Almost-equidistant sets

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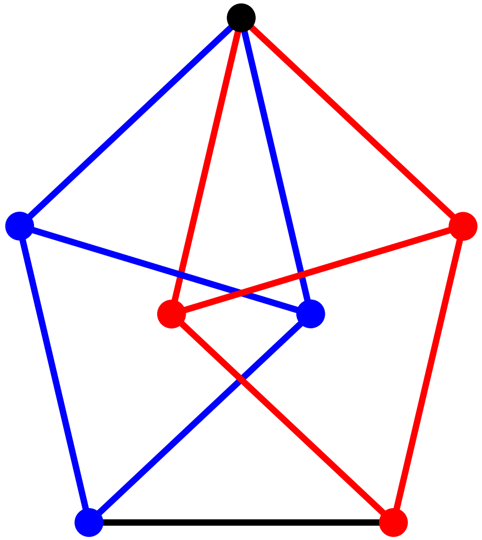
Definitions

A set P of points in $\mathbb{R}^d$ is *almost-equidistant* if for any 3 points there are two at unit distance



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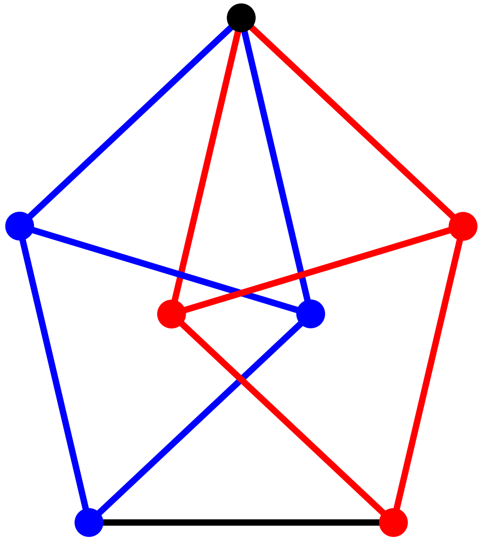
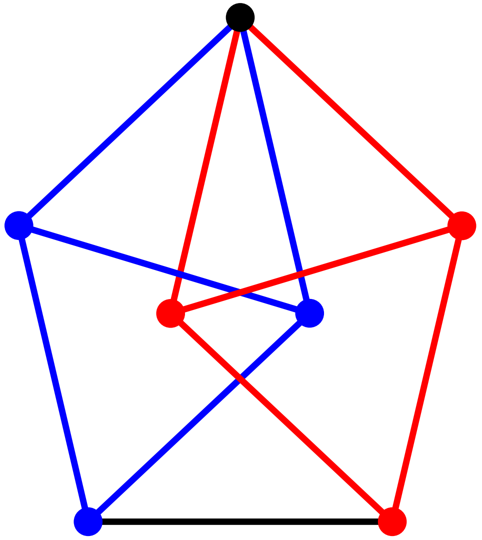


$f(d)$ . . . maximum size of an almost-equidistant set in $\mathbb{R}^d$

Results for small d

Theorem (Bezdek, Naszódi, Visy '03 and Talata '07):
$f(2) = 7$. unique example.



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Theorem (Györey '04):
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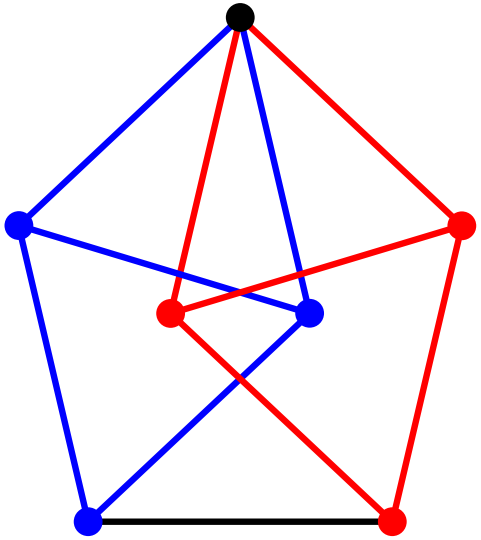

