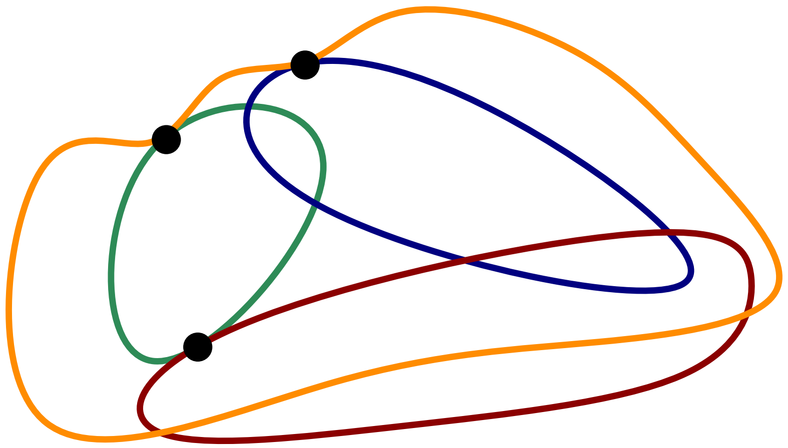
EuroCG 2022 Perugia

Arrangements of Pseudocircles: On Digons and Triangles

Stefan Felsner, Sandro Roch, Manfred Scheucher

Pseudocircle arrangements

Example:



Pseudocircle arrangements

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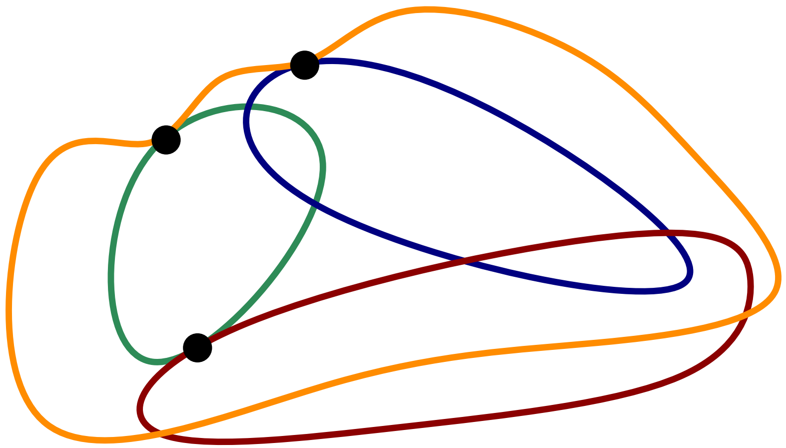
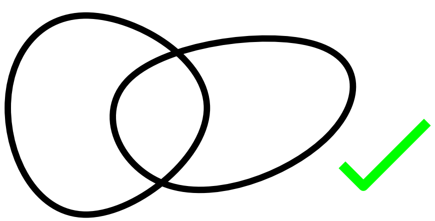
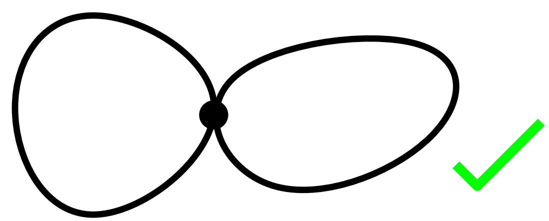


Each two pseudocircles...



or



...either cross exactly twice, ...

...have a single touching

Pseudocircle arrangements

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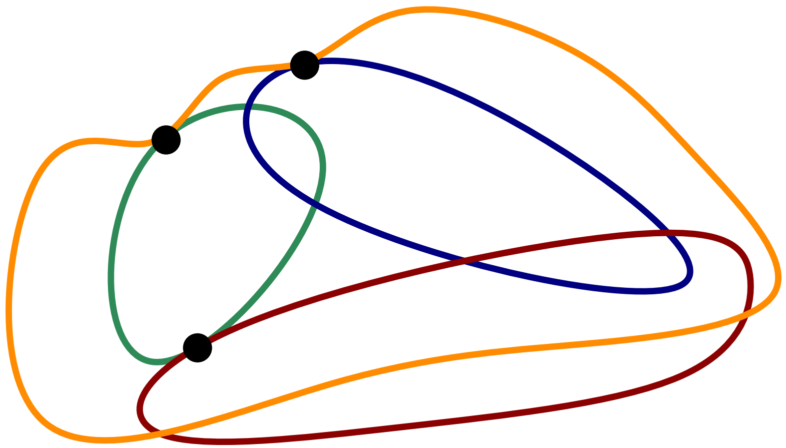

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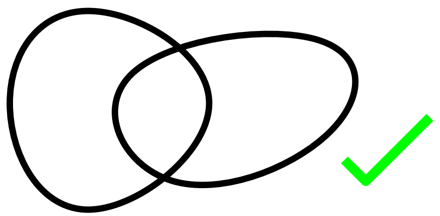

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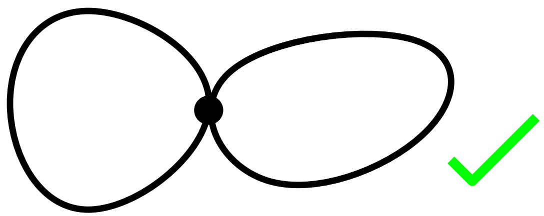

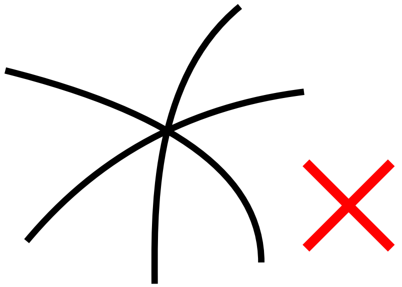

...either cross exactly twice, ...

