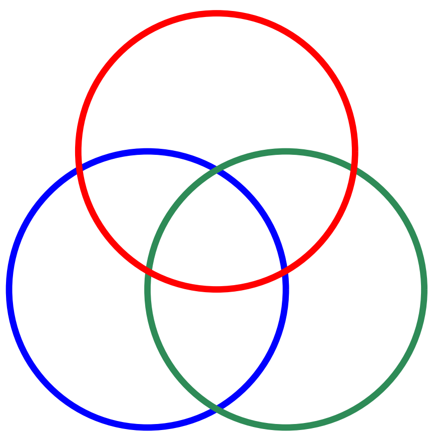
Arrangements of Pseudocircles

Stefan Felsner and Manfred Scheucher

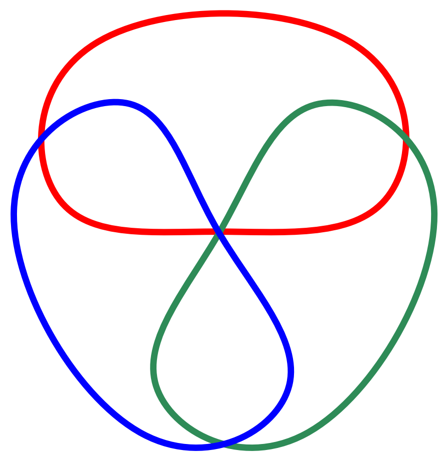
Definitions

pseudocircle . . . simple closed curve

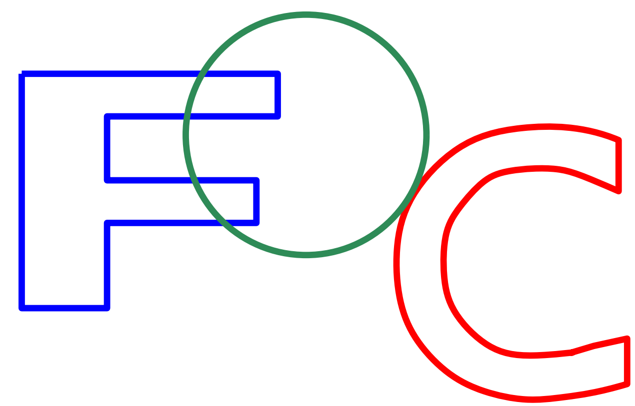
arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross



arrangement



arrangement

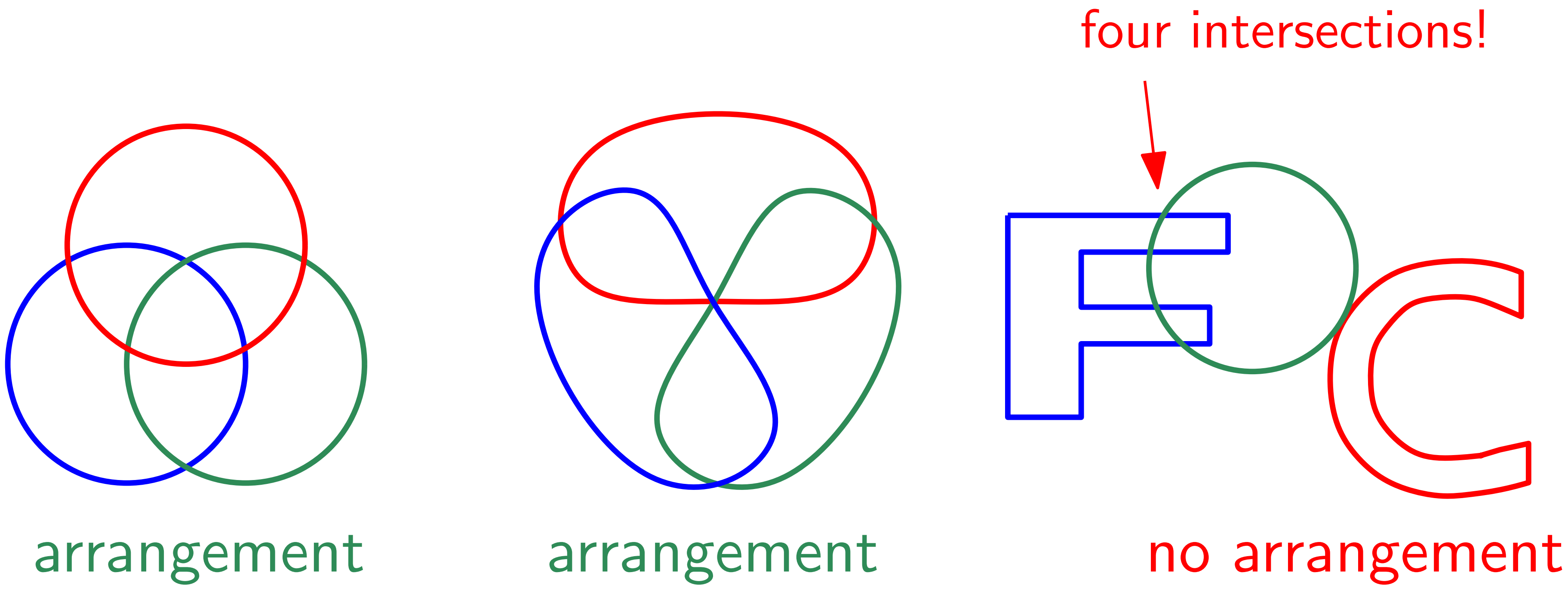


no arrangement

Definitions

pseudocircle . . . simple closed curve

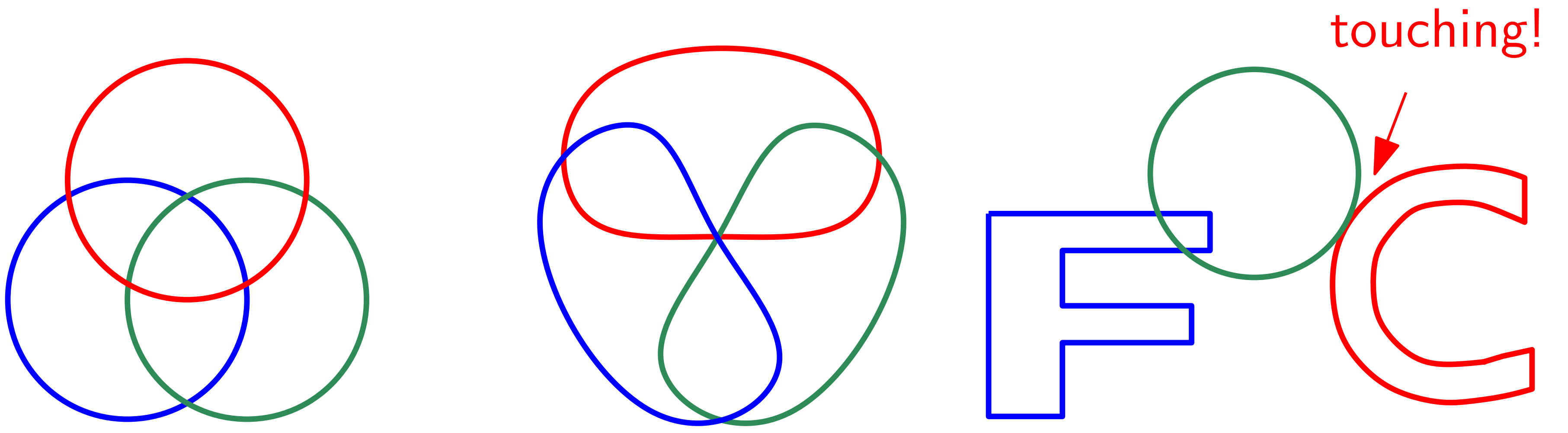
arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross



Definitions

pseudocircle . . . simple closed curve

arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross

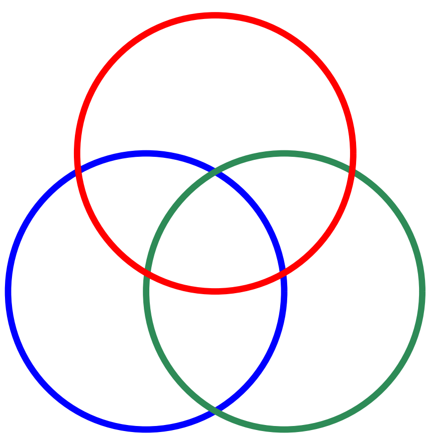


arrangement — arrangement — no arrangement

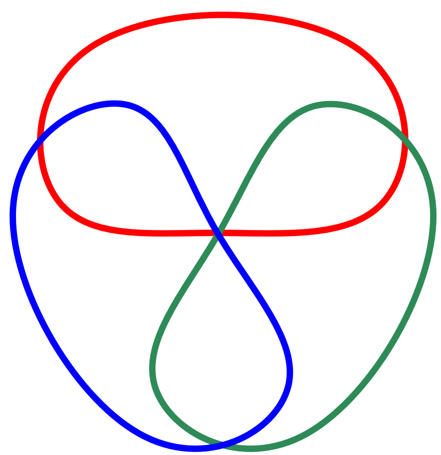
Definitions

pseudocircle . . . simple closed curve

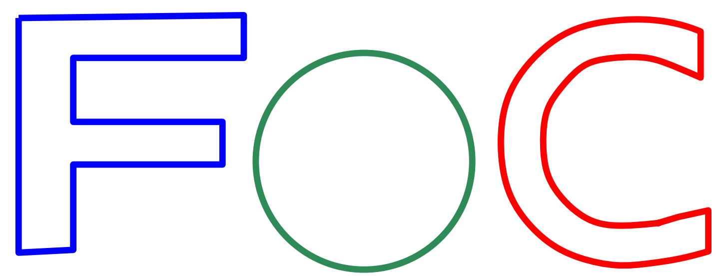
arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross



arrangement



arrangement

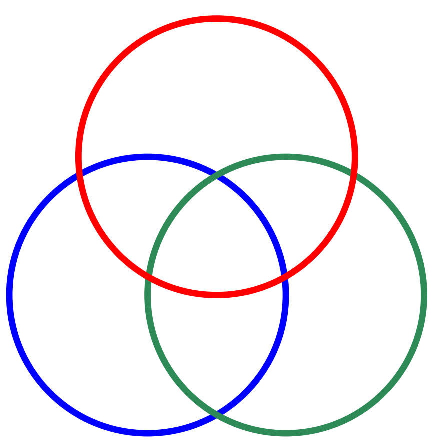


arrangement

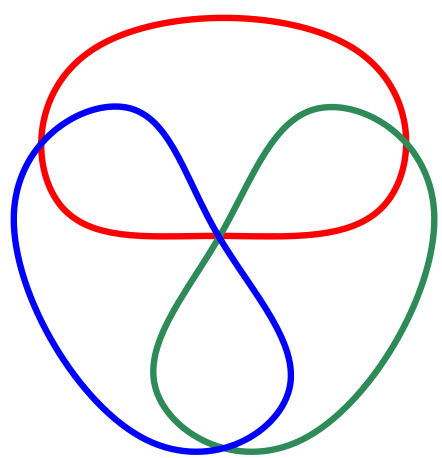
Definitions

simple . . . no 3 pcs. intersect in common point

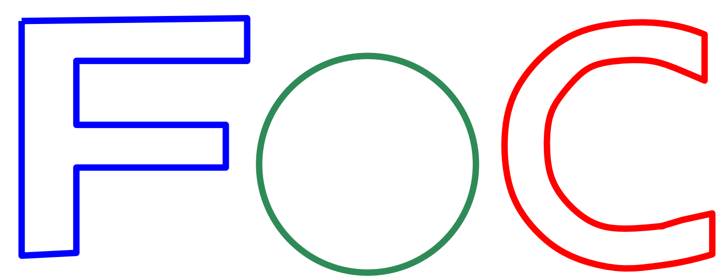
connected . . . intersection graph is connected



simple+connected



not simple

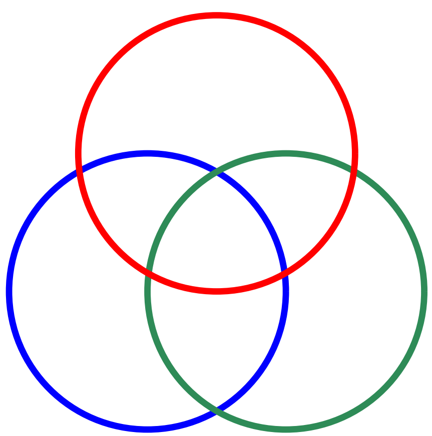


not connected

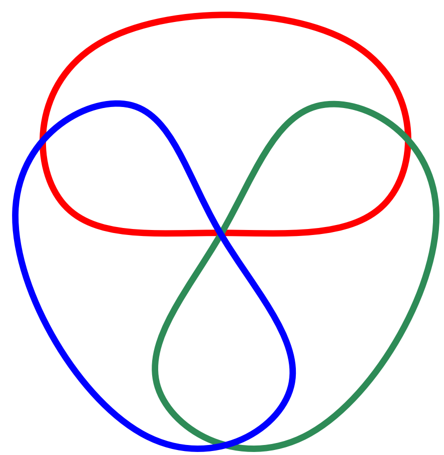
Definitions

simple . . . no 3 pcs. intersect in common point

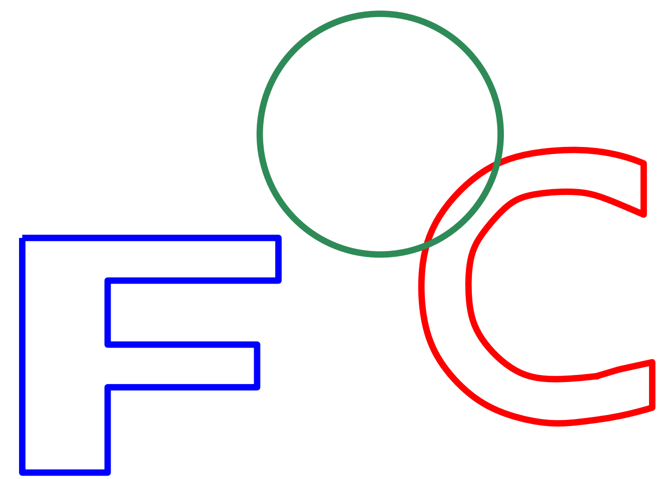
connected . . . intersection graph is connected



simple+connected



not simple



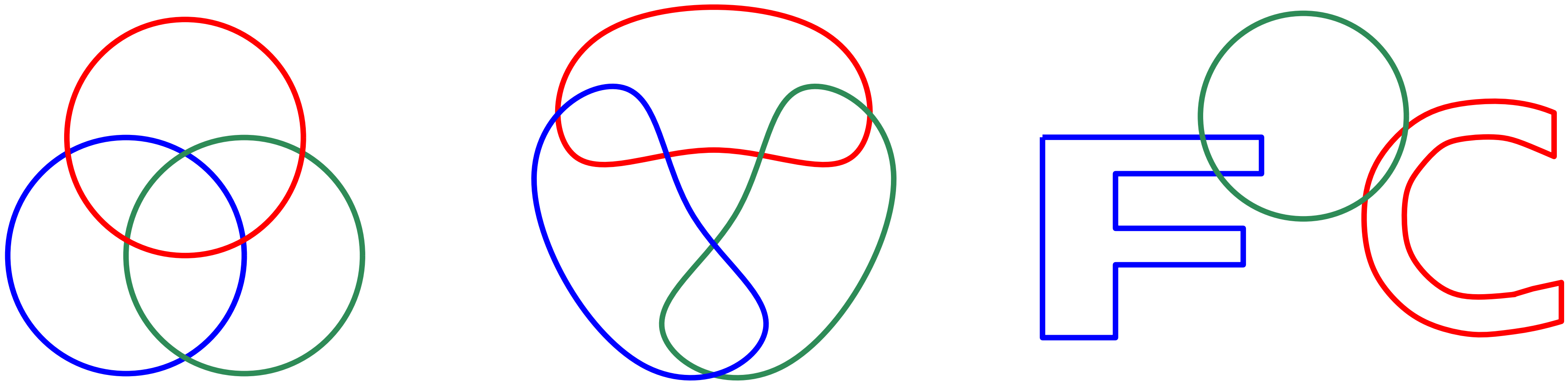
not connected

Definitions

simple . . . no 3 pcs. intersect in common point

connected . . . intersection graph is connected

assumptions throughout presentation

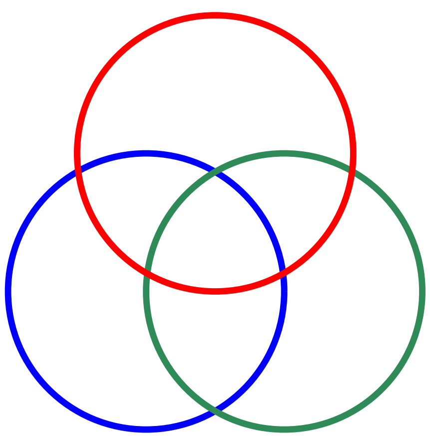


Definitions

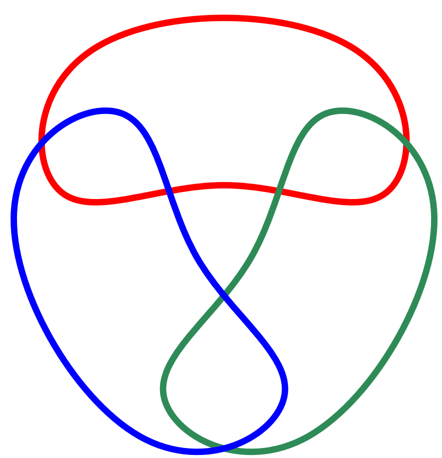
simple . . . no 3 pcs. intersect in common point

connected . . . intersection graph is connected

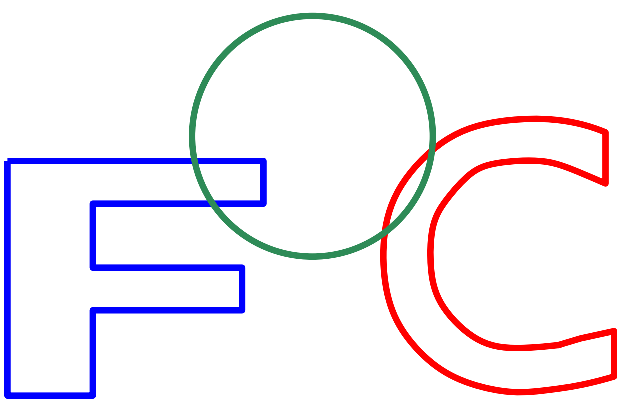
assumptions throughout presentation



Krupp



NonKrupp



3-Chain

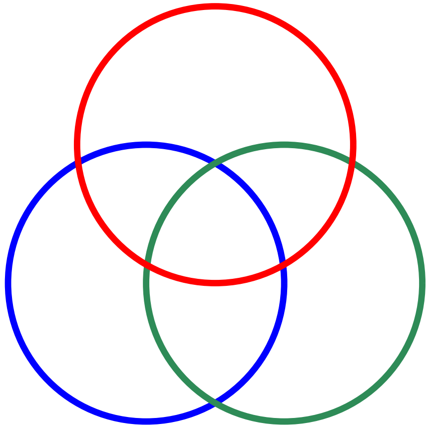
Definitions

simple . . . no 3 pcs. intersect in common point

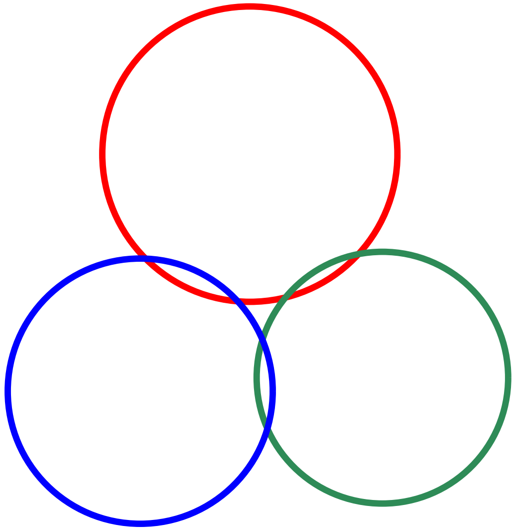
connected . . . intersection graph is connected

assumptions throughout presentation

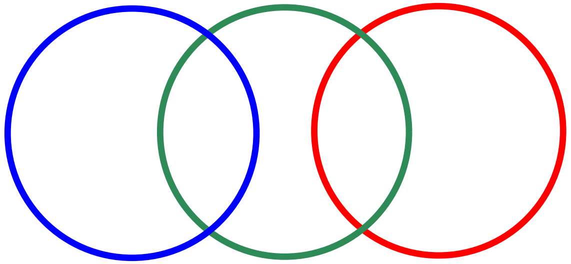
circleable . . . ∃ isomorphic arrangement of circles



Krupp



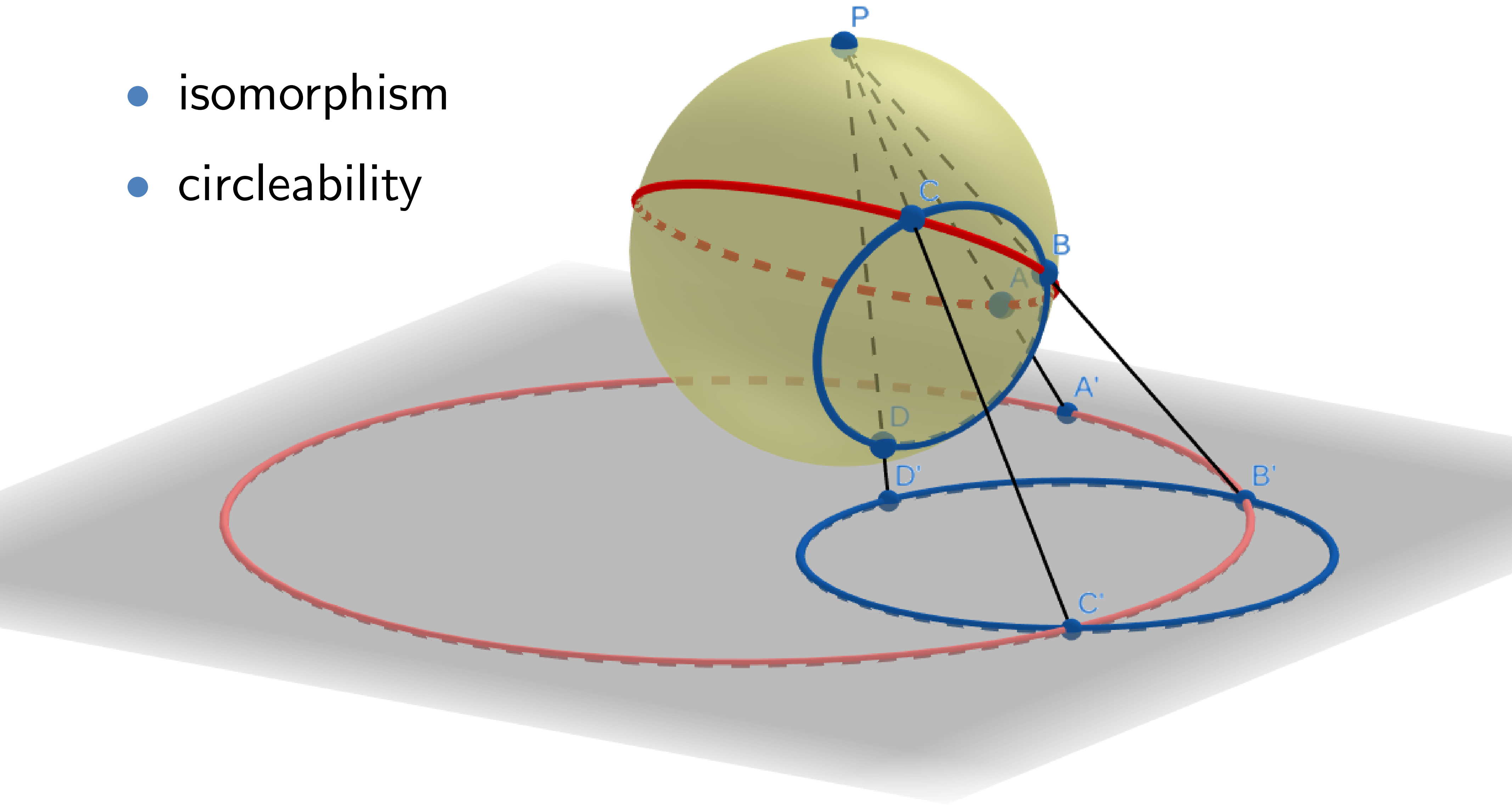
NonKrupp



3-Chain

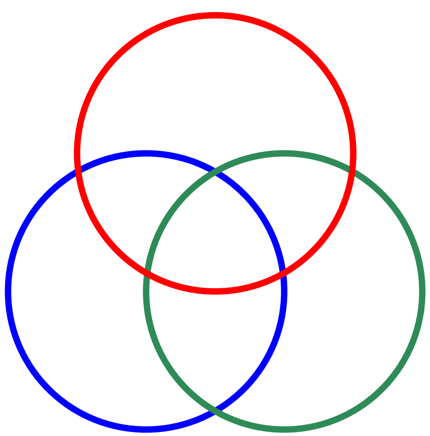
Plane VS Sphere

- isomorphism
- circleability

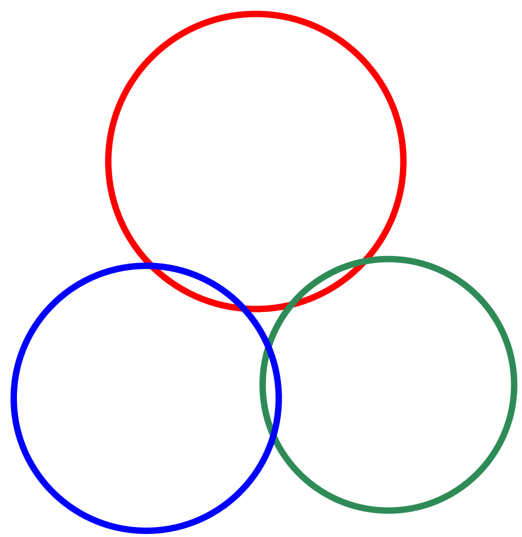


Classes of Arrangements

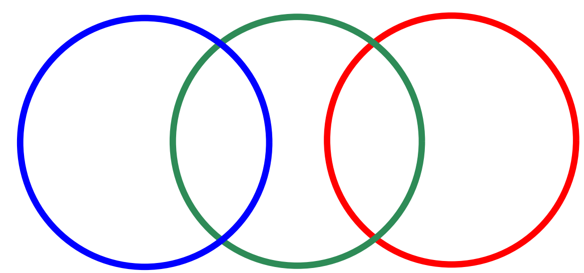
connected . . . graph of arrangement is connected



