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ON DISJOINT HOLES IN POINT SETS

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Abstract. Given a set of points $S \subseteq \mathbb{R}^2$, a subset $X \subseteq S$, $|X| = k$, is called *k-gon* if all points of X lie on the boundary of conv(X), and *k-hole* if, in addition, no point of $S \setminus X$ lies in conv(X). We use computer assistance to show that every set of 17 points in general position admits two *disjoint* 5-holes, that is, holes with disjoint respective convex hulls. This answers a question of Hosono and Urabe (2001).

In a recent article, Hosono and Urabe (2018) present new results on interior-disjoint holes – a variant, which also has been investigated in the last two decades. Using our program, we show that every set of 15 points contains two interior-disjoint 5-holes. Moreover, our program can be used to verify that every set of 17 points contains a 6-gon within significantly smaller computation time than the original program by Szekeres and Peters (2006).

1. Introduction

Throughout this paper every set of points in the plane $S \subseteq \mathbb{R}^2$ is *in general position*, i.e, no three points lie on a line. A subset $X \subseteq S$ of size $|X| = k$ is a *k-gon* if all points of X lie on the boundary of the convex hull of X. A classical result from the 1930s by Erdős and Szekeres asserts that, for fixed $k \in \mathbb{N}$, every set of $\binom{2k-4}{k-2} + 1$ points contains a k-gon [**11**]. They also constructed point sets of size 2^{k-2} with no k-gon. There were several small improvements on the upper bound by various researchers in the last decades, each of order $4^{k-o(k)}$, until Suk [**27**] significantly improved the upper bound to $2^{k+o(k)}$ in 2017. However, the precise minimum number $g(k)$ of points needed to guarantee the existence of a k-gon is still unknown for $k \geq 7$ (cf. [**28**])[1].

In the 1970s, Erdős [**10**] asked whether every sufficiently large point set contains a *k-hole*, that is, a k-gon with no other points of S inside its convex hull. Harborth [**16**] showed that every set of 10 points contains a 5-hole and Horton [**17**] introduced a construction of large point sets without 7-holes. The question, whether 6-holes exist in sufficiently large point sets, remained open until 2007, when Nicolas [**23**] and Gerken [**14**] independently showed that point sets with large k-gons also contain 6-holes. Today it is known that every set of 463 points contains a 6-hole [**22**], while sets of 29 points exist which do not have 6-holes [**25**].

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[1] Erdős offered \$500 for a proof of Szekeres' conjecture that $g(k) = 2^{k-2} + 1$.

In 2001, Hosono and Urabe [**18**] and Bárány and Károlyi [**5**] started the investigation of disjoint holes, where two holes X_1, X_2 of a given point set S are said to be *disjoint* if their respective convex hulls are disjoint. This led to the following question: What is the smallest number $h(k_1, \ldots, k_l)$ such that every set of $h(k_1, \ldots, k_l)$ points determines a k_i-hole for every $i = 1, \ldots, l$, such that the holes are pairwise disjoint [**20**]?

For two parameters, the value $h(k_1, k_2)$ has been determined for all $k_1, k_2 \leq 5$ except for $h(5,5)$ [**18, 19, 20, 6**]. Table 1 summarizes the currently best bounds for two-parametric values. Concerning the value $h(5,5)$, the best bounds are $17 \leq h(5,5) \leq 19$. The lower bound $h(5,5) \geq 17$ is witnessed by a set of 16 points with no two disjoint 5-holes (cf. [**20**]), and the upper bound $h(5,5) \leq 19$ was shown by Bhattacharya and Das [**7**] by an elaborate case distinction.

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17*

Table 1. Values of $h(k_1, k_2)$. The entry marked with star (*) is new.

As our main result of this paper, we determine the precise value of $h(5,5)$. The proof is based on a SAT model which we describe in Section 3.

Theorem 1 (Computer-assisted)**.** *Every set of* 17 *points contains two disjoint* 5*-holes, hence* $h(5,5) = 17$*.*

In the full version [**26**] we also summarize the current status for multi-parametric values and present improved bounds for three-parametric values values $h(k_1, k_2, k_3)$ with $k_1, k_2, k_3 \leq 5$ and for the values $h(5, \ldots, 5)$ (with s parameters).