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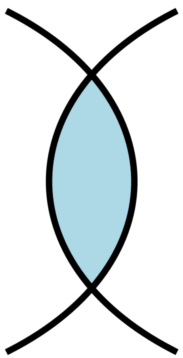
No intersection of ≥ 3 pseudocircles in single point

Grünbaum's conjecture on digons

Conjecture: In every pseudocircle arrangement, there are at most $2n-2$ digons. (Grünbaum, 1972)



digon

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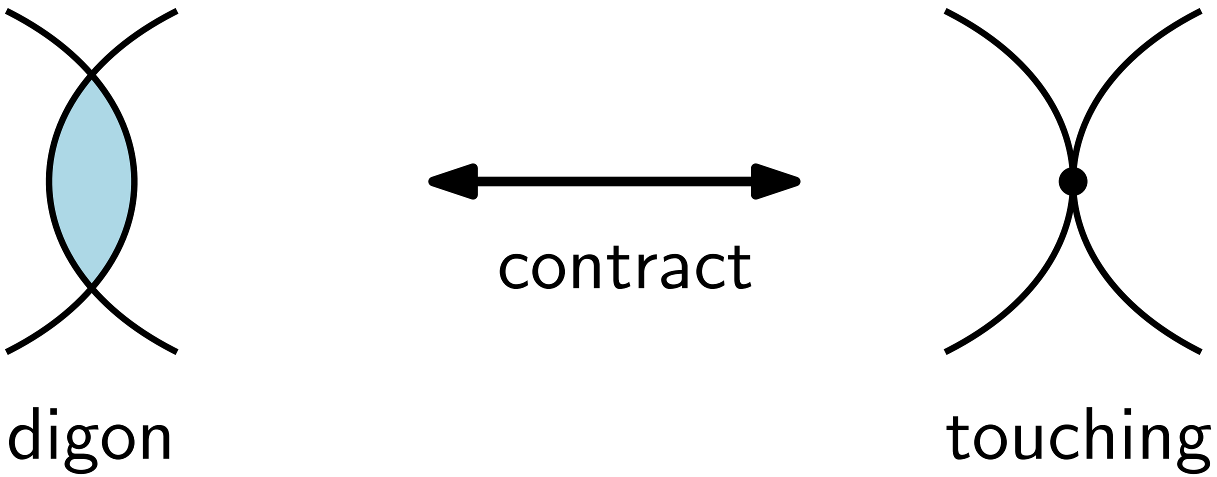

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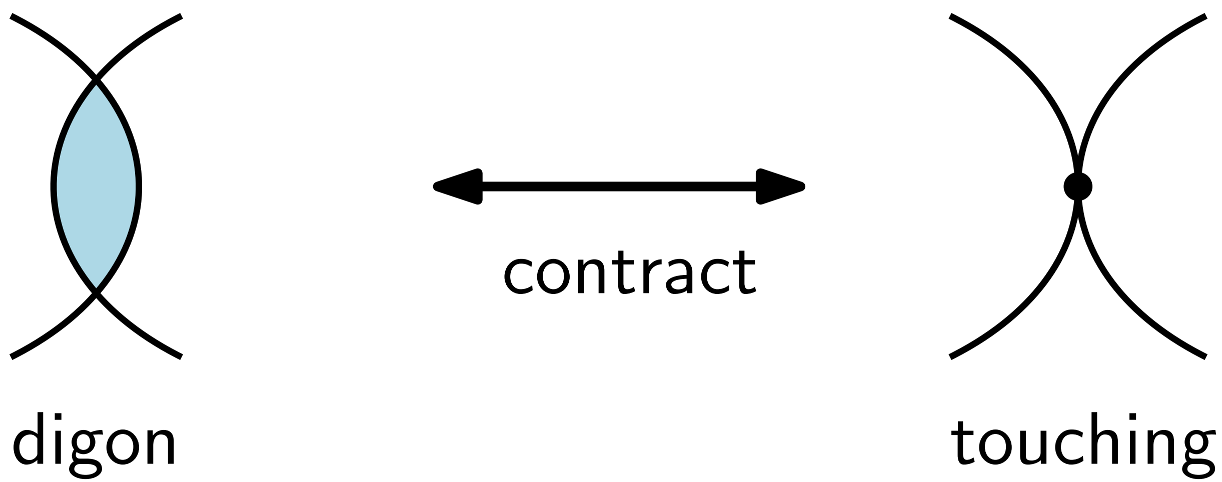
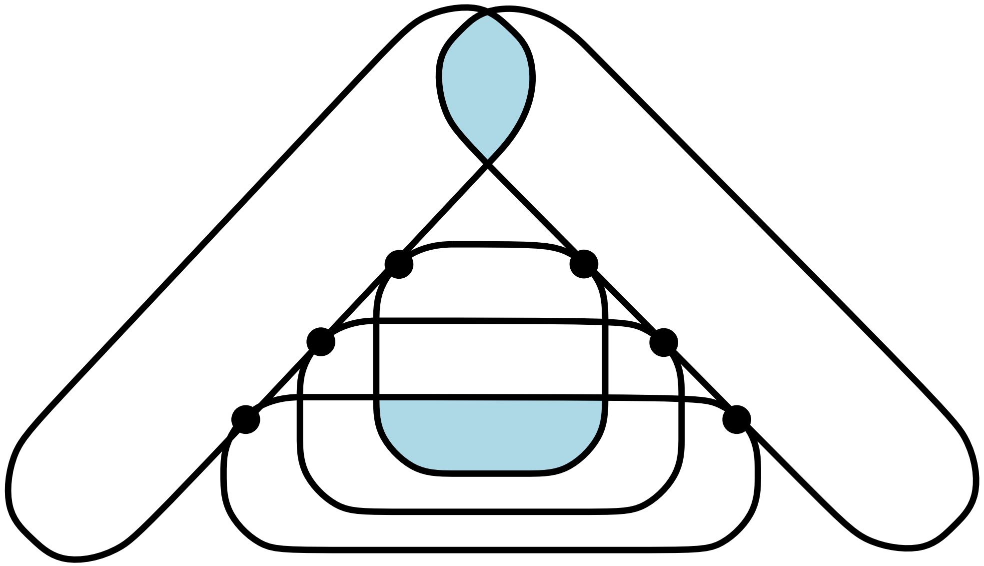


Equivalent: At most $2n-2$ touchings.

