Arrangements of Pseudocircles: Triangles and Drawings

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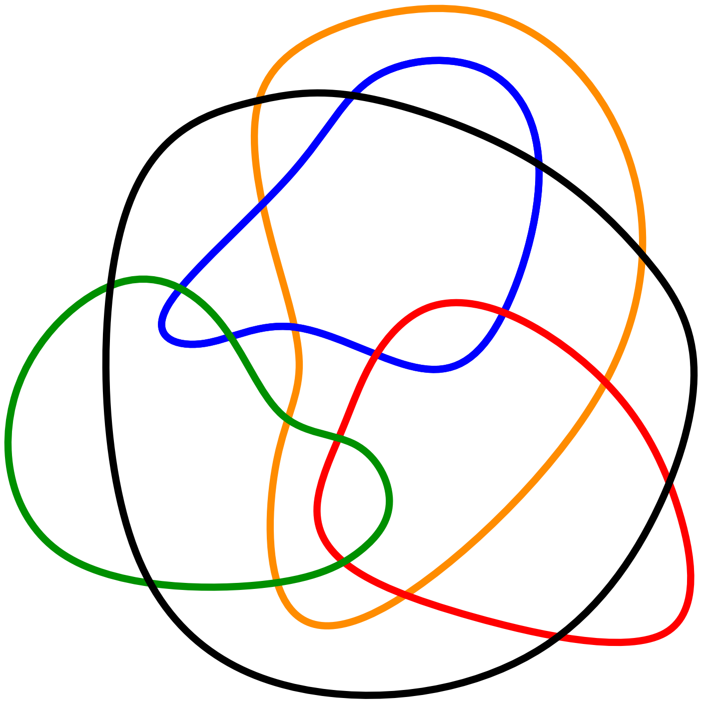
Arrangements of Pseudocircles

pseudocircle . . . simple closed curve

intersecting . . . each 2 pcs cross twice

simple . . . no 3 pcs intersect in common point



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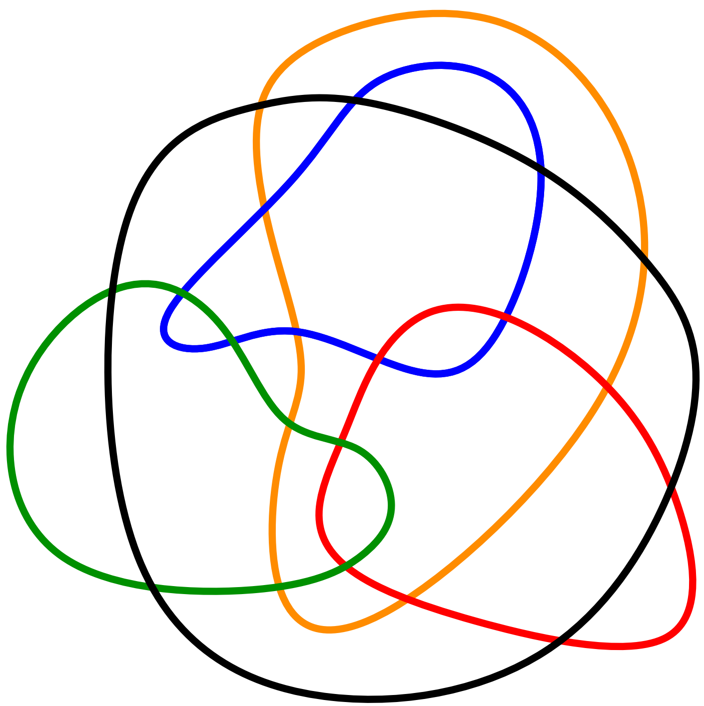
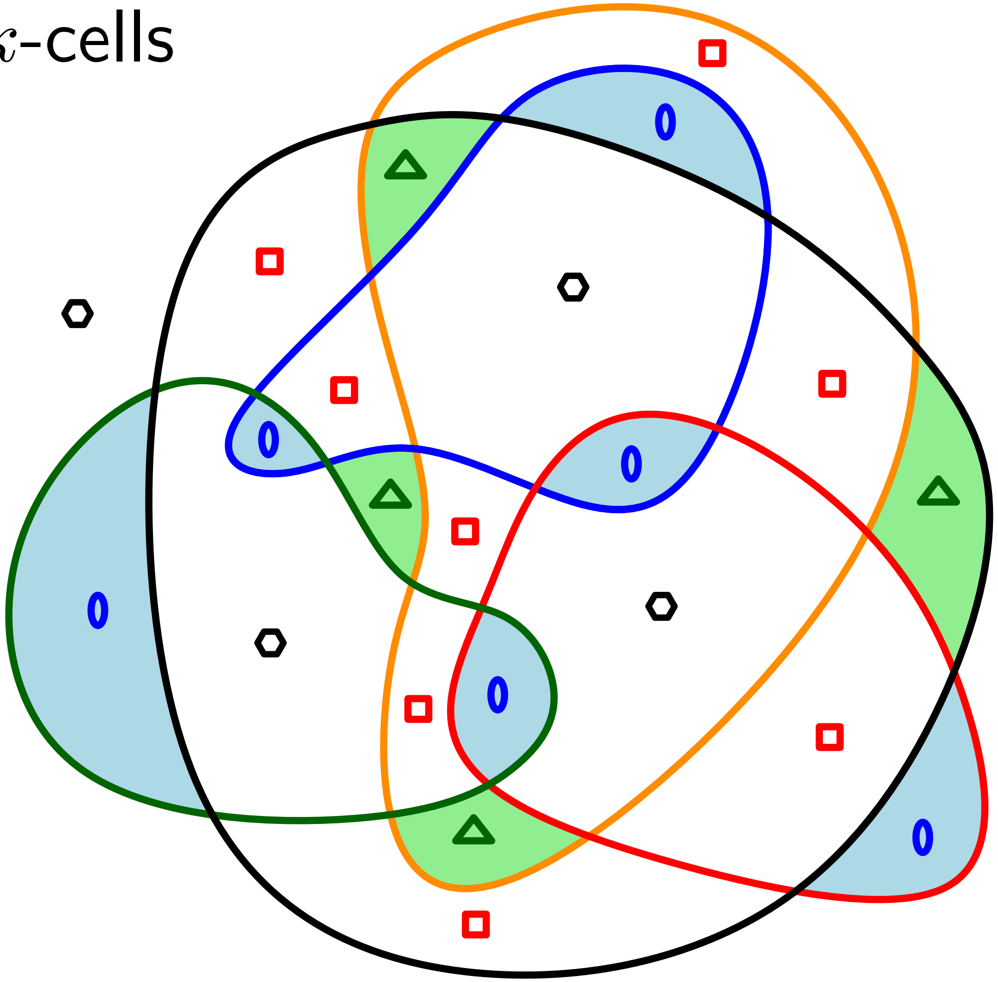
assumptions throughout presentation