2. Encoding with triple orientations

In this section we describe how point sets and disjoint holes can be encoded only using triple orientations. This combinatorial description allows us to get rid of the actual point coordinates and to only consider a discrete parameter-space. This is essential for our SAT model of the problem.

2.1. Triple orientations

Given a set of points $S = \{s_1, \ldots, s_n\}$ with $s_i = (x_i, y_i)$, we say that the triple (a, b, c) is *positively (negatively) oriented* if

$$\chi_{abc} := \operatorname{sgn} \det \begin{pmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{pmatrix} \in \{-1, 0, +1\}$$

is positive (negative). It is easy to see, that convexity is a combinatorial rather than a geometric property since k-gons can be described only using triple orientations: If $s_1, \ldots, s_k$ are the vertices of a convex polygon (ordered along the boundary), then the cyclic order of the other points around s_i is $s_{i+1}, s_{i+2}, \ldots, s_{i-1}$ for every i (indices modulo k). Similarly, one can describe containment: s_0 lies inside a triangle s_1, s_2, s_3 (clockwise ordered) if the (clockwise) cyclic order around s_0 is precisely s_1, s_2, s_3. To describe disjointness solely using triple orientations, suppose that a line ℓ separates point sets A and B. Then we can find another line ℓ' that contains a point $a \in A$ and a point $b \in B$ and separates $A \setminus \{a\}$ and $B \setminus \{b\}$. We have $\chi_{aba'} \leq 0$ for all $a' \in A$ and $\chi_{abb'} \geq 0$ for all $b' \in B$, or the other way round. Altogether, disjoint holes can be described solely using triple orientations.

Note that, even though there are uncountable possibilities to choose n points from the Euclidean plane for fixed $n \in \mathbb{N}$, there are only finitely many equivalence classes of point sets when point sets inducing the same orientation triples are considered equal. As introduced by Goodman and Pollack [**15**], these equivalence classes (sometimes also with unlabeled points) are called *order types*.

2.2. An abstraction of point sets

Consider a set $S = \{s_1, \ldots, s_n\}$ where $s_1, \ldots, s_n$ have increasing x-coordinates. Using the unit paraboloid duality transformation, which maps point $s = (a, b)$ to line $s^* : y = 2ax - b$, we obtain the arrangement of dual lines $S^* = \{s_1^*, \ldots, s_n^*\}$, where the dual lines $s_1^*, \ldots, s_n^*$ have increasing slopes. By the increasing x-coordinates and the properties of the duality (cf. [**24**, Chapter 6.5] or [**9**, Chapter 1.4]), the following three statements are equivalent:

(i) The points s_i, s_j, s_k are positively oriented.
(ii) The point s_k lies above the line $\overline{s_i s_j}$.
(iii) The intersection-point of the two lines s_i^* and s_j^* lies above the line s_k^*.

Due to Felsner and Weil [**13**] (see also [**4**]), for every 4-tuple s_i, s_j, s_k, s_l with $i < j < k < l$ the sequence

$$\chi_{ijk},\ \chi_{ijl},\ \chi_{ikl},\ \chi_{jkl}$$

(index-triples in lexicographic order) changes its sign at most once. These conditions are the *signotope axioms*. Note that the signotope axioms are necessary conditions but not sufficient for point sets. There exist χ-configurations which fulfill the conditions above that are **not** induced by any point set, and in fact, deciding whether there exists a realizing point set is $\exists\mathbb{R}$-complete (see e.g. [**12**]).

2.3. Increasing coordinates and cyclic order

In the following, we see why we can assume w.l.o.g. that in every point set $S = \{s_1, \ldots, s_n\}$ the following three conditions hold: (i) the points $s_1, \ldots, s_n$ have increasing x-coordinates; (ii) in particular, s_1 is an extremal point; (iii) the points $s_2, \ldots, s_n$ are sorted around s_1. When modeling a computer program, we use these constraints to restrict the search space (without affecting the output).

