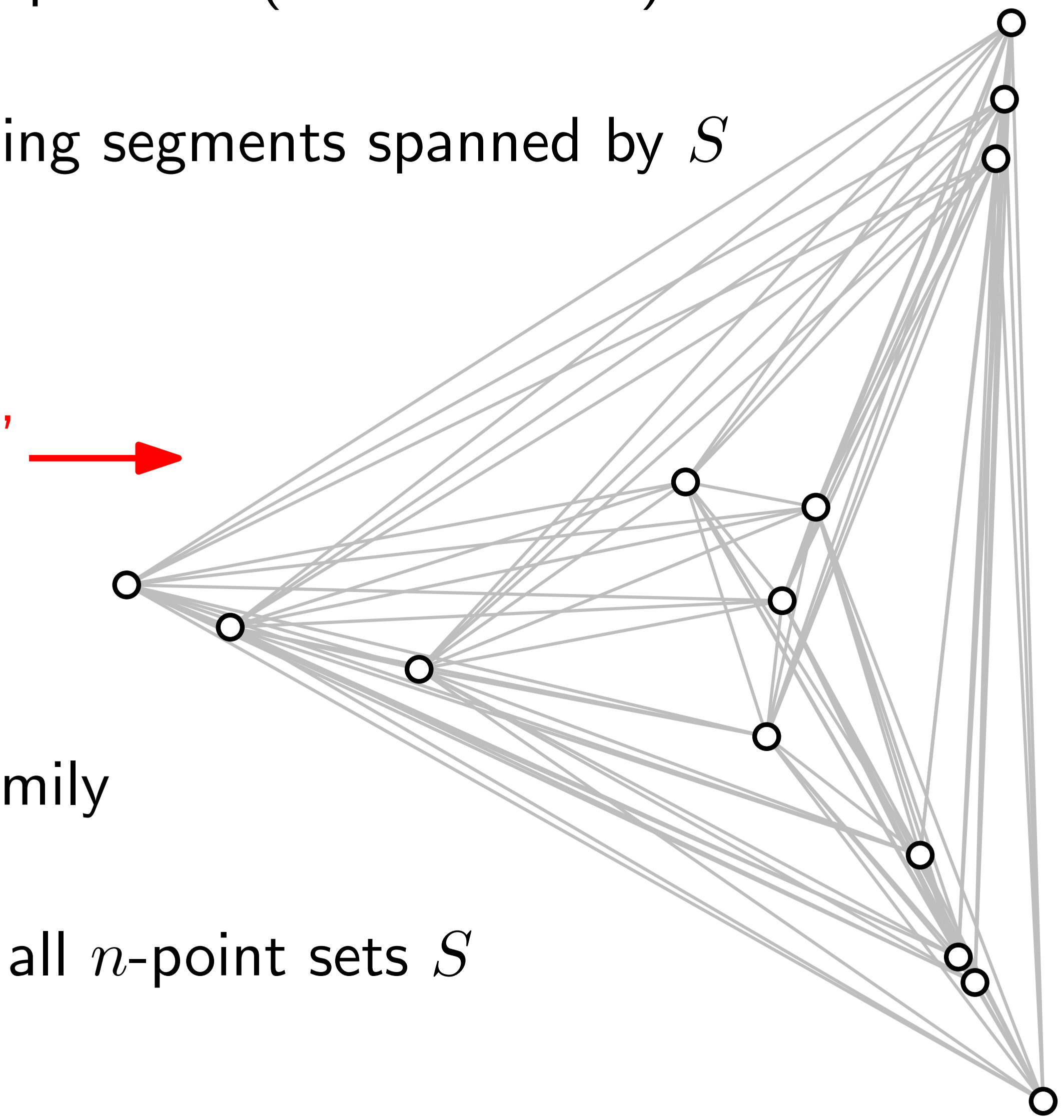
this set shows $\mathrm{cf}(10) \leq 3$,
which is actually tight