Krupp



NonKrupp



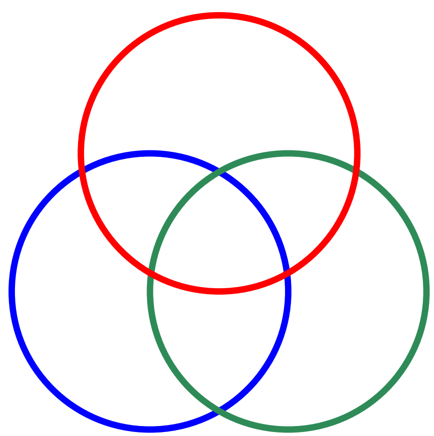
3-Chain

Classes of Arrangements

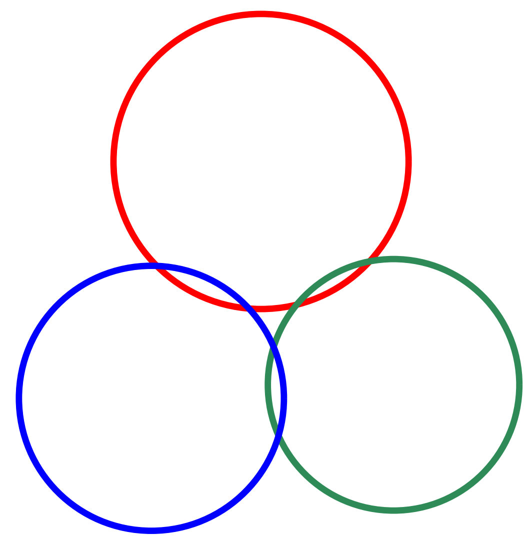
connected . . . graph of arrangement is connected

↑

intersecting . . . any 2 pseudocircles cross twice



Krupp



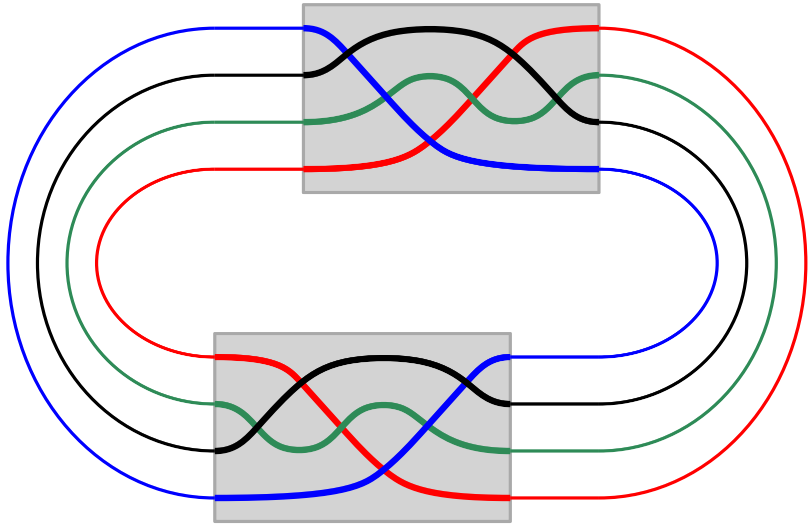
NonKrupp

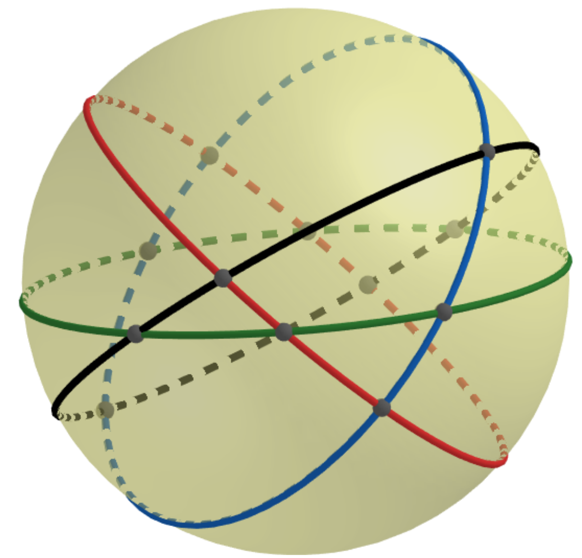
Classes of Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp



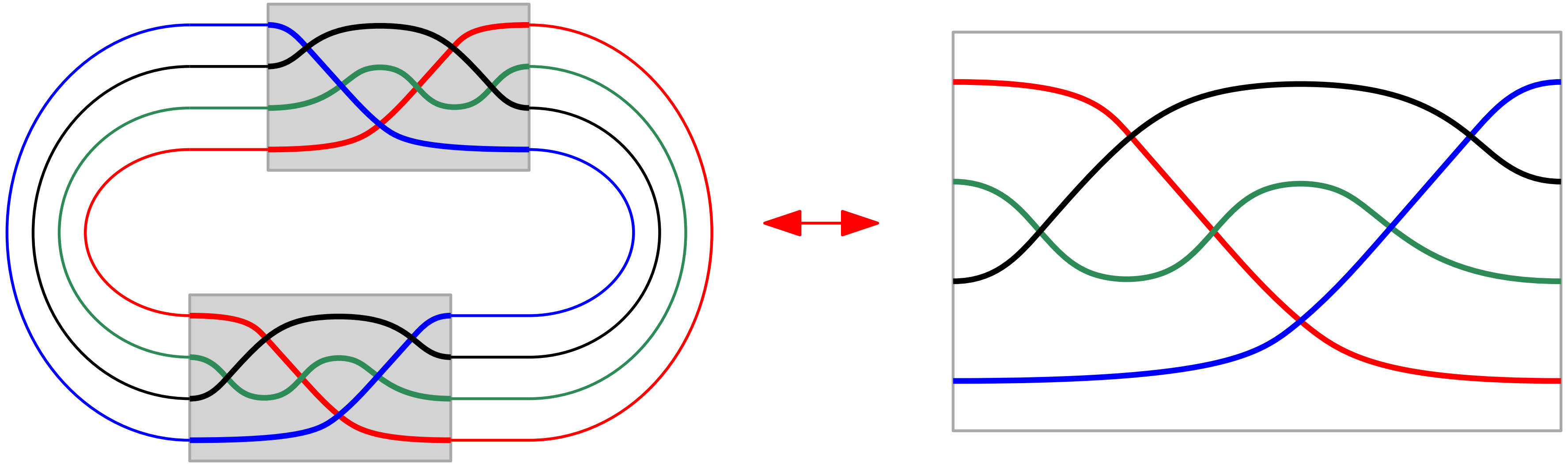


Classes of Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp



Classes of Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp

digon-free . . . no cell bounded by two pcs.

Classes of Arrangements

connected . . . graph of arrangement is connected

intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp

digon-free . . . no cell bounded by two pcs.

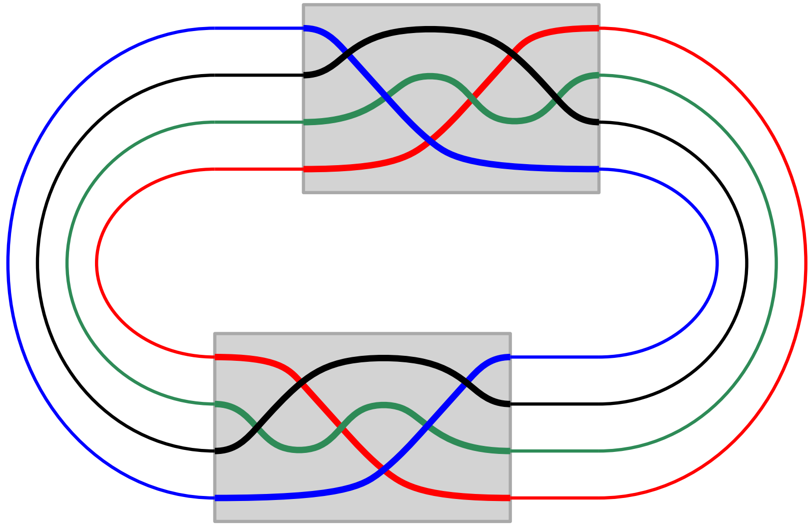
cylindrical . . . $\exists$ two cells separated by each of the pcs.

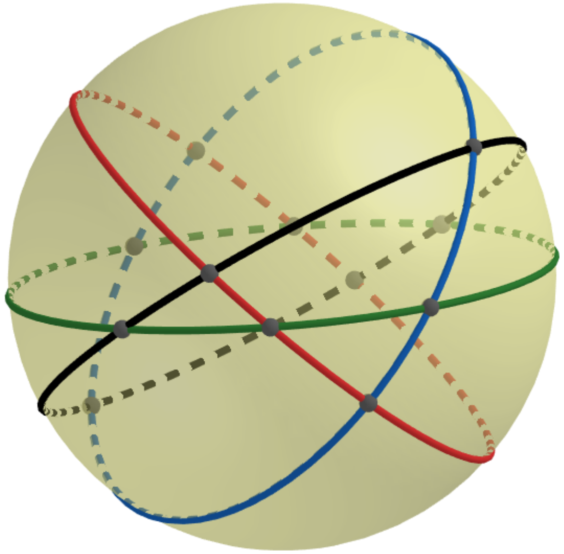
Classes of Arrangements

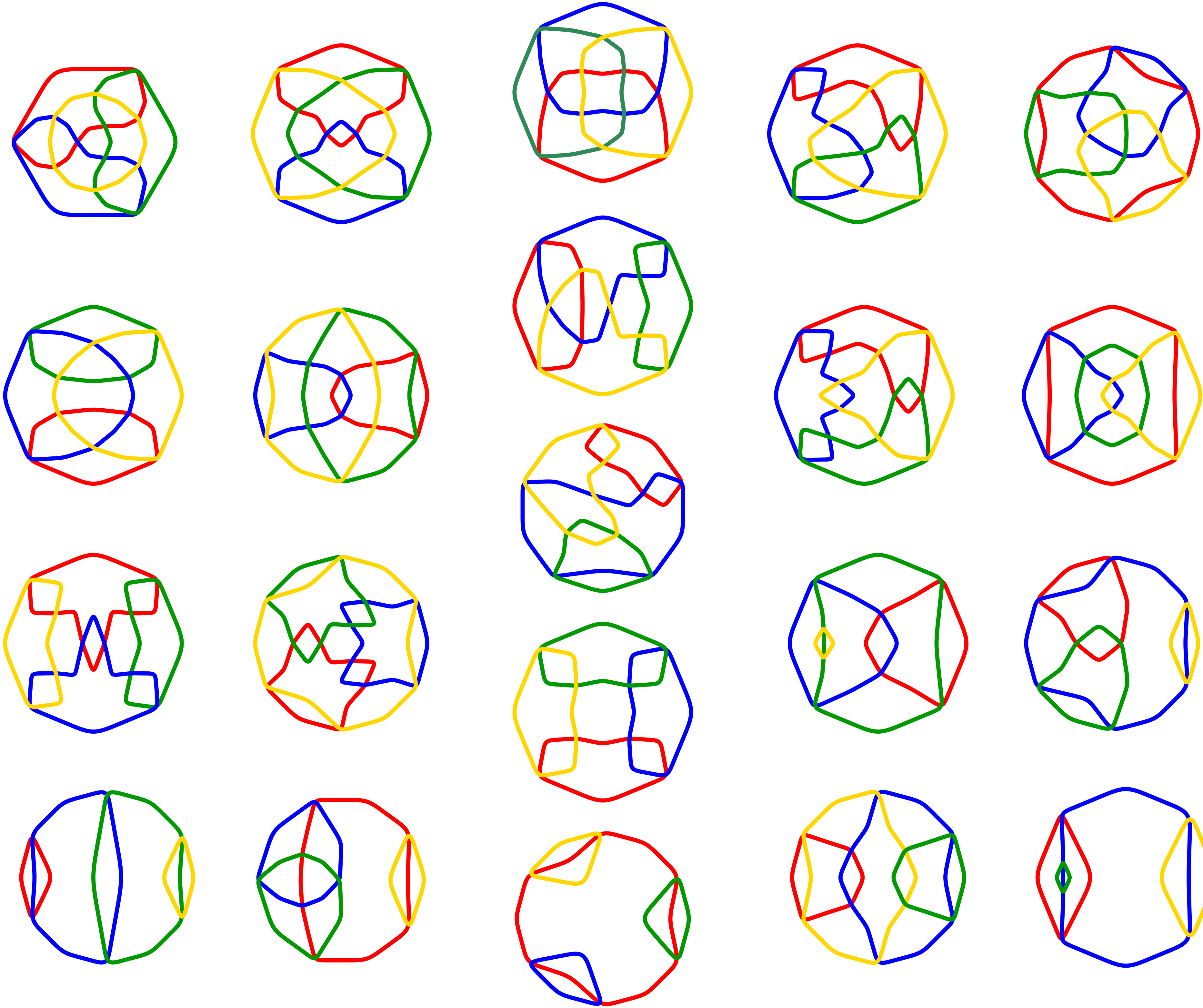
connected . . . graph of arrangement is connected

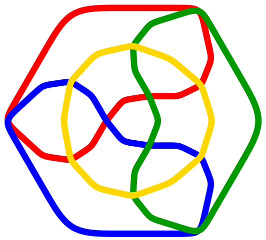
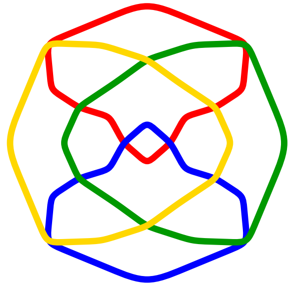
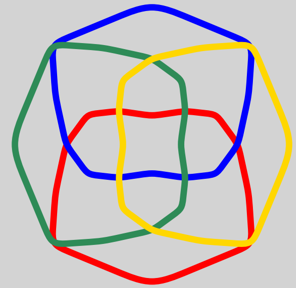
intersecting . . . any 2 pseudocircles cross twice

arr. of great-pseudocircles . . . any 3 pcs. form a Krupp

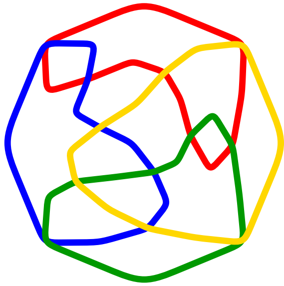
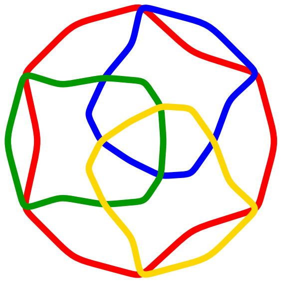
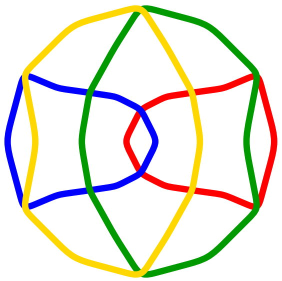
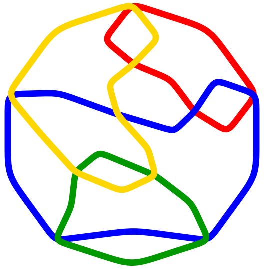
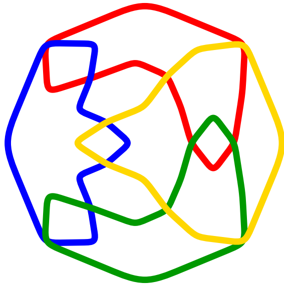






1 greatcircle arr.

8 intersecting arrangements

21 connected arrangements

Enumeration of Arrangements

n	3	4	5	6	7
connected	3	21	984	609 423	?
+digon-free	1	3	30	4 509	?
intersecting	2	8	278	145 058	447 905 202
+digon-free	1	2	14	2 131	3 012 972
great-p.c.s	1	1	1	4	11

Table: # of combinatorially different arragements of n pcs.

Enumeration of Arrangements

n	3	4	5	6	7
connected	3	21	984	609 423	?
+digon-free	1	3	30	4 509	?
intersecting	2	8	278	145 058	447 905 202
+digon-free	1	2	14	2 131	3 012 972
great-p.c.s	1	1	1	4	11

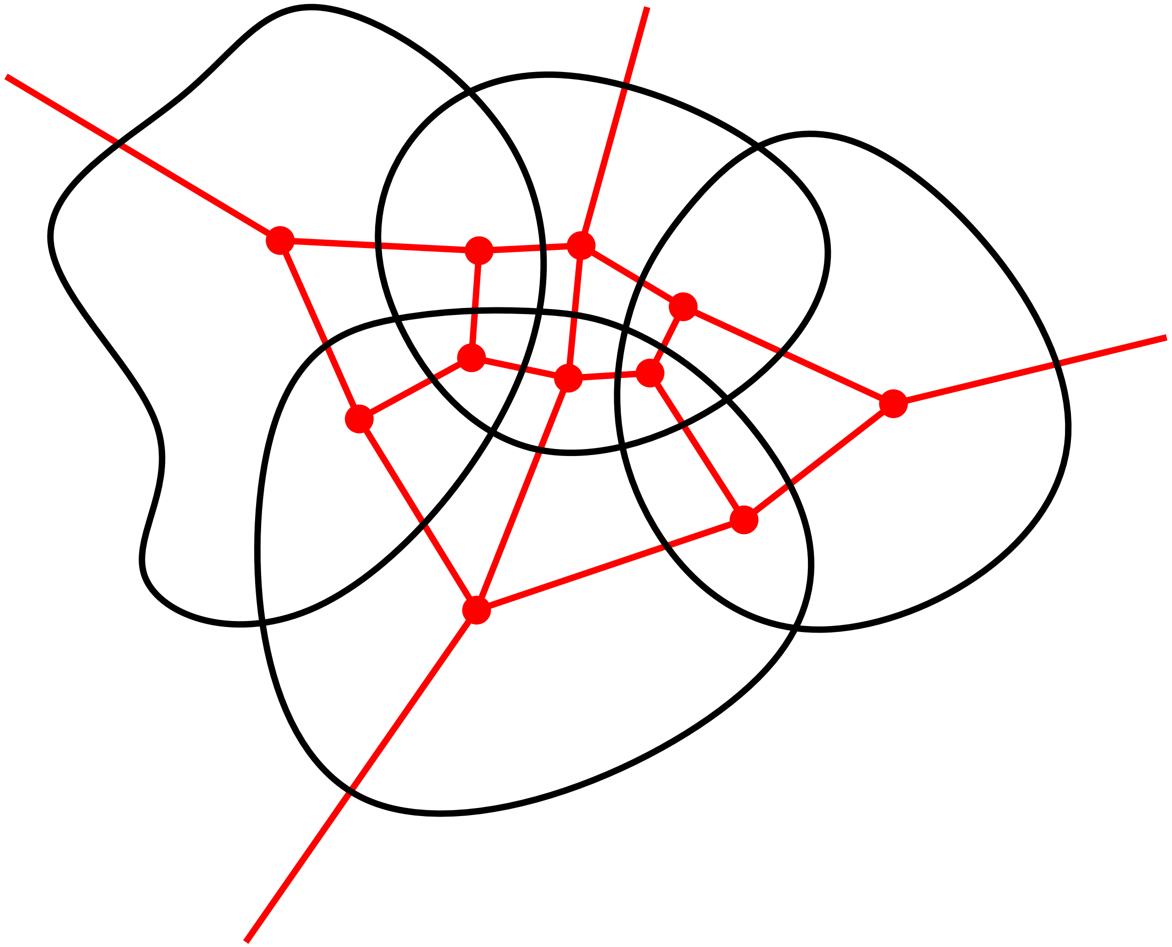
Table: # of combinatorially different arragements of n pcs.

arrangements of pcs: $2^{\Theta(n^2)}$

arrangements of circles: $2^{\Theta(n \log n)}$

Counting Arrangements of Pseudocircles

- dual graph is quadrangulation on $O(n^2)$ vertices



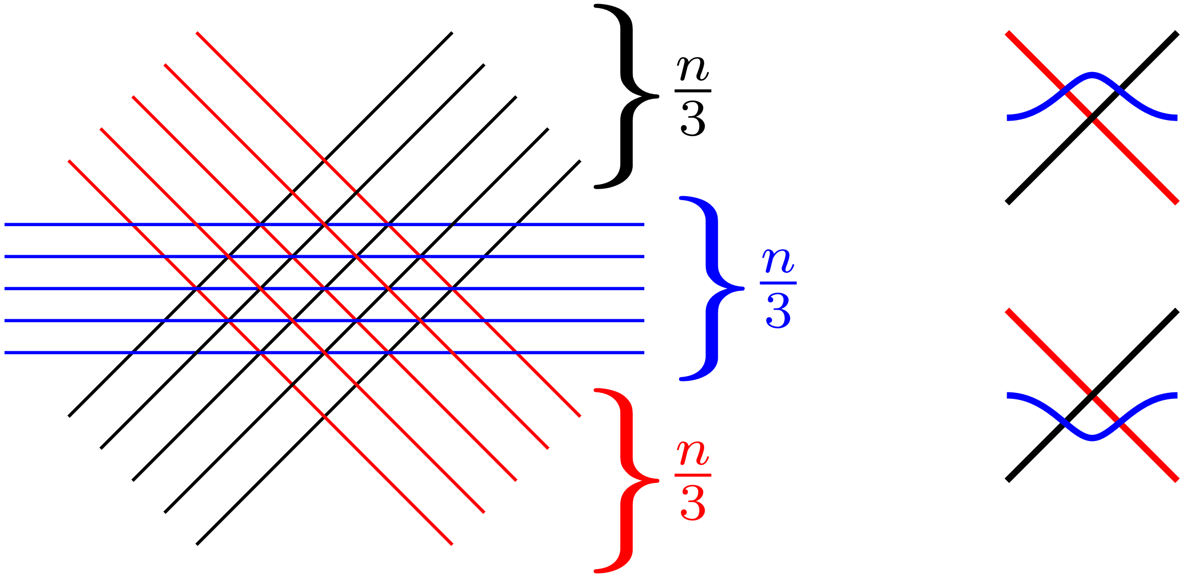
Counting Arrangements of Pseudocircles

- dual graph is quadrangulation on $O(n^2)$ vertices
- Tutte'62: $2^{\Theta(m)}$ triangulations on m vertices
- $\Rightarrow$ Upper bound: $2^{O(n^2)}$ non-isomorphic arrangements

Counting Arrangements of Pseudocircles

- dual graph is quadrangulation on $O(n^2)$ vertices
- Tutte'62: $2^{\Theta(m)}$ triangulations on m vertices
- $\Rightarrow$ Upper bound: $2^{O(n^2)}$ non-isomorphic arrangements

- Lower bound: $2^{\Omega(n^2)}$ non-isomorphic arrangements



Counting Arrangements of Pseudocircles

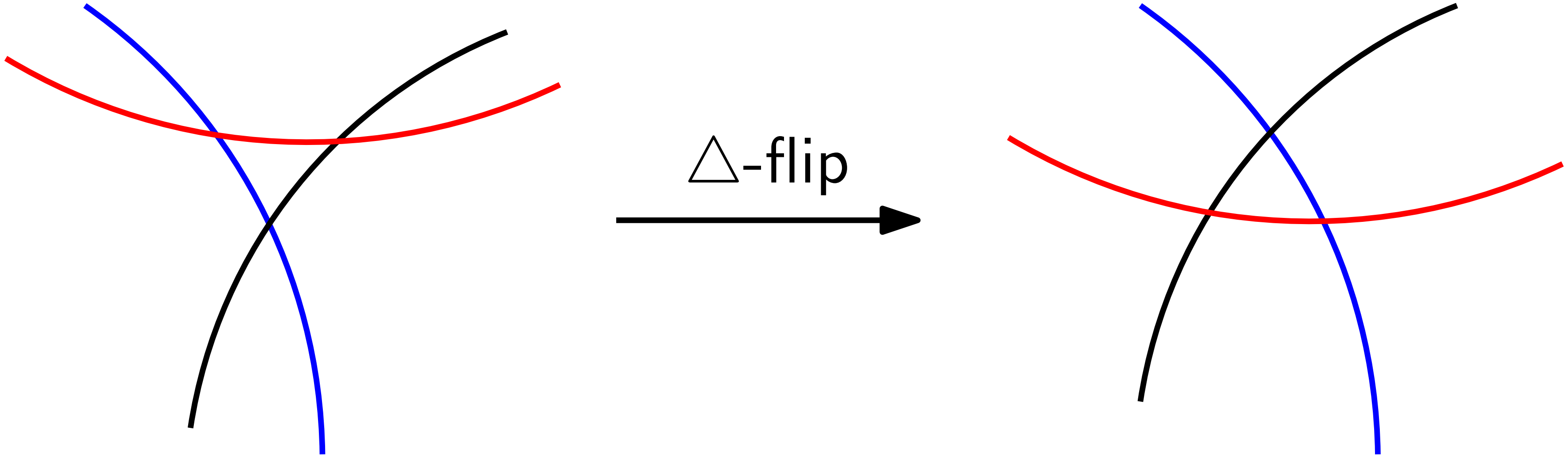
- dual graph is quadrangulation on $O(n^2)$ vertices
- Tutte'62: $2^{\Theta(m)}$ triangulations on m vertices
- $\Rightarrow$ Upper bound: $2^{O(n^2)}$ non-isomorphic arrangements

- Lower bound: $2^{\Omega(n^2)}$ non-isomorphic arrangements

Theorem: There are $2^{\Theta(n^2)}$ arrangements on n pcs.

