Arrangements of Pseudocircles:

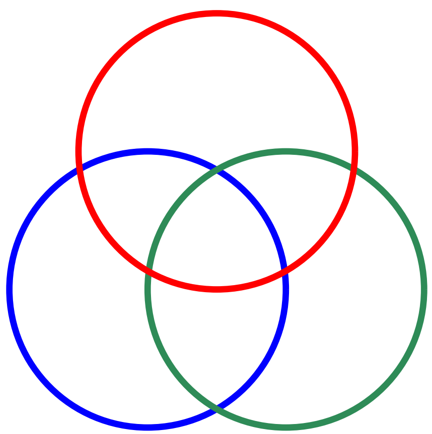
On Circularizability

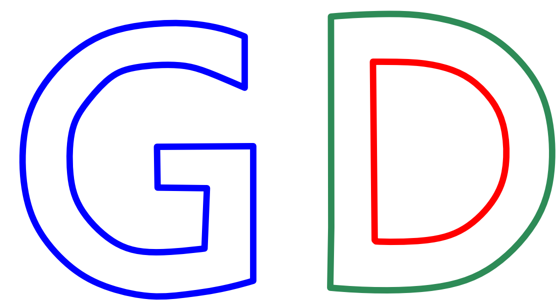
Stefan Felsner and Manfred Scheucher

Definitions

pseudocircle . . . simple closed curve

arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross





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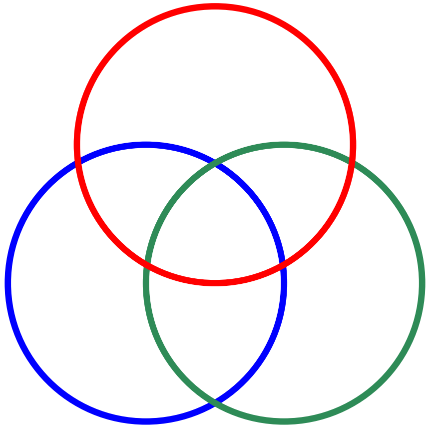
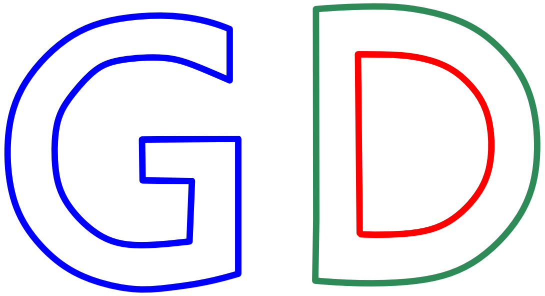
simple . . . no 3 pcs. intersect in common point

connected . . . intersection graph is connected





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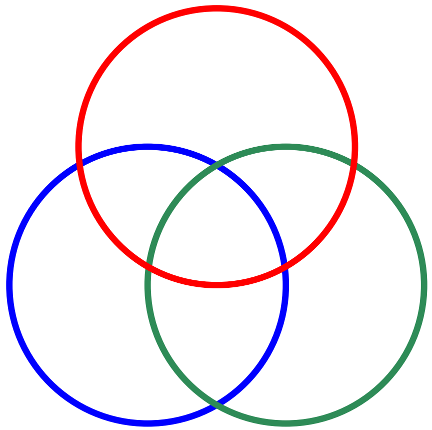

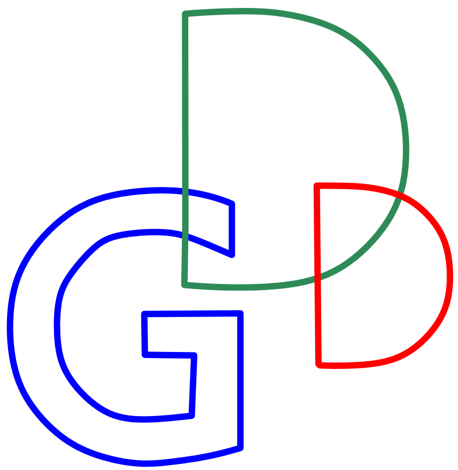

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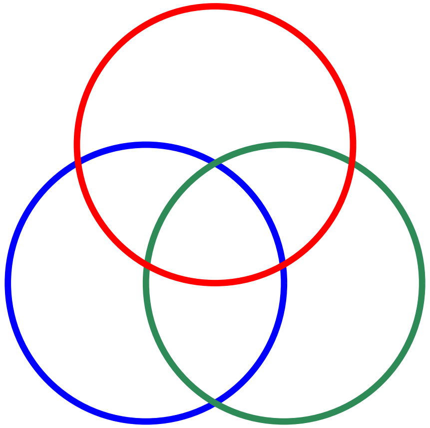
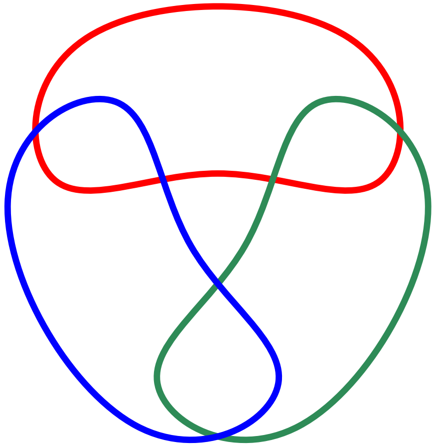
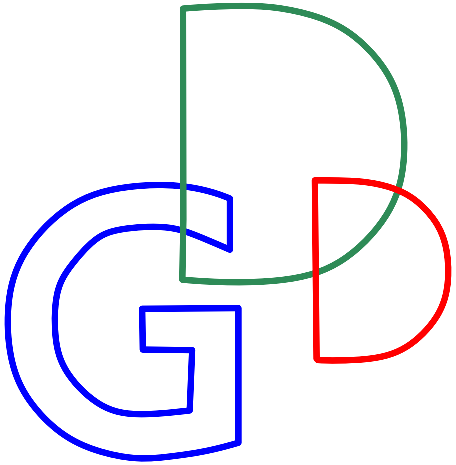