Cells in Arrangements

digon, *triangle*, *quadrangle*, *pentagon*, . . . , *k-cell*

p_k . . . # of k-cells



$p_2 = 6$

$p_3 = 4$

$p_4 = 8$

$p_5 = 0$

$p_6 = 4$

Outline

1.) min. # of triangles

1.1.) in digon-free arrangements

1.2.) in arrangements with digons

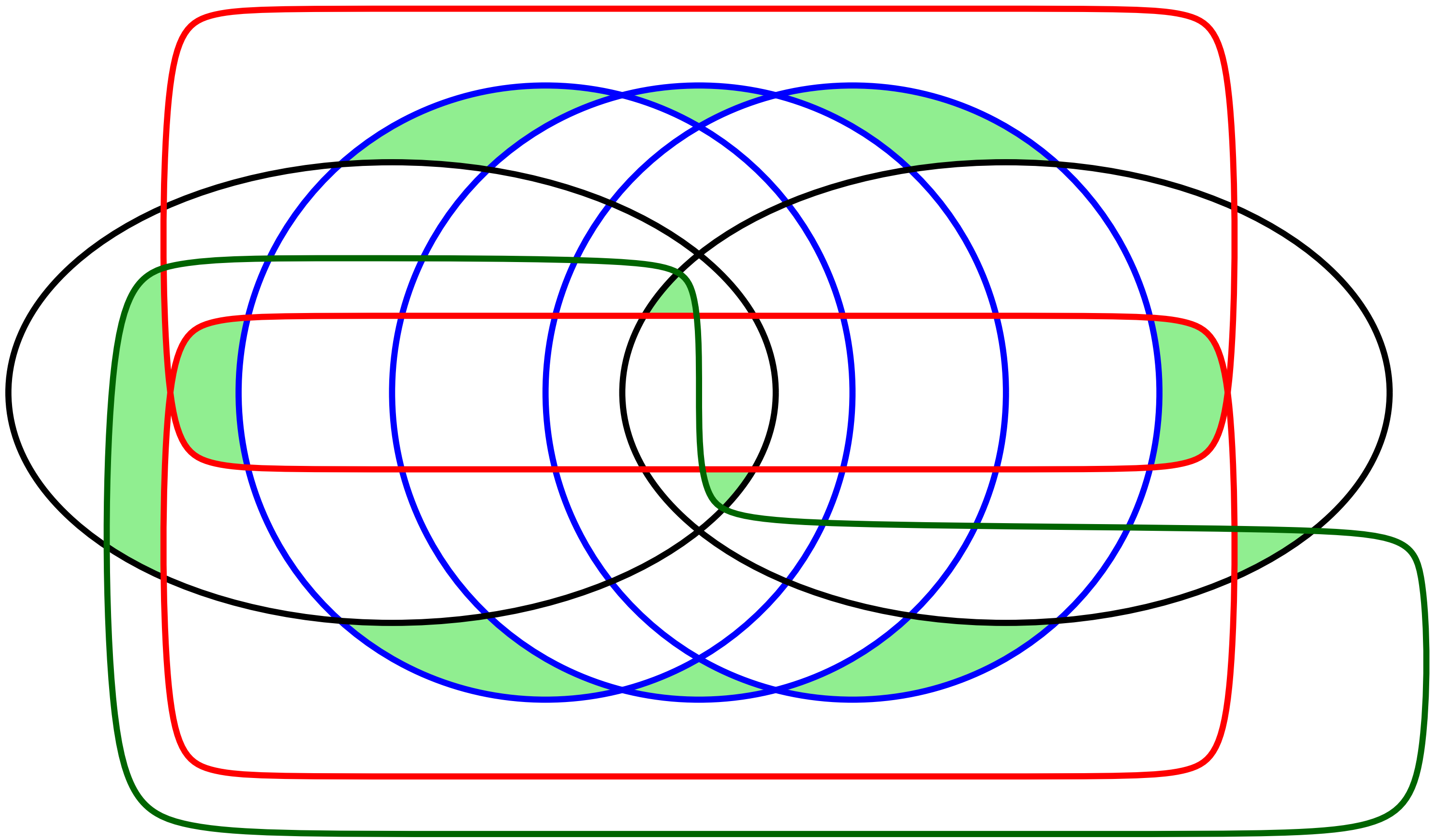
2.) max. # of triangles

3.) visualization of arrangements

Triangles in Digon-free Arrangements

Grünbaum's Conjecture ('72):

- $p_3 \geq 2n - 4$?



Triangles in Digon-free Arrangements

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- $p_3 \geq 2n - 4$?

Known:

- $p_3 \geq 4n/3$ [Hershberger and Snoeyink '91]
- $p_3 \geq 4n/3$ for non-simple arrangements, tight for infinite family [Felsner and Kriegel '98]

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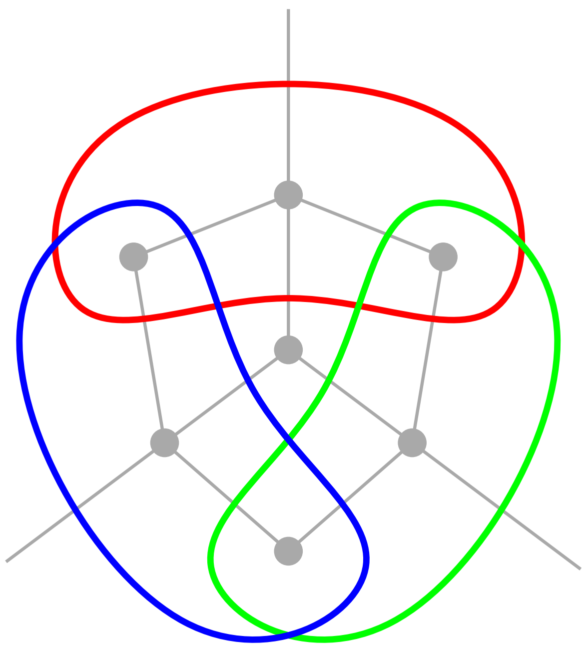
Our Contribution:

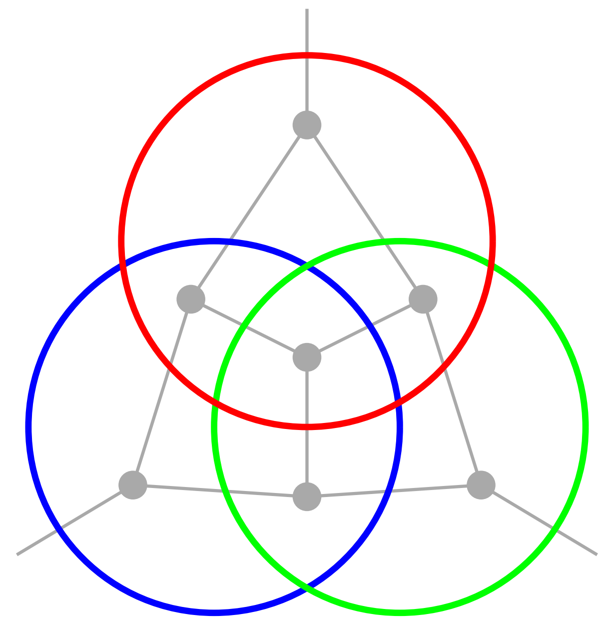
- disprove Grünbaum's Conjecture
- $p_3 < 1.\overline{45}n$
- **New Conjecture:** $4n/3$ is tight

Enumeration of Arrangements

n	2	3	4	5	6	7
sphere	1	2	8	278	145 058	447 905 202
+digon-free	0	1	2	14	2 131	3 012 972

Table: # of combinatorially different arragements of n pcs.





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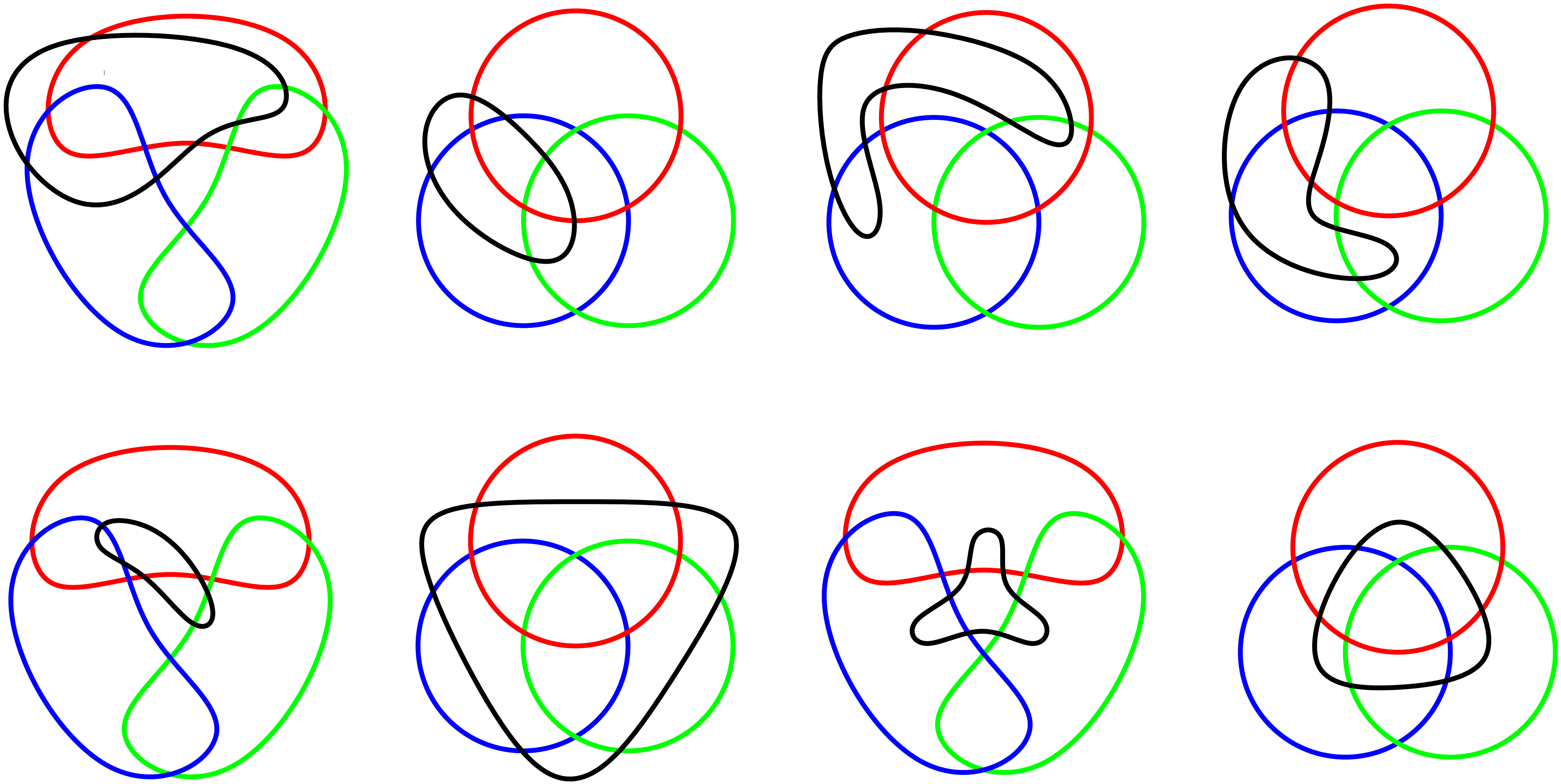