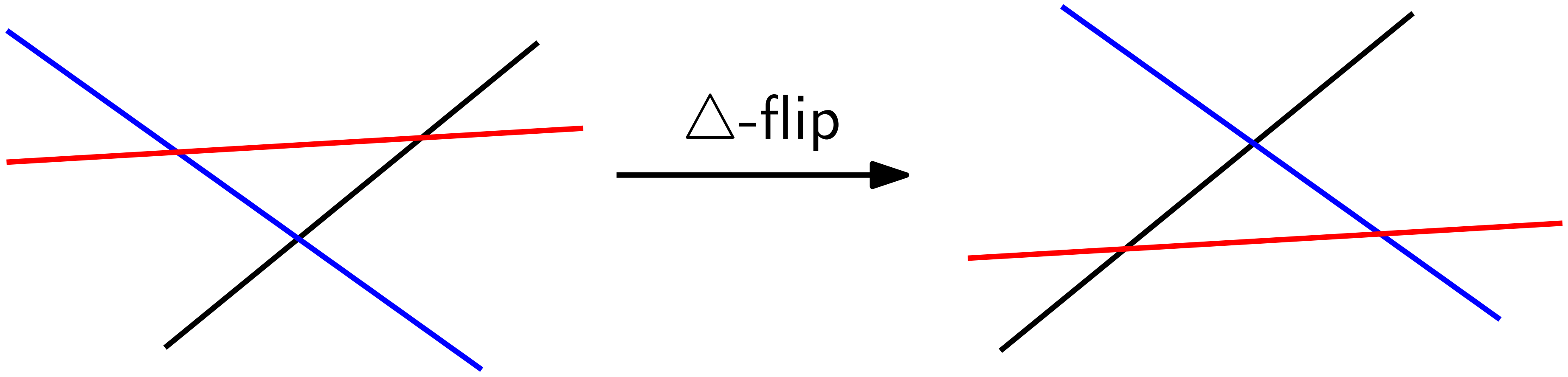
Counting Arrangements of Circles

- Upper bound: arrangement changes if a triangle ”flips”



Counting Arrangements of Circles

- Upper bound: arrangement changes if a triangle ”flips”
- we sketch the proof for *line*-arrangements



Counting Arrangements of Circles

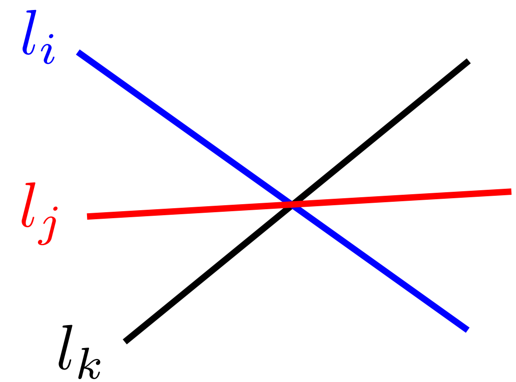
- lines $l_1, \ldots, l_n$ given by $l_i : y_i = a_i x + b_i$

Counting Arrangements of Circles

- lines $l_1, \ldots, l_n$ given by $l_i : y_i = a_i x + b_i$

- l_i, l_j, and l_k meet in a common point

$$\iff \det \begin{pmatrix} 1 & 1 & 1 \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{pmatrix} = 0$$

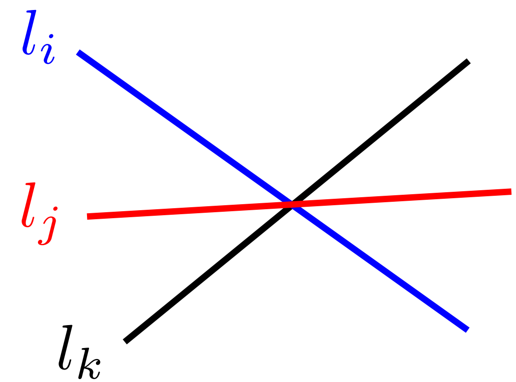


Counting Arrangements of Circles

- lines $l_1, \ldots, l_n$ given by $l_i : y_i = a_i x + b_i$

- l_i, l_j, and l_k meet in a common point

$$\iff \det \begin{pmatrix} 1 & 1 & 1 \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{pmatrix} = 0$$



- system of $\binom{n}{3}$ quadratic polynomials in $2n$ variables
- simple arr. $\Leftrightarrow$ all polynomials non-zero

Counting Arrangements of Circles

- **Milnor–Thom Theorem:**
  the number of cells in $\mathbb{R}^d$ induced by zero set of m polynomials of degree $\leq D$ is at most $(50Dm/d)^d$

Counting Arrangements of Circles

$2n$

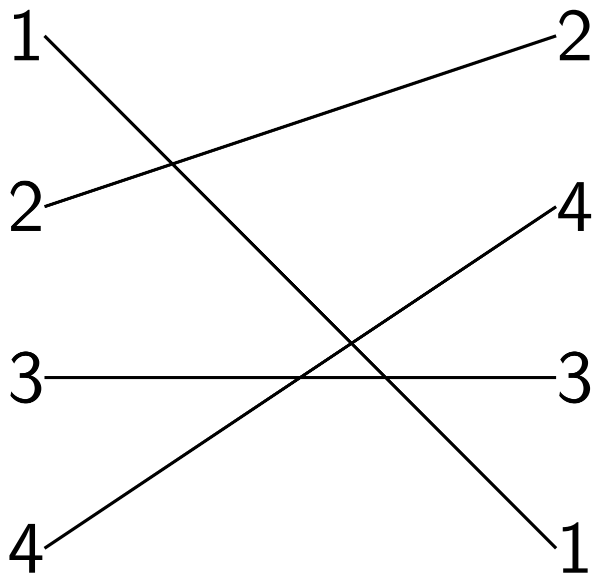
- **Milnor–Thom Theorem:**
  the number of cells in $\mathbb{R}^d$ induced by zero set of m polynomials of degree $\leq D$ is at most $(50Dm/d)^d$

n^3 $\quad$ 2 $\quad$ $n^{O(n)} = 2^{O(n \log n)}$

- Upper bound: $2^{O(n \log n)}$

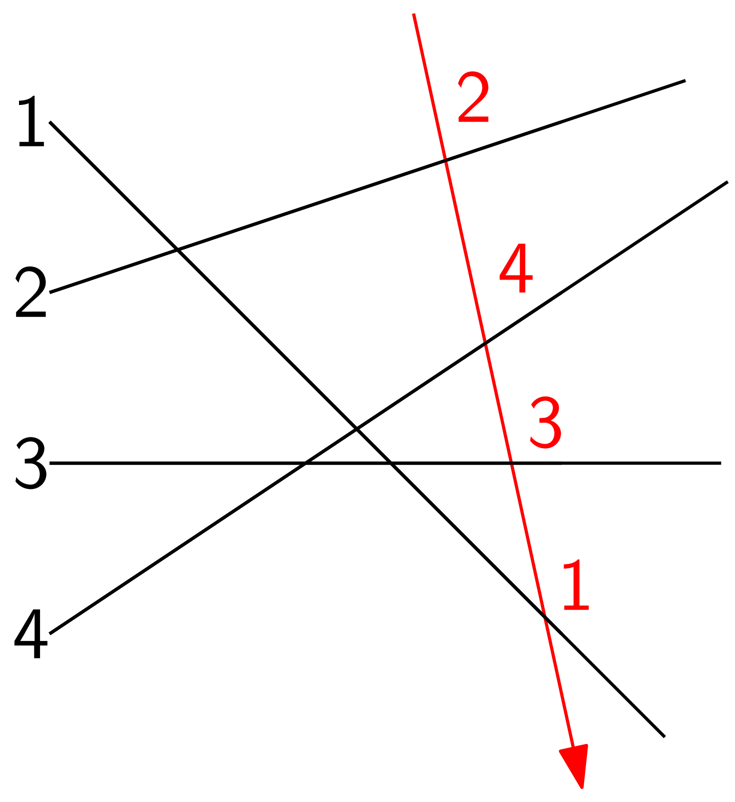
Counting Arrangements of Circles

- Lower bound: # of permutations



Counting Arrangements of Circles

- Lower bound: # of permutations



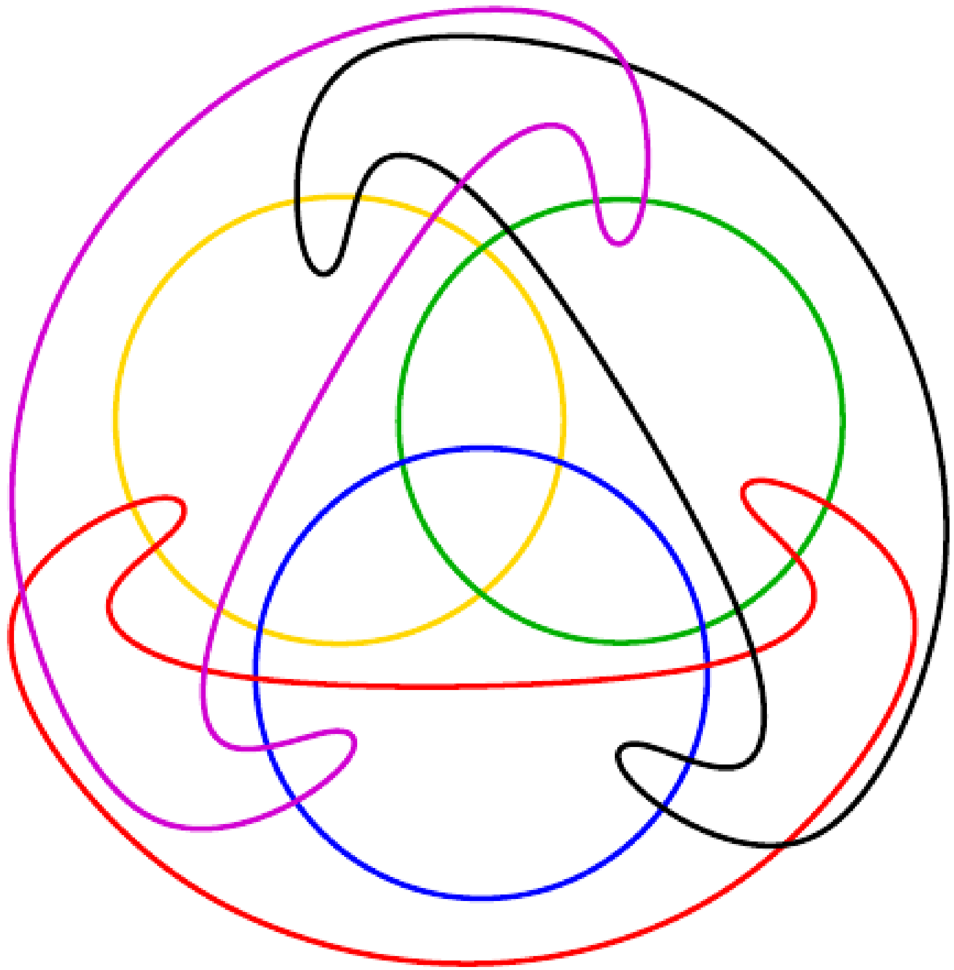
Counting Arrangements of Circles

Theorem: There are $2^{\Theta(n \log n)}$ arrangements on n circles.

Part I: Circleability

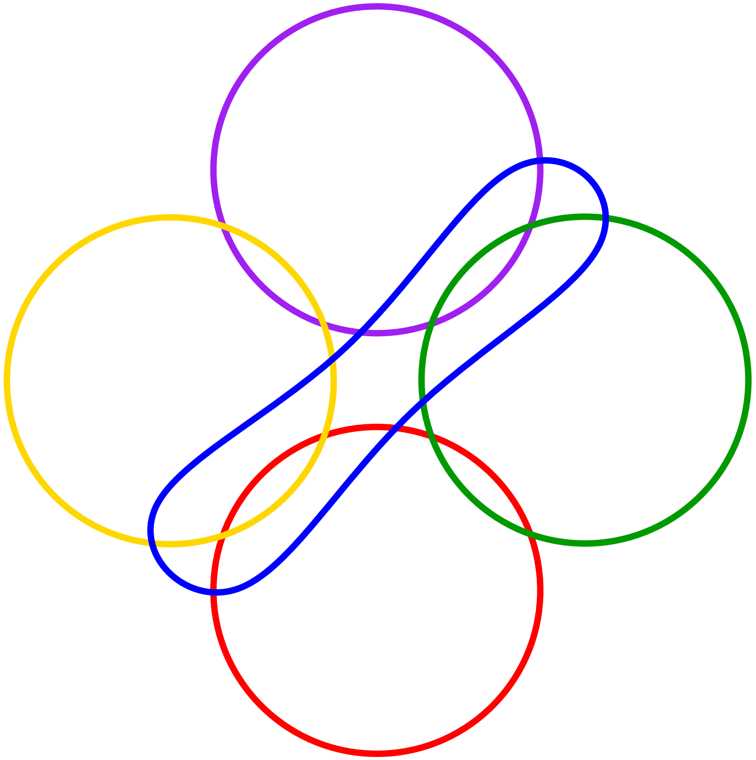
Circleability Results

- non-circleability of <u>intersecting</u> $n = 6$ arrangement [Edelsbrunner and Ramos '97]



Circleability Results

- non-circleability of intersecting $n = 6$ arrangement [Edelsbrunner and Ramos '97]
- non-circleability of $n = 5$ arrangement [Linhart and Ortner '05]



Circleability Results

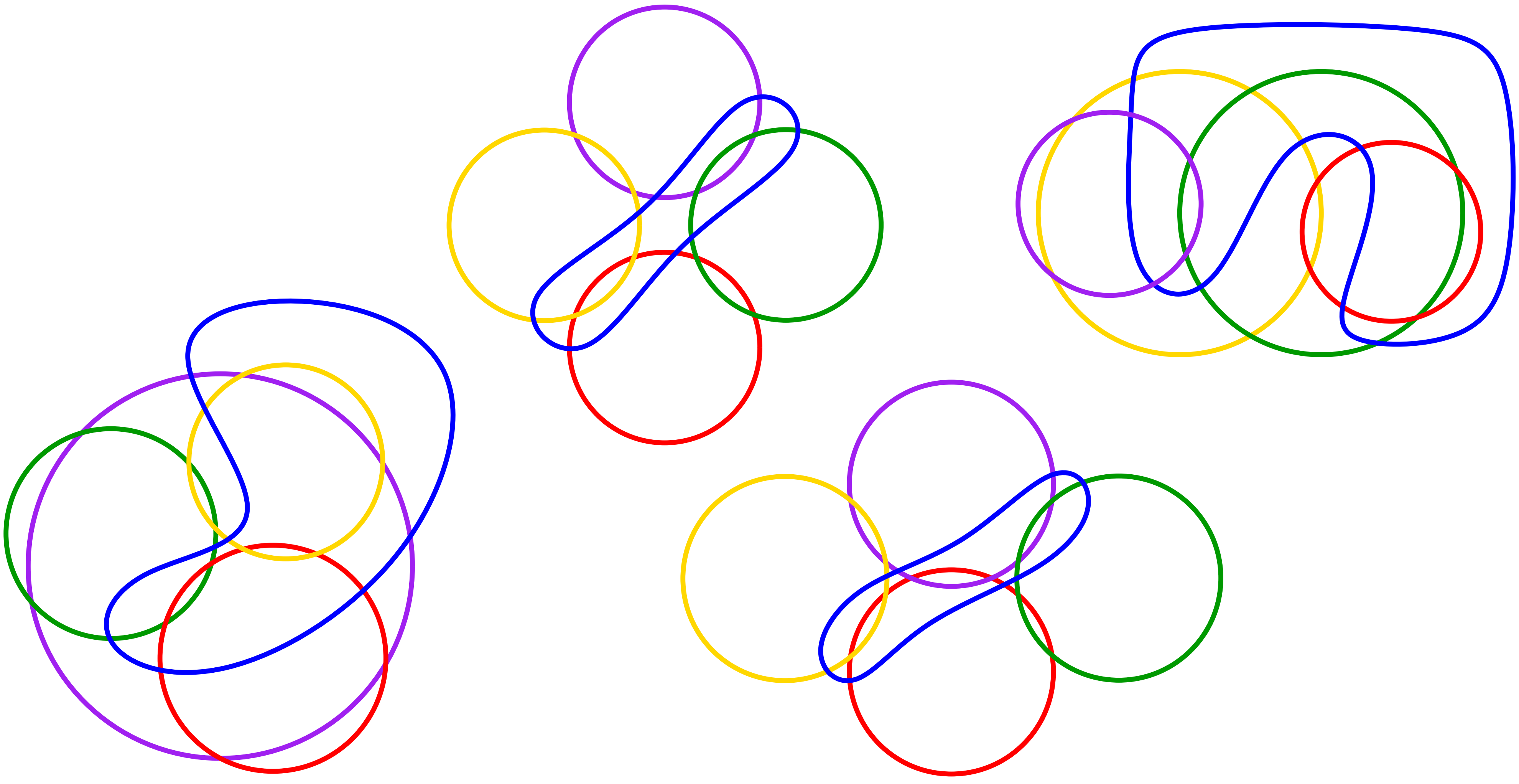
- non-circleability of intersecting $n = 6$ arrangement [Edelsbrunner and Ramos '97]
- non-circleability of $n = 5$ arrangement [Linhart and Ortner '05]
- circleability of all $n = 4$ arrangements [Kang and Müller '14]

Circleability Results

- non-circleability of intersecting $n = 6$ arrangement [Edelsbrunner and Ramos '97]
- non-circleability of $n = 5$ arrangement [Linhart and Ortner '05]
- circleability of all $n = 4$ arrangements [Kang and Müller '14]
- NP-hardness of circleability [Kang and Müller '14]

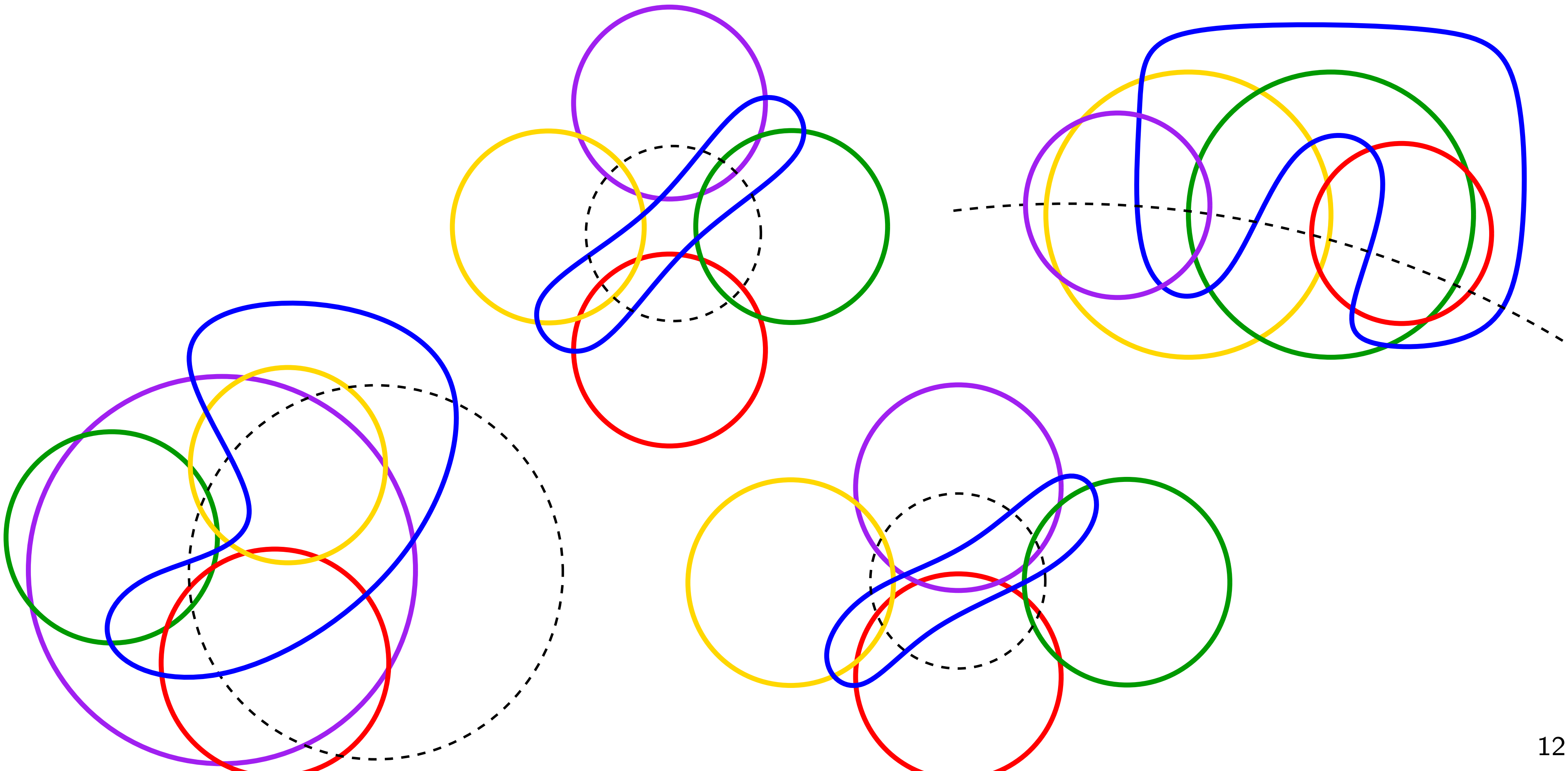
Circleability Results

Theorem. There are exactly 4 non-circleable $n = 5$ arrangements (984 classes).