Grünbaum's conjecture on digons

Cylindrical pseudocircle arrangement:
Exist two cells separated by each pseudocircle



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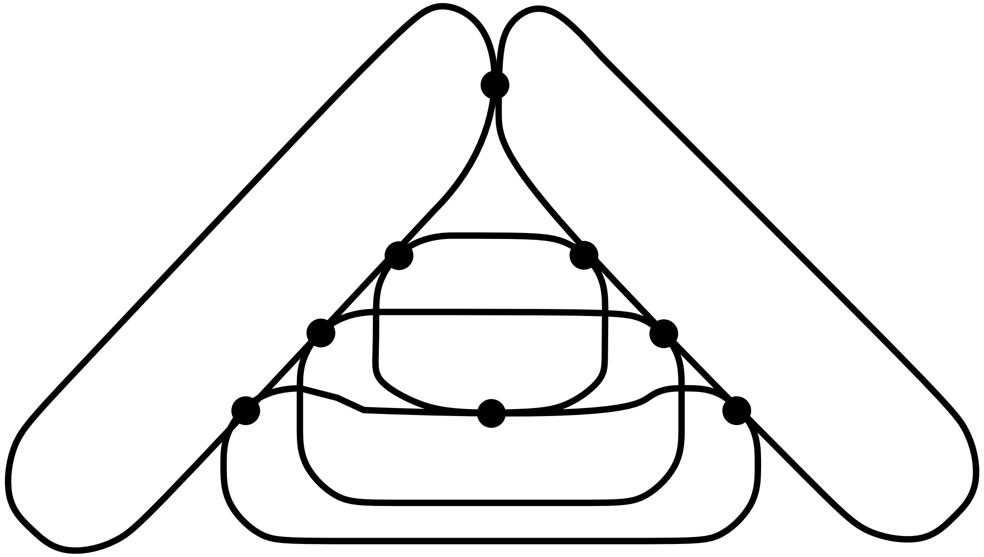

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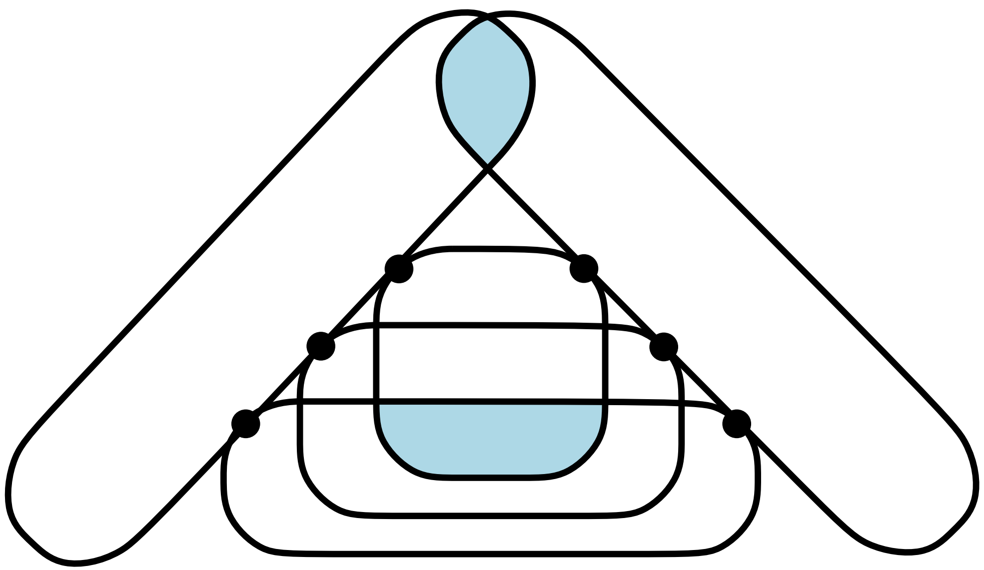


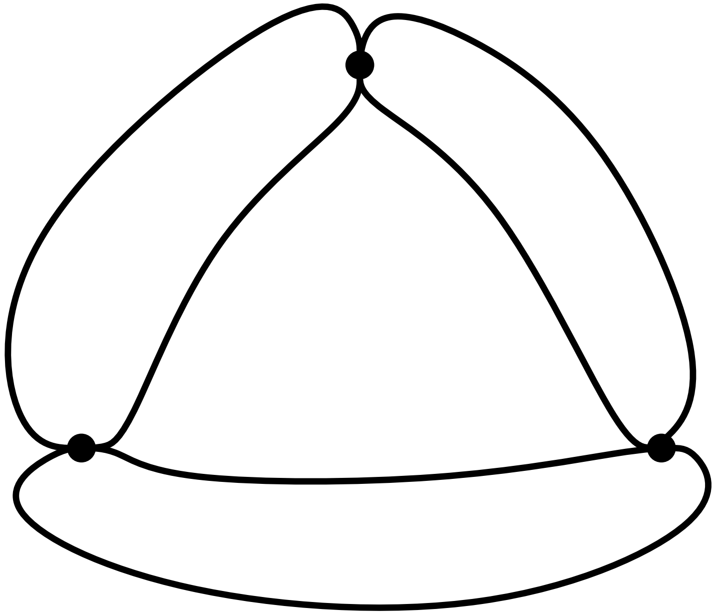
Agarwal et al. (2004):

- Cylindrical case: At most $2n - 2$ touchings
- General case: At most $O(n)$ touchings