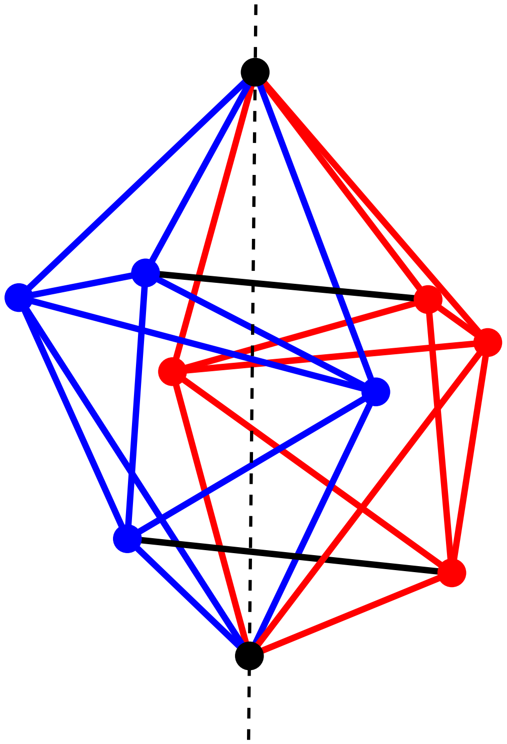

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new, simple proofs
(computer-assisted)

Theorem: $f(4) \in \{12, 13\}$.

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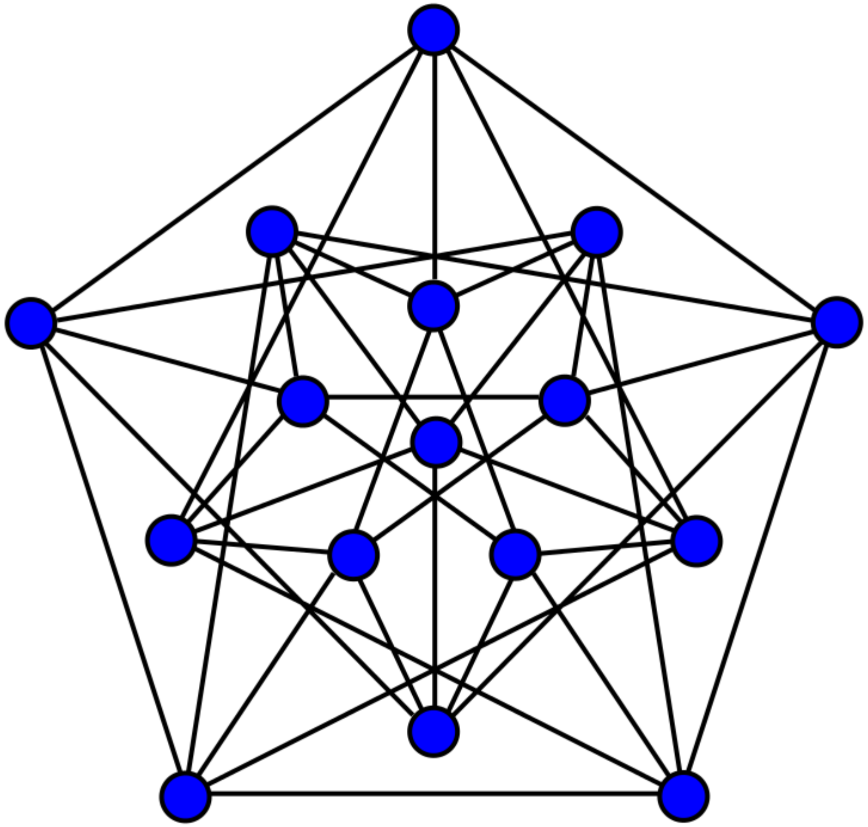
conjectured

Results for small d

- dimension $d = 5$: Clebsch graph witnesses $f(5) \geq 16$ [Larman and Rogers '72]



source of image: http://de.wikipedia.org/wiki/Clebsch-Graph

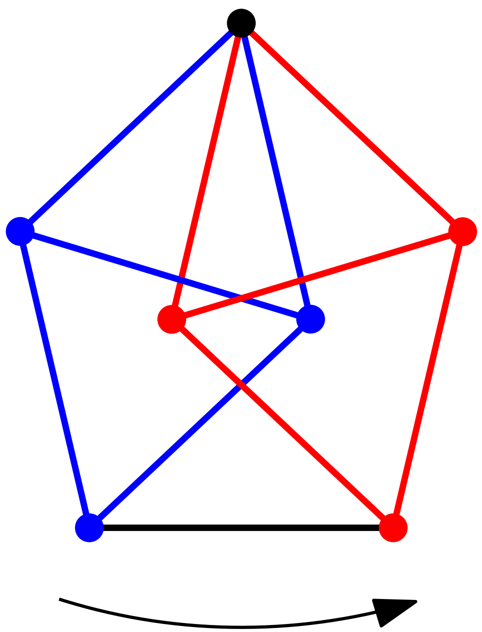
Results for small d

- dimension $d = 5$: Clebsch graph witnesses $f(5) \geq 16$ [Larman and Rogers '72]

d	1	2	3	4	5	6	7	8	9
lower bnd.	4	7	10	12	16	18	20	24	24
upper bnd.	4	7	10	13	20	26	34	41	49