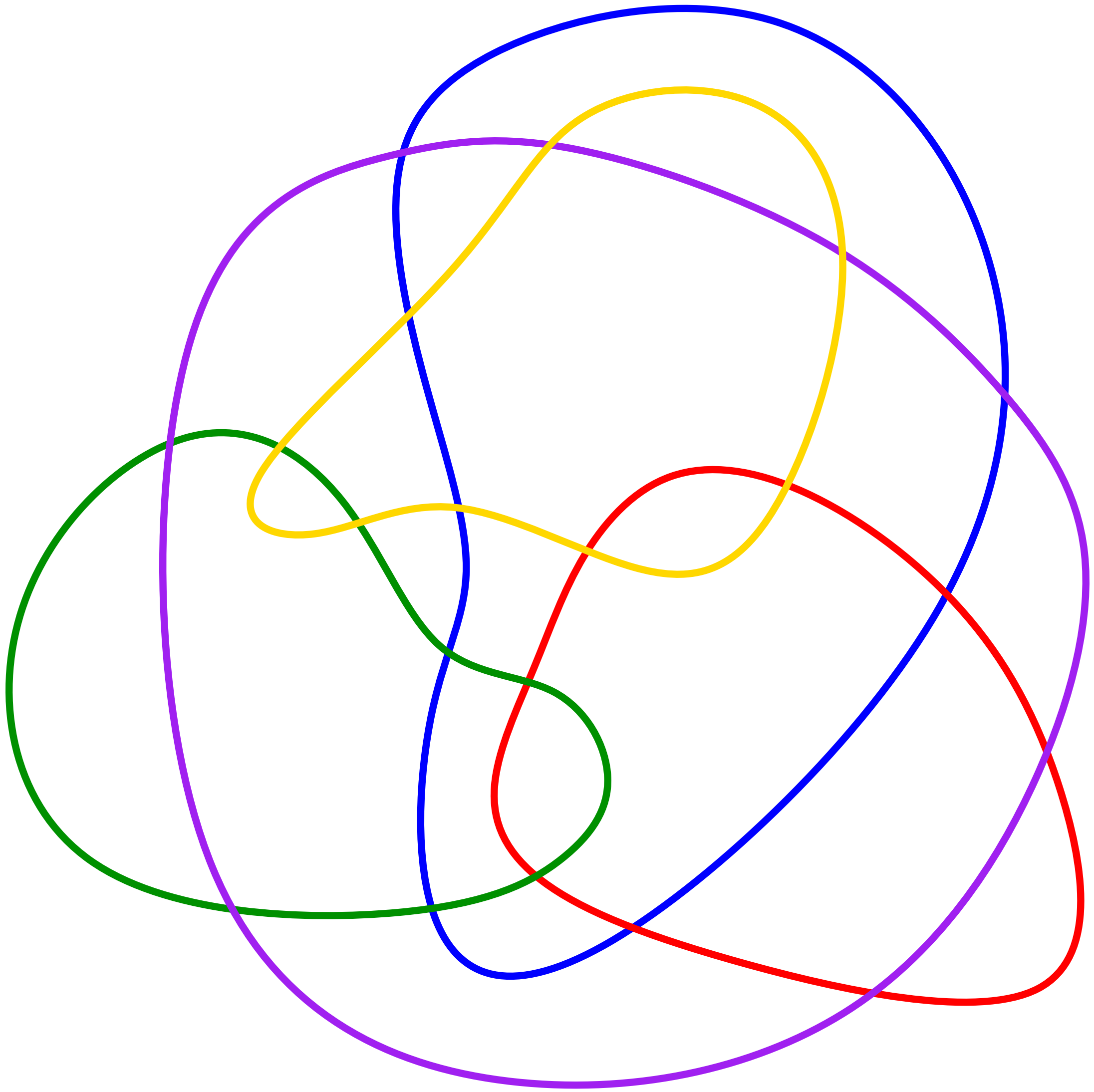


Circleability Results

Theorem. There are exactly 4 non-circleable $n = 5$ arrangements (984 classes).

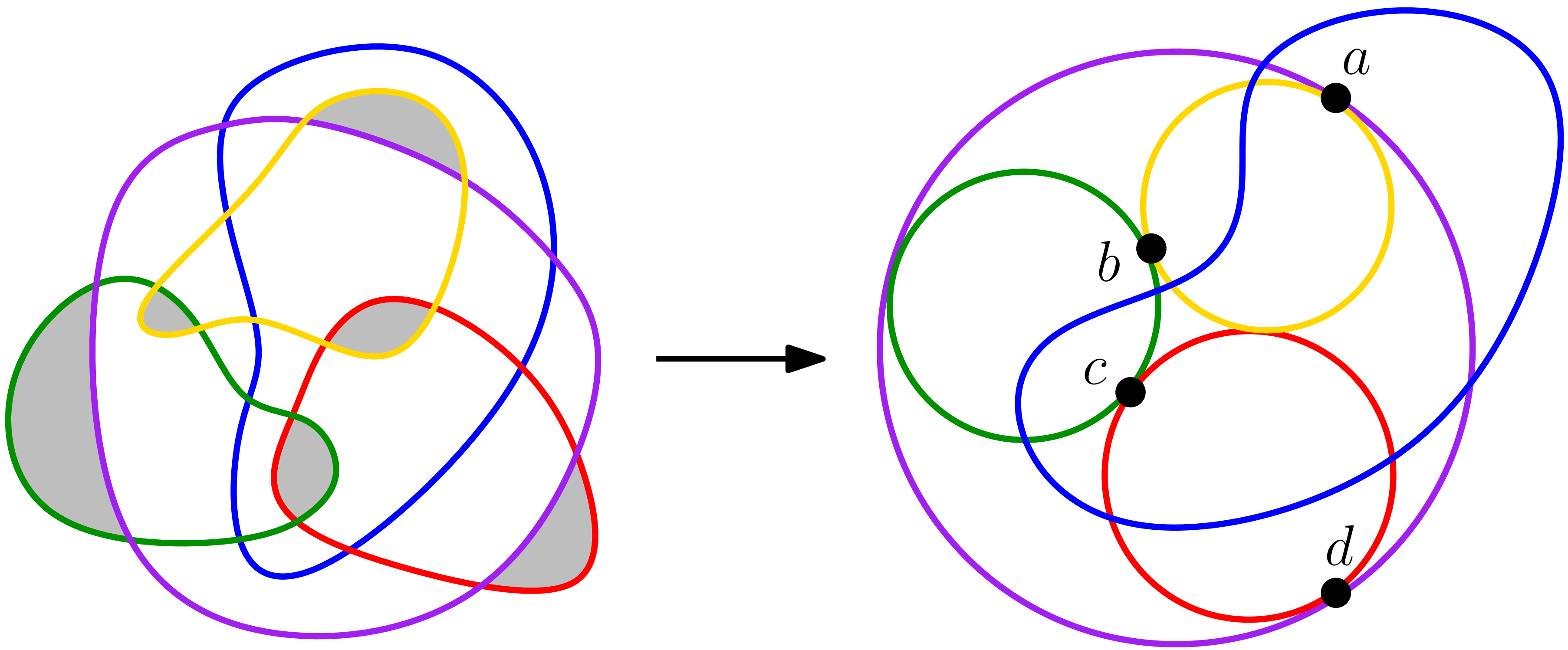


Non-Circleability of $\mathcal{N}_5^1$



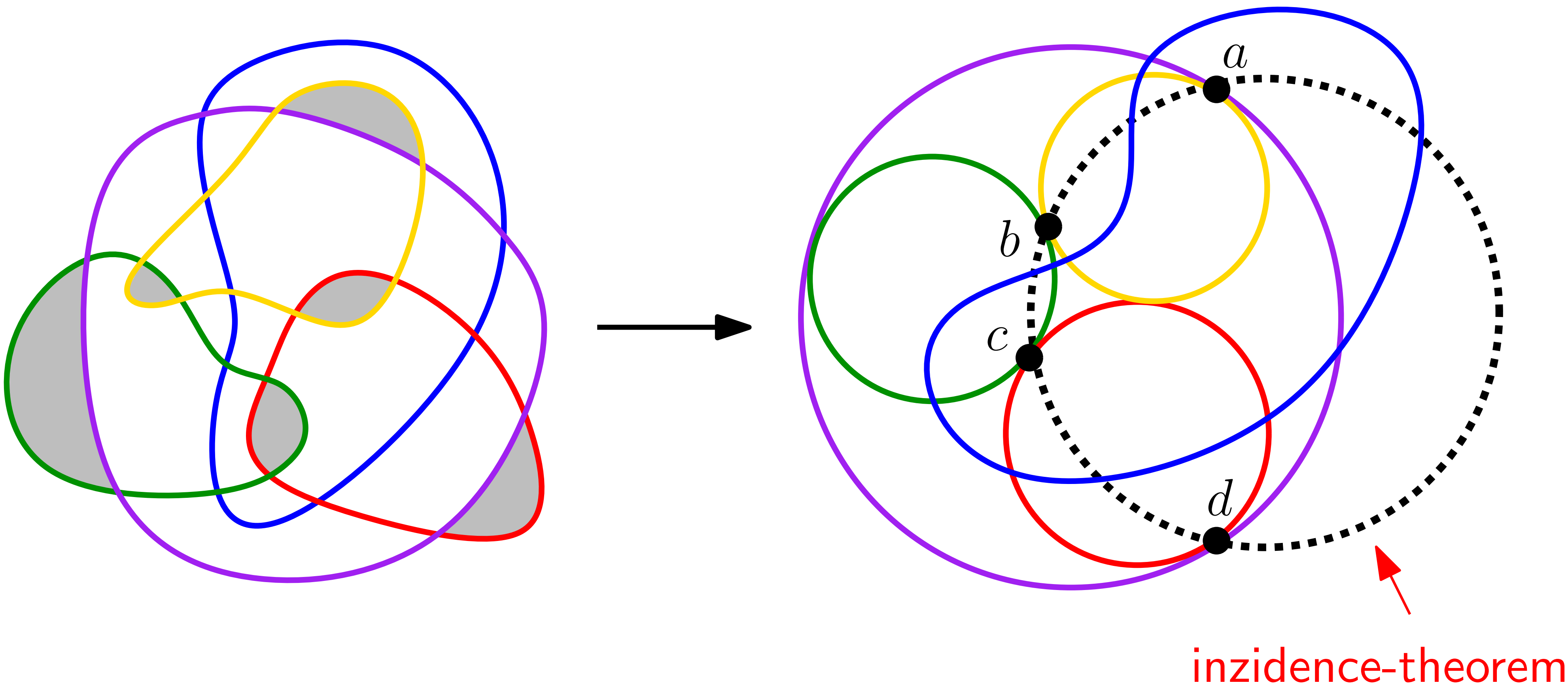
Non-Circleability of $\mathcal{N}_5^1$

- assume there is a circle representation of $\mathcal{N}_5^1$
- shrink the yellow, green, and red circle
- cyclic order is preserved (also for blue)



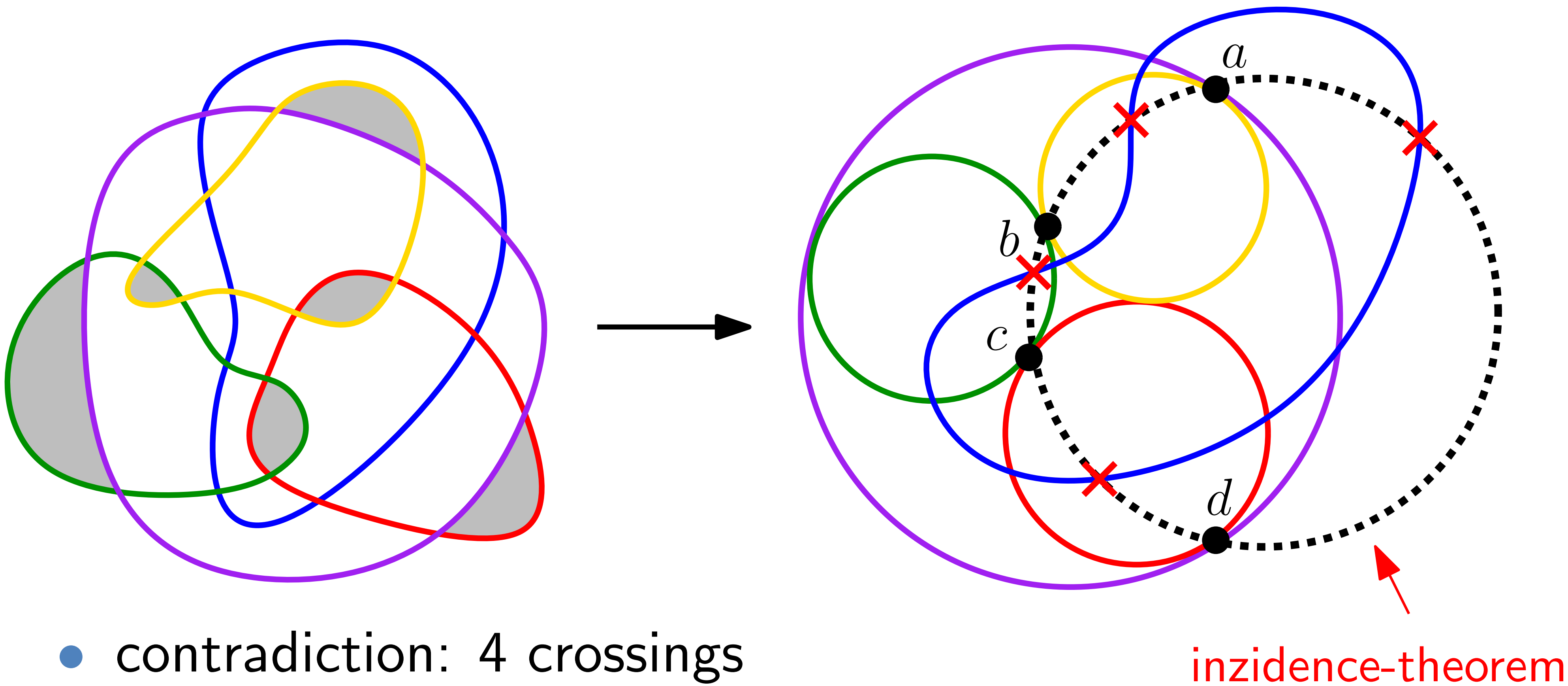
Non-Circleability of $\mathcal{N}_5^1$

- assume there is a circle representation of $\mathcal{N}_5^1$
- shrink the yellow, green, and red circle
- cyclic order is preserved (also for blue)



Non-Circleability of $\mathcal{N}_5^1$

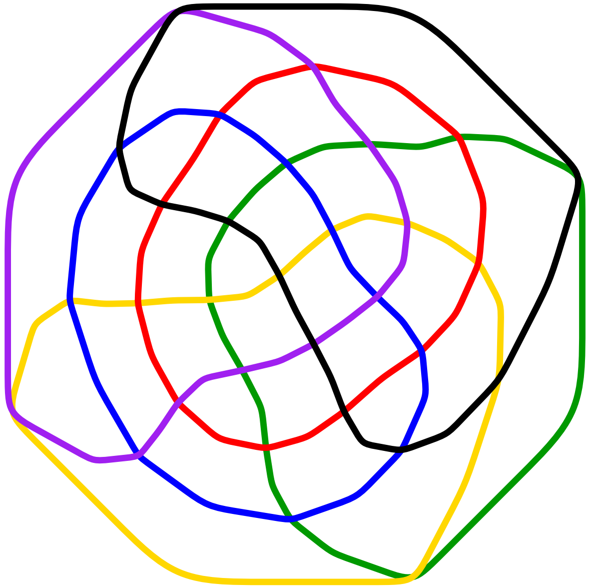
- assume there is a circle representation of $\mathcal{N}_5^1$
- shrink the yellow, green, and red circle
- cyclic order is preserved (also for blue)

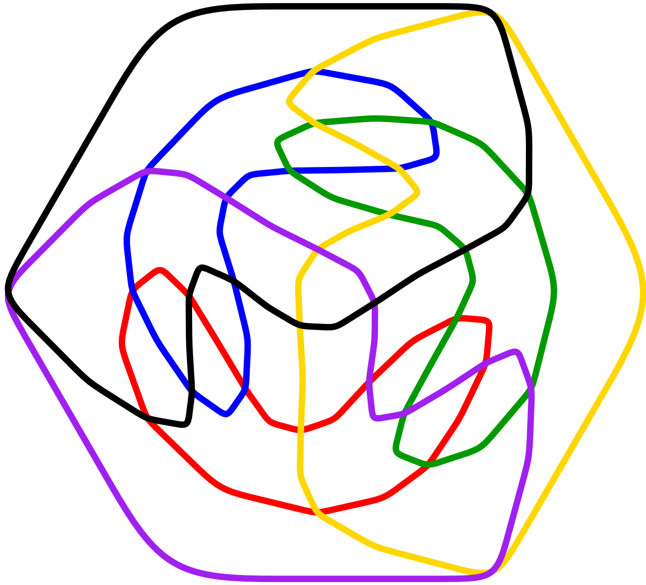


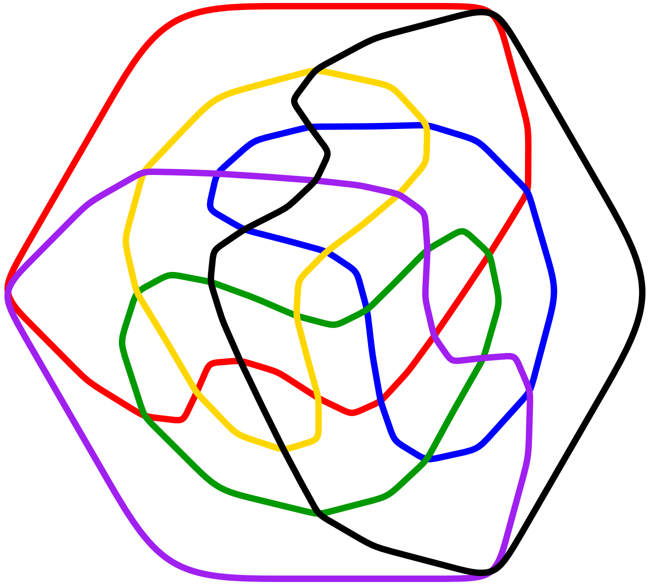
- contradiction: 4 crossings

Circleability Results

Theorem. There are exactly 3 non-circleable digon-free intersecting $n = 6$ arrangements (2131 classes).

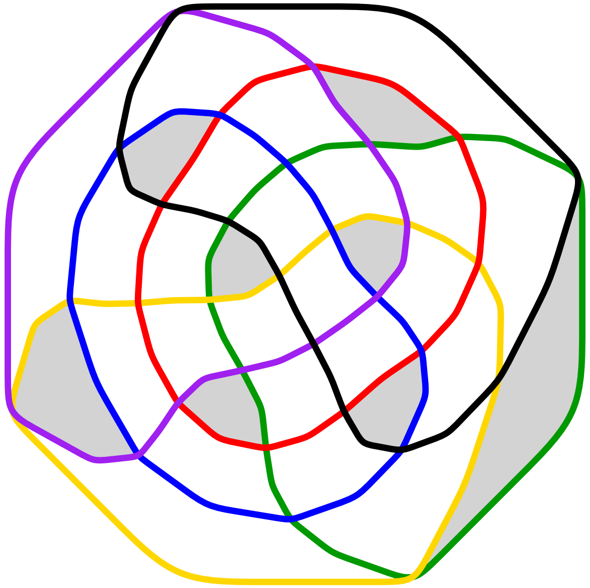






Circleability Results

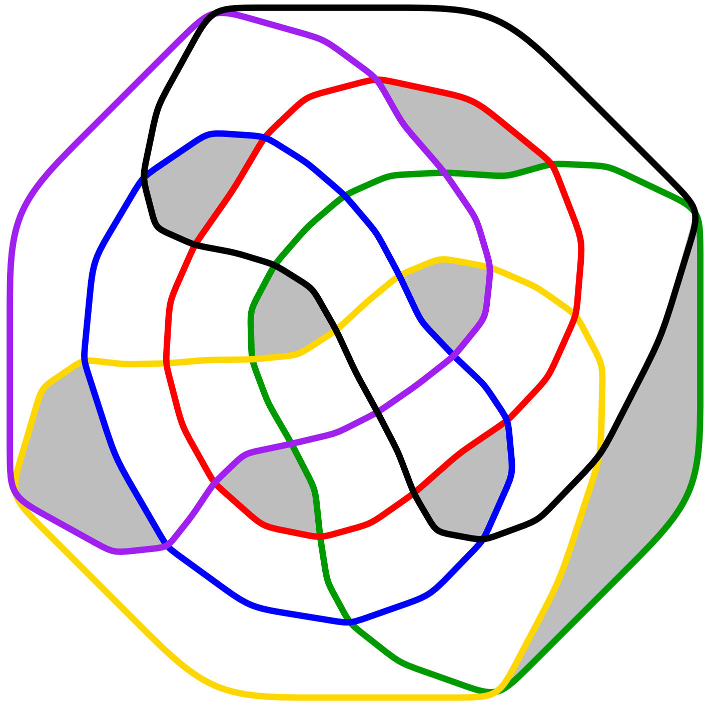
Theorem. There are exactly 3 non-circleable digon-free intersecting $n = 6$ arrangements (2131 classes).



$\mathcal{N}_6^{\triangle}$ is unique digon-free intersecting with 8 triangular cells

Grünbaum Conjecture: $p_3 \geq 2n - 4$

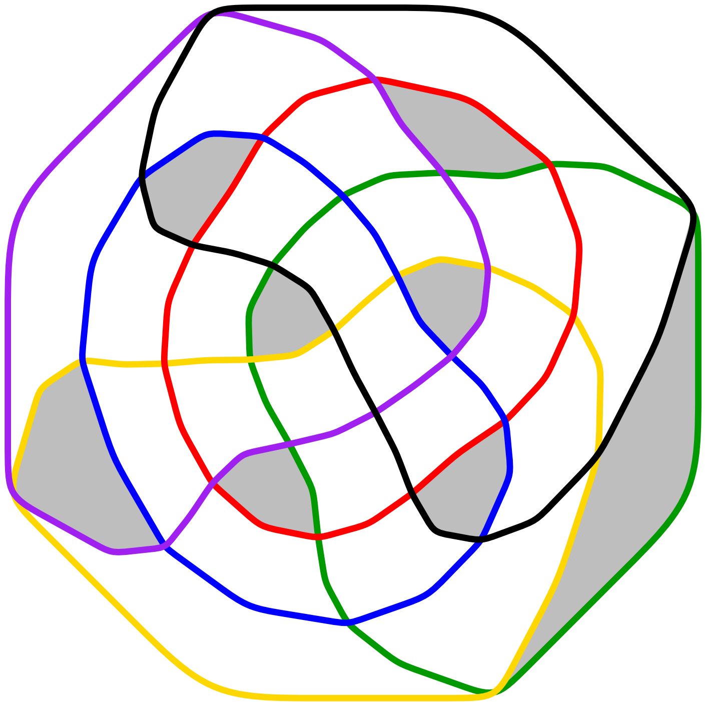
Non-Circleability Proof of $\mathcal{N}_6^{\triangle}$



Proof.

based on sweeping argument in 3-D

Non-Circleability Proof of $\mathcal{N}_6^{\triangle}$

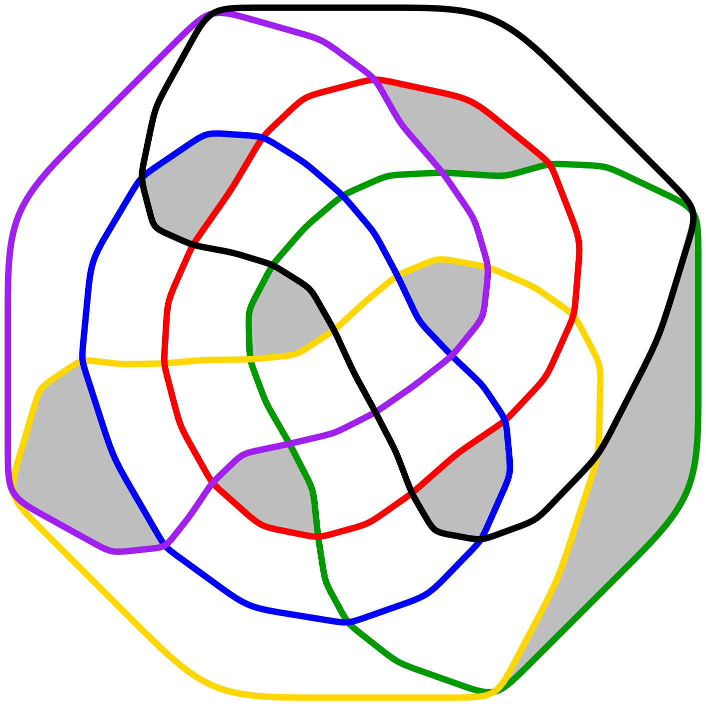


Proof.

$C_1, \ldots, C_6$ . . . circles (on $\mathbb{S}^2$)

$E_1, \ldots, E_6$ . . . planes (in $\mathbb{R}^3$)

Non-Circleability Proof of $\mathcal{N}_6^\triangle$



Proof.