Grünbaum's conjecture on digons

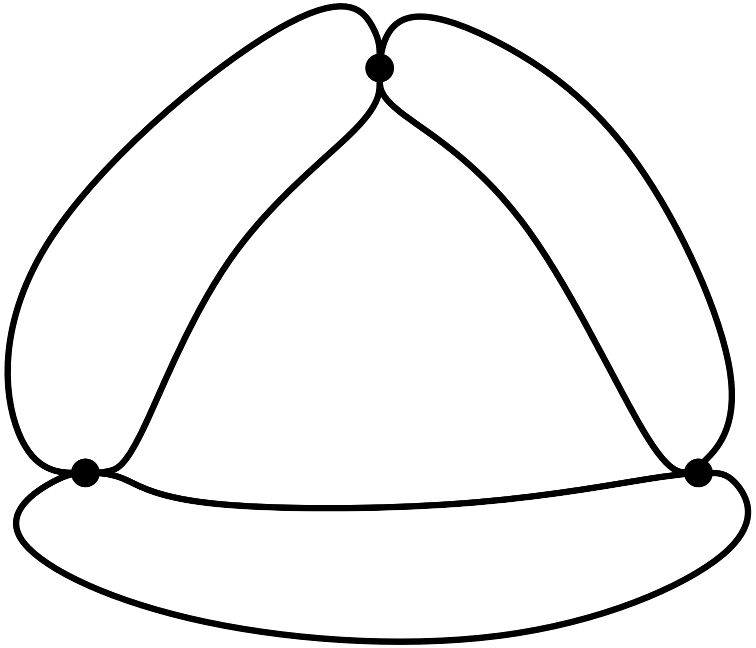
Theorem (Felsner, R., Scheucher)

If three pseudocircles pairwise touch, then the arrangement has at most $2n-2$ touchings.



Grünbaum's conjecture on digons

Sketch of proof:



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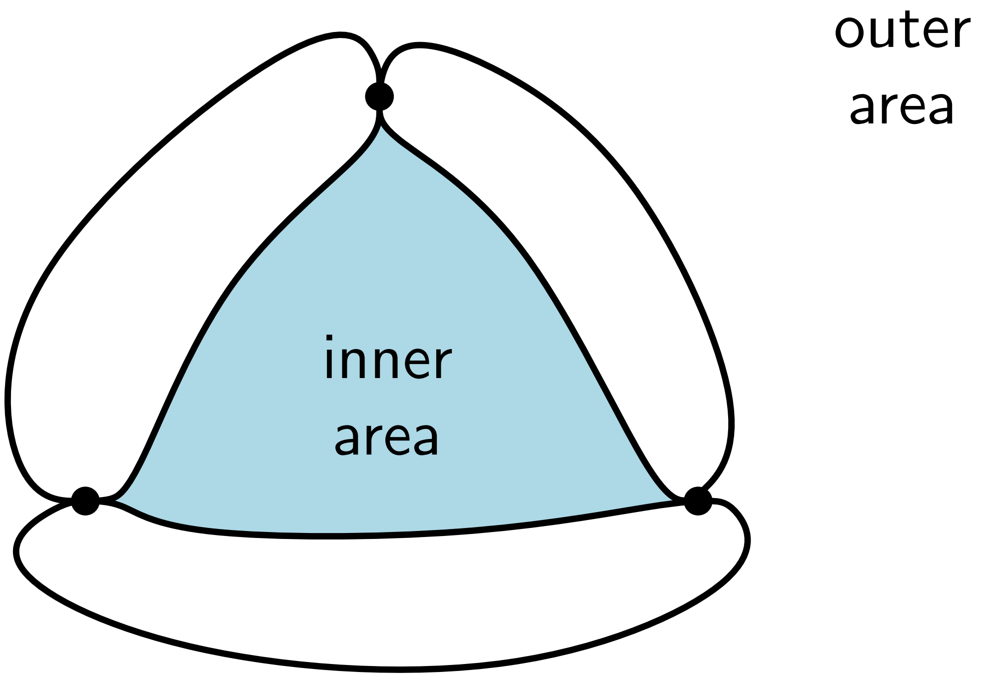

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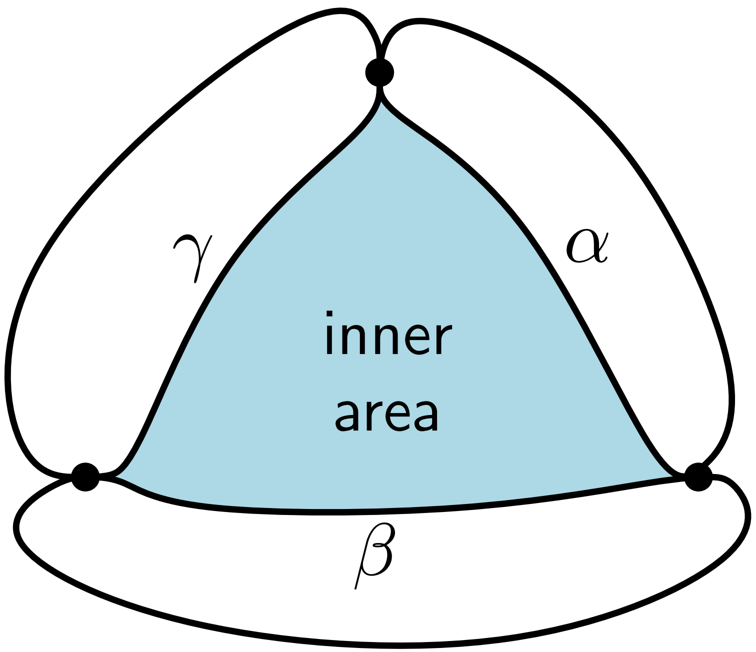