Lemma 1. *Let $S = \{s_1, \ldots, s_n\}$ be a point set where s_1 is extremal and $s_2, \ldots, s_n$ are sorted around s_1. Then there is a point set $\tilde{S} = \{\tilde{s_1}, \ldots, \tilde{s_n}\}$ of the same order type as S such that $\tilde{s_1}, \ldots, \tilde{s_n}$ have increasing x-coordinates.*

Proof. We can assume $s_1 = (0, 0)$ and $x_i, y_i > 0$ for $i \geq 2$ – otherwise we can apply an affine-linear transformation. Moreover, x_i/y_i is increasing for $i \geq 2$ since $s_2, \ldots, s_n$ are sorted around s_1. Since S is in general position, there is an $\varepsilon > 0$

such that S and $S' := \{(0,\varepsilon)\} \cup \{s_2, \ldots, s_n\}$ are of the same order type. We apply the projective transformation $(x,y) \mapsto (x/y, -1/y)$ to S' to obtain $\tilde{S}$. By the multilinearity of the determinant, we obtain

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix} = y_i \cdot y_j \cdot y_k \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ x_i/y_i & x_j/y_j & x_k/y_k \\ -1/y_i & -1/y_j & -1/y_k \end{pmatrix}.$$

Since the points in S' have positive y-coordinates, S' and $\tilde{S}$ have the same order type. Moreover, as $\tilde{x_i} = x'_i/y'_i$ is increasing, $\tilde{S}$ fulfills all desired properties. □

3. SAT model

The basic idea to prove Theorem 1 is to assume – towards a contradiction – that a point set $S = \{s_1, \ldots, s_{17}\}$ with no two disjoint 5-holes exists. We formulate a SAT instance, where Boolean variables indicate whether triples are positively or negatively oriented and clauses encode the necessary conditions described in Section 2. Using a SAT solver we verify that the instance has no solution and conclude that the set S does not exist. This contradiction completes the proof.

It is folklore that satisfiability is NP-hard in general, thus it is challenging for SAT solvers to terminate in reasonable time for certain SAT instances. We now highlight the two crucial parts of our SAT model: First, due to Lemma 1, we can assume w.l.o.g. that the points are sorted from left to right and also around the first point s_1. Second, we teach the solver that every set of 10 points gives a 5-hole, that is, $h(5) = 10$ [**16**]. By dropping either of these two constraints (which only give additional information to the solver and do not affect the solution space), none of the tested SAT solvers terminated within days.

3.1. A detailled description of our SAT model

For the sake of readability, we refer to points also by their indices. Moreover, we use the relation "$a < b$" simultaneously to indicate a larger index, a larger x-coordinate, and the later occurence in the cyclic order around s_1.

(1) Alternating axioms. For every triple (a,b,c), we introduce the variable $O_{a,b,c}$ to indicate whether the triple (a,b,c) is positively oriented. Since we have

$$\chi_{a,b,c} = \chi_{b,c,a} = \chi_{c,a,b} = -\chi_{b,a,c} = -\chi_{a,c,b} = -\chi_{c,b,a},$$

we formulate clauses to assert

$$O_{a,b,c} = O_{b,c,a} = O_{c,a,b} \neq O_{b,a,c} = O_{a,c,b} = O_{c,b,a}$$

using the fact $A = B \Leftrightarrow (\neg A \vee B) \wedge (A \vee \neg B)$, and $A \neq B \Leftrightarrow (A \vee B) \wedge (\neg A \vee \neg B)$.

(2) Signotope axioms. As described in Section 2.2, for every 4-tuple $a < b < c < d$, the sequence $\chi_{abc}, \chi_{abd}, \chi_{acd}, \chi_{bcd}$ changes its sign at most once. Formally, to forbid such sign-patterns (that is, "$-+-$" and "$+-+$"), we add the constraints

$$O_I \vee \neg O_J \vee O_K \quad \text{and} \quad \neg O_I \vee O_J \vee \neg O_K$$

for every lexicographically ordered triple of index triples, that is, $\{I, J, K\} \subset \binom{\{a,b,c,d\}}{3}$ with $I \prec J \prec K$.

(3) Sorted around first point. Since the points are sorted from left to right and also around s_1, all triples $(1, a, b)$ are positively oriented for $1 < a < b$.