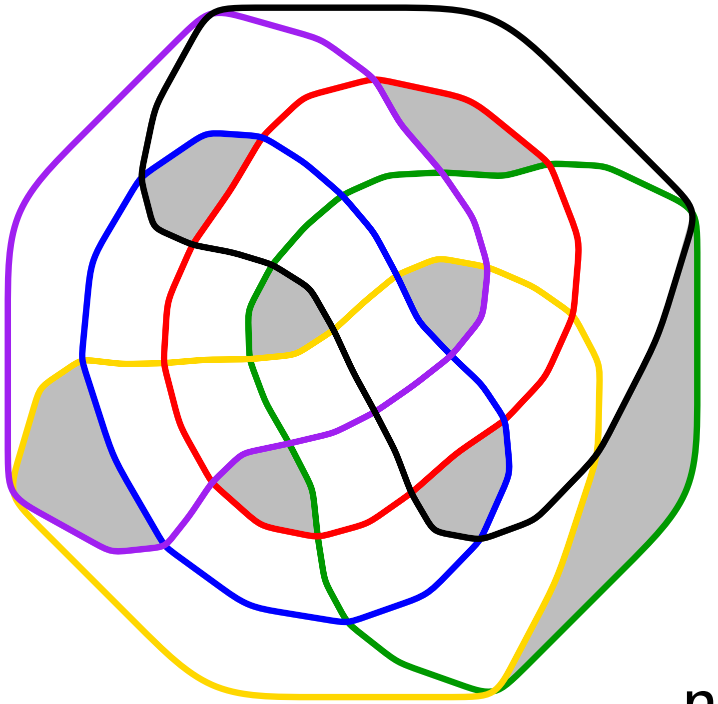
$C_1, \ldots, C_6$ . . . circles (on $\mathbb{S}^2$)

$E_1, \ldots, E_6$ . . . planes (in $\mathbb{R}^3$)

move planes away from the origin

E_i moves to $t \cdot E_i$ as $t \to \infty$

Non-Circleability Proof of $\mathcal{N}_6^{\triangle}$



Proof.

$C_1, \ldots, C_6$ . . . circles (on $\mathbb{S}^2$)

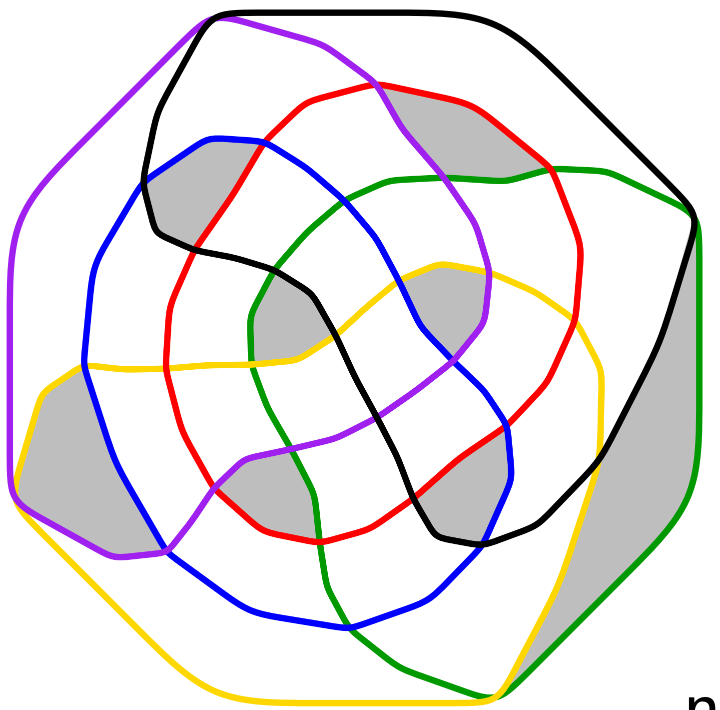
$E_1, \ldots, E_6$ . . . planes (in $\mathbb{R}^3$)

move planes away from the origin

no great-circle arr. $\Rightarrow$ events occur

not all planes contain the origin

Non-Circleability Proof of $\mathcal{N}_6^{\triangle}$



Proof.

$C_1, \ldots, C_6$ ... circles (on $\mathbb{S}^2$)

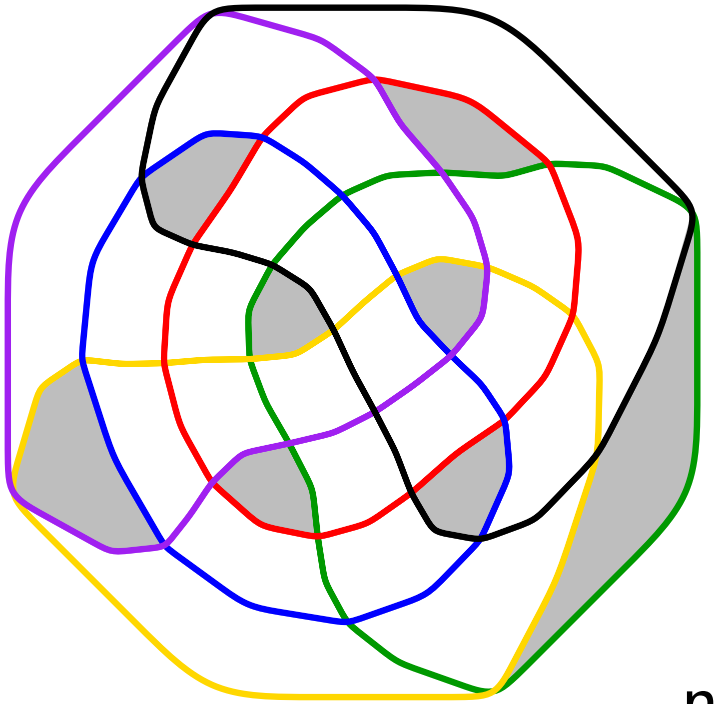
$E_1, \ldots, E_6$ ... planes (in $\mathbb{R}^3$)

move planes away from the origin

no great-circle arr. $\Rightarrow$ events occur

first event is triangle flip ($\nexists$ digons)

Non-Circleability Proof of $\mathcal{N}_6^{\triangle}$



Proof.

$C_1, \ldots, C_6$ ... circles (on $\mathbb{S}^2$)

$E_1, \ldots, E_6$ ... planes (in $\mathbb{R}^3$)

move planes away from the origin

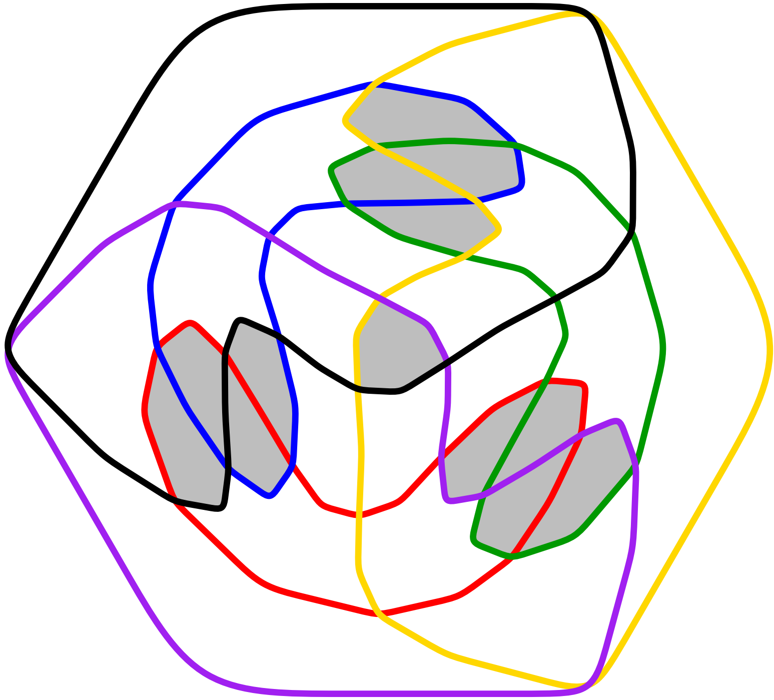
no great-circle arr. $\Rightarrow$ events occur

first event is triangle flip ($\nexists$ digons)

but triangle flip not possible because all triangles in NonKrupp. Contradiction.

$\square$

Non-Circularizability Proof of $\mathcal{N}_6^2$

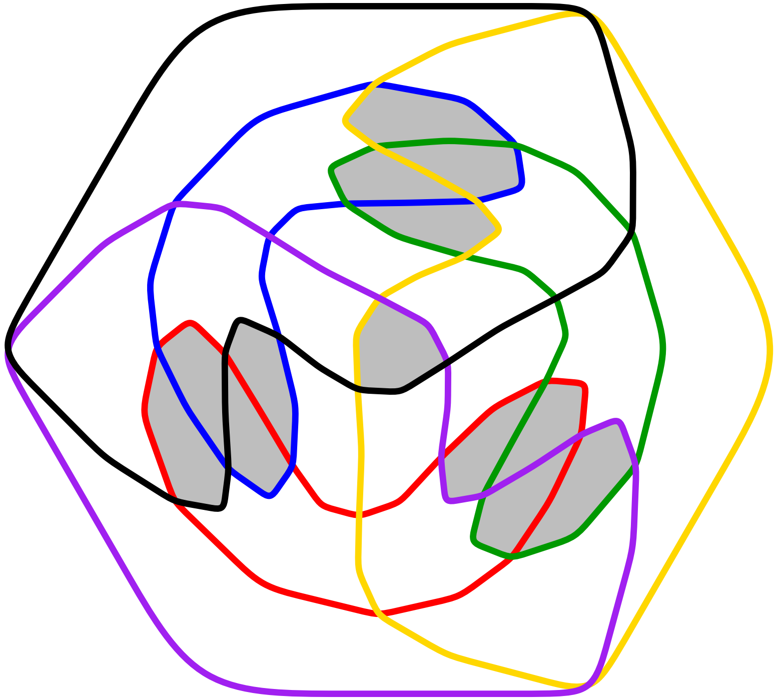


Proof. (similar)

$C_1, \ldots, C_6$ . . . circles

$E_1, \ldots, E_6$ . . . planes

Non-Circularizability Proof of $\mathcal{N}_6^2$



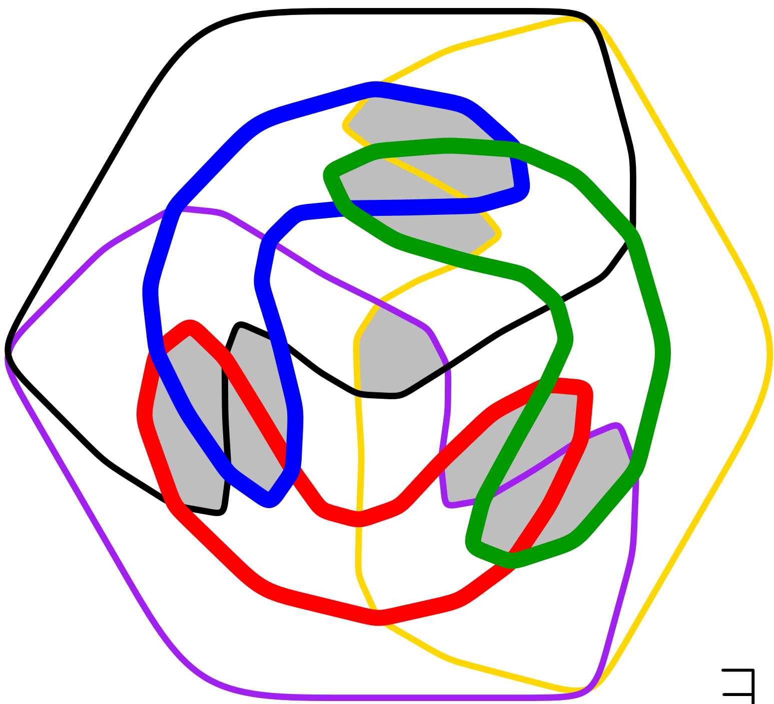
Proof. (similar)

$C_1, \ldots, C_6$ ... circles

$E_1, \ldots, E_6$ ... planes

move planes towards the origin

Non-Circularizability Proof of $\mathcal{N}_6^2$



Proof. (similar)

$C_1, \ldots, C_6$ ... circles

$E_1, \ldots, E_6$ ... planes

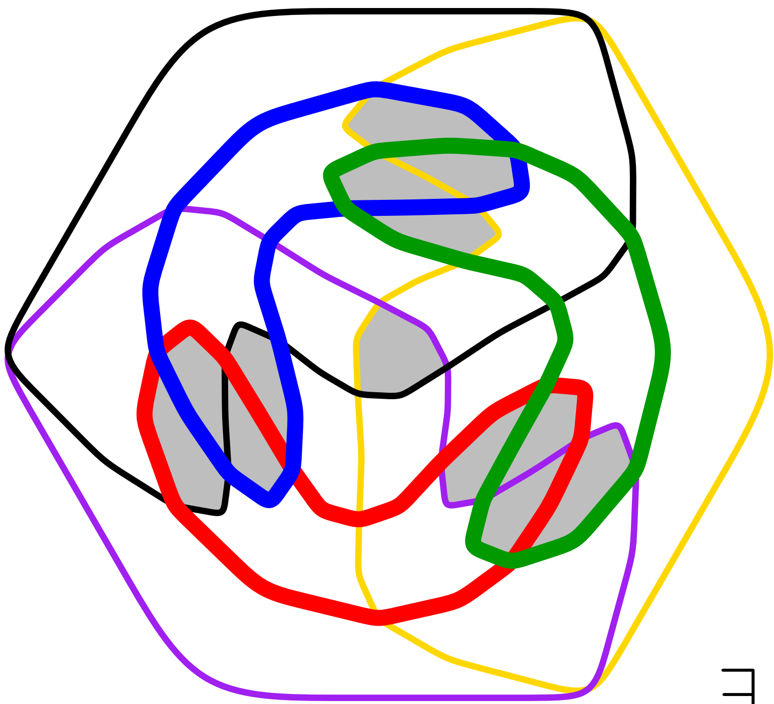
move planes towards the origin

$\exists$ NonKrupp subarr. $\Rightarrow$ events occur

↑

$\exists$ point of intersection
outside the unit-sphere
(will move inside)

Non-Circularizability Proof of $\mathcal{N}_6^2$



Proof. (similar)

$C_1, \ldots, C_6$ ... circles

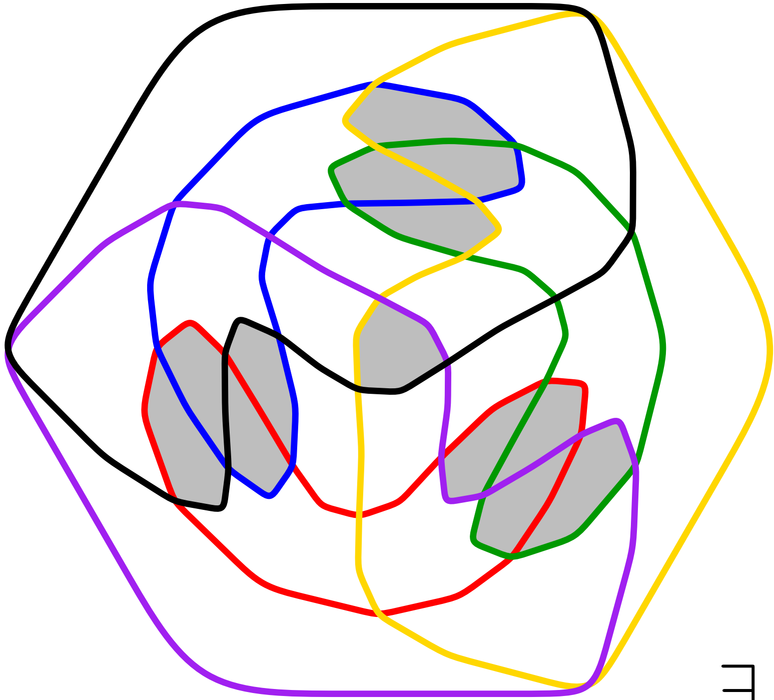
$E_1, \ldots, E_6$ ... planes

move planes towards the origin

$\exists$ NonKrupp subarr. $\Rightarrow$ events occur

first event is triangle flip ($\nexists$ digons)

Non-Circularizability Proof of $\mathcal{N}_6^2$



Proof. (similar)

$C_1, \ldots, C_6$ ... circles

$E_1, \ldots, E_6$ ... planes

move planes towards the origin

$\exists$ NonKrupp subarr. $\Rightarrow$ events occur

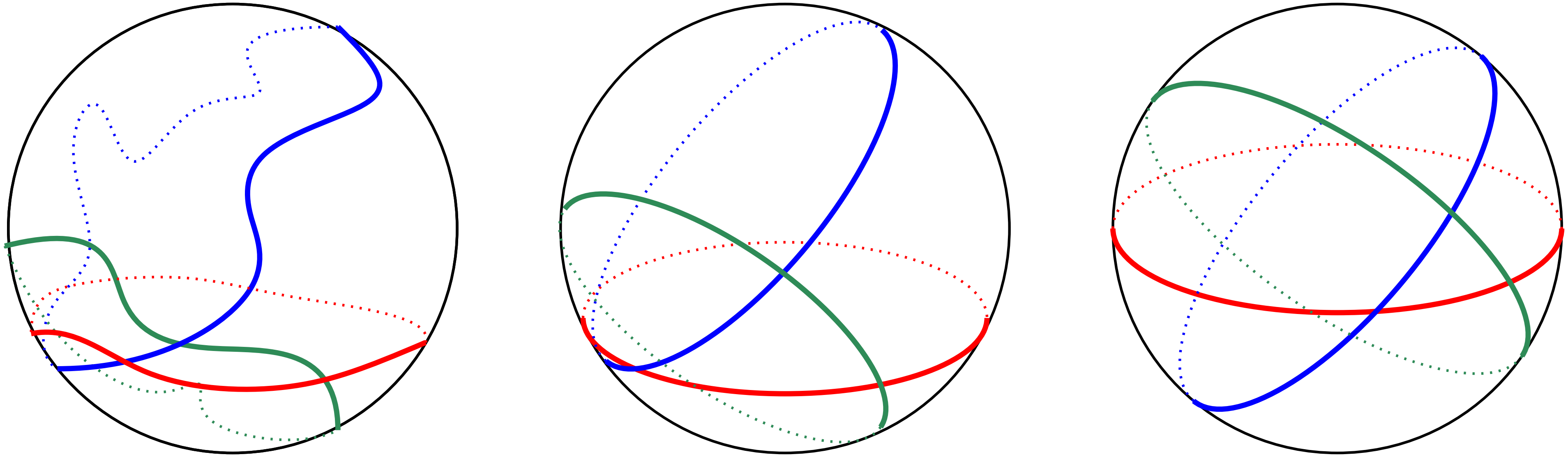
first event is triangle flip ($\nexists$ digons)

but triangle flip not possible because all triangles in Krupp. Contradiction.

$\square$

Great-(Pseudo)Circles

Great-Circle Theorem:
An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.



Great-(Pseudo)Circles

Great-Circle Theorem:

An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Proof.

- $C_1, \ldots, C_n$ ... circles $\qquad E_1, \ldots, E_n$ ... planes

Great-(Pseudo)Circles

Great-Circle Theorem:

An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Proof.

- $C_1, \ldots, C_n$ ... circles $E_1, \ldots, E_n$ ... planes
- move planes towards the origin

Great-(Pseudo)Circles

Great-Circle Theorem:

An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Proof.

- $C_1, \ldots, C_n$ ... circles $\qquad E_1, \ldots, E_n$ ... planes
- move planes towards the origin
- all triples Krupp
  $\Rightarrow$ all intersections remain inside
  $\Rightarrow$ no events

Great-(Pseudo)Circles

Great-Circle Theorem:

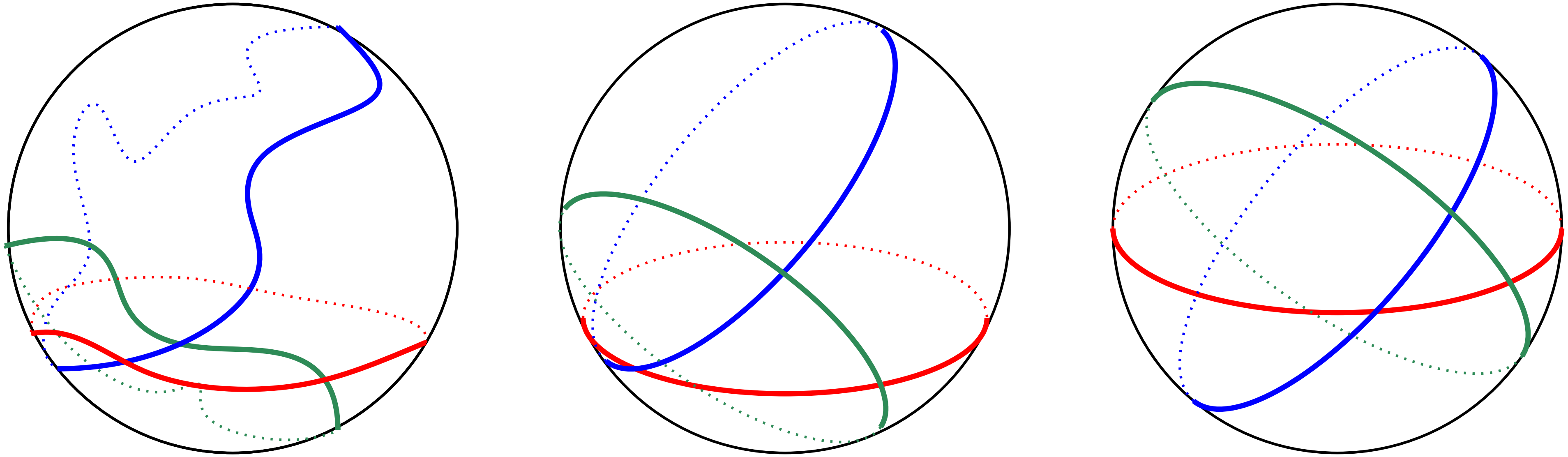
An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Proof.

- $C_1, \ldots, C_n$ . . . circles $\qquad$ $E_1, \ldots, E_n$ . . . planes
- move planes towards the origin
- all triples Krupp
  $\Rightarrow$ all intersections remain inside
  $\Rightarrow$ no events
- we obtain a great-circle arrangement $\square$

Great-(Pseudo)Circles

Great-Circle Theorem:
An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.



Great-(Pseudo)Circles

Great-Circle Theorem:

An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Corollaries:

- $\forall$ non-stretchable arr. of pseudolines
  $\exists$ corresponding non-circleable arr. of pseudocircles

Great-(Pseudo)Circles

Great-Circle Theorem:

An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Corollaries:

- $\forall$ non-stretchable arr. of pseudolines
  $\exists$ corresponding non-circleable arr. of pseudocircles
- deciding circleability is $\exists\mathbb{R}$-complete

(NP $\subseteq \exists\mathbb{R} \subseteq$ PSPACE)

Great-(Pseudo)Circles

Great-Circle Theorem:

An arrangement of great-pcs. is circleable (i.e., has a circle representation) if and only if it has a great-circle repr.