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A small example

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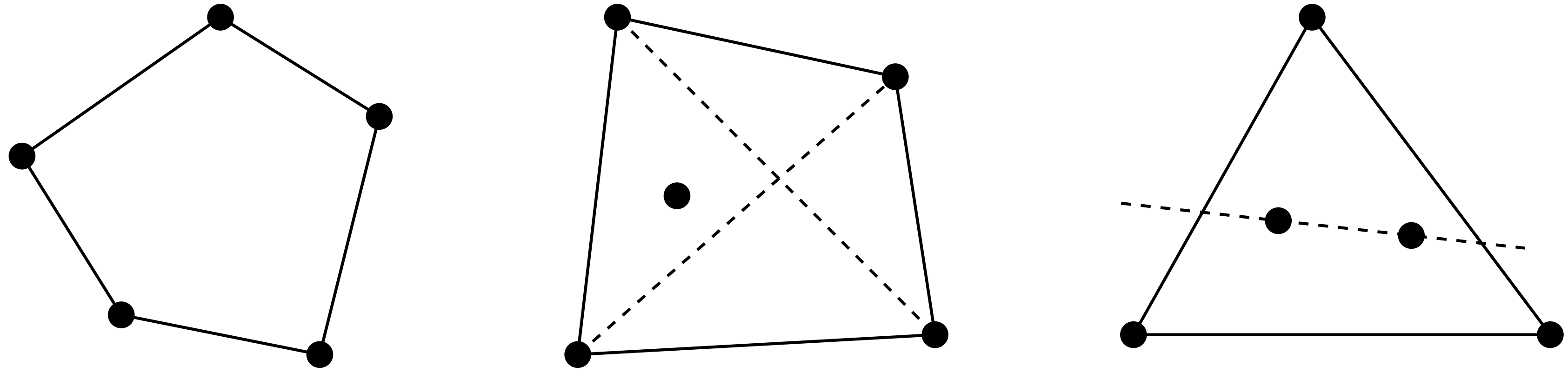

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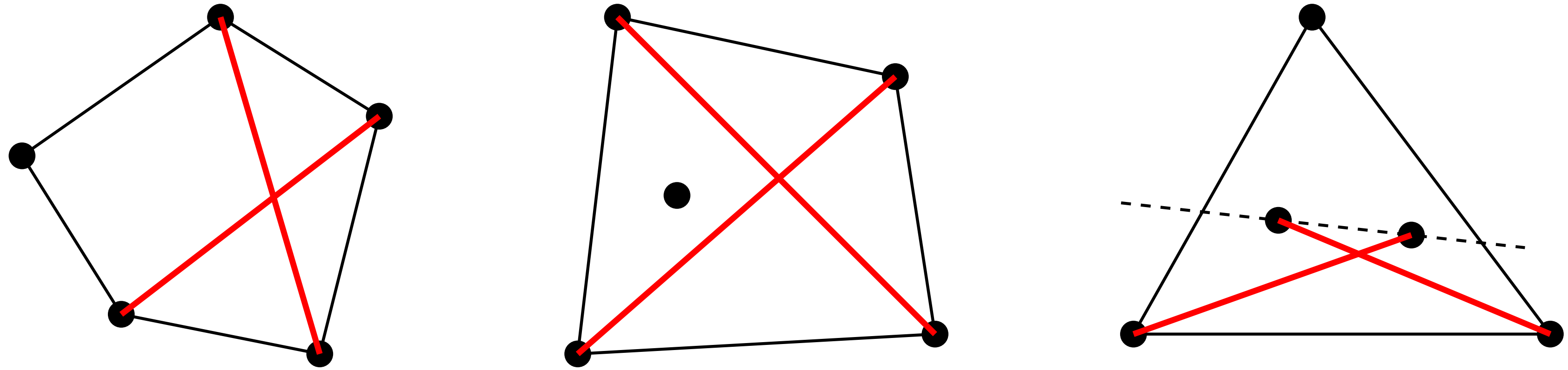

Lower Bounds

- $\mathrm{cf}(n) \geq \Omega(\sqrt{n})$ [Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, and Schulman '94]

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- $\mathrm{cf}(n) \geq n^{1-o(1)}$ [Pach, Rubin, and Tardos '19]