Triangles in Digon-free Arrangements

Theorem. The minimum number of triangles in digon-free arrangements of n pseudocircles is

(i) 8 for $3 \leq n \leq 6$.

(ii) $\lceil \frac{4}{3}n \rceil$ for $6 \leq n \leq 14$.

(iii) $< 1.\overline{45}n$ for all $n = 11k + 1$ with $k \in \mathbb{N}$.

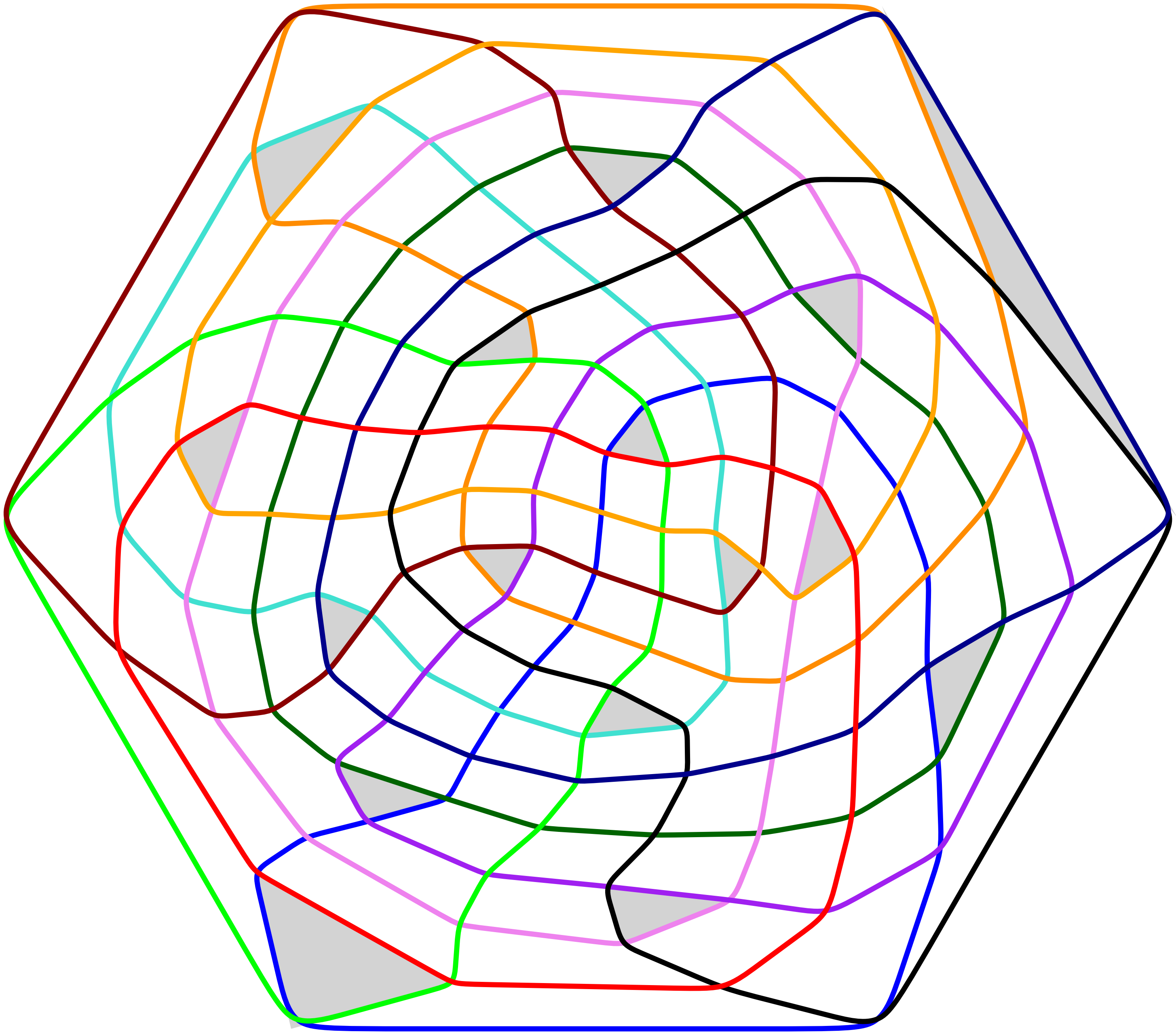


Figure: Arrangement of $n = 12$ pcs with $p_3 = 16$ triangles.