(4) Bounding segments. For a 4-tuple a, b, c, d, we introduce the auxiliary variable $E_{a,b;c,d}$ to indicate whether the segment ab spanned by a and b bounds the convex hull $\operatorname{conv}(\{a, b, c, d\})$. Since the segment ab bounds $\operatorname{conv}(\{a, b, c, d\})$ if and only if c and d lie on the same side of the line $\overline{ab}$, we add the constraints

$$\begin{aligned} \neg E_{a,b;c,d} \vee O_{a,b,c} \vee \neg O_{a,b,d}, &\qquad \neg E_{a,b;c,d} \vee \neg O_{a,b,c} \vee O_{a,b,d}, \\ E_{a,b;c,d} \vee O_{a,b,c} \vee O_{a,b,d}, &\qquad E_{a,b;c,d} \vee \neg O_{a,b,c} \vee \neg O_{a,b,d}. \end{aligned}$$

(5) 4-Gons and containments. For every 4-tuple $a < b < c < d$, we introduce the auxiliary variable $G^4_{a,b,c,d}$ to indicate whether the points $\{a, b, c, d\}$ form a 4-gon. Moreover we introduce the auxiliary variable $I_{i;a,b,c}$ for every 4-tuple a, b, c, i with $a < b < c$ and $a < i < c$ to indicate whether the point i lies inside the triangular convex hull $\operatorname{conv}(\{a, b, c\})$.

Four points $a < b < c < d$, sorted from left to right, form a 4-gon if and only if both segments ab and cd bound the convex hull $\operatorname{conv}(\{a, b, c, d\})$. Moreover, if $\{a, b, c, d\}$ does not form a 4-gon, then either b lie inside the triangular convex hull $\operatorname{conv}(\{a, c, d\})$ or c lies inside $\operatorname{conv}(\{a, b, d\})$. Pause to note that a and d are the left- and rightmost points, respectively, and that not both points b and c can lie in the interior of $\operatorname{conv}(\{a, b, c, d\})$. Formally, we assert

$$\begin{aligned} G^4_{a,b,c,d} &= E_{a,b;c,d} \wedge E_{c,d;a,b}, \\ I_{b;a,c,d} &= \neg E_{a,b;c,d} \wedge E_{c,d;a,b}, \\ I_{c;a,b,d} &= E_{a,b;c,d} \wedge \neg E_{c,d;a,b}. \end{aligned}$$

(6) 3-Holes. For every triple of points $a < b < c$, we introduce the auxiliary variable $H^3_{a,b,c}$ to indicate whether the points $\{a, b, c\}$ form a 3-hole. Since three points $a < b < c$ form a 3-hole if and only if every other point i lies outside the triangular convex hull $\operatorname{conv}(\{a, b, c\})$, we add the constraint

$$H^3_{a,b,c} = \bigwedge_{i \in S \setminus \{a,b,c\}} \neg I_{i;a,b,c}.$$

(7) 5-Holes. For every 5-tuple $X = \{a, b, c, d, e\}$ with $a < b < c < d < e$, we introduce the auxiliary variable H^5_X to indicate that the points from X form a 5-hole. It is easy to see that the points from X form a 5-hole if and only if every 4-tuple $Y \in \binom{X}{4}$ forms a 4-gon and if every triple $Y \in \binom{X}{3}$ forms a 3-hole. Hence,

$$H^5_X = \Bigg(\bigwedge_{Y \in \binom{X}{4}} G^4_Y \Bigg) \wedge \Bigg(\bigwedge_{Y \in \binom{X}{3}} H^3_Y \Bigg).$$

(8) Forbid disjoint 5-holes. If that there were two disjoint 5-holes X_1 and X_2 in our point set S, then – as discussed in Section 2 – we could find two points $a \in X_1$ and $b \in X_2$ such that the line $\overline{ab}$ separates $X_1 \setminus \{a\}$ and $X_2 \setminus \{b\}$ – and this is what we have to forbid in our SAT model. Hence, for every pair of two points a, b we introduce the variables

- $L_{a,b}$ to indicate that there exists a 5-hole X containing the point a that lies to the left of the directed line $\overrightarrow{ab}$, that is, the triple (a, b, x) is positively oriented for every $x \in X \setminus \{a\}$, and
- $R_{a,b}$ to indicate that there exists a 5-hole X containing the point b that lies to the right of the directed line $\overrightarrow{ab}$, that is, the triple (a, b, x) is negatively oriented for every $x \in X \setminus \{b\}$.