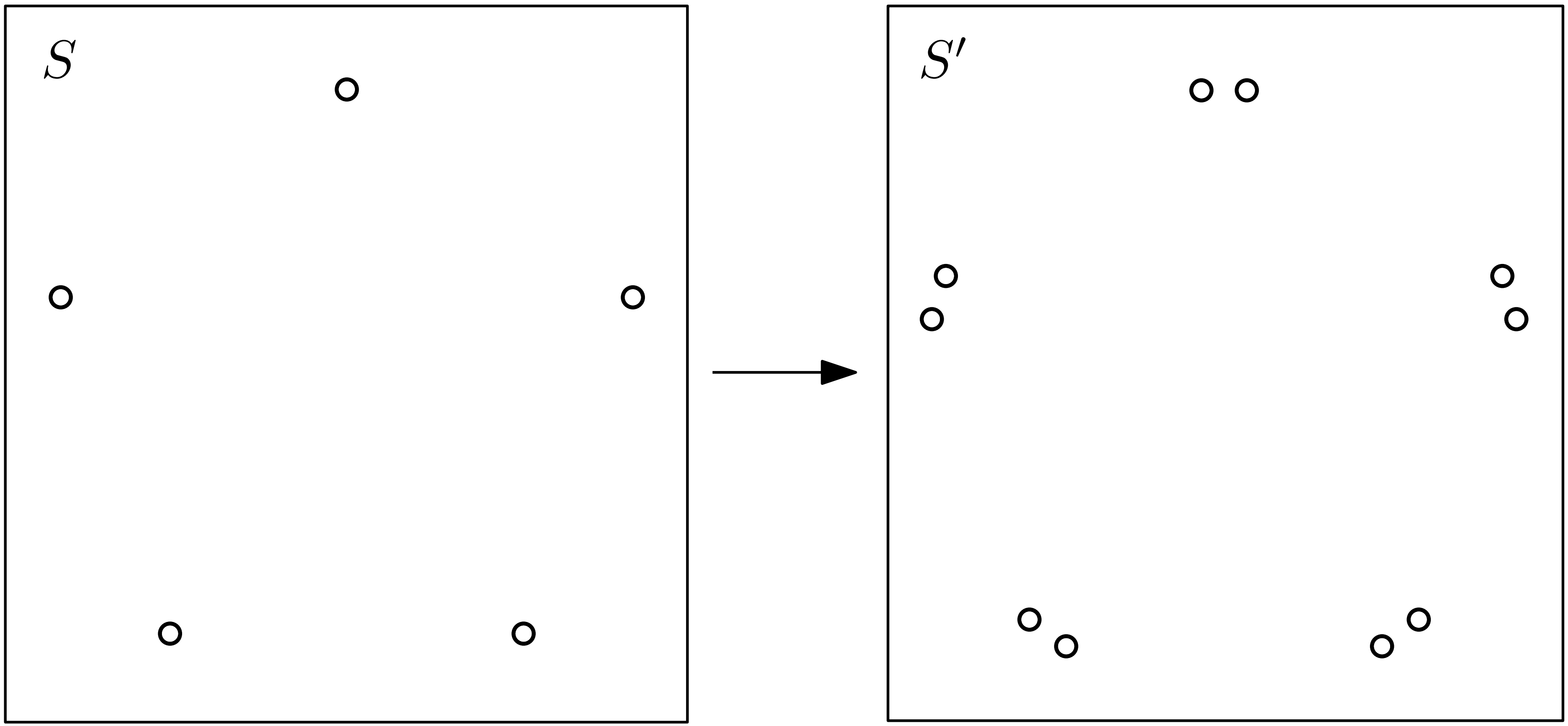
Upper Bounds

- **Theorem (Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, and Schulman '94)**:

$$\mathrm{cf}(n) \leq \mathrm{cf}(n_0) \cdot \left\lceil \frac{n}{n_0} \right\rceil = \underbrace{\frac{\mathrm{cf}(n_0)}{n_0}}_{\text{ratio}} \cdot n + O(1)$$