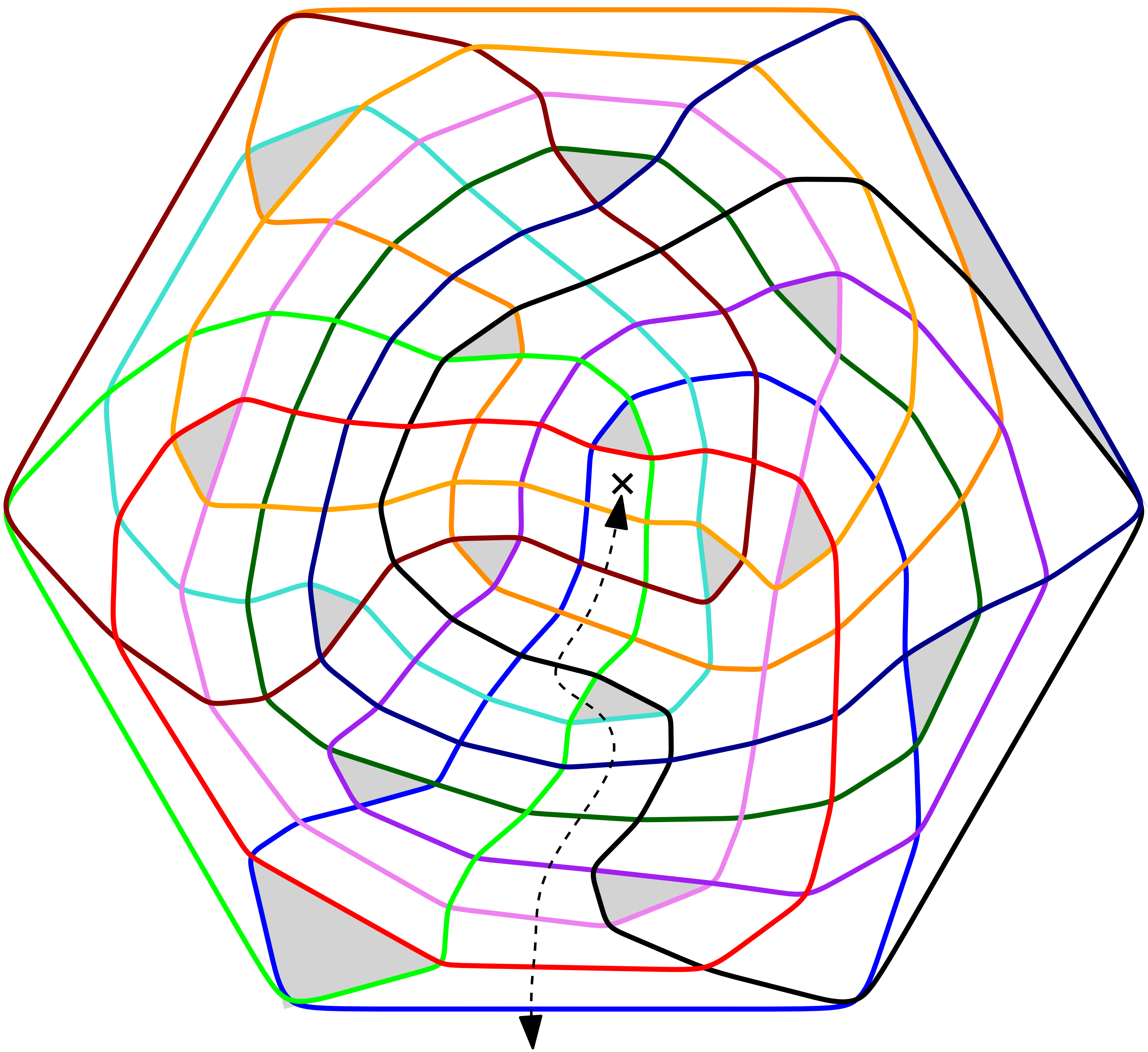
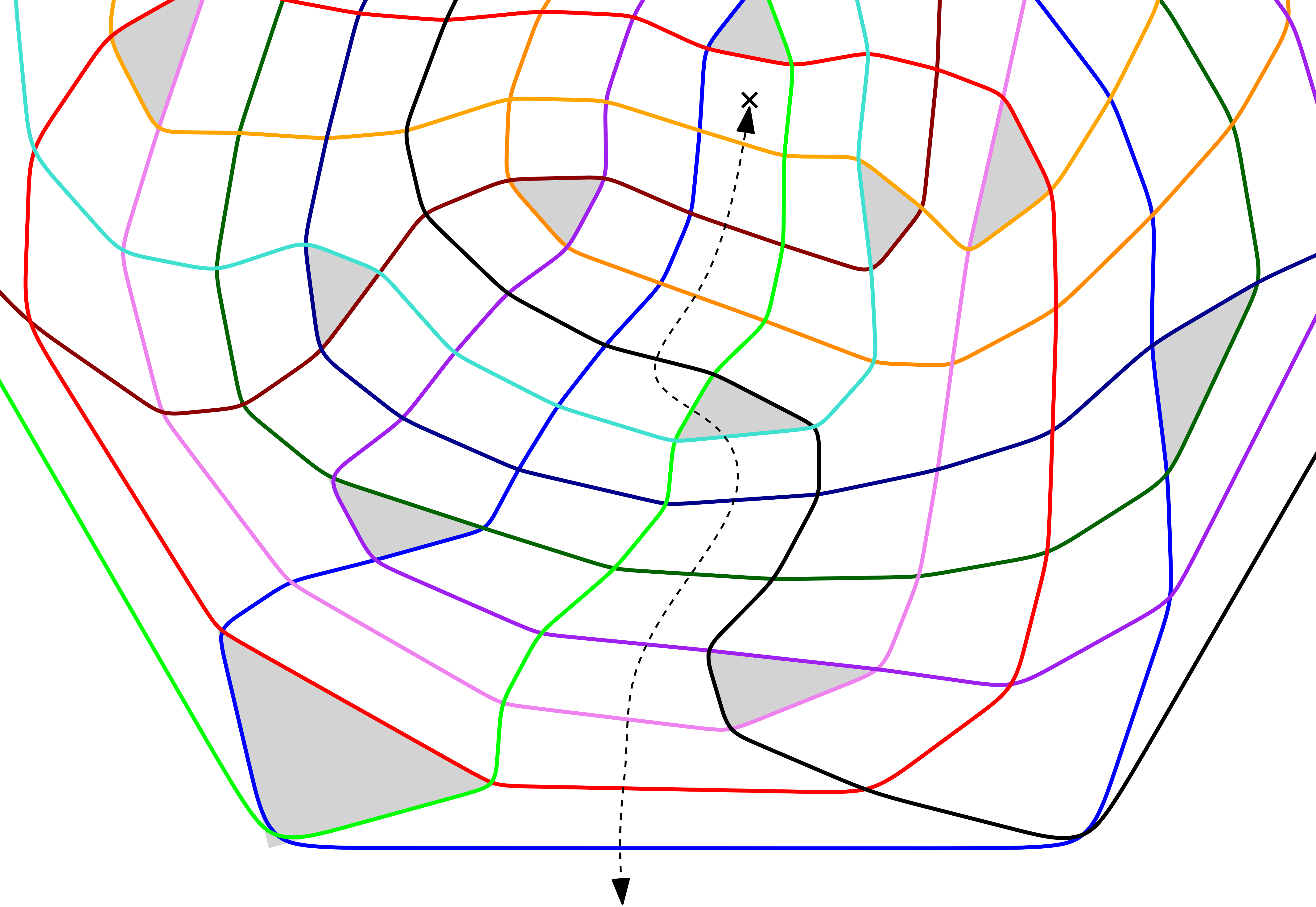
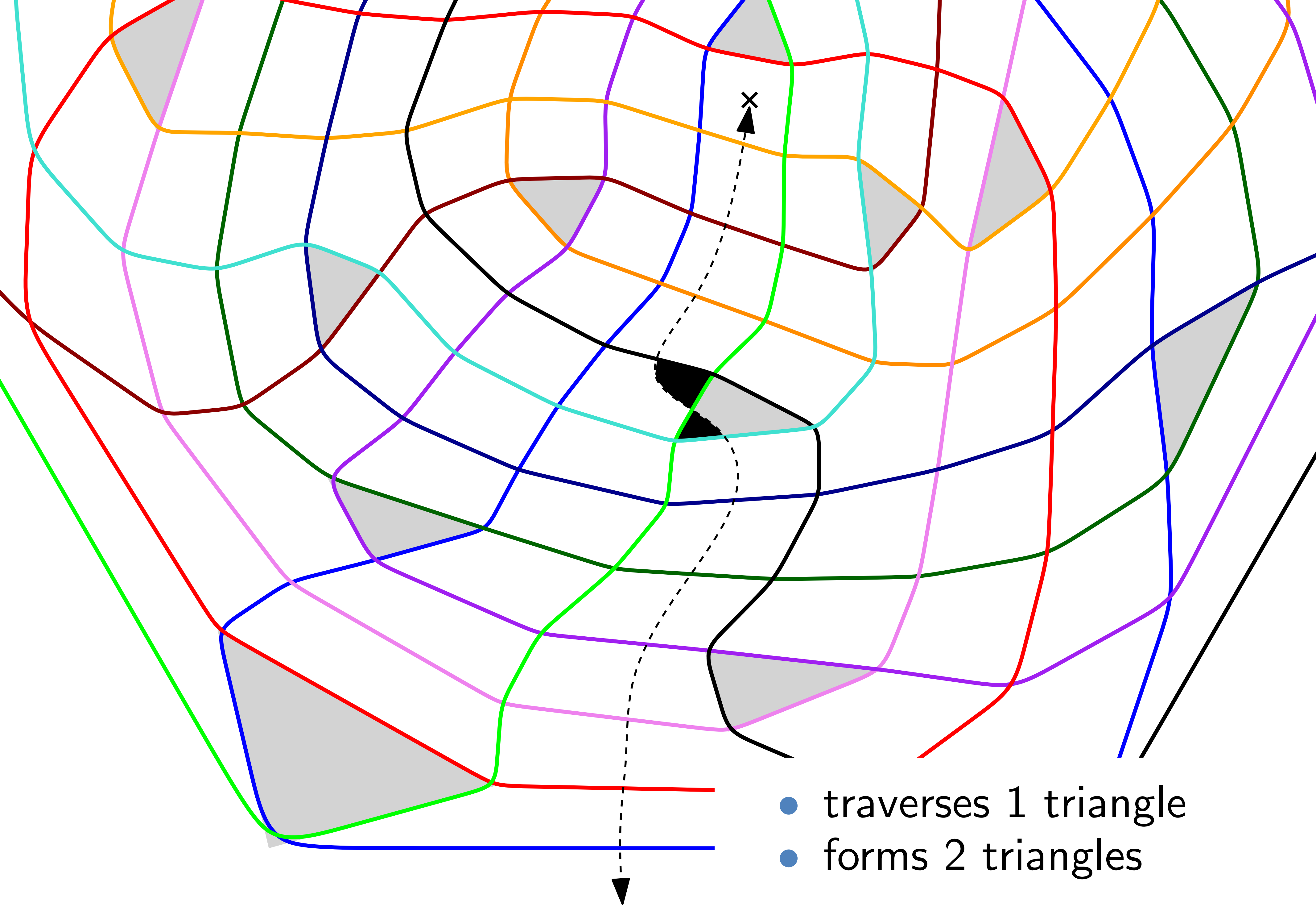


Figure: Arrangement of $n = 12$ pcs with $p_3 = 16$ triangles.