Lower Bound

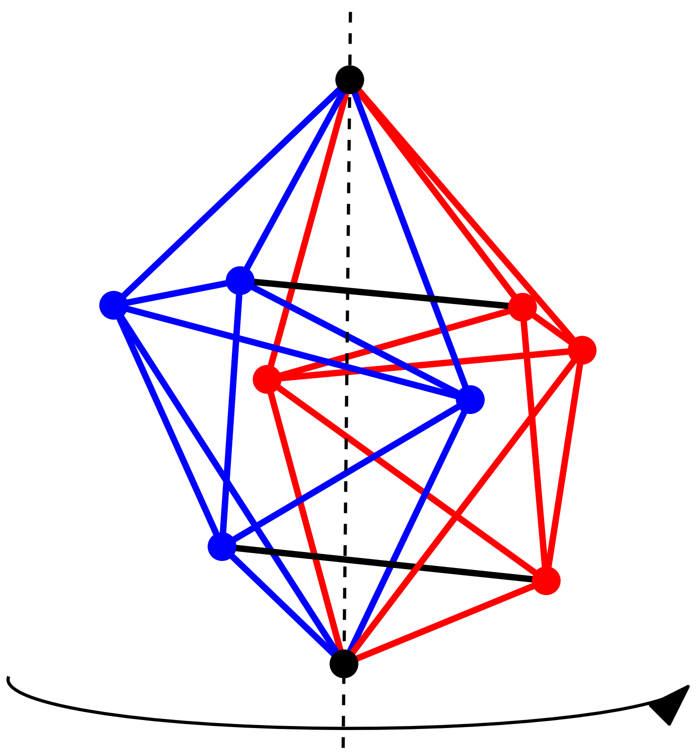
- trivial generalization of Moser spindle gives $2d + 3$



(i) glue 2 d-simplices
(ii) copy with shared top vertex
(iii) rotate until bottom vertices unit distance

Lower Bound

- trivial generalization of Moser spindle gives $2d + 3$
- generalization of 10 points in $\mathbb{R}^3$ gives $2d + 4$



Upper Bound

- no K_{d+2} and complement triangle-free
- $f(d) \leq R(d+2, 3)$
- $R(d+2, 3) \leq O(d^2 / \log d)$ [Atjai, Komlós, Szemerédi '80]

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Theorem: $f(d) \leq O(d^{3/2})$.

Upper Bound: Sketch of the Proof

Our proof is based on Deaett's, who gave a simpler proof of Rosenfeld's result: an almost-equidistant set on a sphere with radius $1/\sqrt{2}$ in $\mathbb{R}^d$ has at most $2d$ points.

Upper Bound: Sketch of the Proof

Lemma: C ... k points in $\mathbb{R}^d$, pairw. at unit distance
center $c := \frac{1}{k}\sum_{p\in C} p$
$A := \mathrm{span}(C - c)$.
Then the set of points equidistant from all points of C is the affine space $c + A^{\perp}$.
Furthermore, the intersection of all unit spheres centered at the points in C is the $(d-k)$-dimensional sphere of radius $\sqrt{(k+1)/(2k)}$ centered at c and contained in $c + A^{\perp}$.