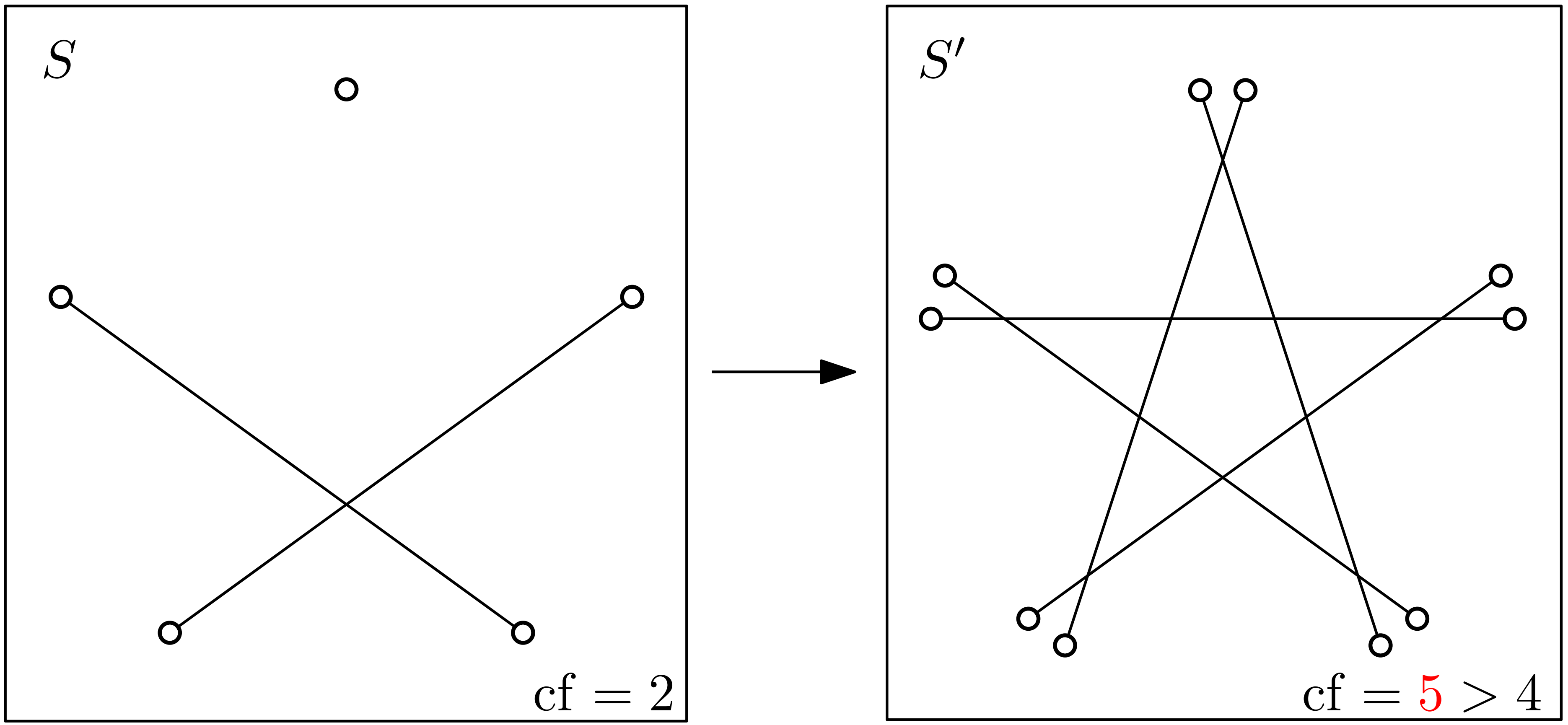
Proof Idea

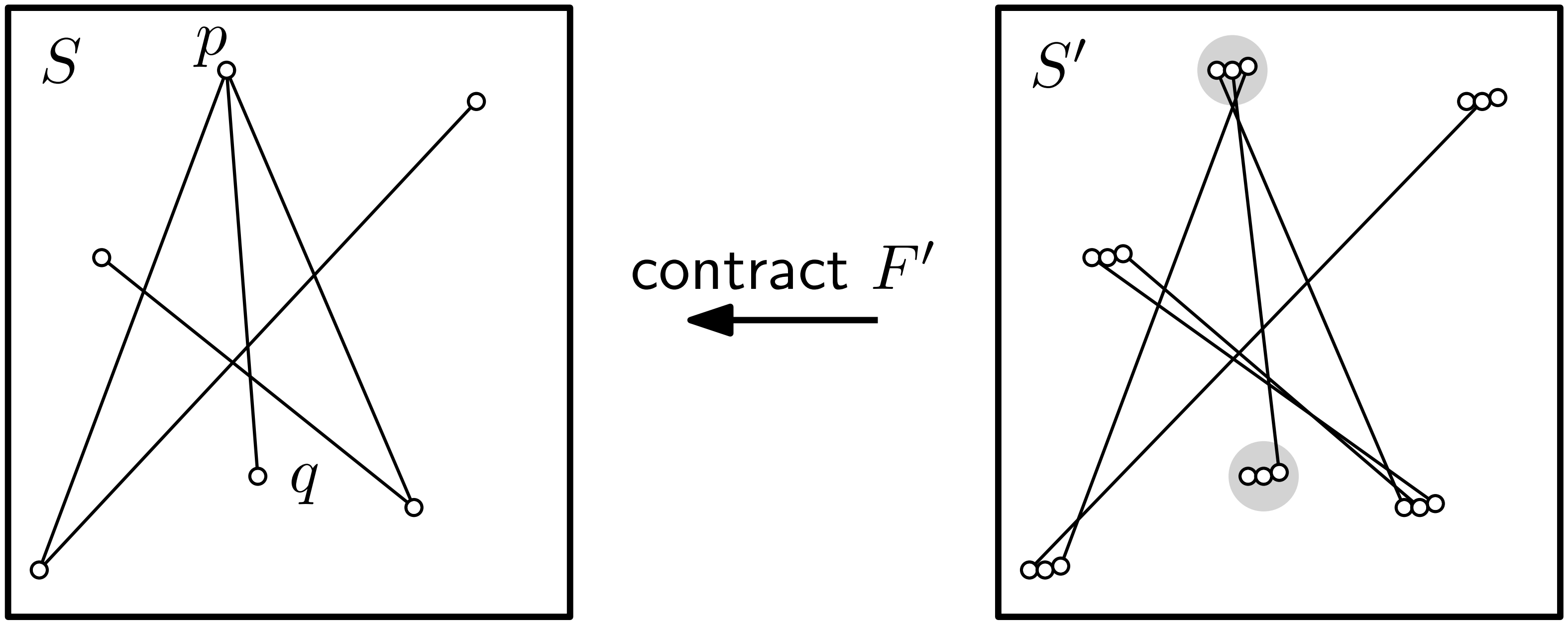
- replacing each point by k perturbed copies