Krupp



NonKrupp



3-Chain

Definitions

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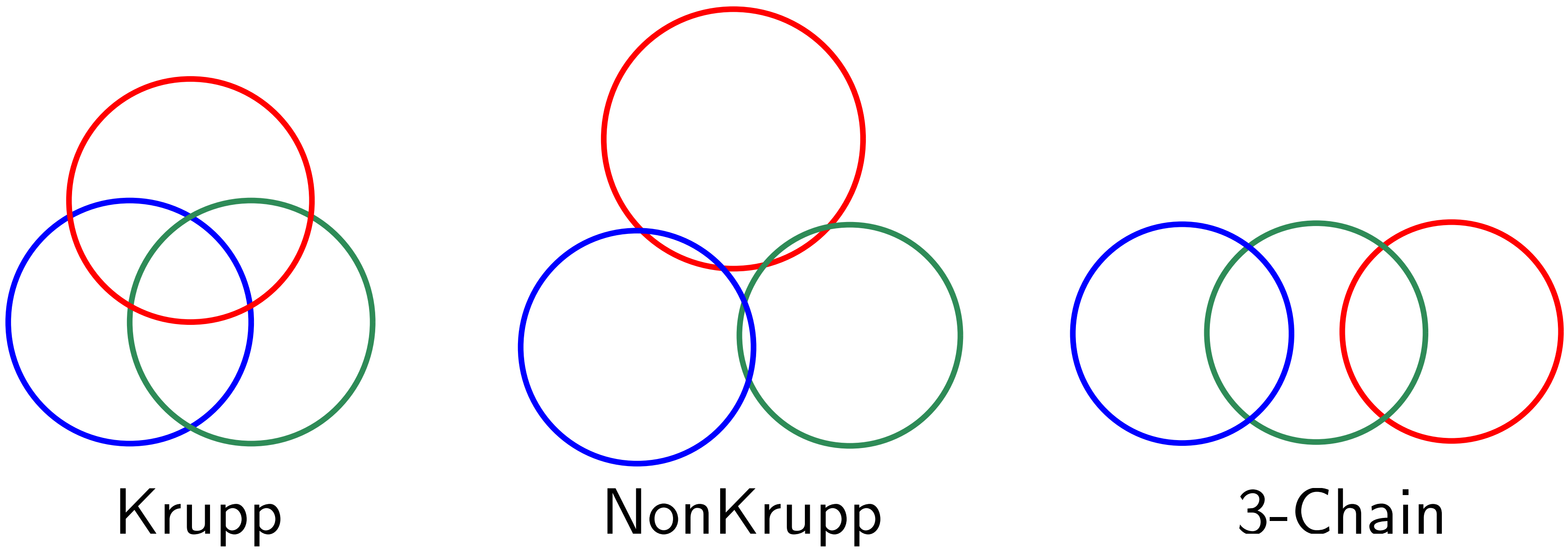
connected . . . intersection graph is connected

assumptions throughout presentation

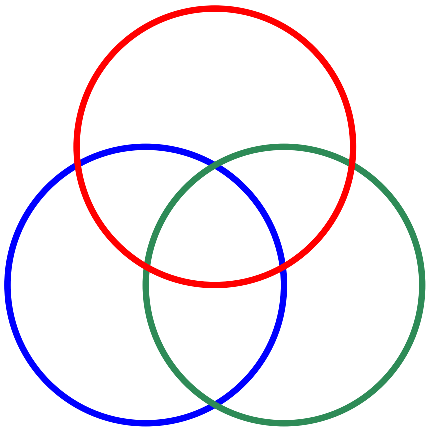
circularizable . . . $\exists$ isomorphic arrangement of circles



Classes of Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice



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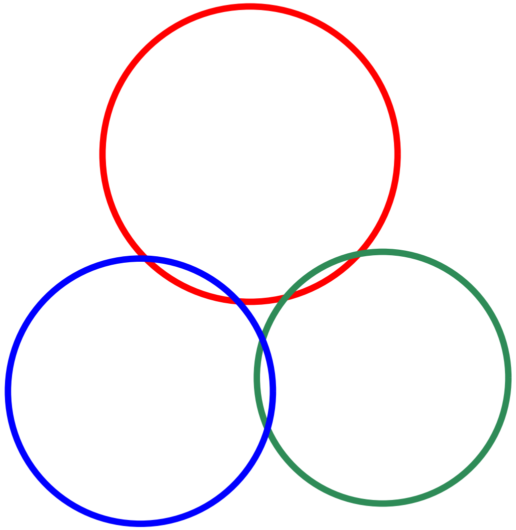

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digon-free . . . no cell bounded by two pcs.

cylindrical . . . $\exists$ two cells separated by each of the pcs.

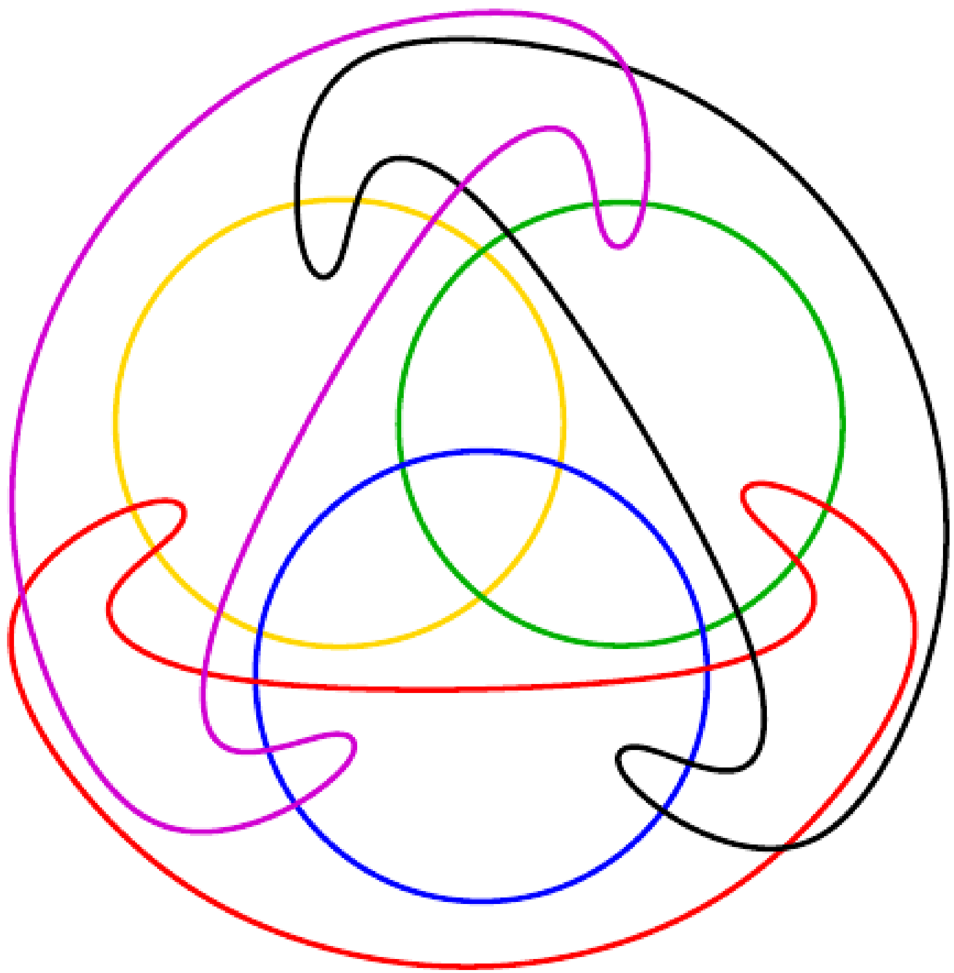
Enumeration of Arrangements

n	3	4	5	6	7
connected	3	21	984	609 423	?
+digon-free	1	3	30	4 509	?
intersecting	2	8	278	145 058	447 905 202
+digon-free	1	2	14	2 131	3 012 972
great-p.c.s	1	1	1	4	11

Table: # of combinatorially different aragements of n pcs.