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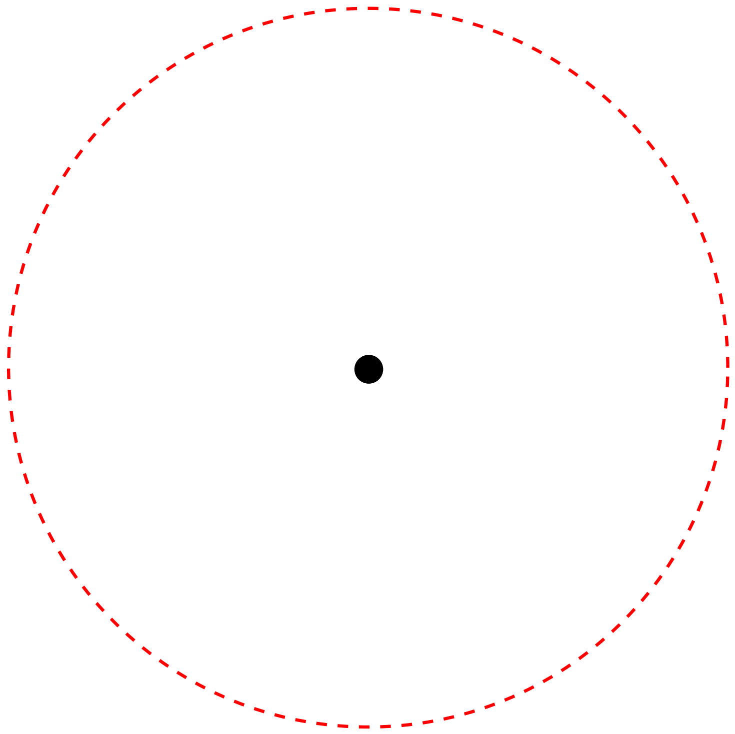
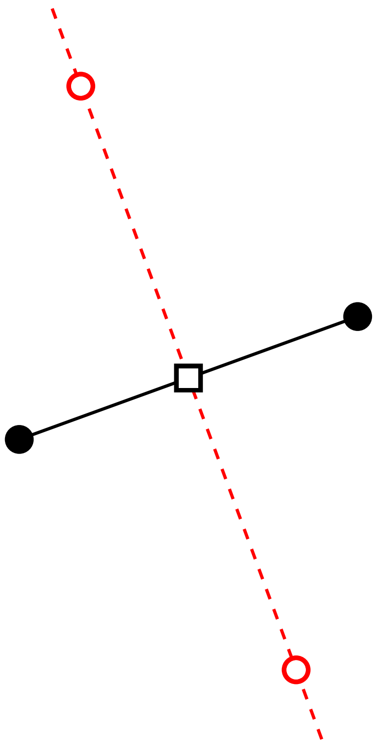


$d = 2, k = 1$



$d = k = 2$

Upper Bound: Sketch of the Proof

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Furthermore, the intersection of all unit spheres centered at the points in C is the $(d-k)$-dimensional sphere of radius $\sqrt{(k+1)/(2k)}$ centered at c and contained in $c + A^{\perp}$.

Corollary: At most $d+1$ points pairwise at unit distance.

Upper Bound: Sketch of the Proof

Theorem: $f(d) \leq O(d^{3/2})$.

Proof: Let $d \geq 2$, $V \subseteq \mathbb{R}^d$ be an almost-equidistant set, $G = (V, E)$ unit-distance graph of V, and let $k := \lfloor 2\sqrt{d} \rfloor$.

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Let $S \subseteq V$ be a set of k points, pairwise at unit distance.

If no such set exists, then

$$|V| < R(k,3) \le \binom{k+3-2}{3-1} < \binom{2\sqrt{d}+1}{2} = 2d + \sqrt{d}.$$

Thus we assume S exists.

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$B := \{x \in V : \|x - s\| = 1 \ \forall s \in S\}$ (common neighbors).

Since V is a.e.d., the set of non-neighbors of any vertex form a clique. By Corollary, size at most $d + 1$.

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Every vertex of $V \setminus B$ is non-neighbor of some vertex of S, thus $|V \setminus B| \leq k(d + 1)$.

Upper Bound: Sketch of the Proof

Apply Lemma to S, B lies on a sphere of radius $\sqrt{(k+1)/(2k)}$ in affine subspace of dimension $d-k+1$.

... and bit of linear algebra to bound $|B|$ ...