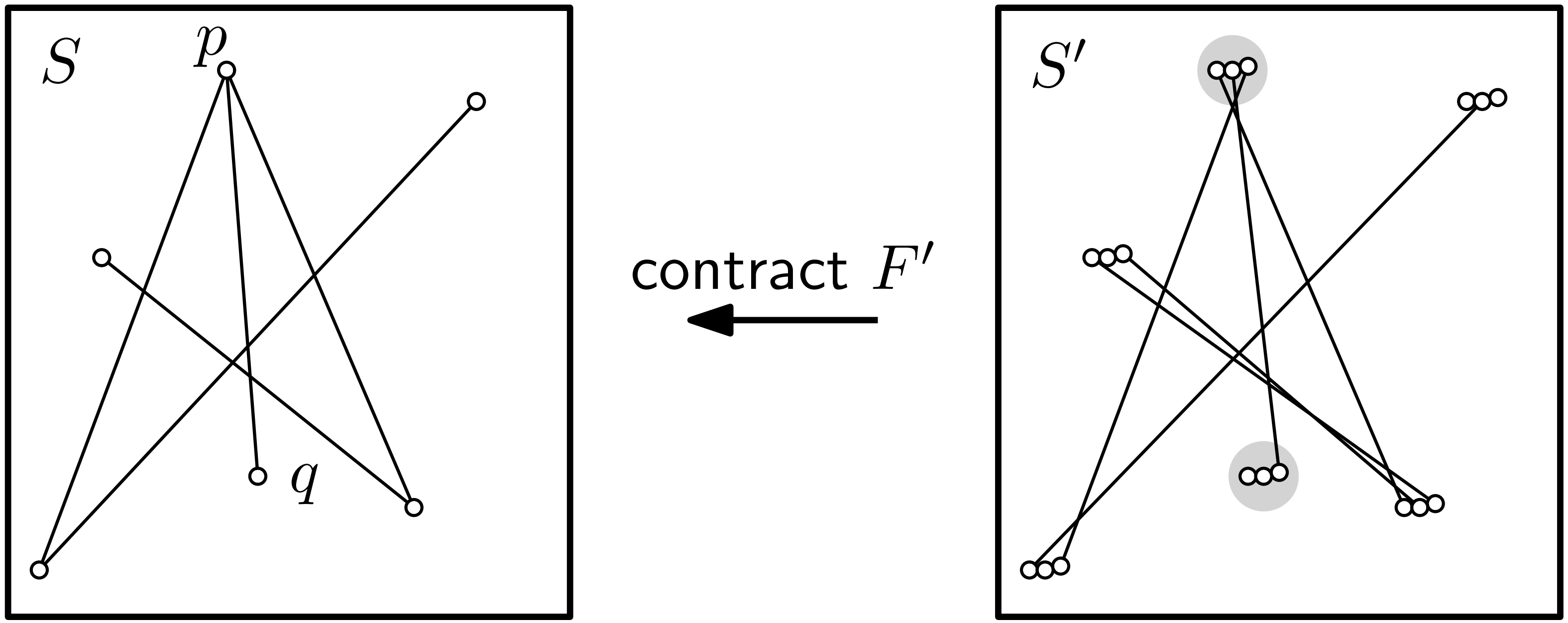
Proof Idea

- replacing each point by k perturbed copies
- need to avoid generating odd cycles of pairwise crossing edges!