Circularizability Results

- non-circularizability of intersecting $n = 6$ arrangement [Edelsbrunner and Ramos '97]



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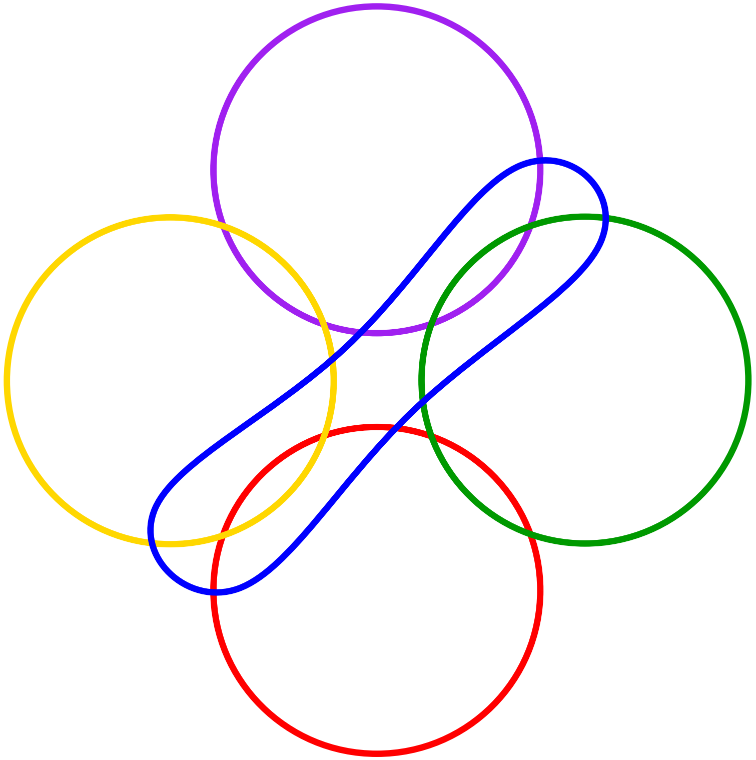

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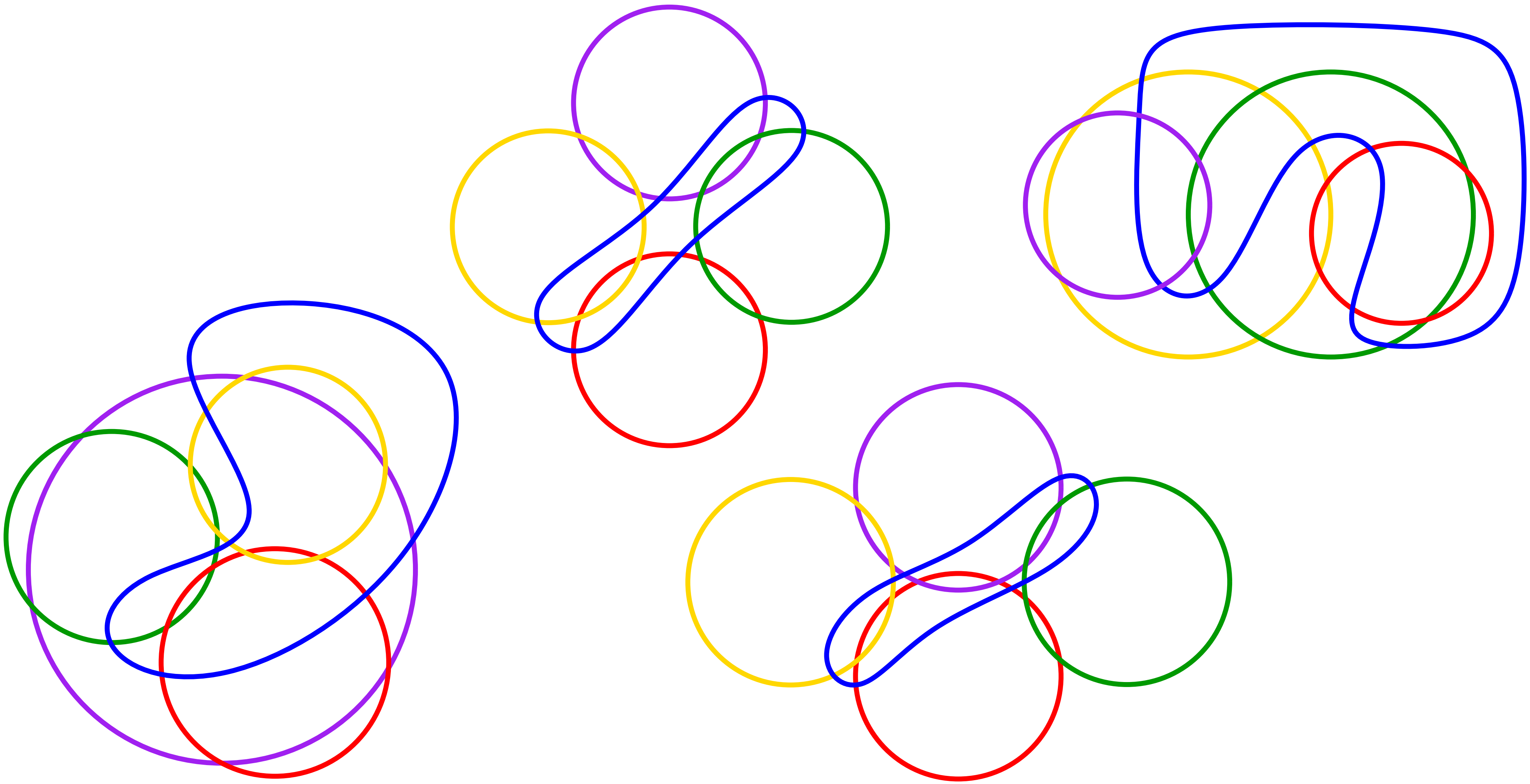
- NP-hardness of circularizability [Kang and Müller '14]

Circularizability Results

Theorem. There are exactly 4 non-circularizable $n = 5$ arrangements (984 classes).