- traverses 1 triangle
- forms 2 triangles

Proof of the Theorem

- arrangement $\mathscr{A}_{12}$ with
  - $n = 12$
  - $p_3 = 16$
  - $\delta = 2$ (formed triangles)
  - $\tau = 1$ (traversed)
- start with $\mathscr{C}_1 := \mathscr{A}_{12}$
- merge $\mathscr{C}_k$ and $\mathscr{A}_{12} \longrightarrow \mathscr{C}_{k+1}$
- $n(\mathscr{C}_k) = 11k + 1$, $p_3(\mathscr{C}_k) = 16k$
- $\frac{16k}{11k+1}$ increases as k increases with limit $\frac{16}{11} = 1.\overline{45}$

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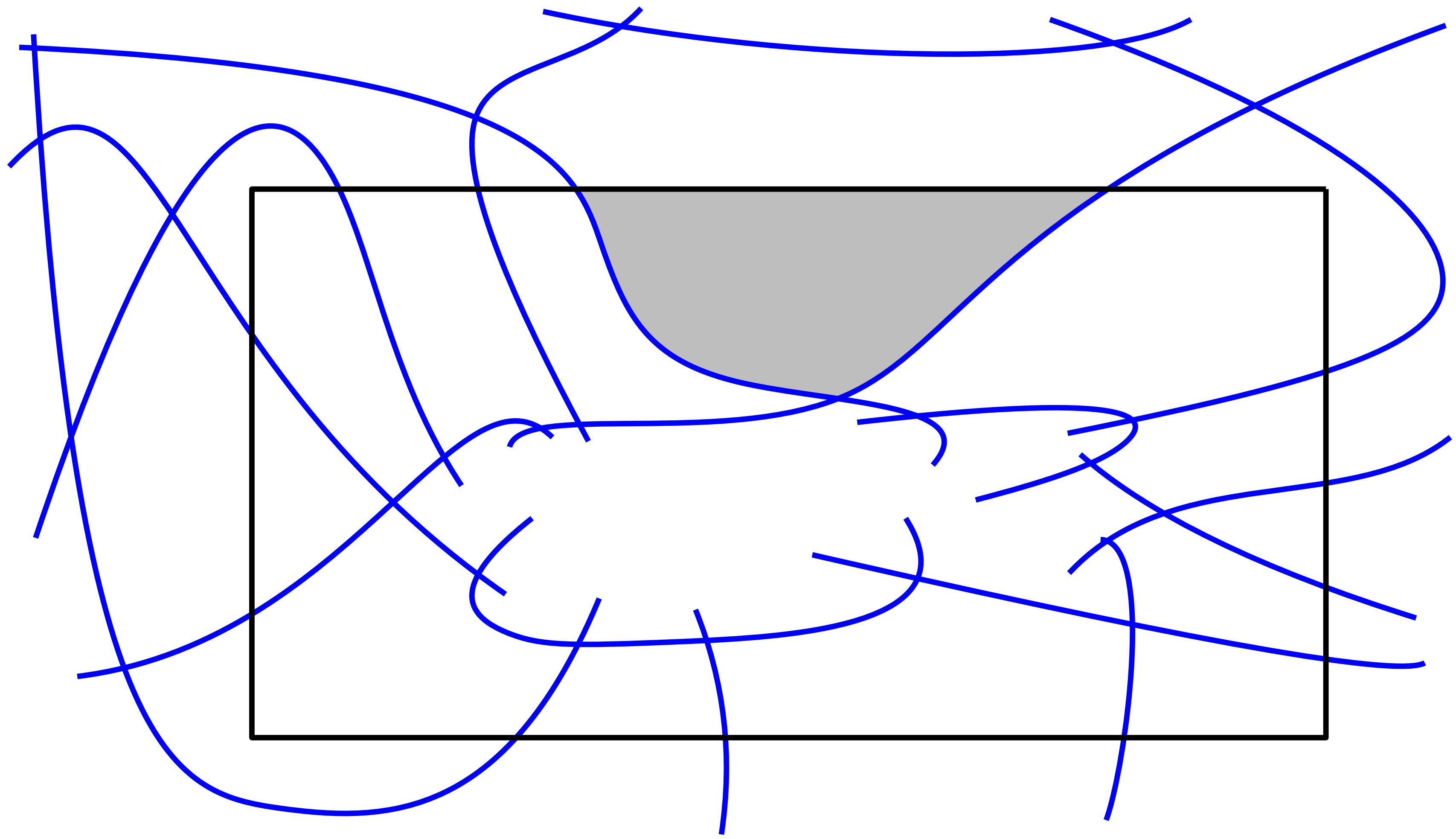

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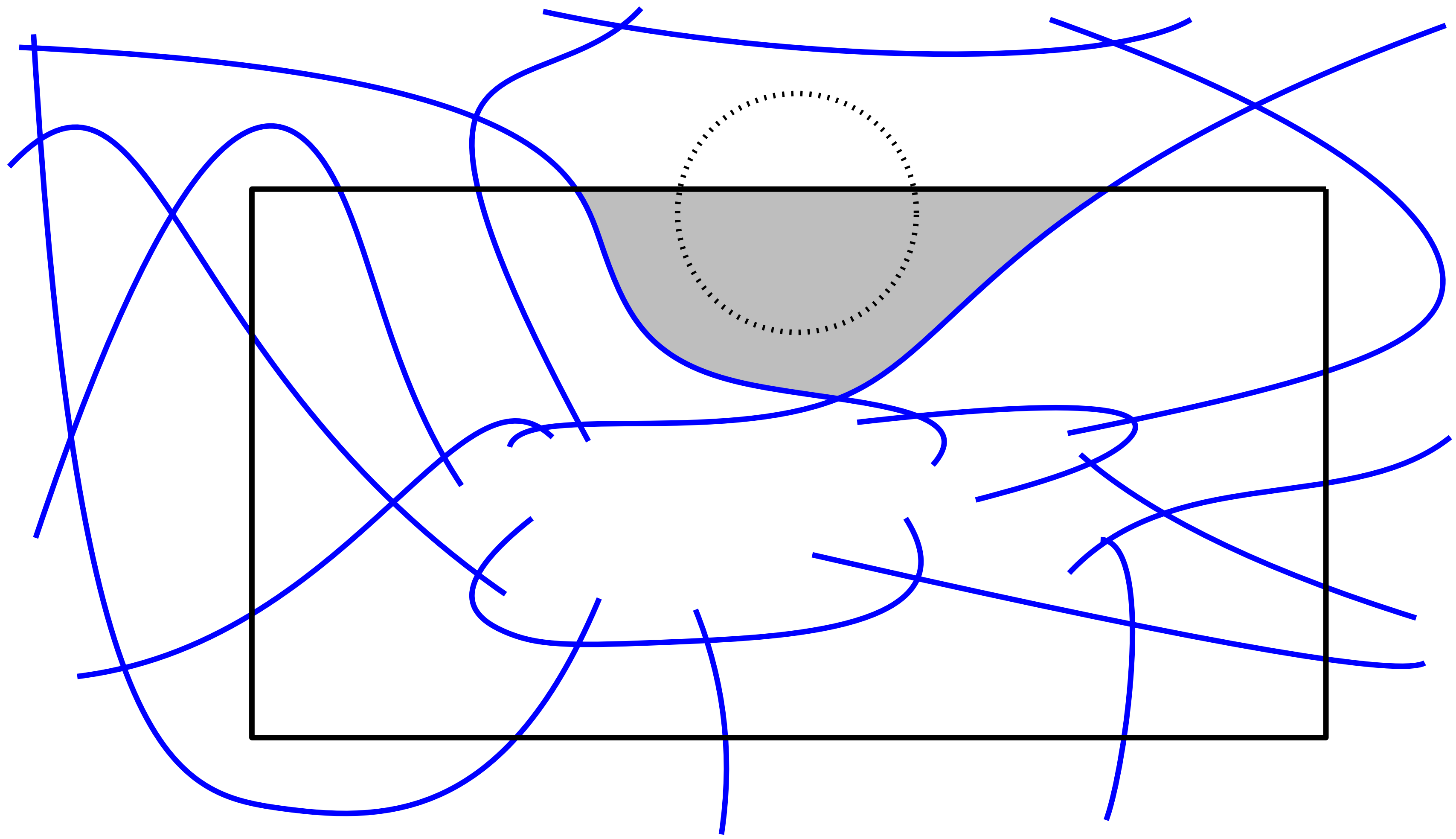