- contraction is a geometric thrackle $\Rightarrow$ no even cycles



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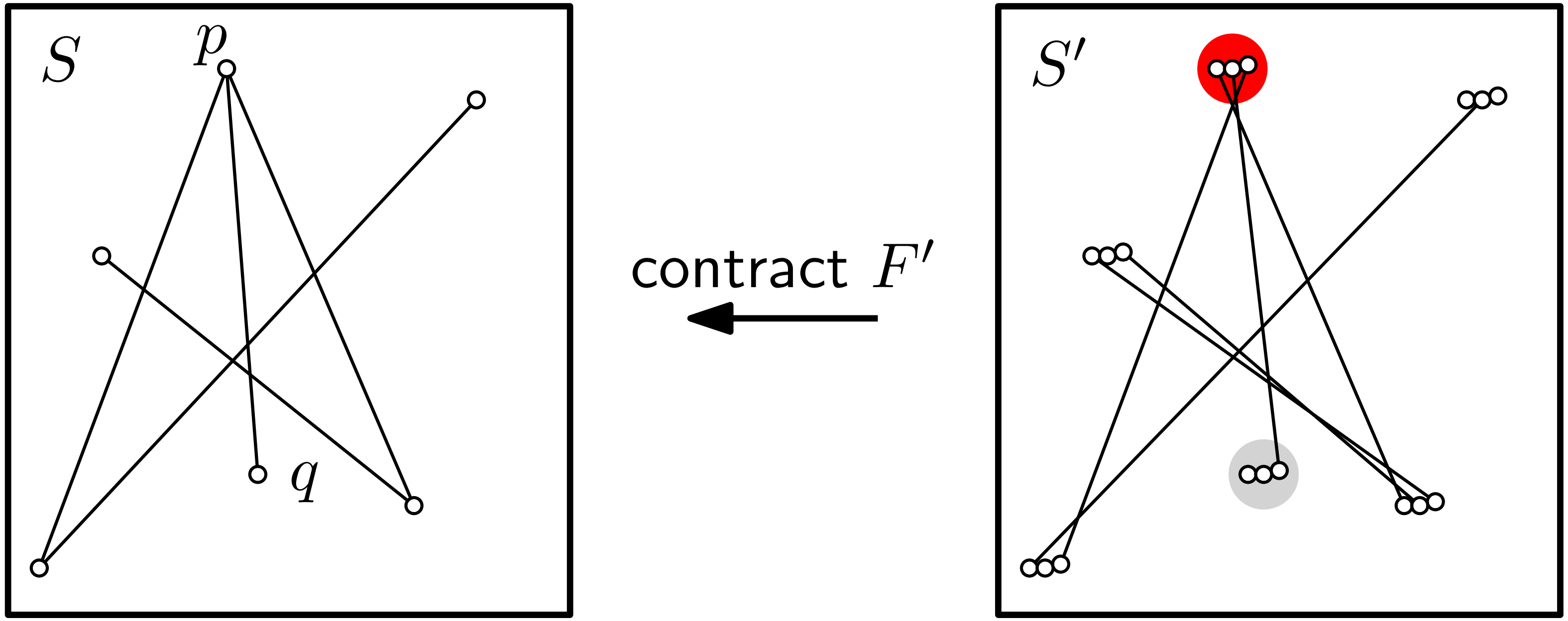
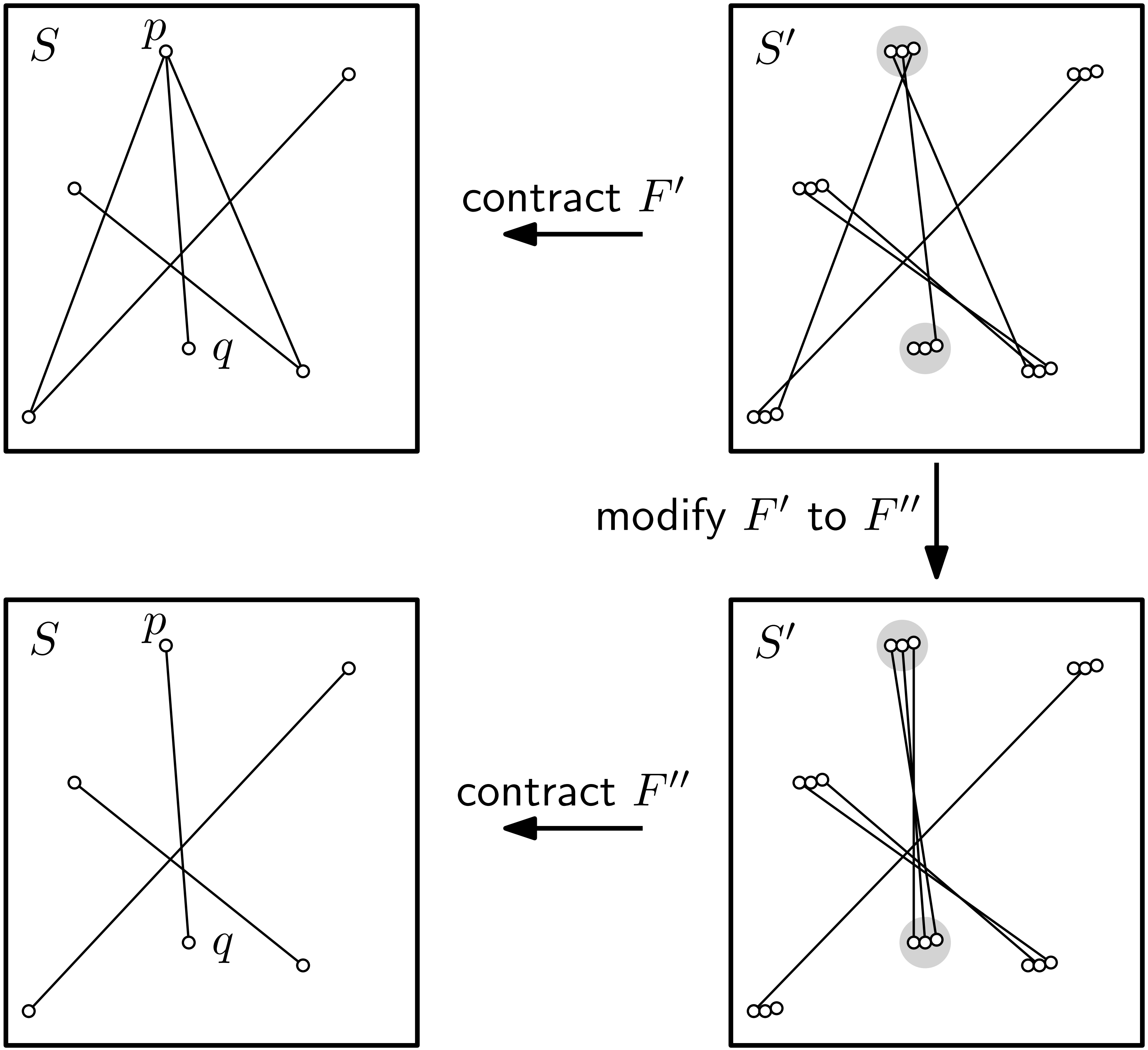