Grünbaum's conjecture on digons

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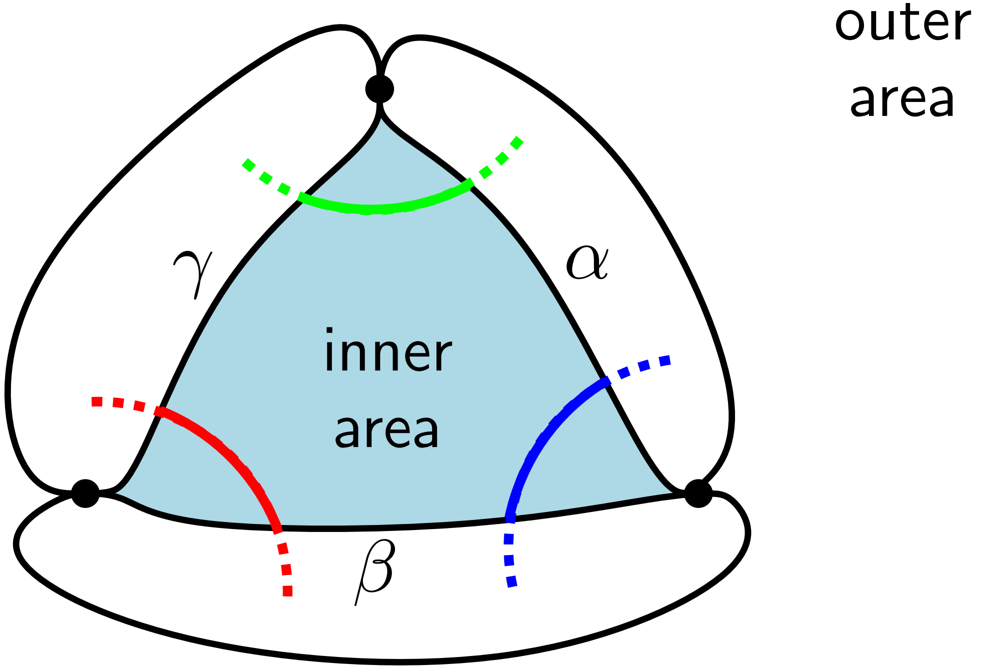
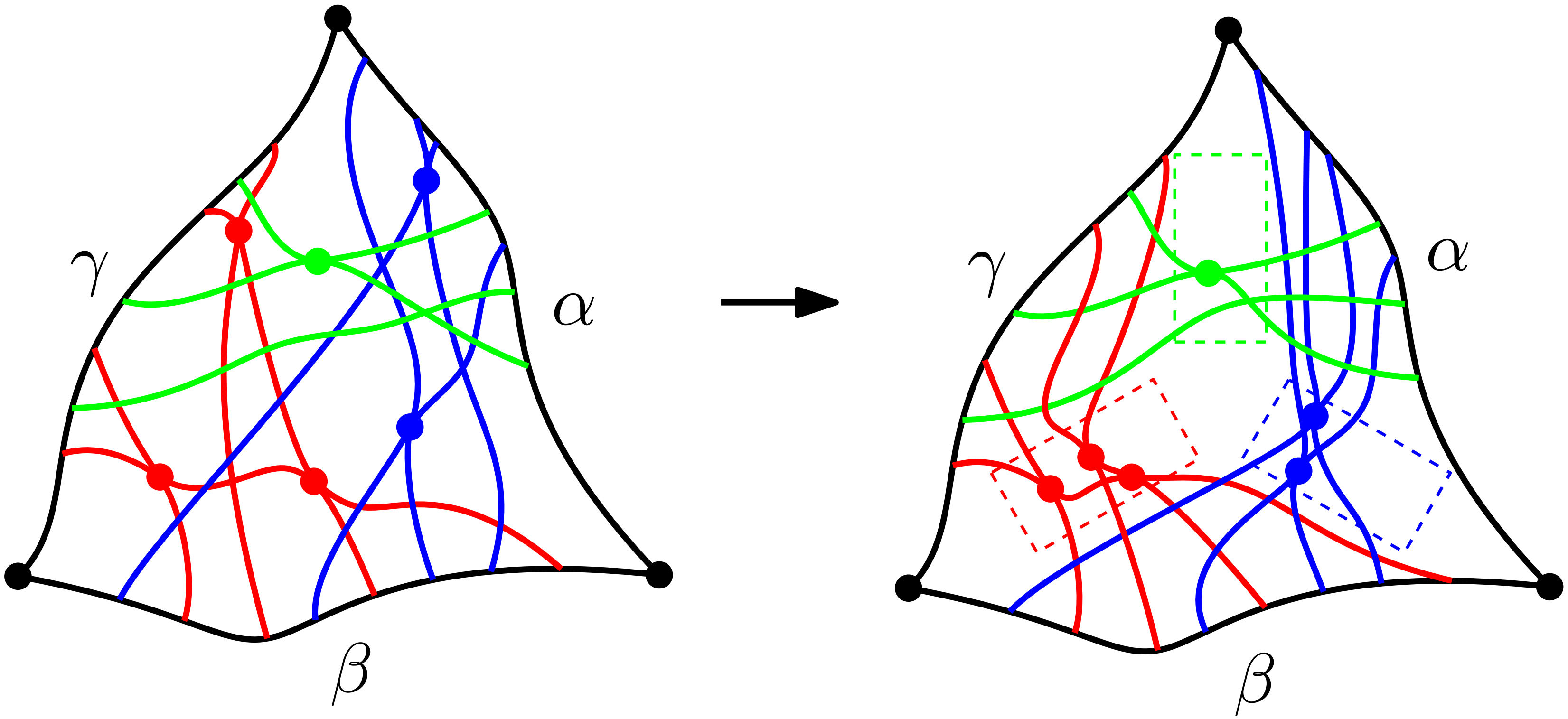


Grünbaum's conjecture on digons

Main idea: Reduction to cylindrical case

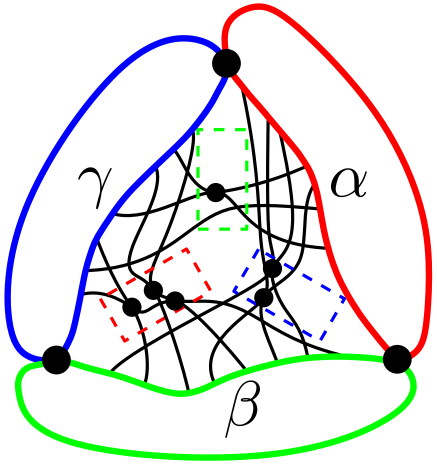
Step 1: Transformation of the inner and outer area