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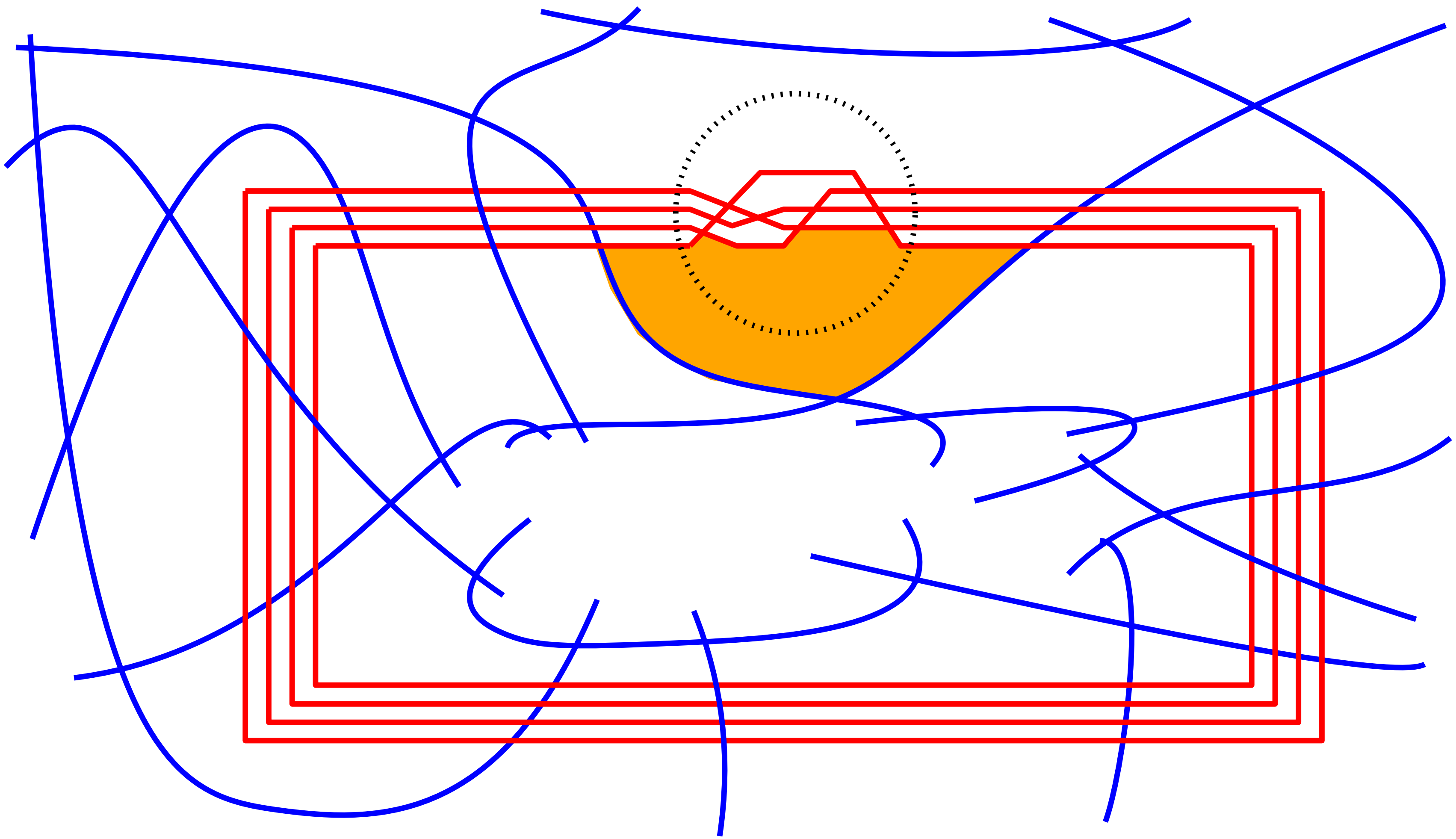

$$\Rightarrow p_3(\mathscr{C}) = p_3(\mathscr{A}) + p_3(\mathscr{B}) + \delta - \tau - 1.$$

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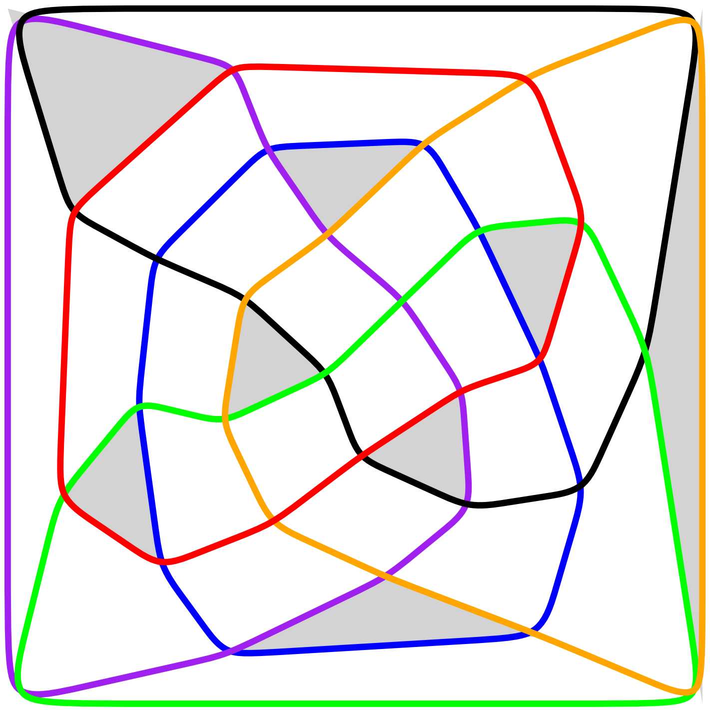
(iii) $< 1.\overline{45} n$ for all $n = 11k + 1$ with $k \in \mathbb{N}$.

Conjecture. $\lceil 4n/3 \rceil$ is tight for infinitely many n.

Triangles in Digon-free Arrangements

- $\exists$ unique arrangement $\mathcal{N}_6$ with $n = 6, p_3 = 8$
- $\mathcal{N}_6$ appears as a subarrangement of every arrangement with $p_3 < 2n - 4$ for $n = 7, 8, 9$
- $\mathcal{N}_6$ is non-circularizable [FS'17]



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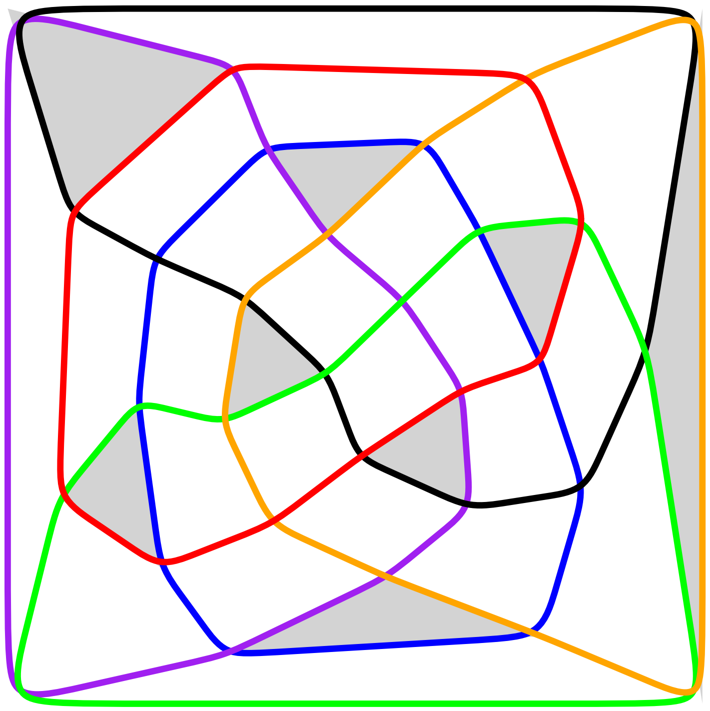
- $\Rightarrow$ Grünbaum's Conjecture might still be true for arrangements of circles!