- contraction is a geometric thrackle $\Rightarrow$ no even cycles
- place all copies of a point almost on horizontal line
  $\Rightarrow$ neighbors either all above or all below
  $\Rightarrow$ bipartite (no odd cycles)
- while $\deg(p) > 1$, replace edges incident to p by one full bundle


S
p
q
S′
contract F′
modify F′ to F″
S
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q
S′
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- $\mathrm{cf}(9) = 2 \Rightarrow \mathrm{cf}(n) \leq \frac{2}{9}n + O(1)$ [Aichholzer & Krasser '06]
- $\mathrm{cf}(24) \leq 5 \Rightarrow \mathrm{cf}(n) \leq \frac{5}{24}n + O(1)$ [Evans & Saeedi '19]

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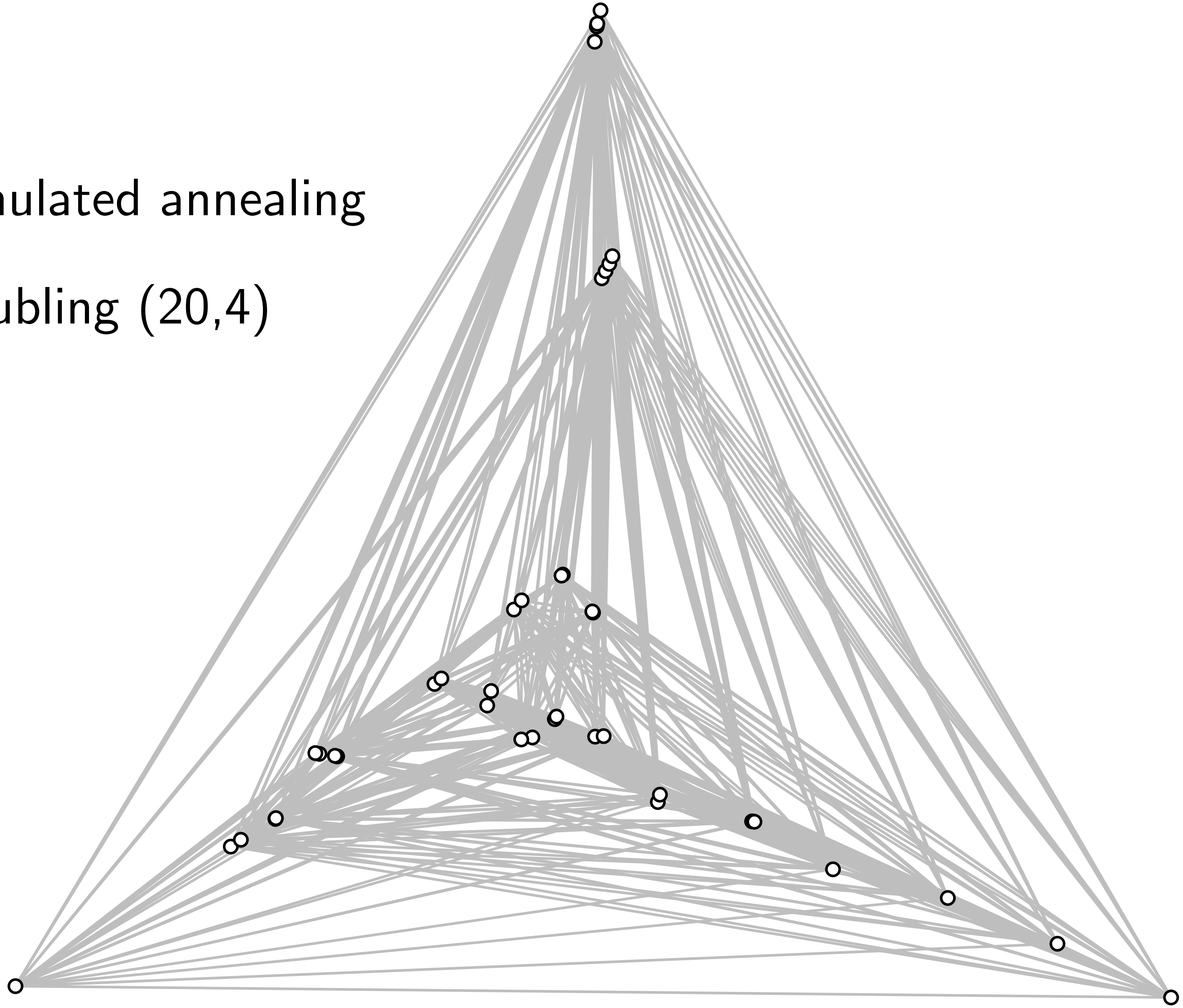
- **BRAND NEW:** $\mathrm{cf}(41) \leq 8 \Rightarrow \mathrm{cf}(n) \leq \frac{8}{41}n + O(1)$