Corollaries:

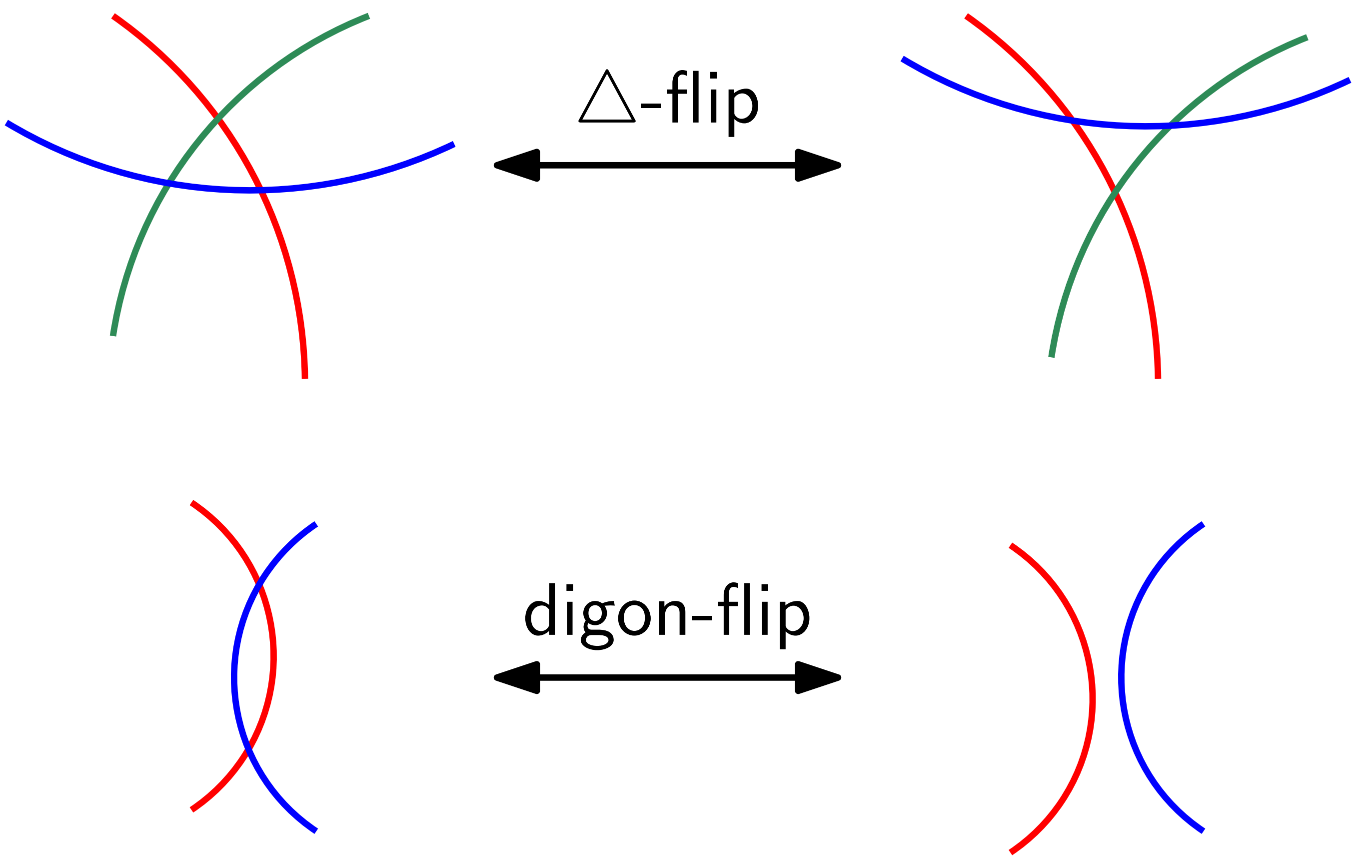
- $\forall$ non-stretchable arr. of pseudolines
  $\exists$ corresponding non-circleable arr. of pseudocircles
- deciding circleability is $\exists\mathbb{R}$-complete
- $\exists$ infinite families of minimal non-circ. arrangements
- $\exists$ arr with a disconnected realization space
- ...

Computational Part

- find circle representations heuristically
- hard instances by hand

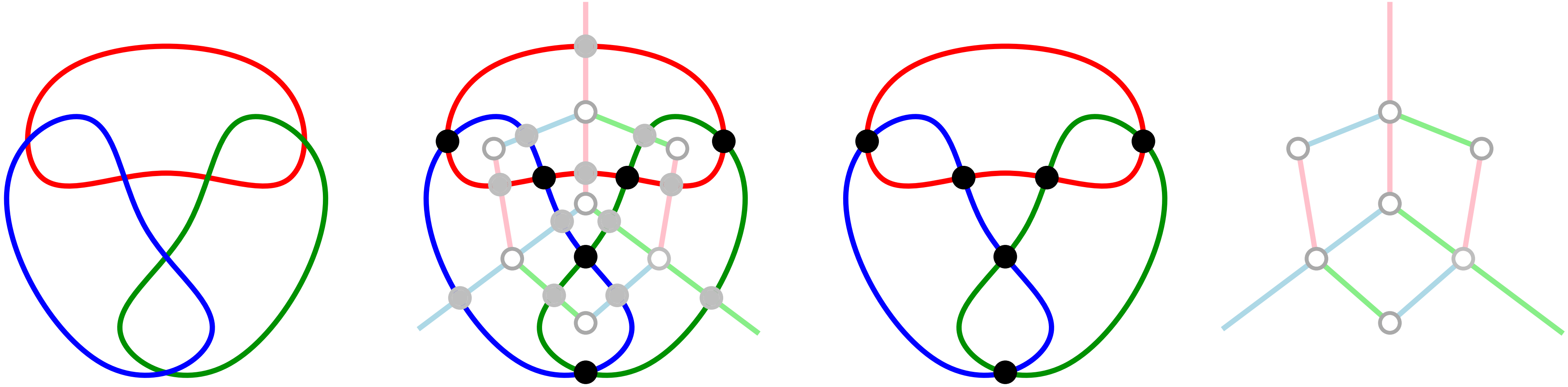
Computational Part

- enumeration via recursive search on flip graph



Computational Part

- connected arrangements encoded via primal-dual graph
- intersecting arrangements encoded via dual graph



arrangement primal-dual gr. primal graph dual graph

Part II: Triangles in Arrangements

Part II: Triangles in Arrangements

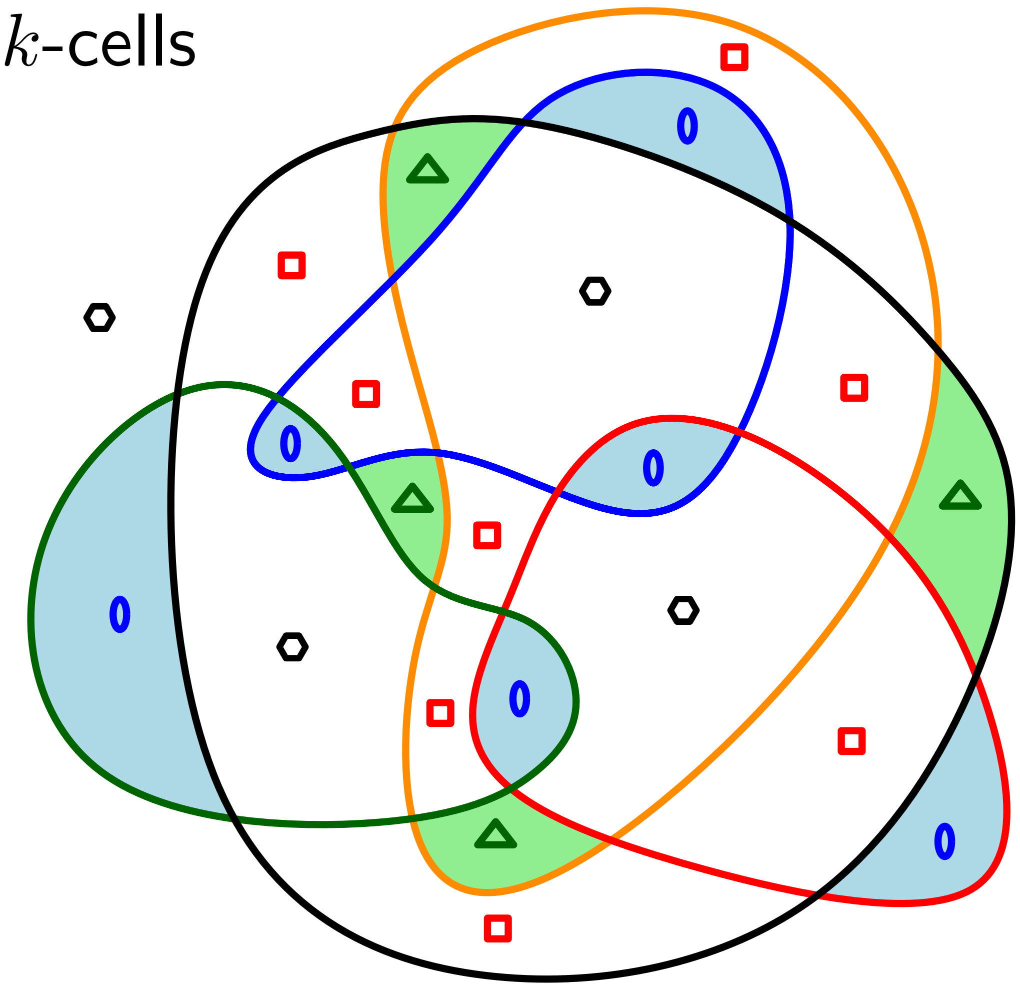
assumption throughout part II:

intersecting . . . any 2 pseudocircles cross twice

Cells in Arrangements

digon, *triangle*, *quadrangle*, *pentagon*, . . . , *k-cell*

p_k . . . # of k-cells



$p_2 = 6$

$p_3 = 4$

$p_4 = 8$

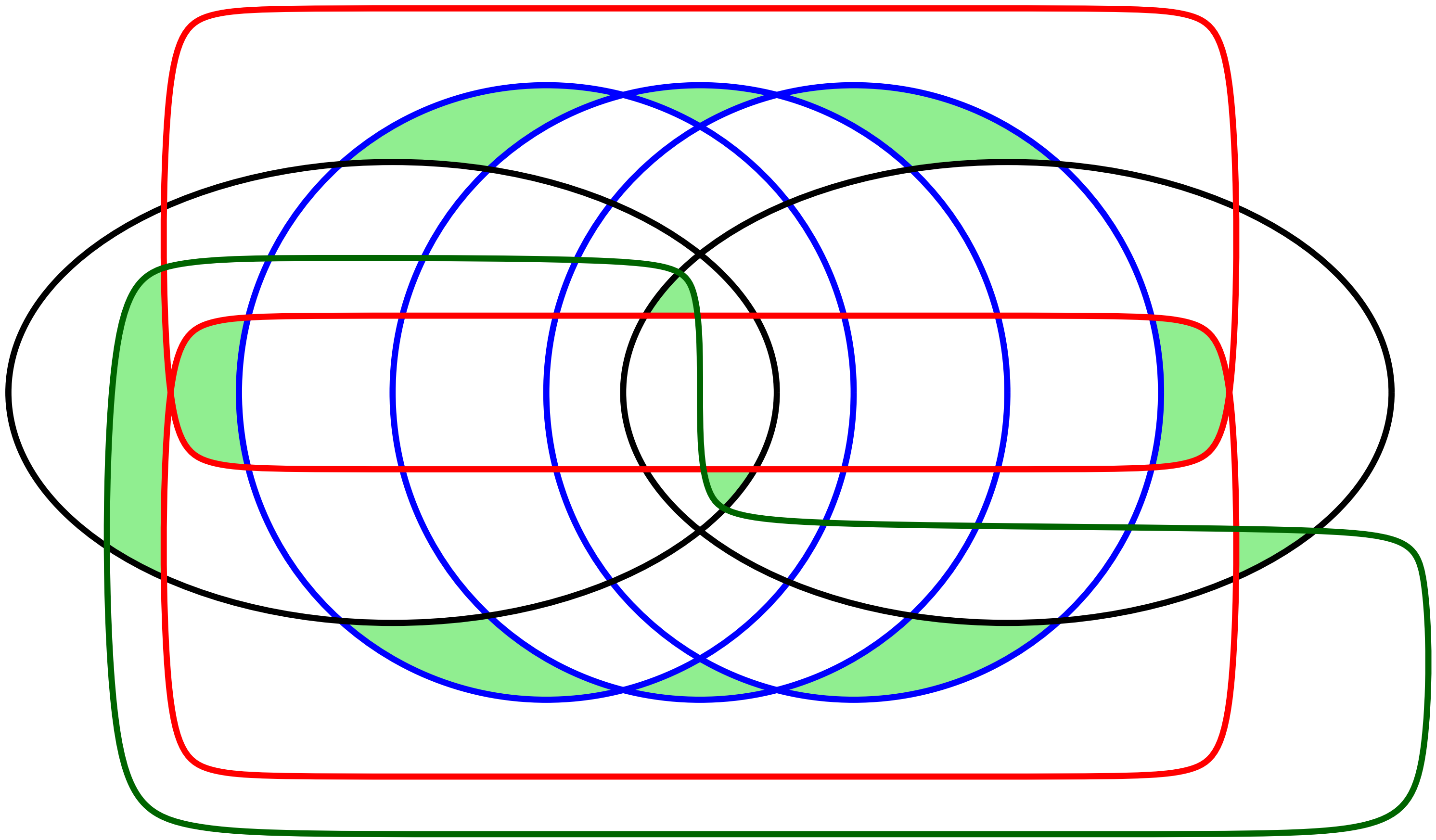
$p_5 = 0$

$p_6 = 4$

Triangles in **Digon-free** Arrangements

Grünbaum's Conjecture ('72):

- $p_3 \geq 2n - 4$?



Triangles in **Digon-free** Arrangements

Grünbaum's Conjecture ('72):

- $p_3 \geq 2n - 4$?

Known:

- $p_3 \geq 4n/3$ [Hershberger and Snoeyink '91]
- $p_3 \geq 4n/3$ for non-simple arrangements,
  tight for infinite family [Felsner and Kriegel '98]

Triangles in **Digon-free** Arrangements

Grünbaum's Conjecture ('72):

- $p_3 \geq 2n - 4$?

Known:

- $p_3 \geq 4n/3$ [Hershberger and Snoeyink '91]
- $p_3 \geq 4n/3$ for non-simple arrangements, tight for infinite family [Felsner and Kriegel '98]

Our Contribution:

- disprove Grünbaum's Conjecture
- $p_3 < 1.\overline{45}n$
- **New Conjecture:** $4n/3$ is tight

Triangles in Digon-free Arrangements

Theorem. The minimum number of triangles in digon-free arrangements of n pseudocircles is

(i) 8 for $3 \leq n \leq 6$.

(ii) $\lceil \frac{4}{3} n \rceil$ for $6 \leq n \leq 14$.

(iii) $< 1.\overline{45}n$ for all $n = 11k + 1$ with $k \in \mathbb{N}$.

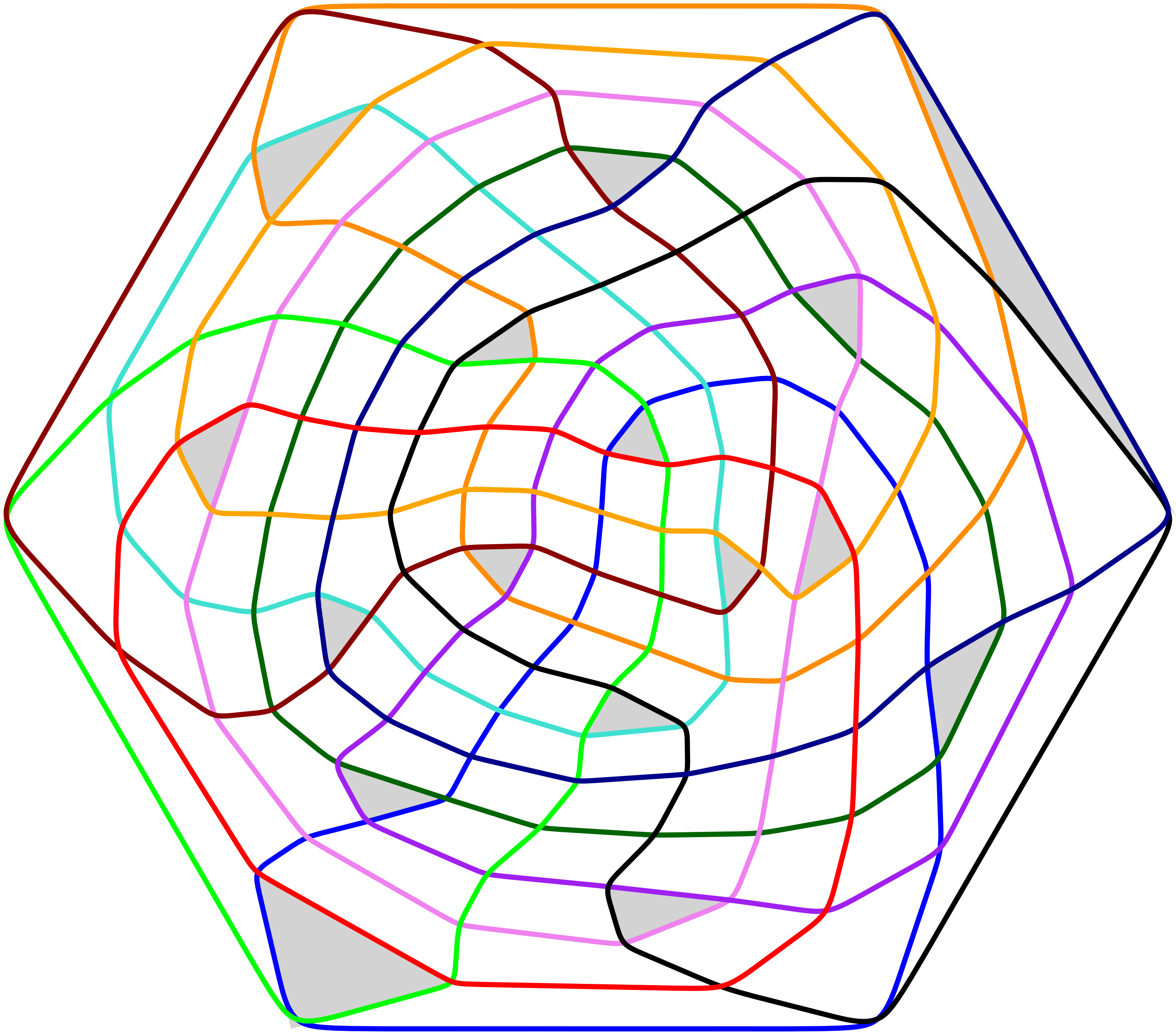


Figure: Arrangement of $n = 12$ pcs with $p_3 = 16$ triangles.

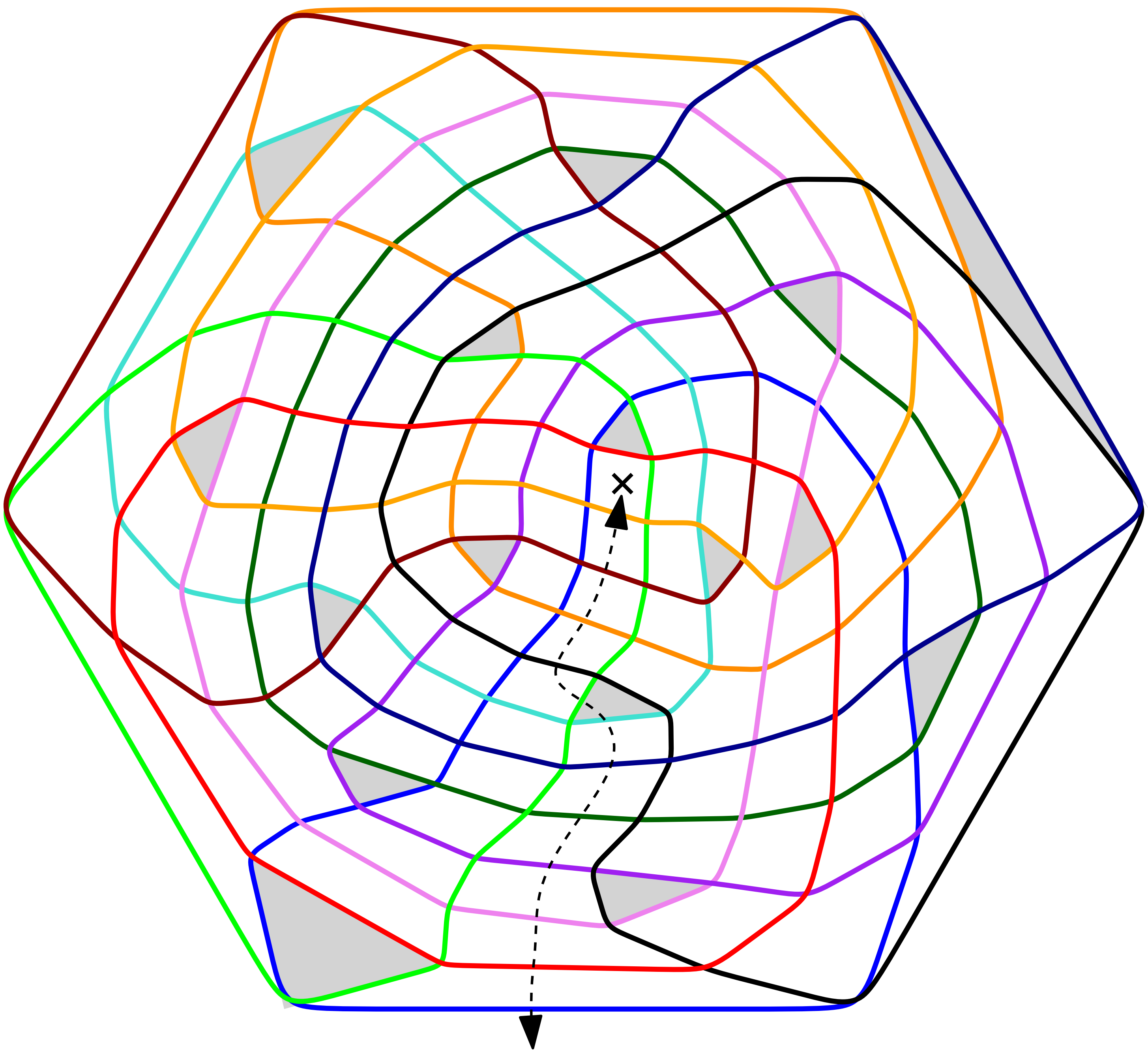
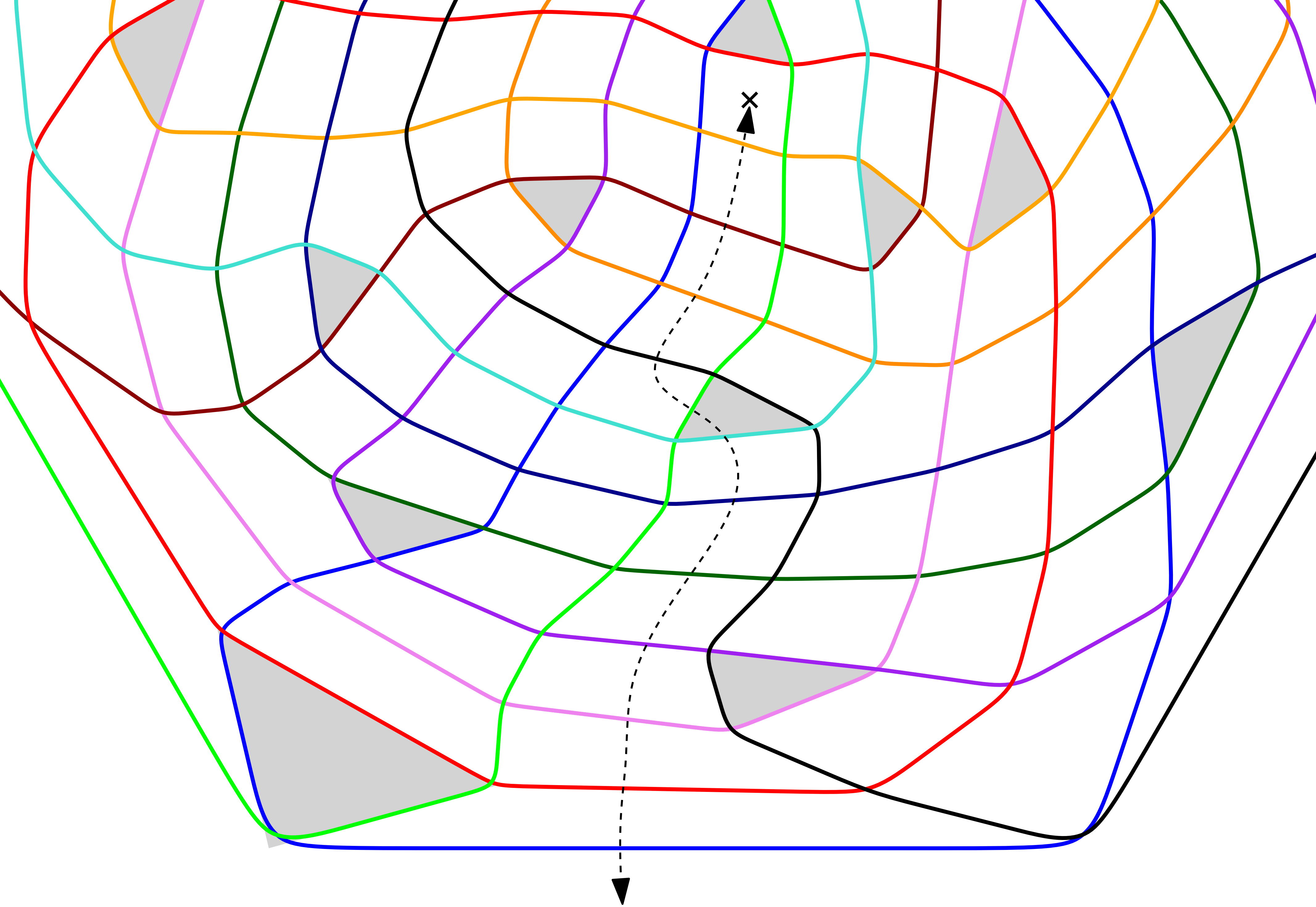
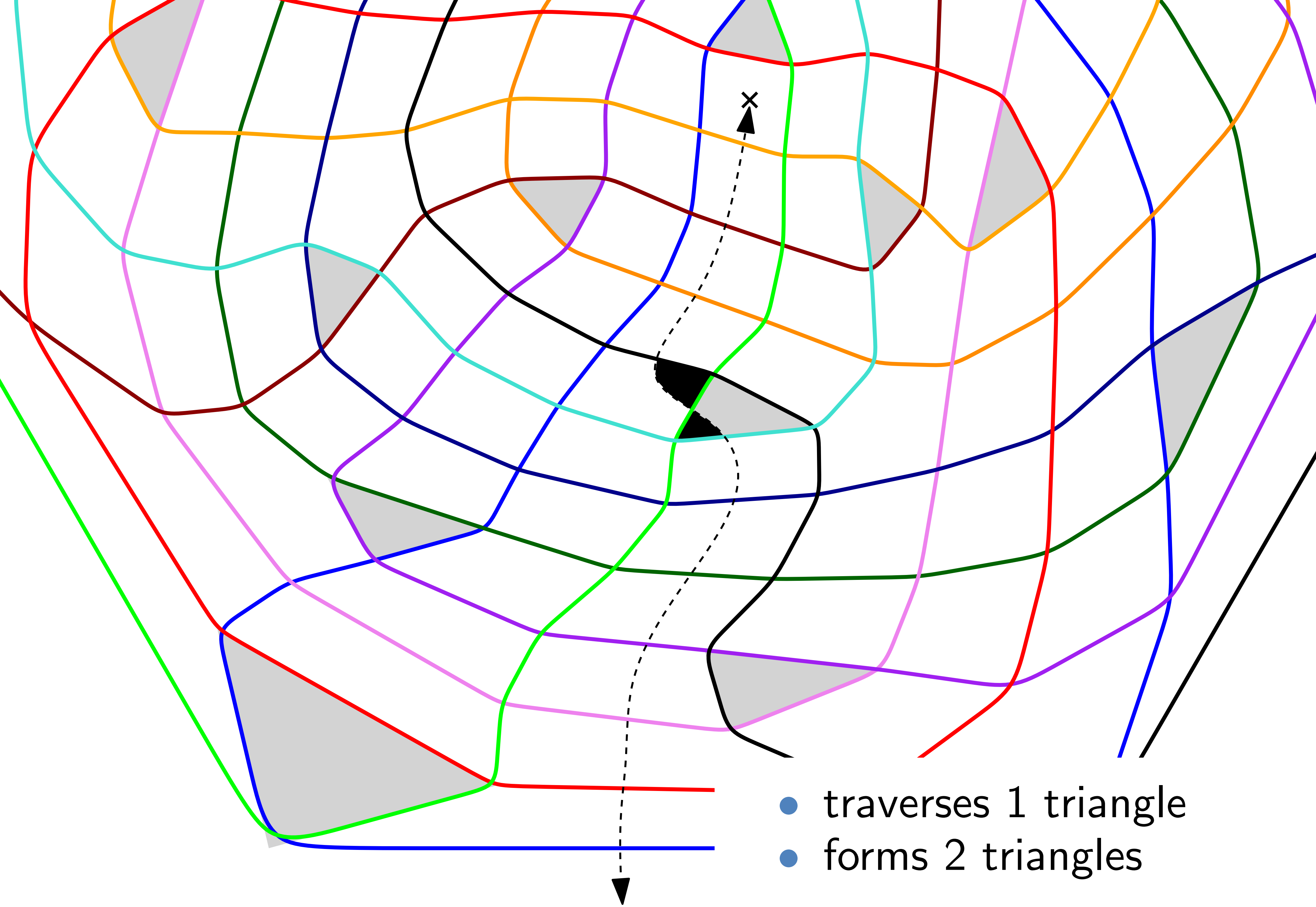