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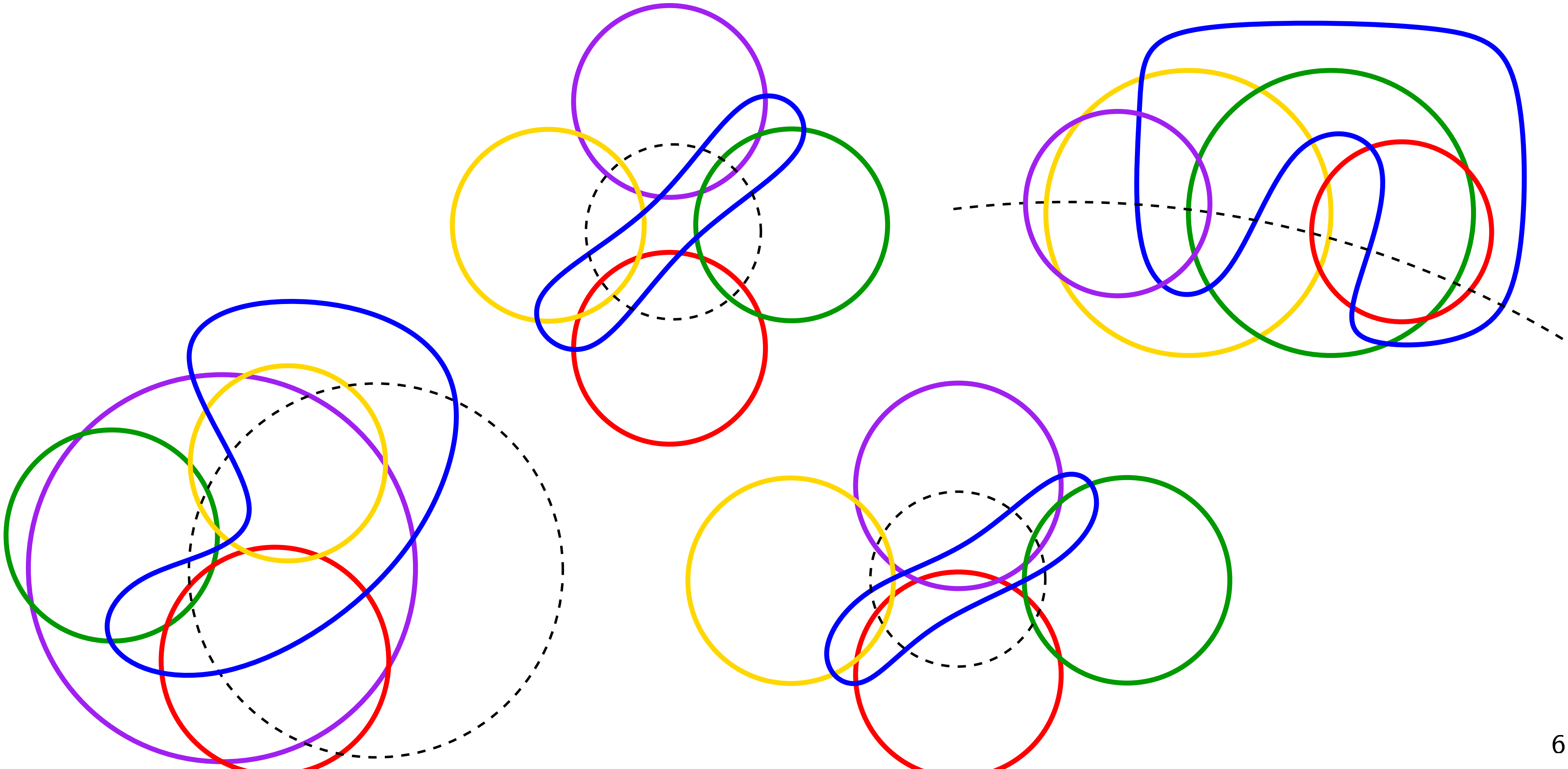
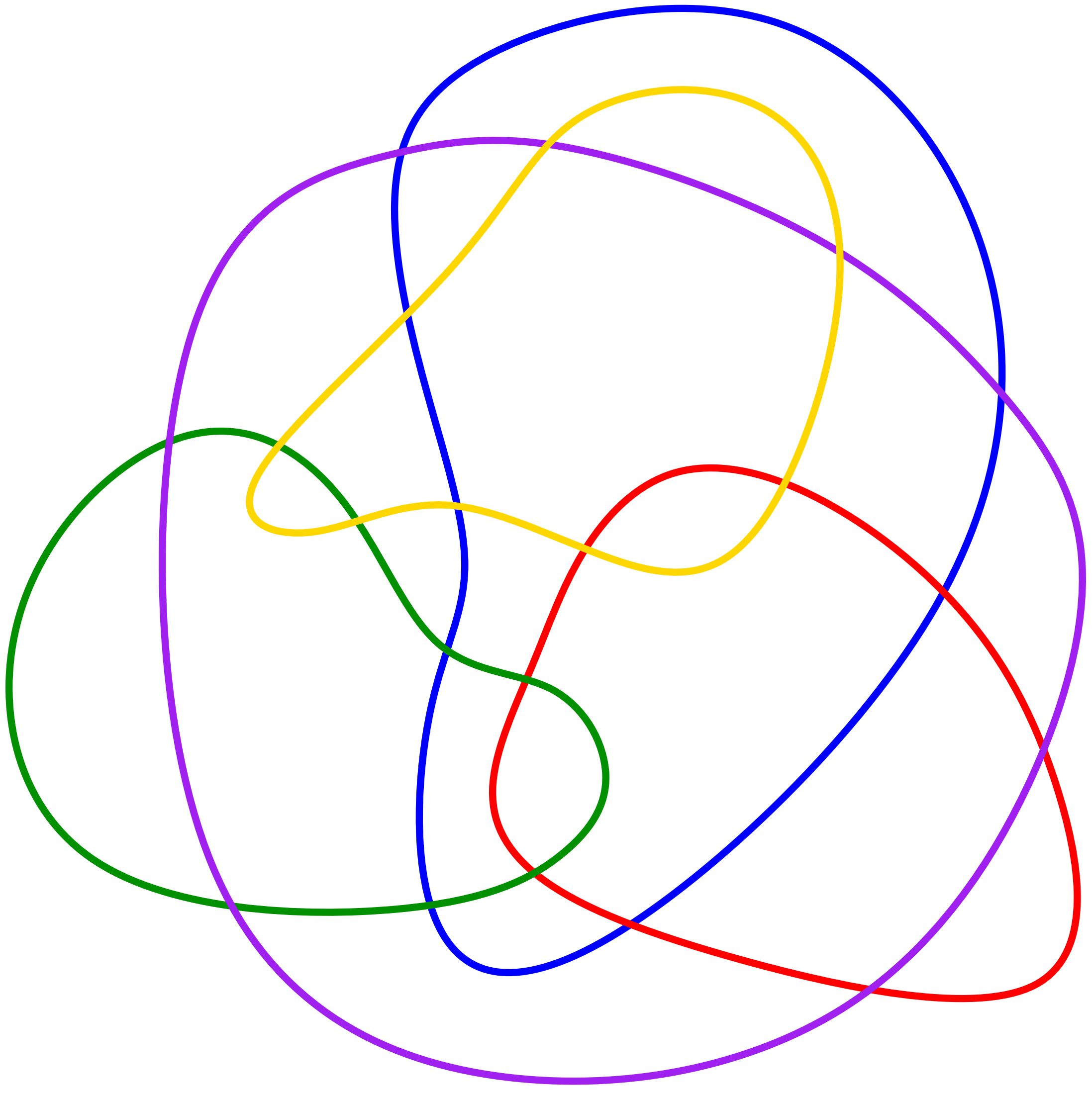
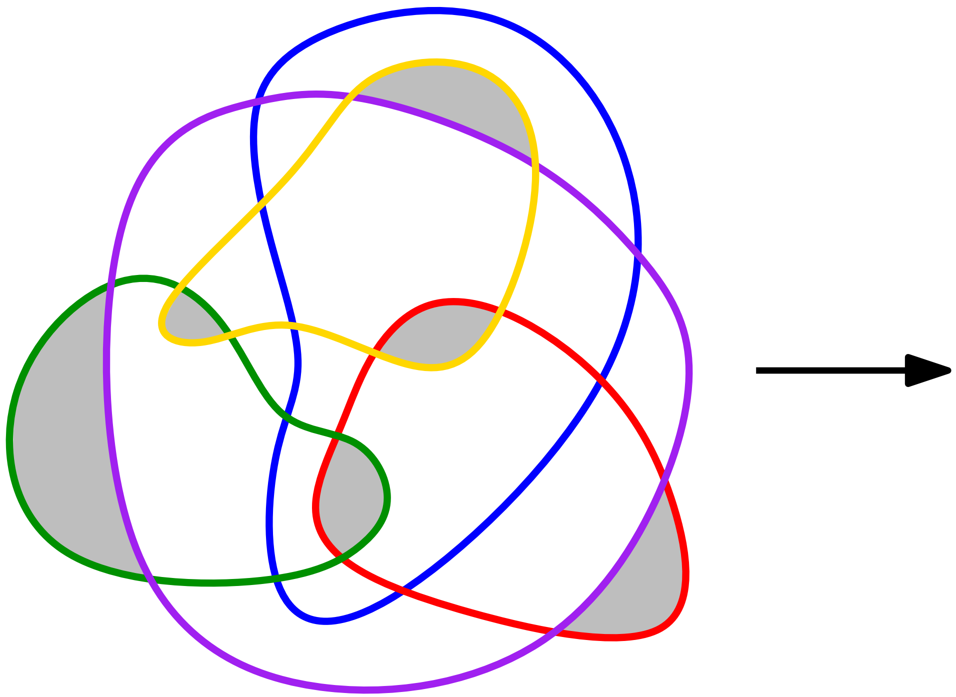