Concentrate intersections between same type in small areas

Grünbaum's conjecture on digons

Step 2: Make arrangement cylindrical



Grünbaum's conjecture on digons

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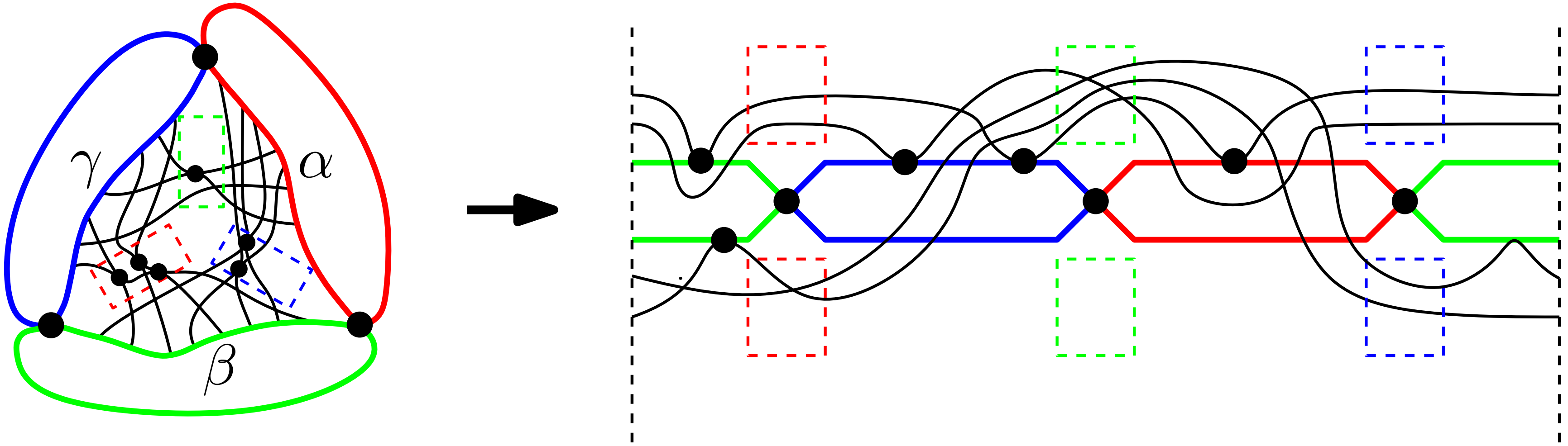

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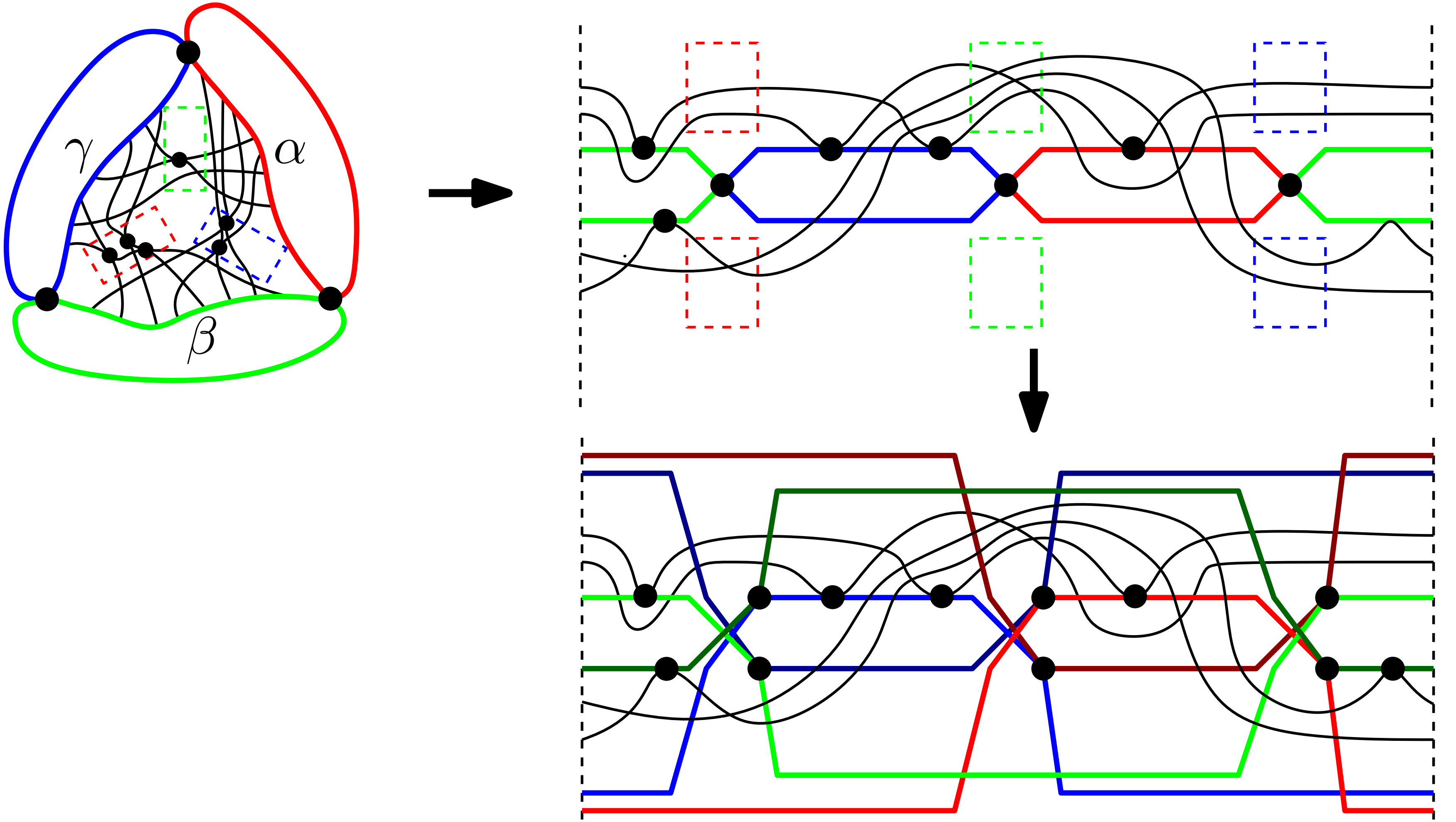