Figure: Arrangement of $n = 12$ pcs with $p_3 = 16$ triangles.

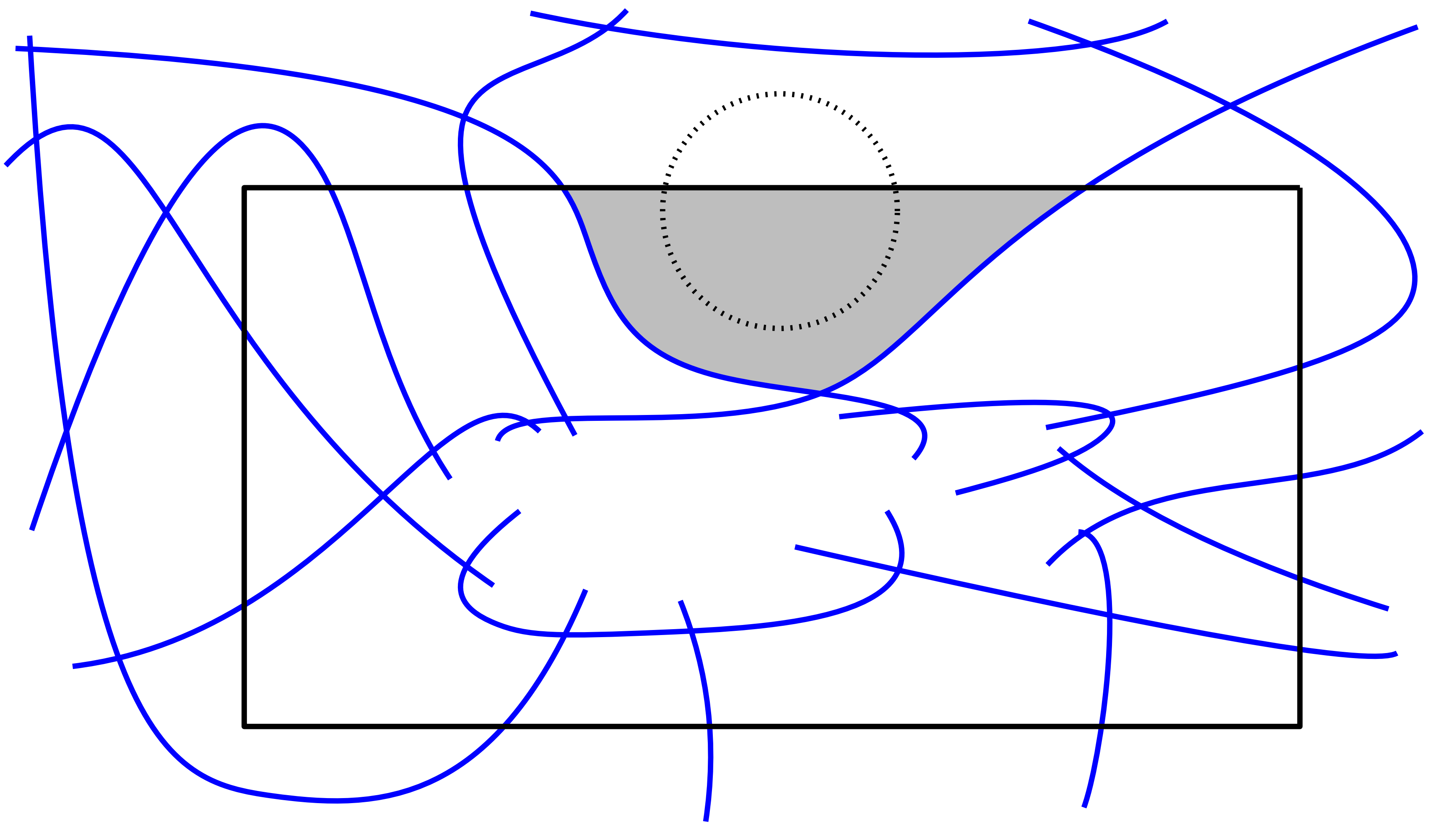




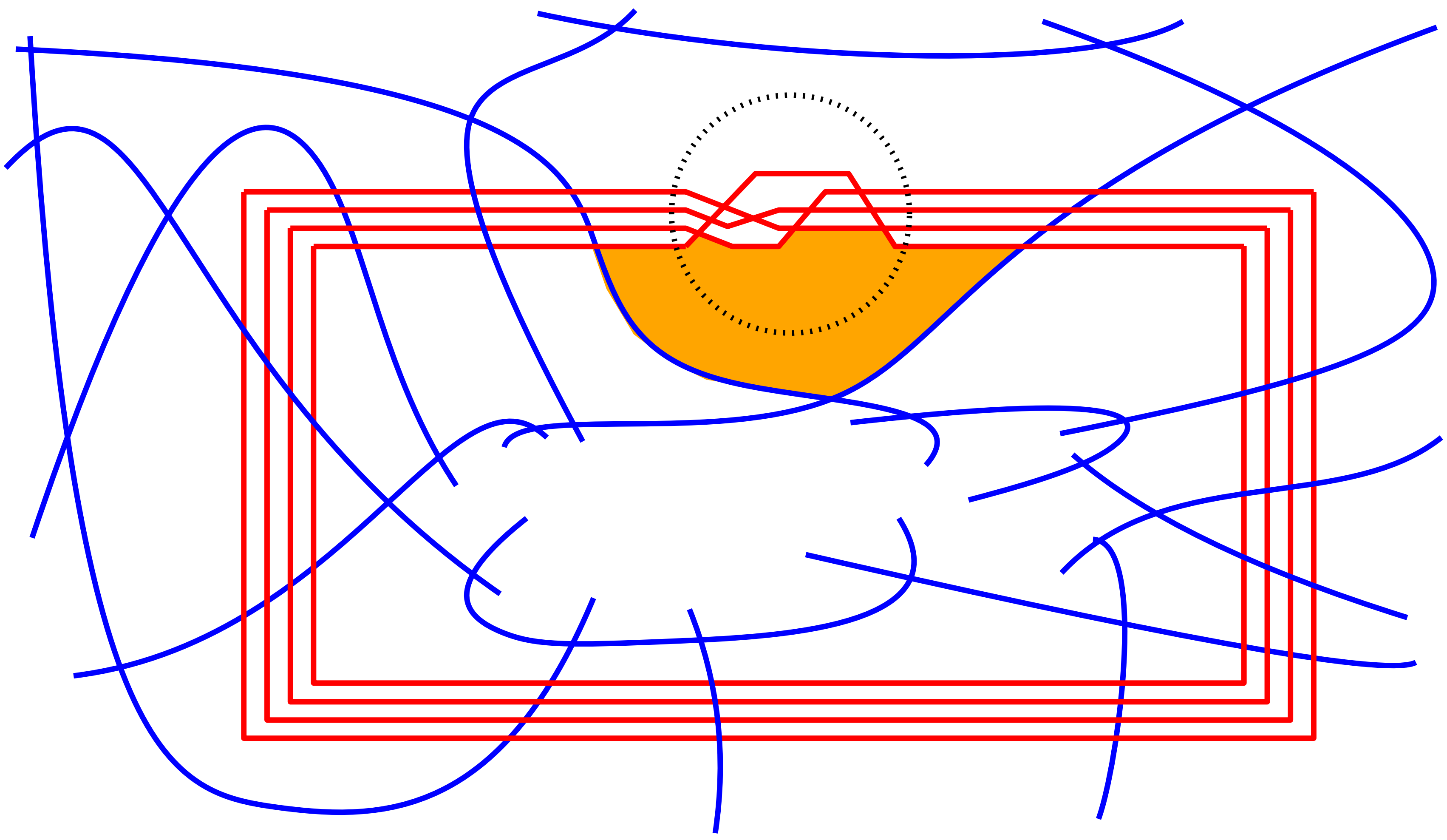
- traverses 1 triangle
- forms 2 triangles

Proof of the Theorem

Proof of the Theorem



Proof of the Theorem



Proof of the Theorem

- start with $\mathscr{C}_1 := \mathscr{A}_{12}$
- merge $\mathscr{C}_k$ and $\mathscr{A}_{12} \longrightarrow \mathscr{C}_{k+1}$
- $n(\mathscr{C}_k) = 11k + 1$, $p_3(\mathscr{C}_k) = 16k$
- $\frac{16k}{11k+1}$ increases as k increases with limit $\frac{16}{11} = 1.\overline{45}$

Proof of the Theorem

- start with $\mathscr{C}_1 := \mathscr{A}_{12}$
- merge $\mathscr{C}_k$ and $\mathscr{A}_{12} \longrightarrow \mathscr{C}_{k+1}$

maintain the path!

- $n(\mathscr{C}_k) = 11k + 1$, $p_3(\mathscr{C}_k) = 16k$
- $\frac{16k}{11k+1}$ increases as k increases with limit $\frac{16}{11} = 1.\overline{45}$

Triangles in Digon-free Arrangements

Theorem. The minimum number of triangles in digon-free arrangements of n pseudocircles is

(i) 8 for $3 \leq n \leq 6$.

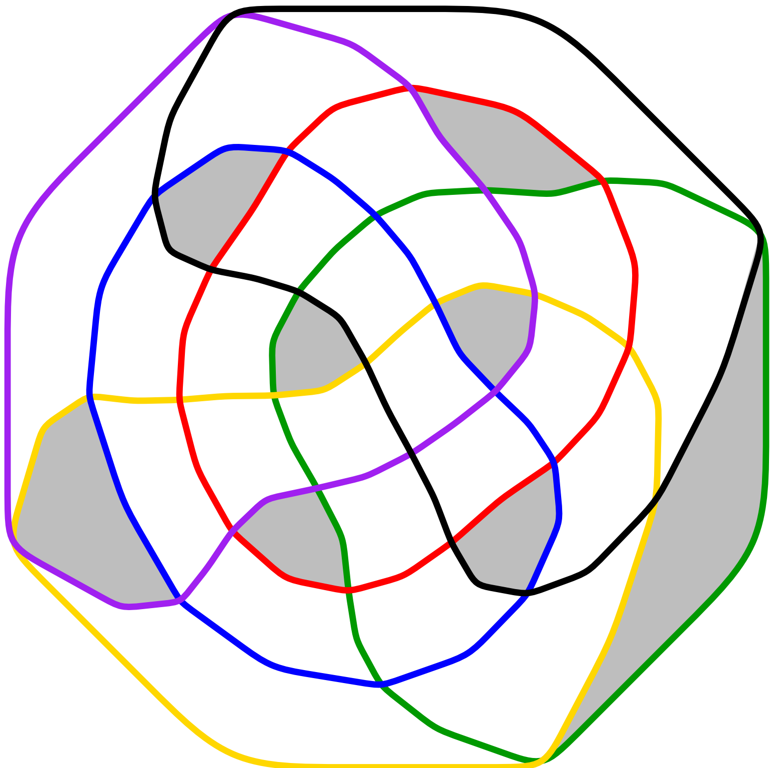
(ii) $\lceil \frac{4}{3} n \rceil$ for $6 \leq n \leq 14$.

(iii) $< 1.\overline{45}n$ for all $n = 11k + 1$ with $k \in \mathbb{N}$.

Conjecture. $\lceil 4n/3 \rceil$ is tight for infinitely many n.

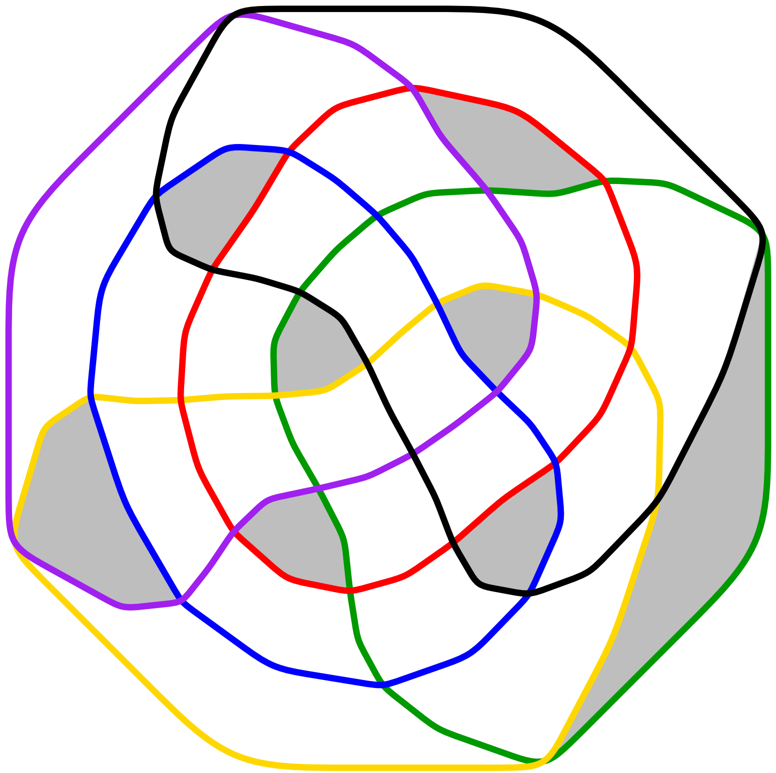
Triangles in Digon-free Arrangements

- $\exists$ unique arrangement $\mathcal{N}_6^\triangle$ with $n = 6, p_3 = 8$
- $\mathcal{N}_6^\triangle$ appears as a subarrangement of every arr. with $p_3 < 2n - 4$ for $n = 7, 8, 9$
- $\mathcal{N}_6^\triangle$ is non-circularizable



Triangles in Digon-free Arrangements

- $\exists$ unique arrangement $\mathcal{N}_6^{\triangle}$ with $n = 6, p_3 = 8$
- $\mathcal{N}_6^{\triangle}$ appears as a subarrangement of every arr. with $p_3 < 2n - 4$ for $n = 7, 8, 9$
- $\mathcal{N}_6^{\triangle}$ is non-circularizable
- $\Rightarrow$ Grünbaum's Conjecture might still be true for arrangements of circles!



Triangles in Arrangements **with Digons**

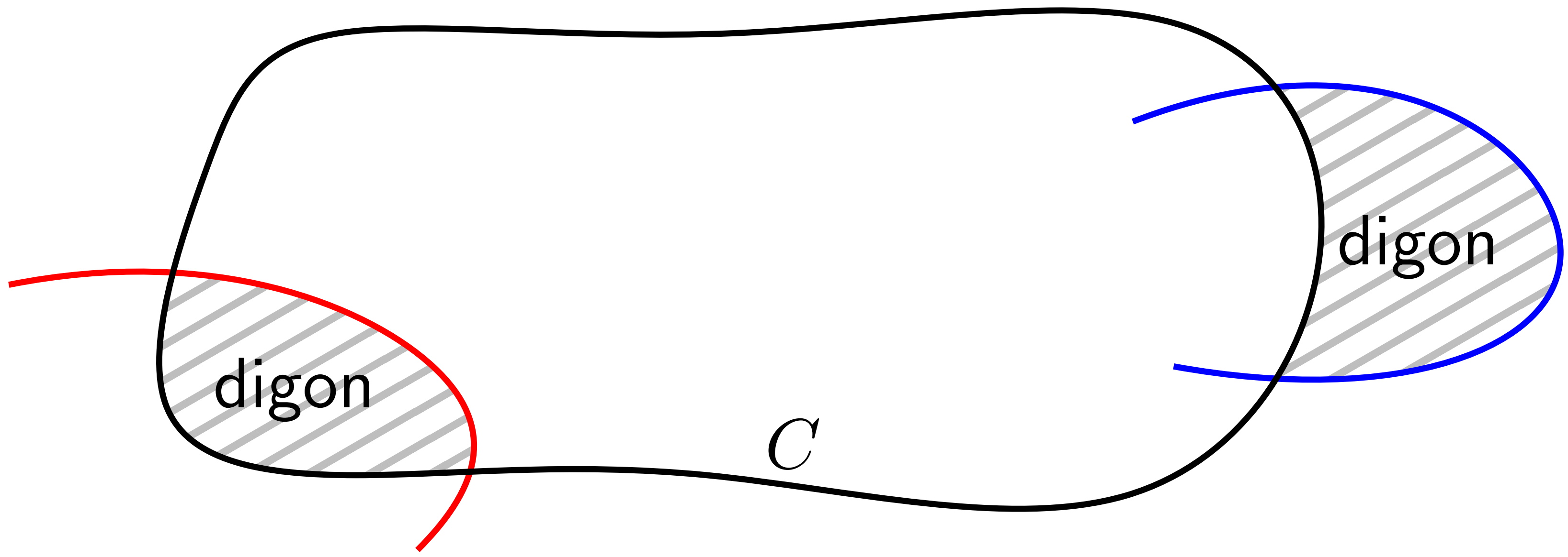
Theorem. $p_3 \geq 2n/3$

Triangles in Arrangements **with Digons**

Theorem. $p_3 \geq 2n/3$

Proof.

- C ... pseudocircle in $\mathscr{A}$ ← intersecting
- All incident digons lie on the same side of C.

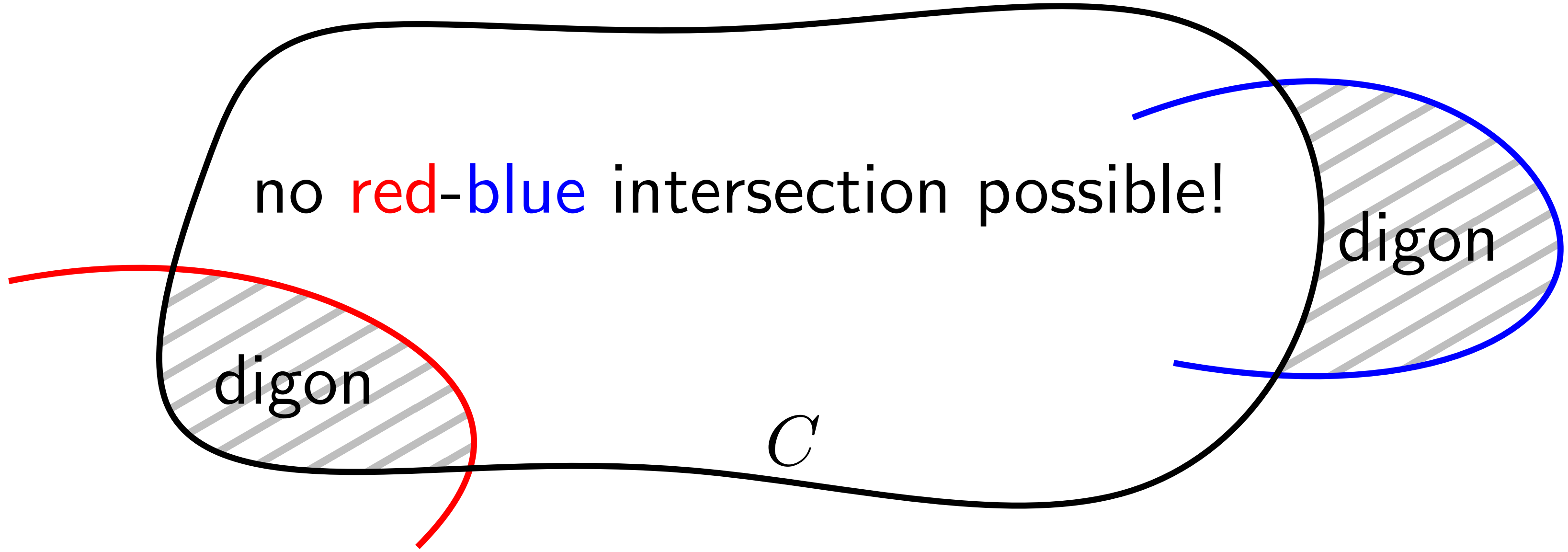


Triangles in Arrangements **with Digons**

Theorem. $p_3 \geq 2n/3$

Proof.

- C . . . pseudocircle in $\mathscr{A}$ intersecting
- All incident digons lie on the same side of C.