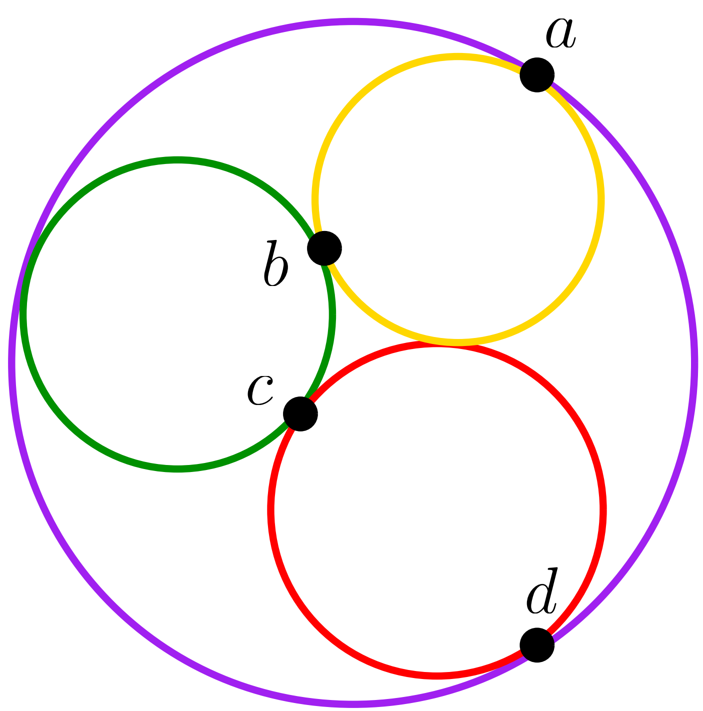
Noncircularizability of $\mathcal{N}_5^1$



Noncircularizability of $\mathcal{N}_5^1$

- assume there is a circle representation of $\mathcal{N}_5^1$
- shrink the yellow, green, and red circle
- cyclic order is preserved (also for blue)





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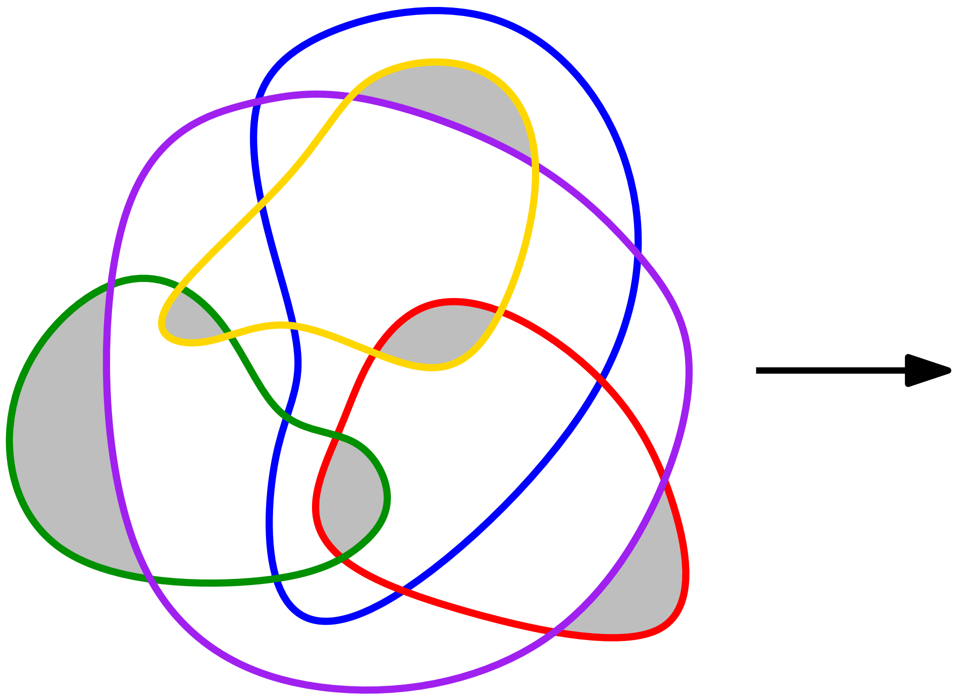