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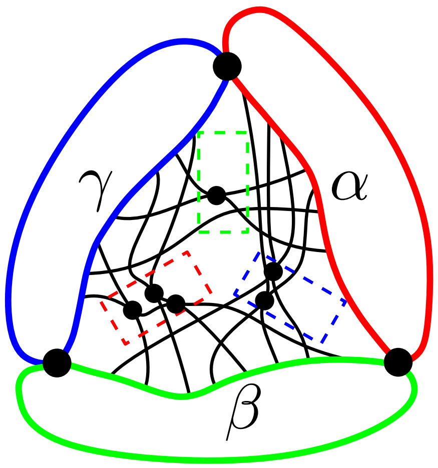

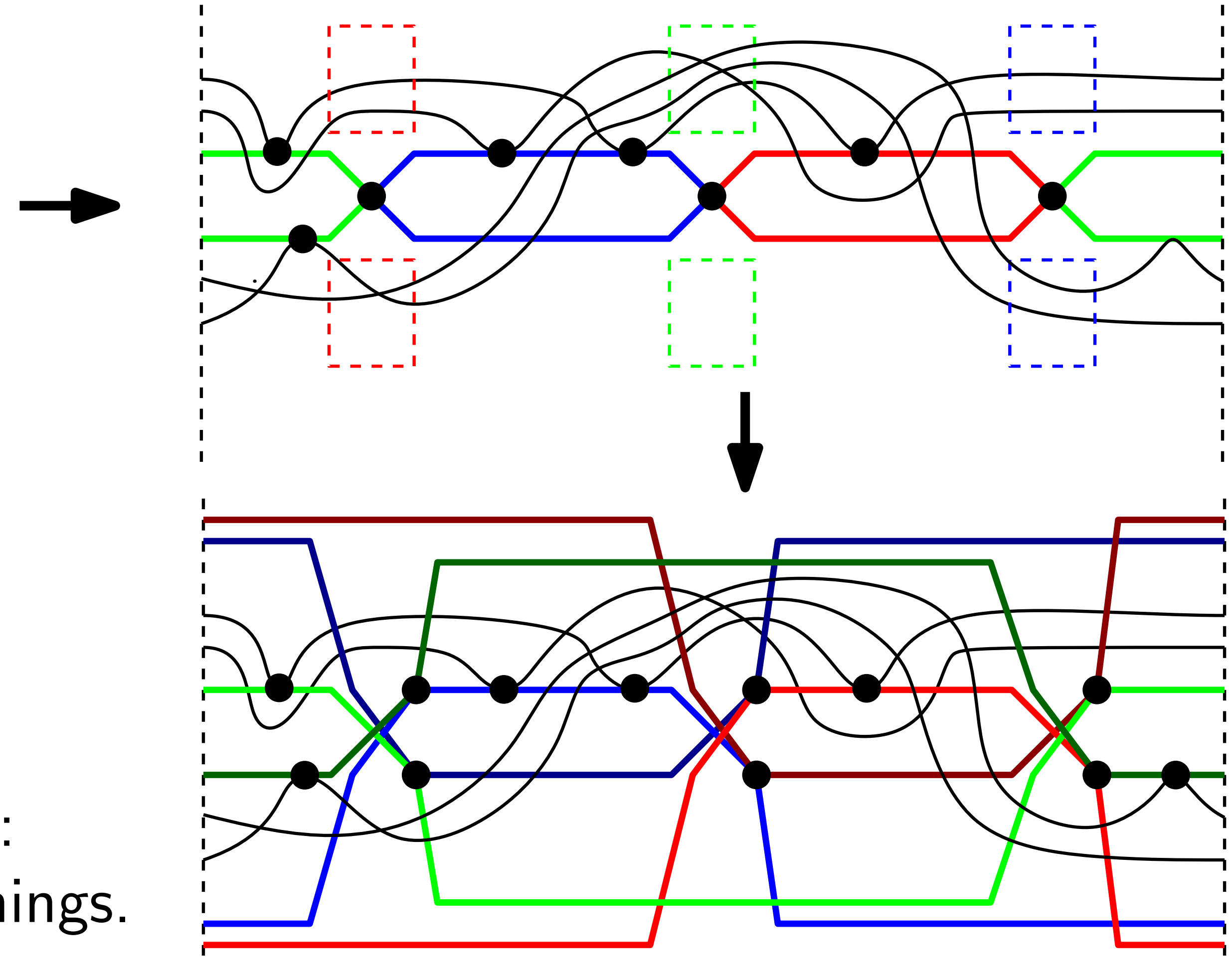

Agarwal et al. (2004):
At most $2n-2$ touchings.

Triangles in digon and touching free arrangements

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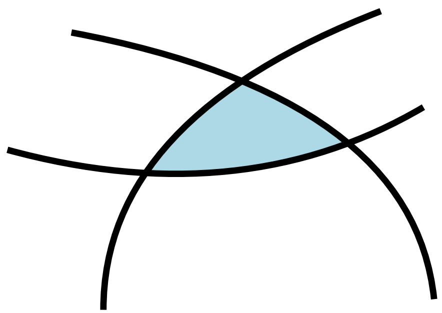
Conjecture (Grünbaum 1972): Arrangements without digons and touchings have $p_3 \geq 2n-4$ triangles.