41 points with $\mathrm{cf} = 8$

- simulated annealing
- doubling (20,4)



Exact Bounds

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- $\text{cf}(4) = 1$, $\text{cf}(5) = 2$ (K_5 non-planar)
- $\text{cf}(9) = 2$, $\text{cf}(10) = 3$ [Aichholzer & Krasser '06]
- **NEW:** $\text{cf}(14) = 3$, $\text{cf}(15) = 4$

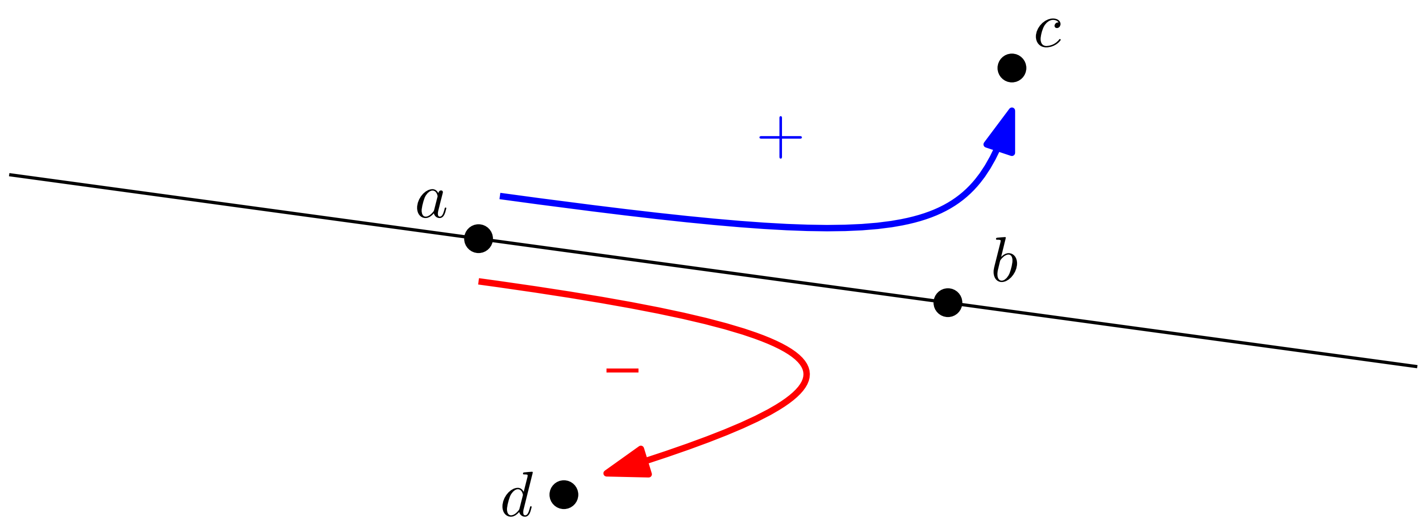
$\text{cf}(n) = 1$ for $n \leq 4$,
$\text{cf}(n) = 2$ for $5 \leq n \leq 9$,
$\text{cf}(n) = 3$ for $10 \leq n \leq 14$,
$\text{cf}(n) = 4$ for $15 \leq n \leq 20$

k	1	2	3	4	5	6	7	8
$\|S_k\|$	4	9	14	≥20	≥25	≥29	≥34	≥41

Largest point sets S_k with $\text{cf}(S_k) = k$.

Proving $\mathrm{cf}(15) = 4$

- crossings only depend on triple-orientations



Proving $\mathrm{cf}(15) = 4$

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- proof #1: via exhaustive enumeration of order types
- proof #2: via SAT model

Thank you very much for your attention!