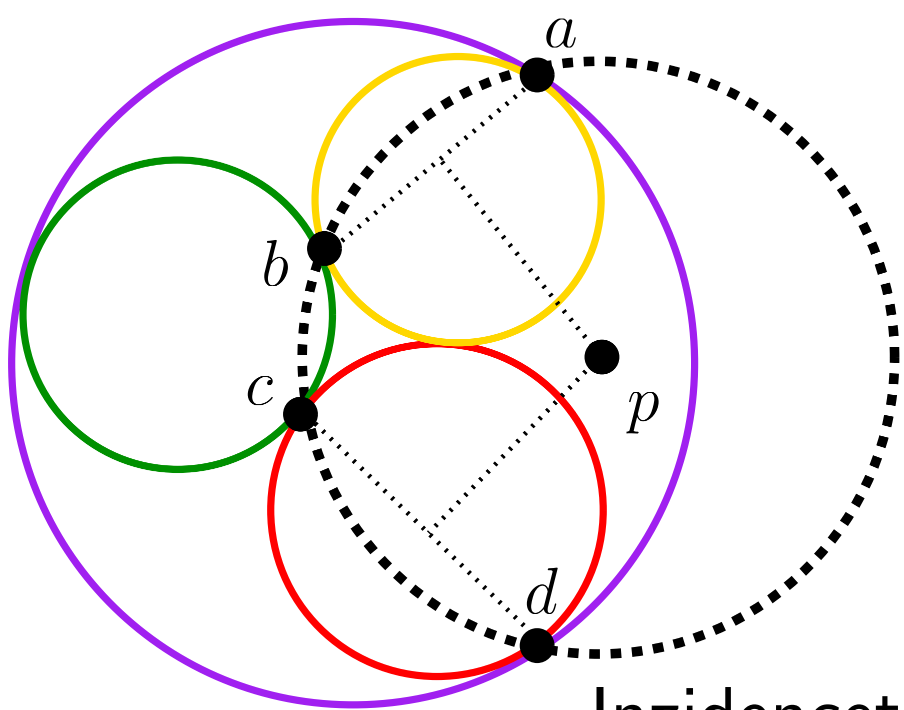

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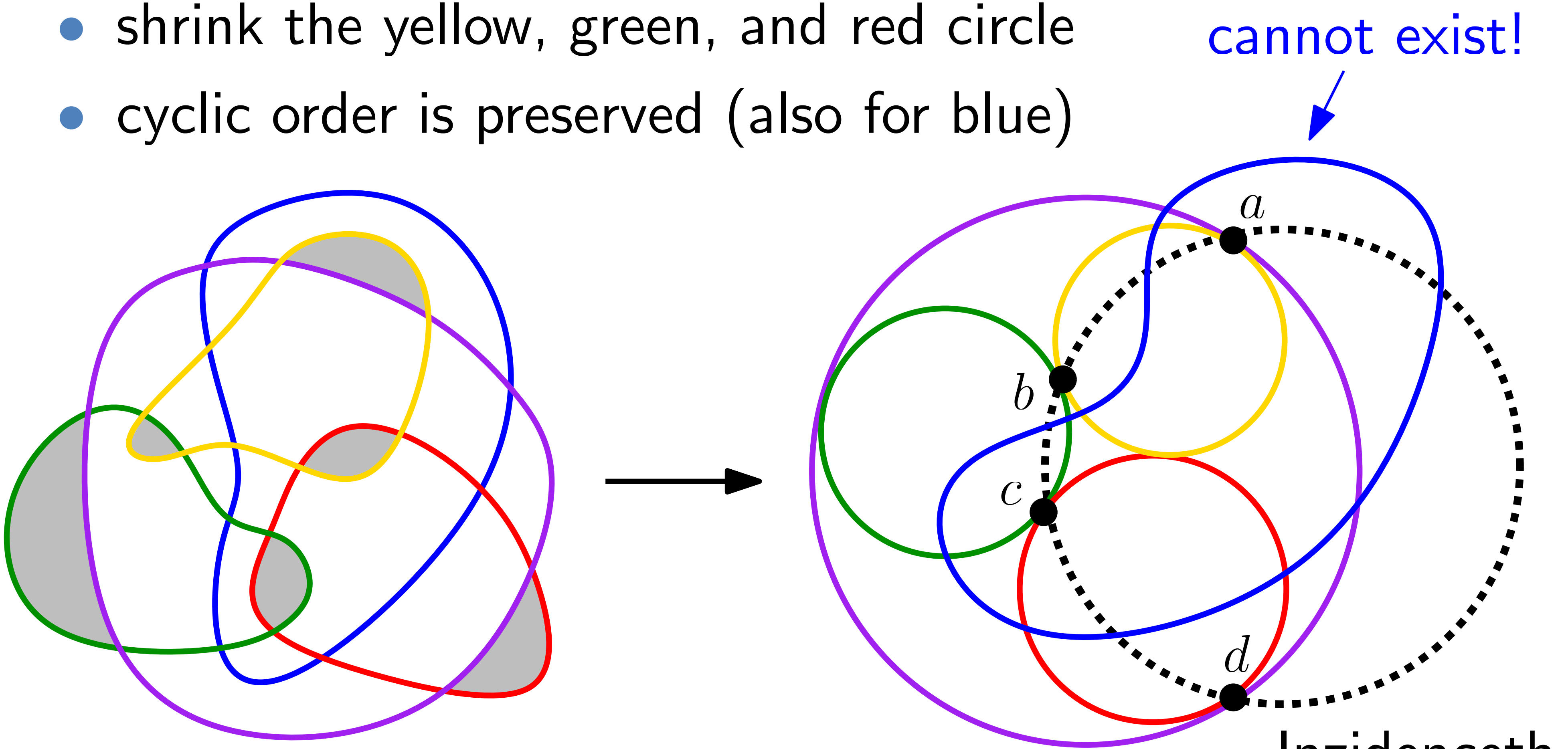
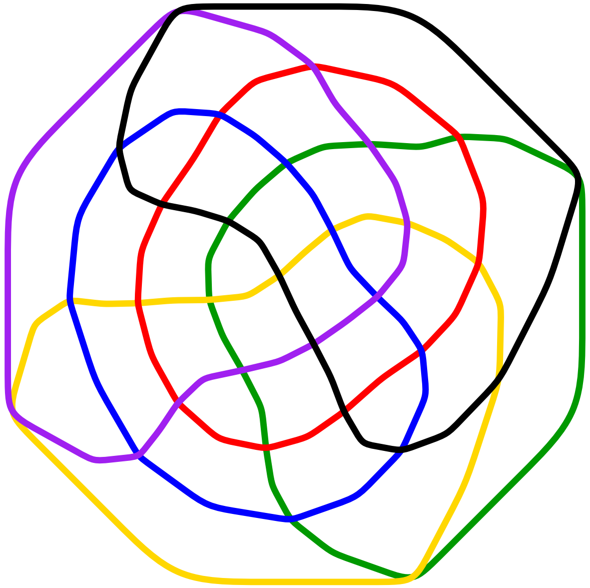
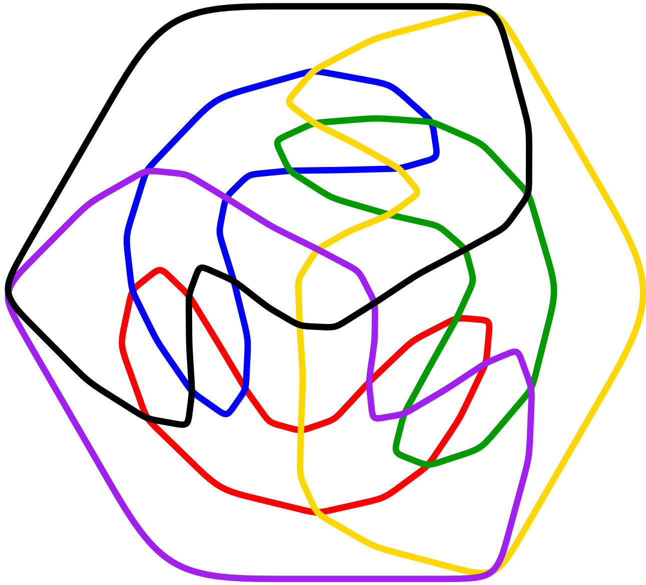
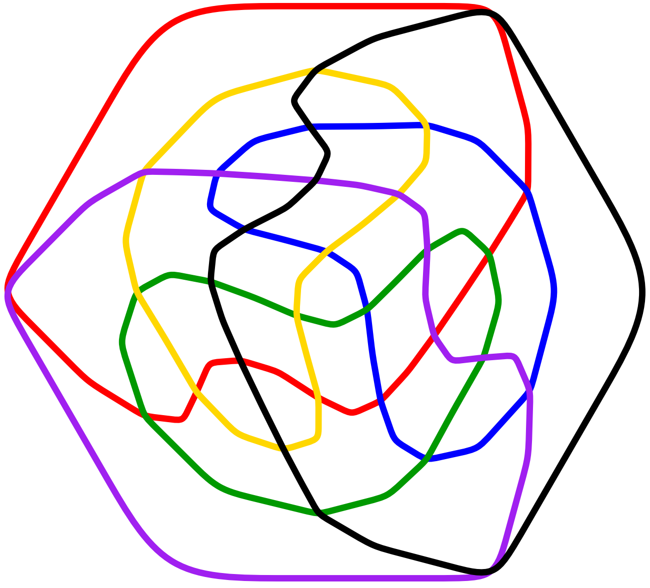


- blue and black: 4 crossings – contradiction

Circularizability Results

Theorem. There are exactly 3 non-circularizable digon-free intersecting $n = 6$ arrangements (2131 classes).