For every 5-tuple X with $a \in X$ and $b \notin X$ we assert

$$L_{a,b} \ \vee \ \neg H_X \ \vee \ \left(\bigvee_{c \in X \setminus \{a\}} \neg O_{a,b,c} \right),$$

and for every 5-tuple X with $a \notin X$ and $b \in X$ we assert

$$R_{a,b} \ \vee \ \neg H_X \ \vee \ \left(\bigvee_{c \in X \setminus \{b\}} O_{a,b,c} \right).$$

To forbid 5-holes on both sides of the line $\overline{ab}$, we assert $\neg L_{a,b} \vee \neg R_{a,b}$.

(9) Harborth's result. Harborth [**16**] has shown that every set of 10 points gives a 5-hole, that is, $h(5) = 10$. Consquently, there is a 5-hole X_1 in the set $\{1, \ldots, 10\}$, and if $X_1 \subset \{1, \ldots, 7\}$, then there is another 5-hole X_2 in the set $\{8, \ldots, 17\}$. Analogously, if there is a 5-hole $X_3 \subset \{11, \ldots, 17\}$, then there is another 5-hole X_4 in the set $\{1, \ldots, 10\}$. Hence, we can teach the SAT solver that

- there is a 5-hole X with $X \subset \{1, \ldots, 10\}$,
- there is no 5-hole X with $X \subset \{1, \ldots, 7\}$,
- there is a 5-hole X with $X \subset \{8, \ldots, 17\}$, and
- there is no 5-hole X with $X \subset \{11, \ldots, 17\}$.

The source code of our python program is available online on our website[2].

3.2. Unsatisfiability and verification

Having the satisfiability instance generated, we used the following command to create an unsatisfiability certificate:

```
glucose instance.cnf -certified -certified-output=proof.out
```

The certificate created by glucose [**3**] was then verified using the proof checking tool drat-trim [**29**] by the following command:

```
drat-trim instance.cnf proof.out
```

The execution of each of the two commands (glucose and drat-trim), took about 2 hours and the certificate used about 3.1 GB of disk space.

[2]http://page.math.tu-berlin.de/~scheuch/supplemental/5holes/disjoint_holes

We have also used pycosat [**8**] to prove unsatisfiability:

```
picosat instance.cnf -R proof.out
```

This command ran for about 6 hours and created a certificate of size about 2.1 GB. The verification of the certificate[3] using drat-trim took about 9 hours.

4. Final remarks

Interior-disjoint holes: Two holes X_1, X_2 are *interior-disjoint* if their respective convex hulls are interior-disjoint. This variant has been investigated intensively by various groups of researchers, as they play a role in the study of other geometric objects such as visibility graphs or flip-graph of triangulations on point sets (references and more information are deferred to the full version [**26**]). By slightly adapting the SAT model from Section 3, we managed to show that every set of 15 points contains two interior-disjoint 5-holes; this further improves Theorem 3 from Hosono and Urabe [**21**]. Moreover, this bound is best possible because sets of 14 points with no two interior-disjoint 5-holes exist (cf. [**26**]).

Classical Erdős–Szekeres: The computation time for the computer assisted proof by Szekeres and Peters [**28**] for $g(6) = 17$ was about 1500 hours. By slightly adapting the model from Section 3 (cf. the full version [**26**]); we have been able to confirm $g(6) = 17$ using glucose and drat-trim with about 1 hour of CPU time.

Counting 5-holes: SAT models can also be used to count occurences of certain substructures. For example to find point sets with as few 5-holes as possible, we have introduced variables $X_{abcde;k}$ indicating whether the indices $1 \le a < b < c < d < e \le n$ form the k-th 5-hole in lexicographic order. Using SAT solvers we have been able to show that every set of 16 points has at least 11 5-holes (cf. [**1, 2**]).

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[3] In our experiments, picosat wrote a comment "%RUPD32 ..." as first line in the RUP file. This line had to be removed manually to make the file parsable for drat-trim.

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