Triangles in Arrangements with Digons

Theorem. $p_3 \geq 2n/3$

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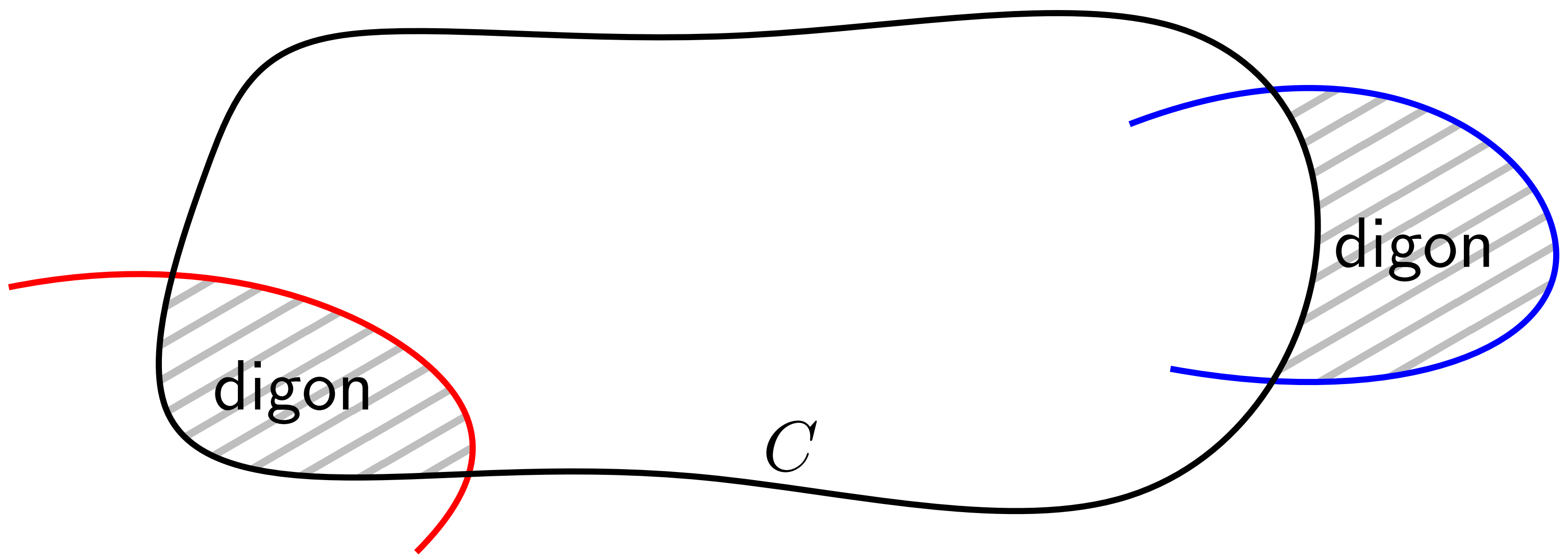
Proof. Let C be a pseudocircle in an arrangement $\mathscr{A}$.

All digons incident to C lie on the same side of C.



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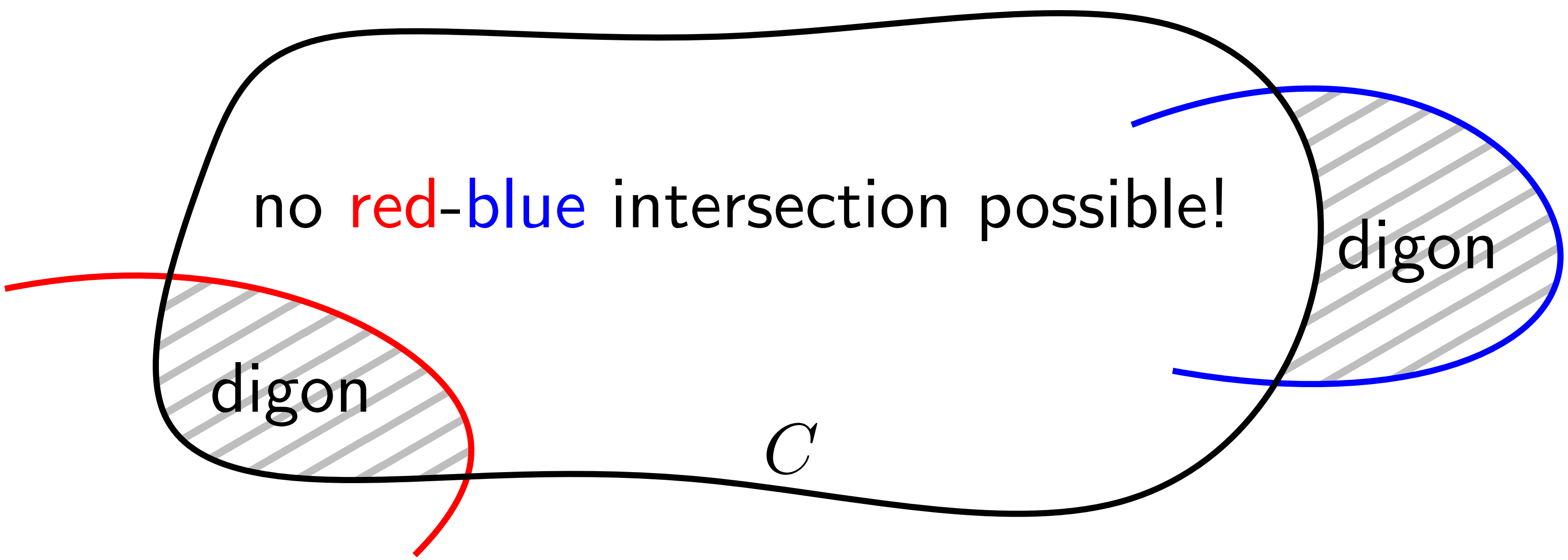


Triangles in Arrangements with Digons

Theorem. $p_3 \geq 2n/3$

Proof. Let C be a pseudocircle in an arrangement $\mathscr{A}$.

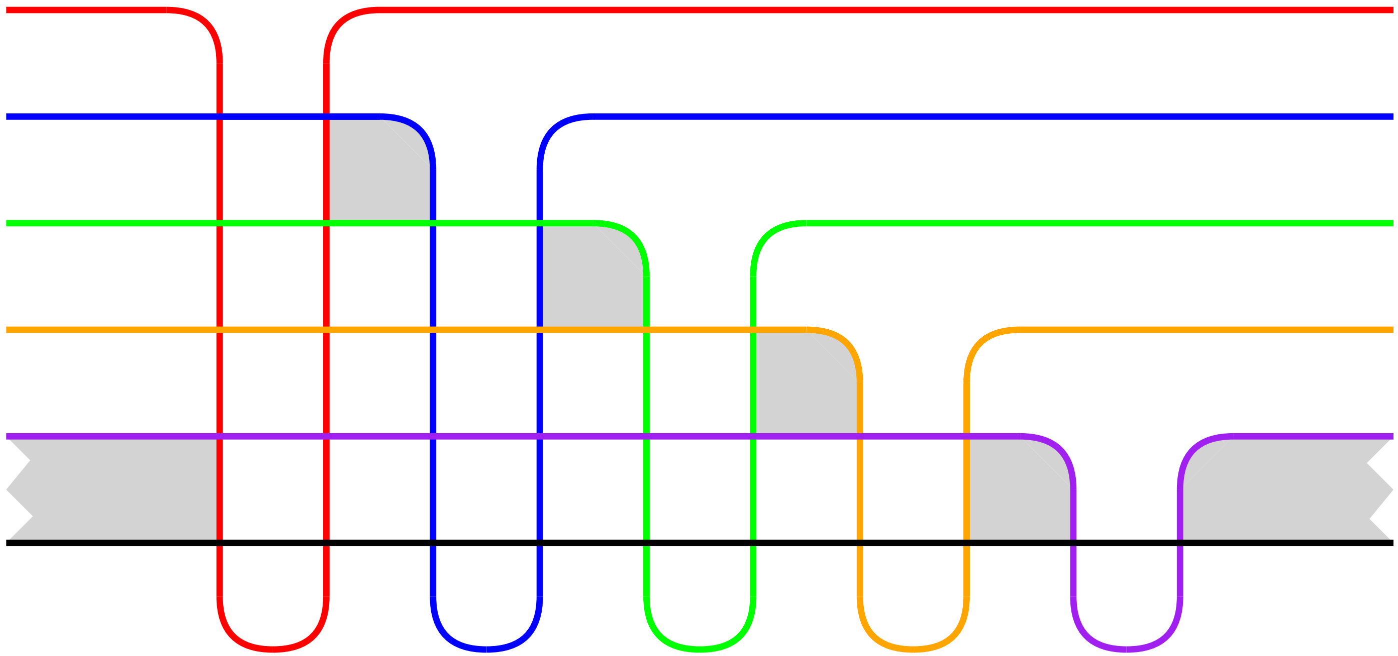
All digons incident to C lie on the same side of C.