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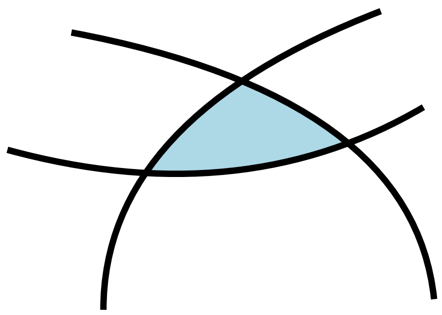
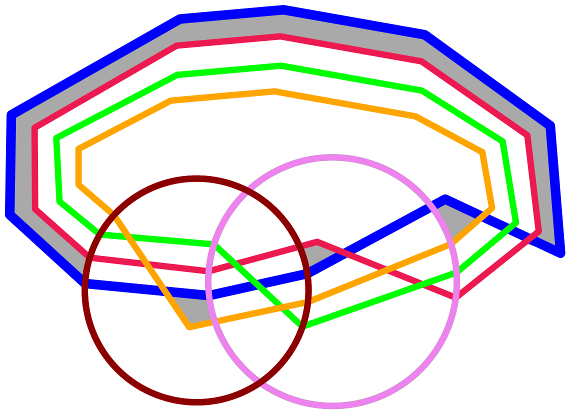
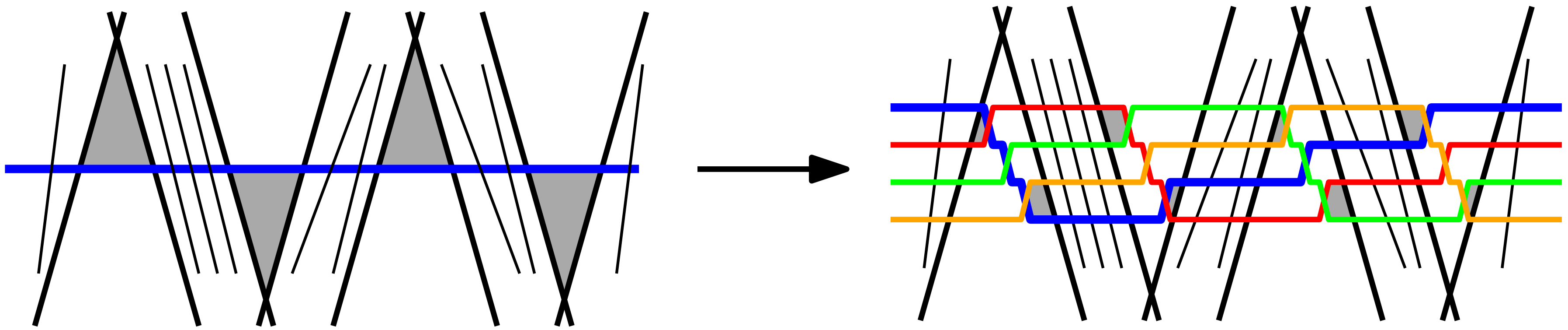


- Snoeyink and Hershberger (1991): $p_3 \geq \frac{4}{3}n$
- Felsner and Scheucher (EuroCG 2017):
  Examples with $p_3 < \frac{16}{11}n$, Grünbaum's conjecture disproved
- Felsner, R., Scheucher (2022):
  Theorem: For $n \geq 6$ there exist examples with $p_3 = \lceil \frac{4}{3}n \rceil$.

Triangles in digon and touching free arrangements

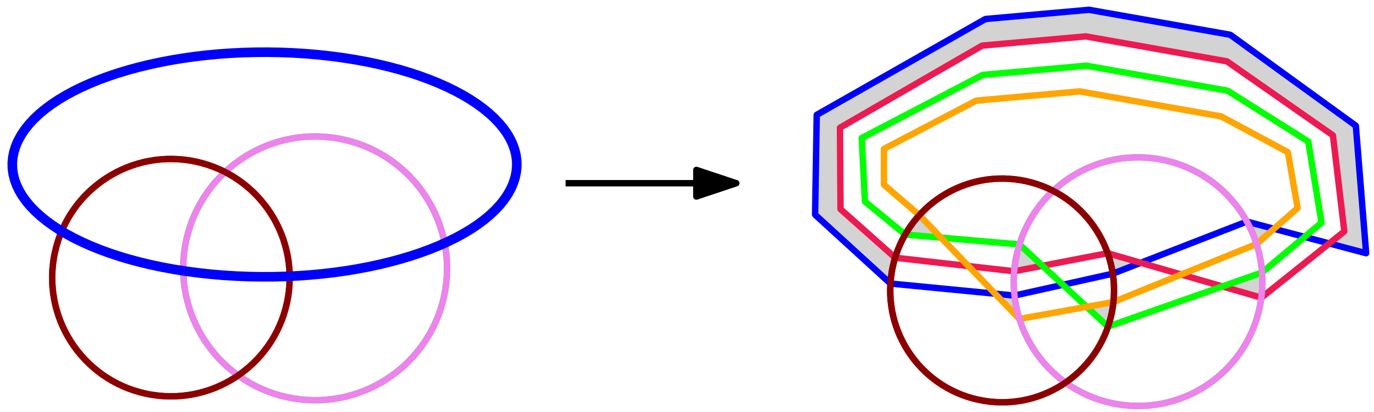
Replace iteratively blue pseudocircle by 4 twisted pseudocircles:



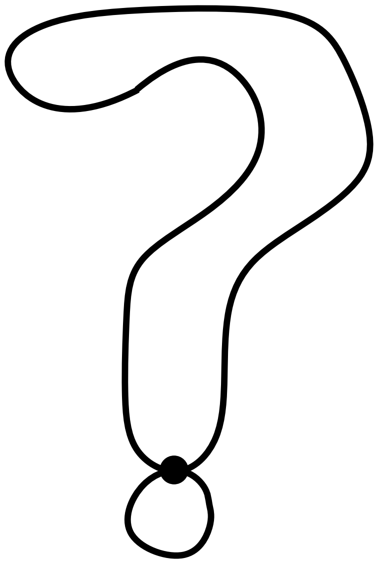


Each iteration increases n by 3 and p_3 by 4.

Triangles in digon and touching free arrangements



Questions?



Theorem 1: If three pseudocircles pairwise touch, then the pseudocircle arrangement has at most $2n - 2$ touchings.

Theorem 2: There exist digon and touching free pseudocircle arrangements with $p_3 = \lceil \frac{4}{3} n \rceil$ triangles.