Triangles in Arrangements **with Digons**

Theorem. $p_3 \geq 2n/3$

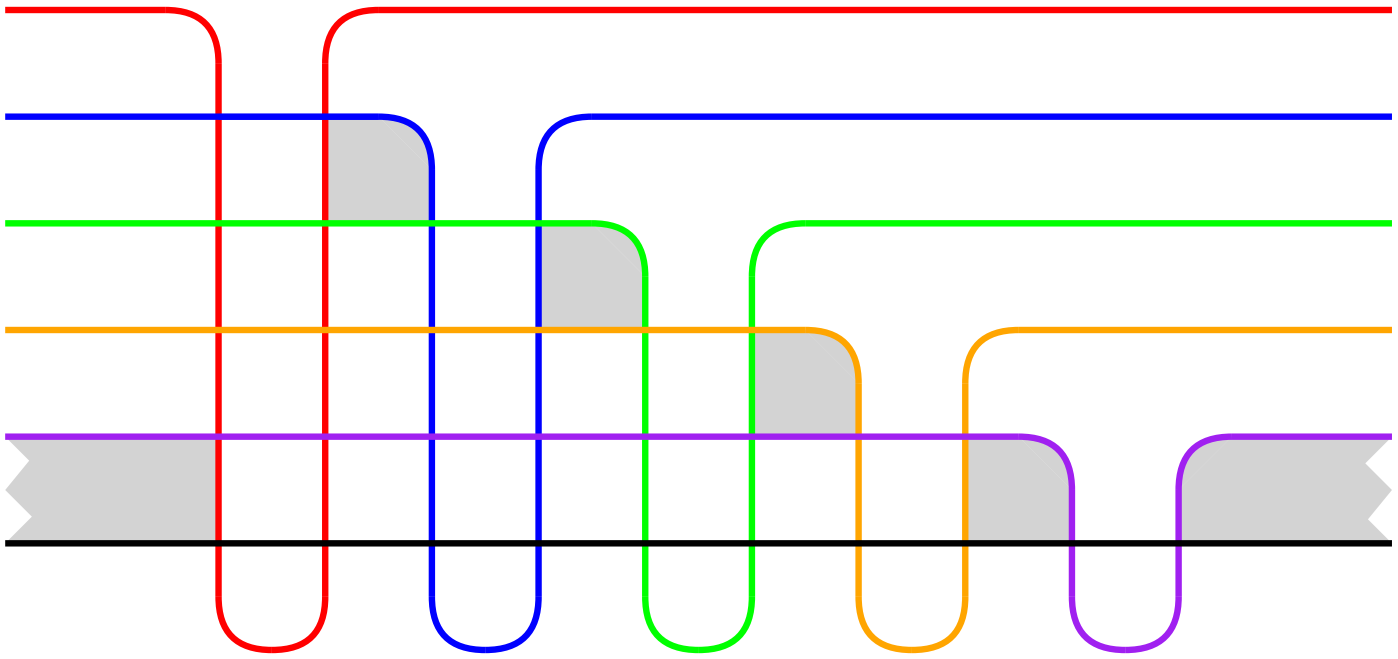
Proof.

- C ... pseudocircle in $\mathscr{A}$
- All incident digons lie on the same side of C.
- $\exists$ two digons or triangles on each side of C [Hershberger and Snoeyink '91] . $\square$

Triangles in Arrangements **with Digons**

Theorem. $p_3 \geq 2n/3$

Conjecture. $p_3 \geq n - 1$



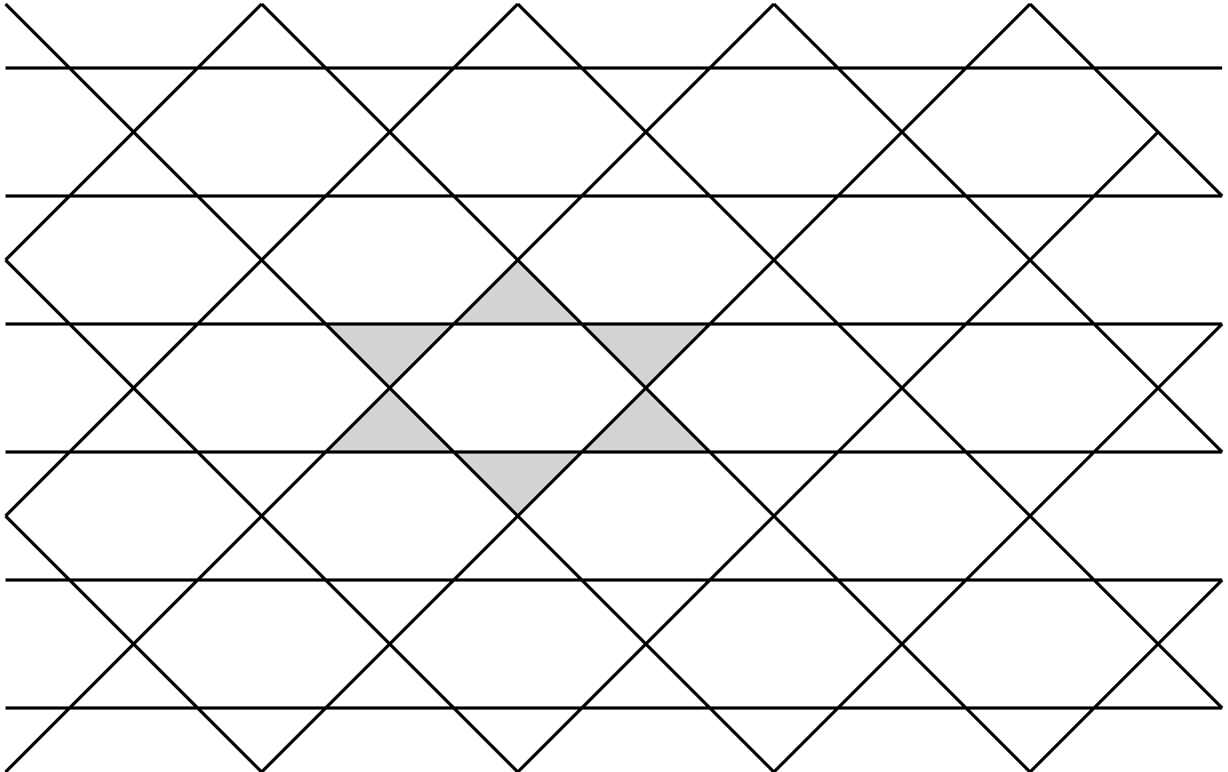
Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

- $\frac{4}{3}\binom{n}{2}$ construction for infinitely many values of n, based on pseudoline arrangements [Blanc '11]



Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

- $\frac{4}{3}\binom{n}{2}$ construction for infinitely many values of n, based on pseudoline arrangements [Blanc '11]
- Question: $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$?

Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

- $\frac{4}{3}\binom{n}{2}$ construction for infinitely many values of n, based on pseudoline arrangements [Blanc '11]

- Question: $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$?

n	2	3	4	5	6	7	8	9	10
simple	0	8	8	13	20	29	≥ 37	≥ 48	≥ 60
+digon-free	-	8	8	12	20	29	≥ 37	≥ 48	≥ 60
$\lfloor \frac{4}{3}\binom{n}{2} \rfloor$	1	4	8	13	20	28	37	48	60

Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

- $\frac{4}{3}\binom{n}{2}$ construction for infinitely many values of n, based on pseudoline arrangements [Blanc '11]

- Question: $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$?

circularizable!

n	2	3	4	5	6	7	8	9	10
simple	0	8	8	13	20	29	≥ 37	≥ 48	≥ 60
+digon-free	-	8	8	12	20	29	≥ 37	≥ 48	≥ 60
$\lfloor \frac{4}{3}\binom{n}{2} \rfloor$	1	4	8	13	20	28	37	48	60

Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

Proof:

$\mathscr{A}$ ... arrangement of $n \geq 4$ pseudocircles

X ... set of crossings (vertices of graph)

Maximum Number of Triangles

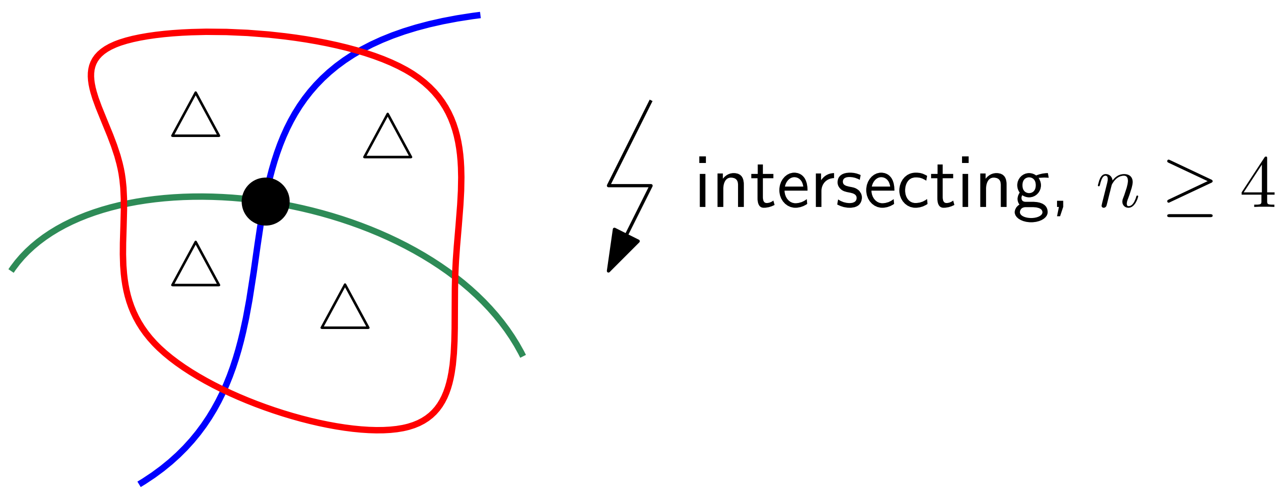
Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

Proof:

$\mathscr{A}$ . . . arrangement of $n \geq 4$ pseudocircles

X . . . set of crossings (vertices of graph)

Claim A: No vertex is incident to 4 triangular cells.



Maximum Number of Triangles

Theorem. $p_3 \leq \frac{2}{3}n^2 + O(n)$

Proof:

$\mathscr{A}$ . . . arrangement of $n \geq 4$ pseudocircles

X . . . set of crossings (vertices of graph)

Claim A: No vertex is incident to 4 triangular cells.

X' . . . crossings incident to *precisely* 3 triangles

Maximum Number of Triangles

Outline:

- We will show $|X'| = O(n)$.

Maximum Number of Triangles

Outline:

- We will show $|X'| = O(n)$.
- $\Rightarrow \#$ of triangles incident to a crossing from X' is $O(n)$.

Maximum Number of Triangles

Outline:

- We will show $|X'| = O(n)$.
- $\Rightarrow$ # of triangles incident to a crossing from X' is $O(n)$.
- each crossing of $Y := X \setminus X'$ is incident to at most 2 triangles
- each remaining triangle is incident to 3 crossings of Y

↑ not incident to any vertex from X'

Maximum Number of Triangles

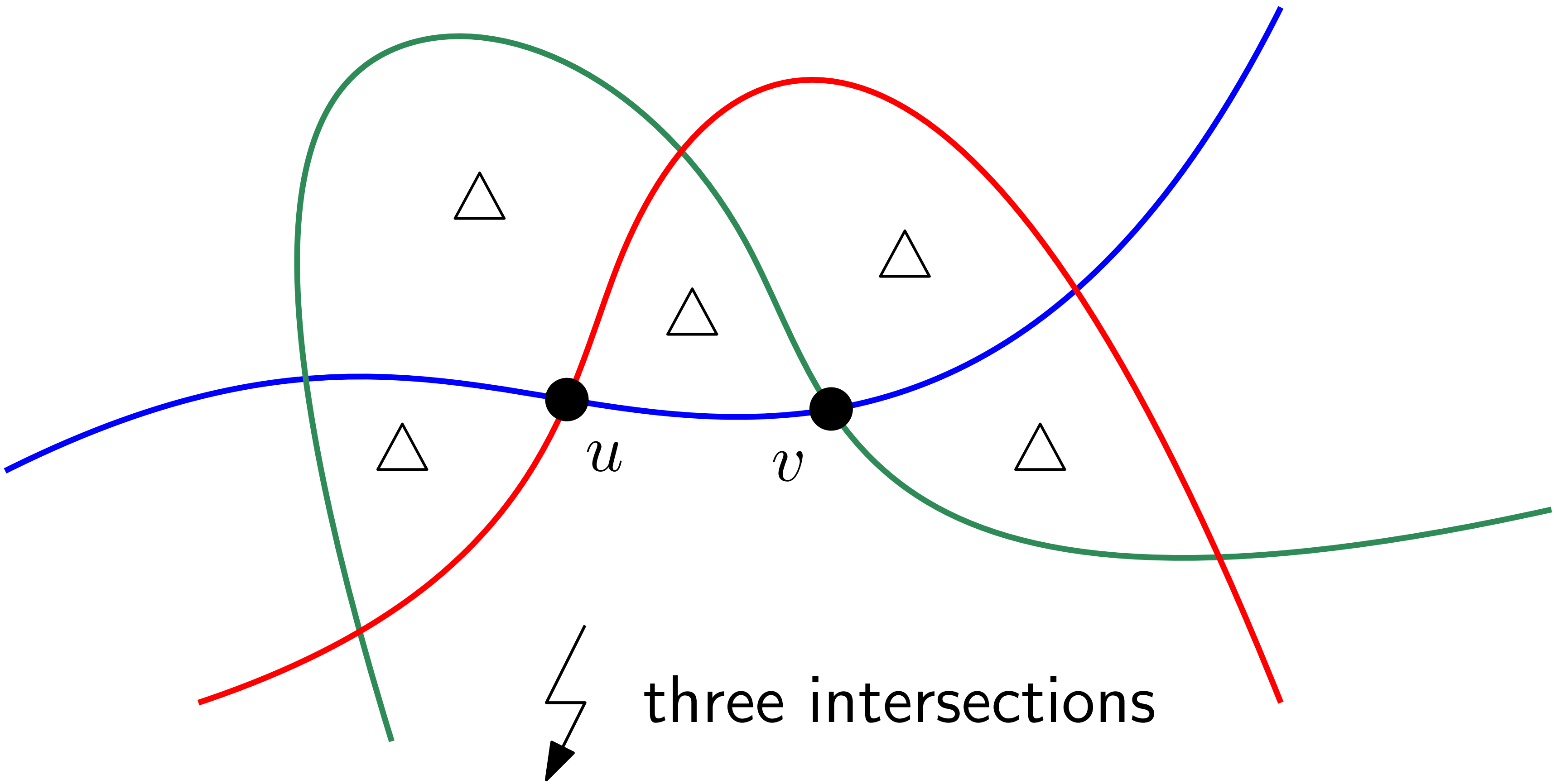
Outline:

- We will show $|X'| = O(n)$.
- $\Rightarrow$ # of triangles incident to a crossing from X' is $O(n)$.
- each crossing of $Y := X \setminus X'$ is incident to at most 2 triangles
- each remaining triangle is incident to 3 crossings of Y
- Since $|Y| \leq |X| = n(n-1)$, we count

$$p_3 \leq \frac{2}{3}|Y| + O(n) \leq \frac{2}{3}n^2 + O(n)$$

Maximum Number of Triangles

Claim B: Two adjacent crossings u, v in X' share two triangles.

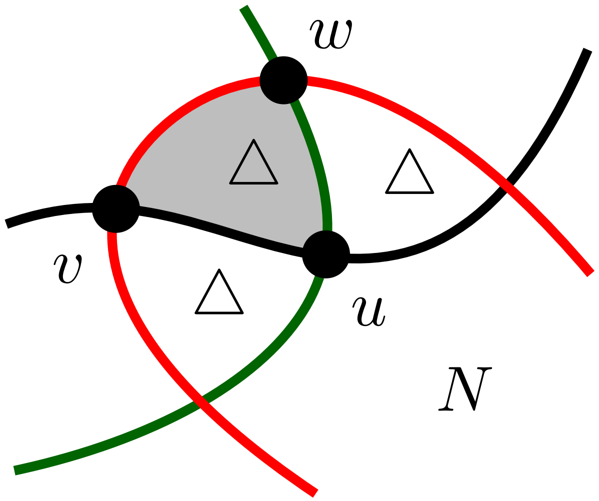


Maximum Number of Triangles

Claim B: Two adjacent crossings u, v in X' share two triangles.

Claim C: Let u, v, w be three distinct crossings in X'. If u is adjacent to both v and w, then v is adjacent to w.

(If both edges uv and uw are incident to two triangles, then uvw form a triangle)

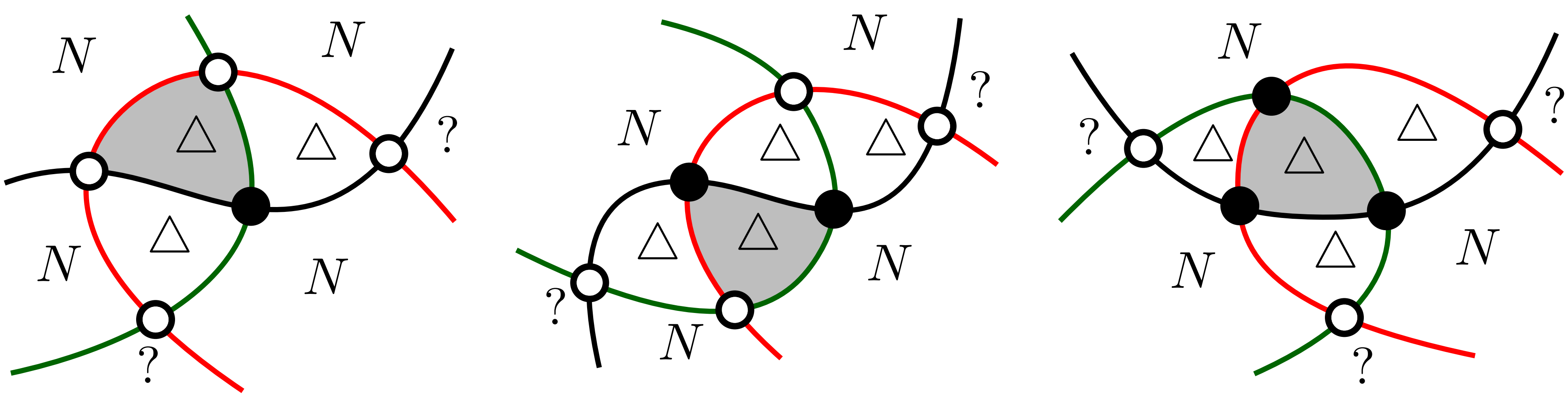


Maximum Number of Triangles

Claim B: Two adjacent crossings u, v in X' share two triangles.

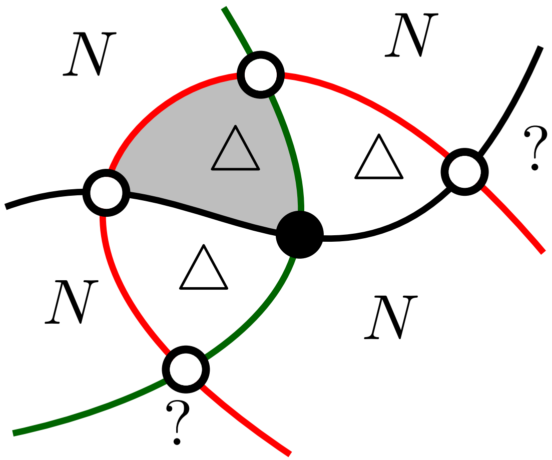
Claim C: Let u, v, w be three distinct crossings in X'.
If u is adjacent to both v and w, then v is adjacent to w.

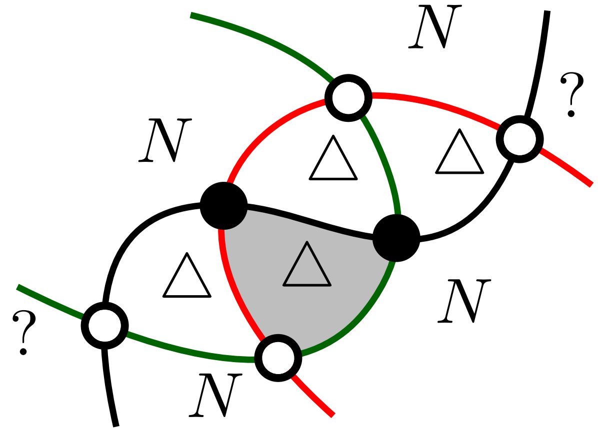
$\Rightarrow$ each connected comp. of the graph induced by X' is either singleton, edge, or triangle.

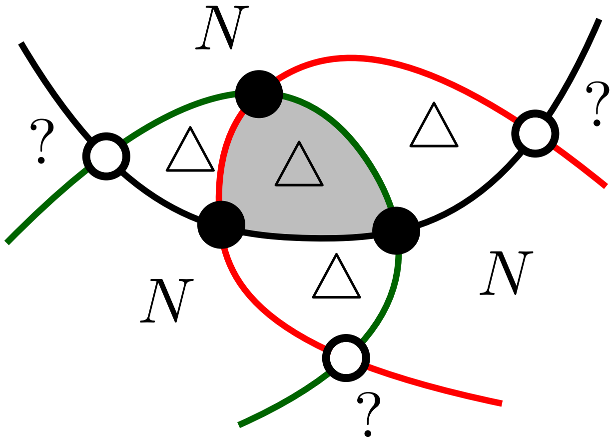


Maximum Number of Triangles

We can convert crossings of X' into digons using △-flips!

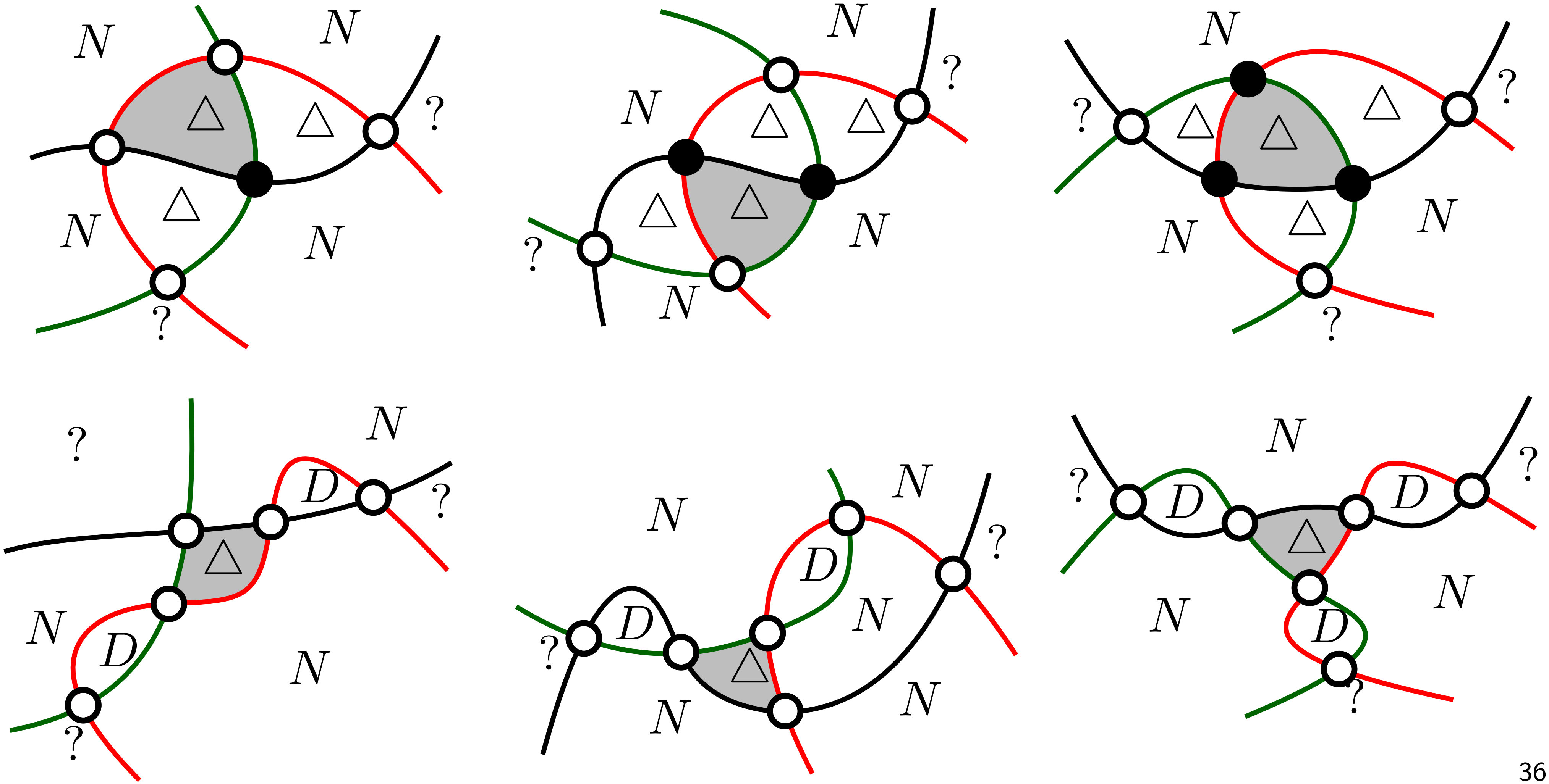






Maximum Number of Triangles

We can convert crossings of X' into digons using △-flips!



Maximum Number of Triangles

We can convert crossings of X' into digons using $\triangle$-flips!

There are at most $O(n)$ digons
[Agarwal, Nevo, Pach, Pinchasi, Sharir, Smorodinsky 2004]

$\Rightarrow$ at most $O(n)$ flips

$\Rightarrow$ $|X'|$ at most $O(n)$

$\Rightarrow p_3 \leq \frac{2}{3}n^2 + O(n)$ $\square$

Thank you for your attention!