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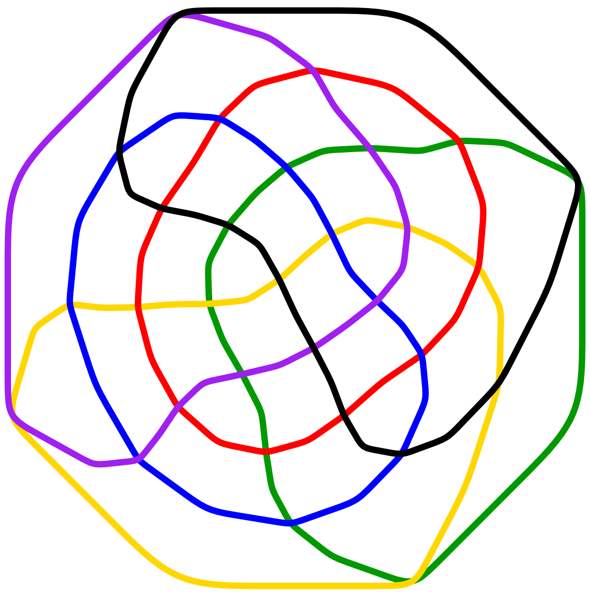

$\mathcal{N}_6^{\triangle}$ is unique digon-free intersecting with 8 triangular cells

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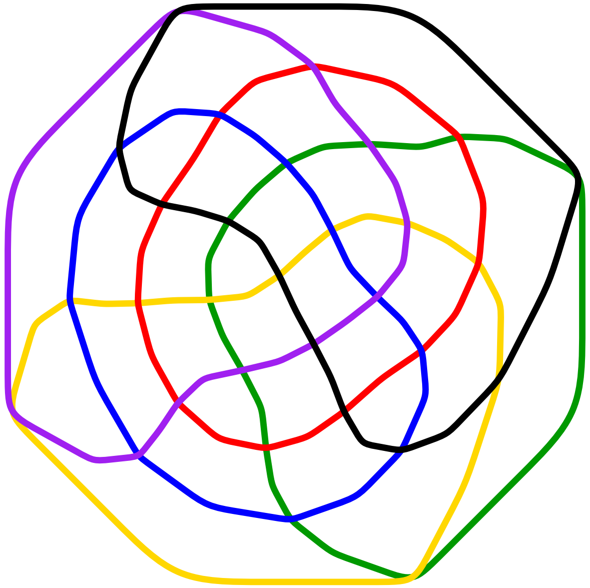

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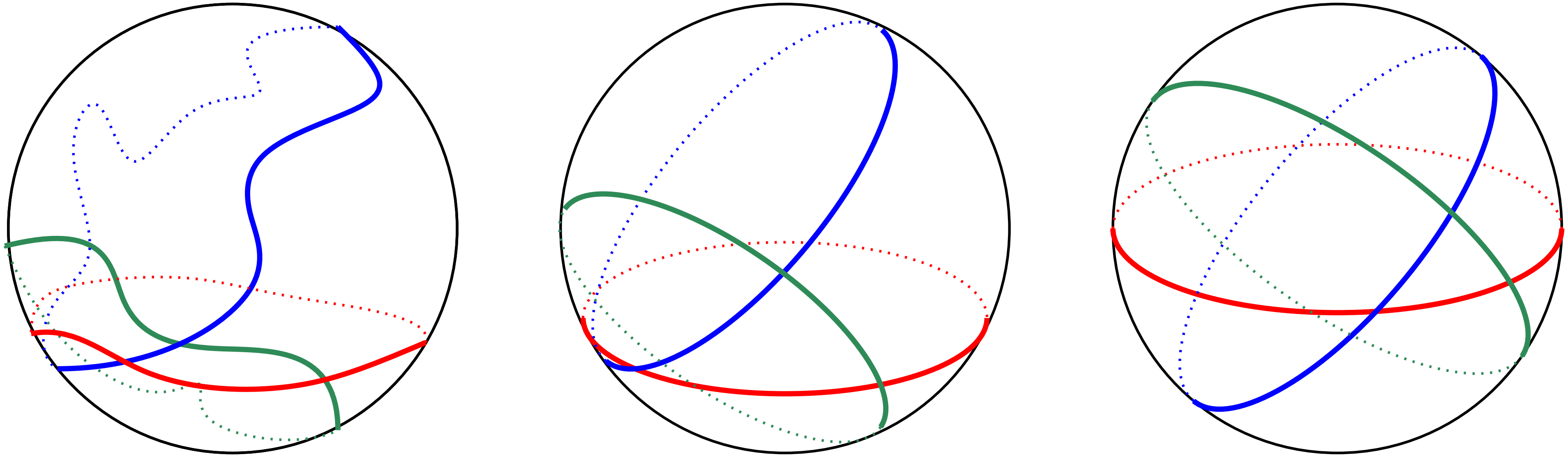
non-circularizability proof based on sweeping argument in 3-D

Great-(Pseudo)Circles

Great-Circle Theorem:

An arr. of great-pcs. is circularizable (i.e., has a circle representation) if and only if it has a great-circle repr.



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Proof.

$C_1, \ldots, C_n$ . . . circles on sphere realizing the arrangement

$E_1, \ldots, E_n$ . . . planes spanned by $C_1, \ldots, C_n$

for $t \geq 1$, sweep E_i to $\frac{1}{t} E_i$ (towards origin)

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all triples are Krupp, thus intersections remain inside sphere during sweep, thus no flip

as $t \to \infty$, we obtain great-circle arrangement $\square$

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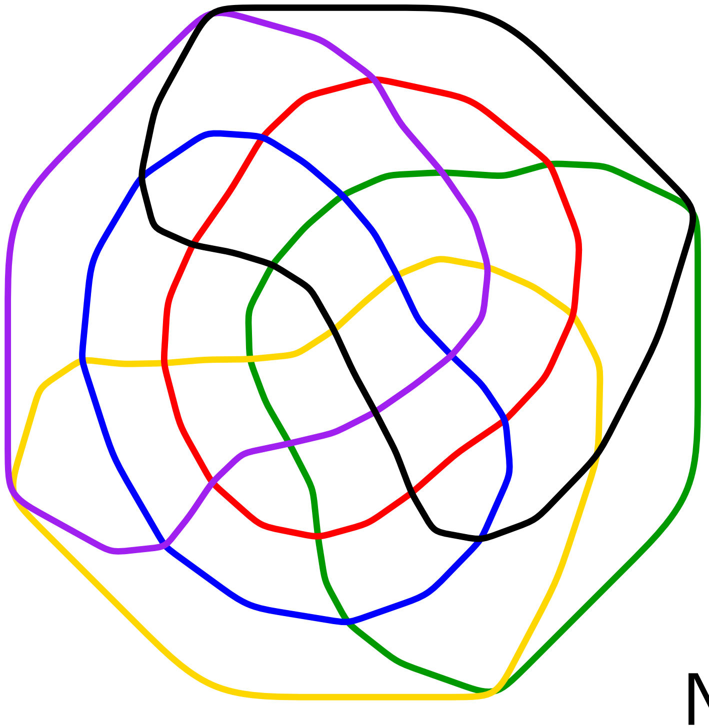
Non-Circularizability Proofs of $\mathcal{N}_6^{\triangle}$



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No greatcircle arr., thus events occur