Upper Bound

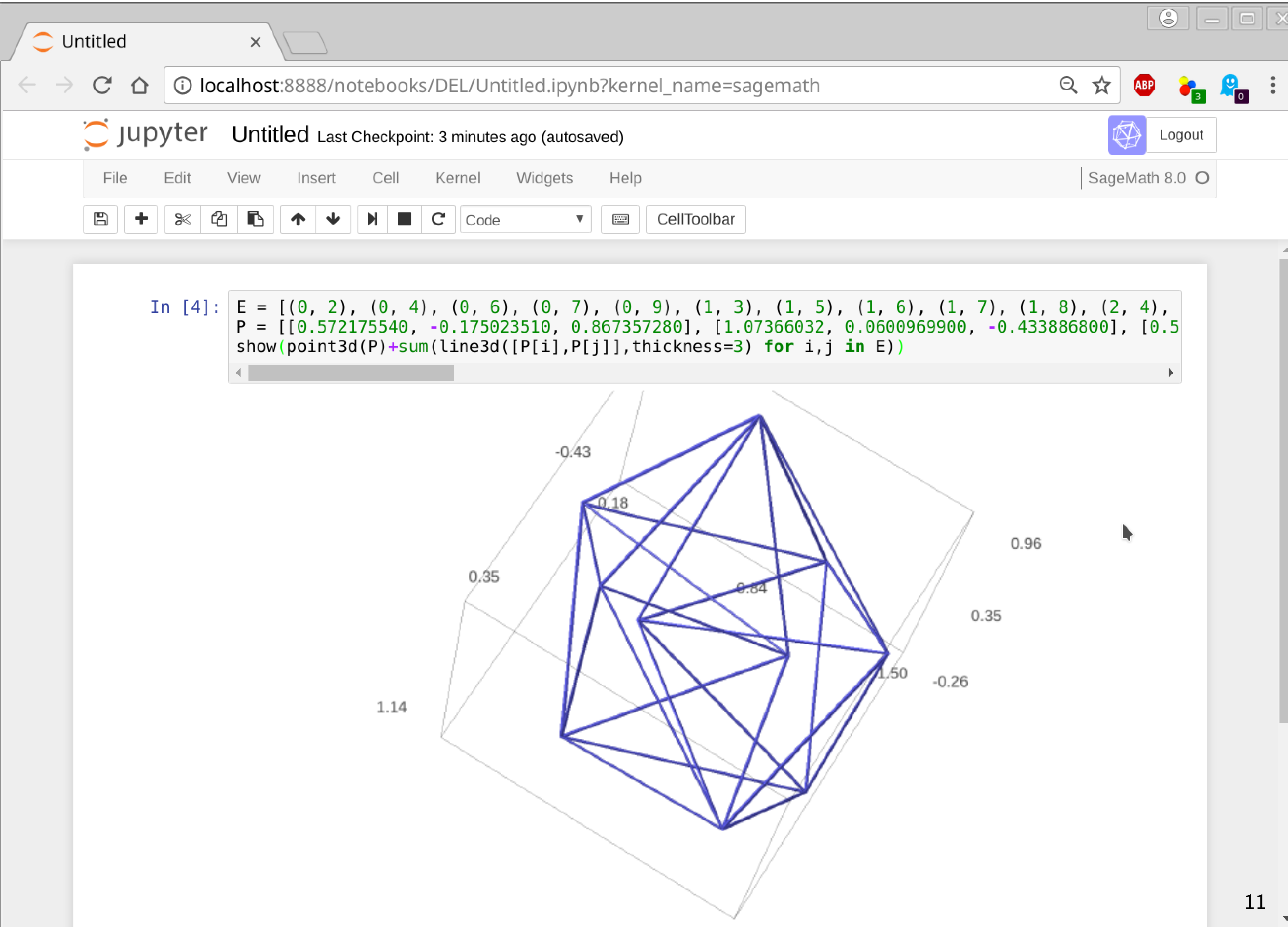
Recent improvements:

use ideas from our proof

- Polyanskii: $O(n^{13/9})$
- Kupavskii, Mustafa, Swanepoel: $O(n^{4/3})$
- Polyanskii: $O(n^{4/3})$ (new proof)

Remark

- 3D visualization via SageMath


Untitled
localhost:8888/notebooks/DEL/Untitled.ipynb?kernel_name=sagemath
jupyter Untitled Last Checkpoint: 3 minutes ago (autosaved)
Logout
File Edit View Insert Cell Kernel Widgets Help
SageMath 8.0
Code
CellToolbar
In [4]:
E = [(0, 2), (0, 4), (0, 6), (0, 7), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8), (2, 4),
P = [[0.572175540, -0.175023510, 0.867357280], [1.07366032, 0.0600969900, -0.433886800], [0.5
show(point3d(P)+sum(line3d([P[i],P[j]],thickness=3) for i,j in E))
-0.43
0.18
0.35
0.84
0.96
0.35
1.50
-0.26
1.14

Thank you for your attention!