There are at least two digons or triangles on each side of C [Hershberger and Snoeyink '91] .

Triangles in Arrangements with Digons

Theorem. $p_3 \geq 2n/3$

Conjecture. $p_3 \geq n - 1$



Maximum Number of Triangles

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- Question: $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$?

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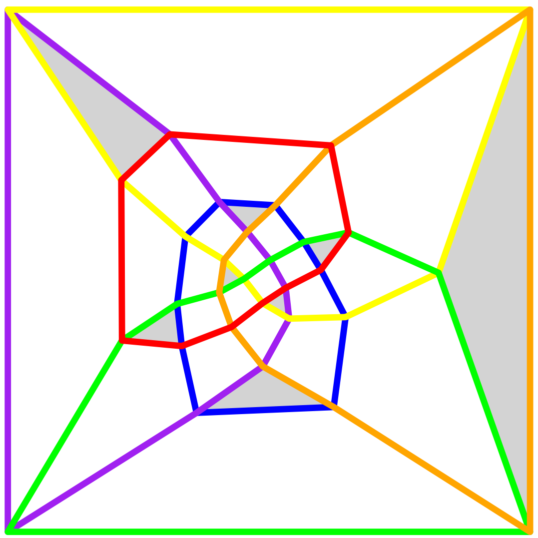
- Question: $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$?

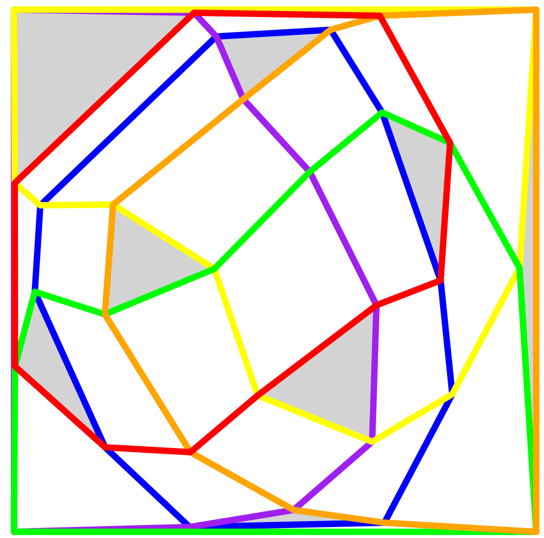
n	2	3	4	5	6	7	8	9	10
simple	0	8	8	13	20	29	≥ 37	≥ 48	≥ 60
+digon-free	-	8	8	12	20	29	≥ 37	≥ 48	≥ 60
$\lfloor \frac{4}{3}\binom{n}{2} \rfloor$	1	4	8	13	20	28	37	48	60

Visualization

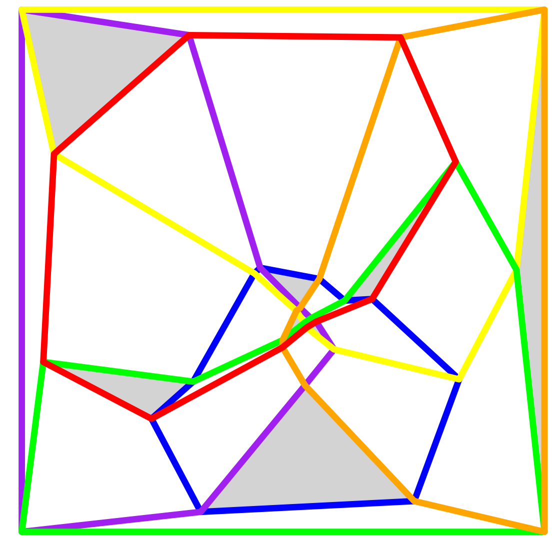
- iterated weighted Tutte embeddings
- large face $\Rightarrow$ shorten edge $\Rightarrow$ smaller face in next iter.



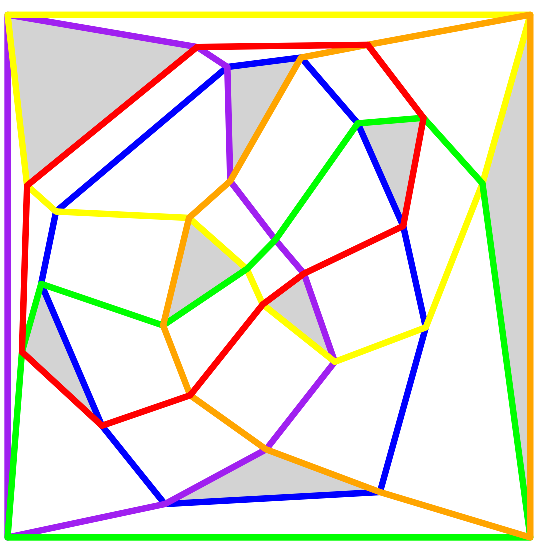
iteration 1



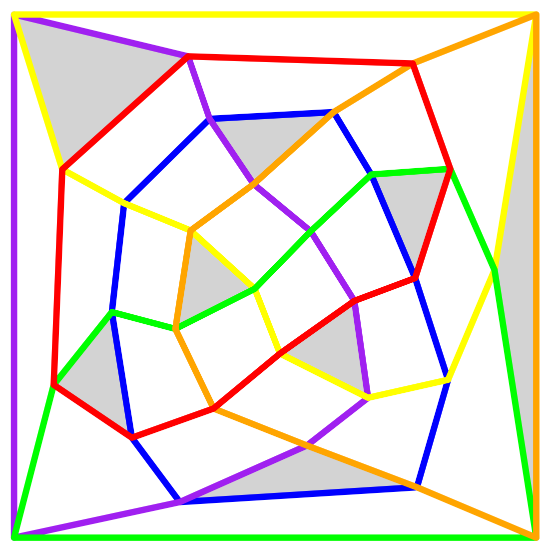
iteration 2



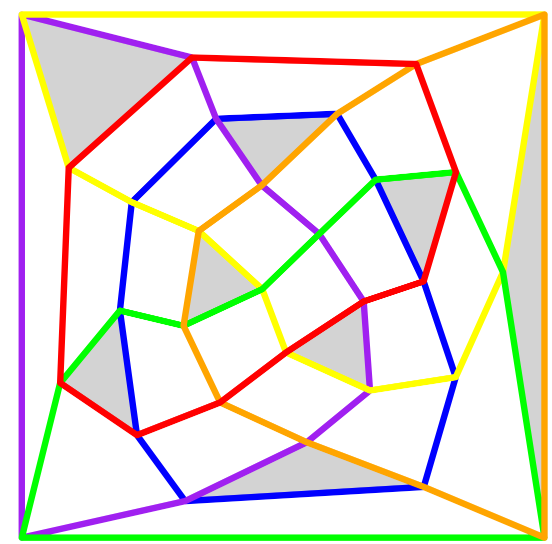
iteration 3



iteration 5



iteration 10



iteration 50

Visualization

- iterated weighted Tutte embeddings
- smoothen curves using B-splines