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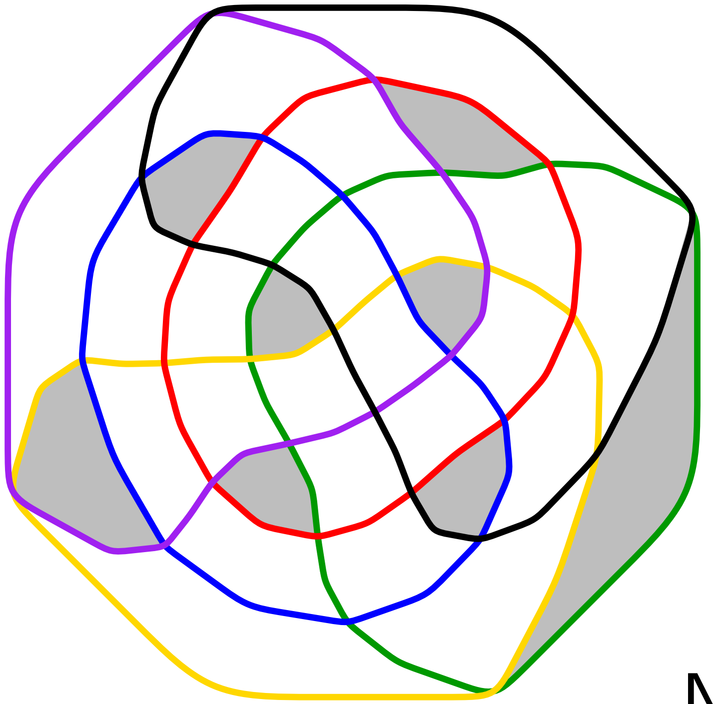

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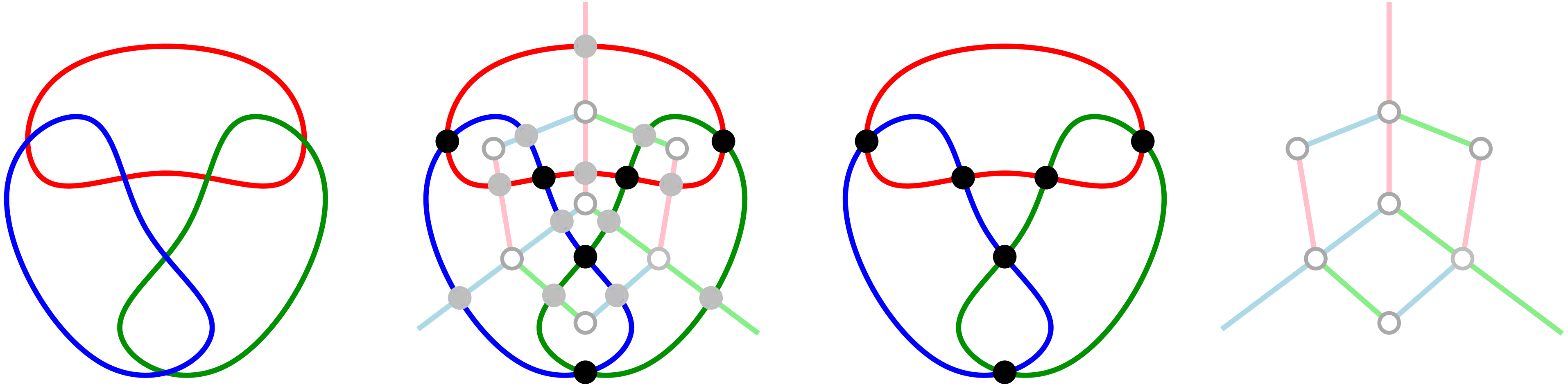
first event is triangle flip (no digons)

but triangle flip not possible as all triangles in NonKrupp $\square$

Computational Part

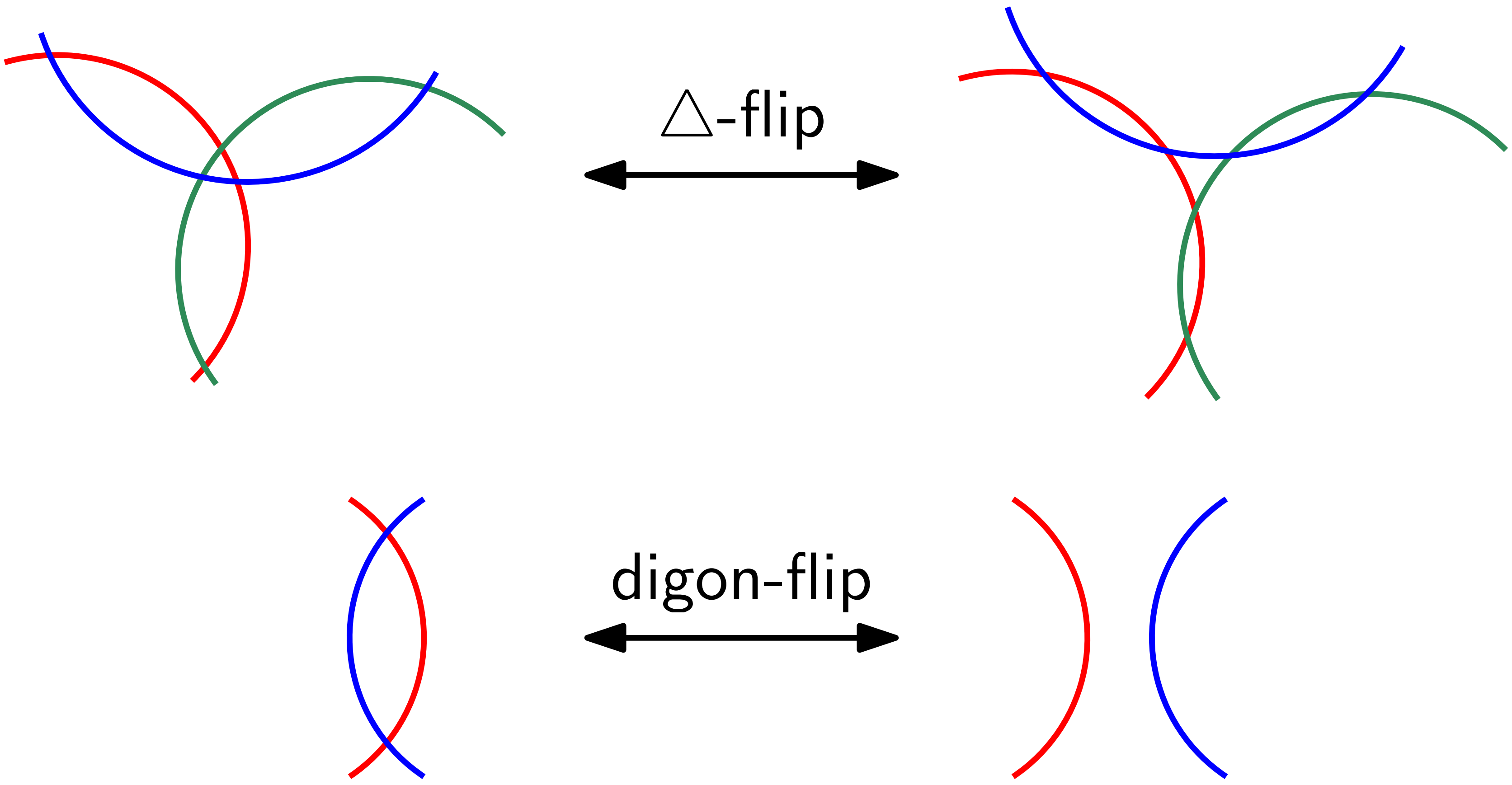
- connected arrangements encoded via primal-dual graph
- intersecting arrangements encoded via dual graph



arrangement — primal-dual gr. — primal graph — dual graph

Computational Part

- enumeration via recursive search on flip graph



Computational Part

- circle representations heuristically
- hard instances by hand

Thank you for your attention!