Arrangements of Pseudocircles: On Digons and Triangles*

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Abstract

The investigation of arrangements of pseudolines and their cell structure goes back to Levi in the 1920's. In Grünbaum's monograph from the 1970's, he started the investigation of arrangements of pseudocircles and posed several interesting problems and conjectures, some of which are still open. Here we discuss the cell-structure of arrangements of pairwise intersecting pseudocircles.

First, we discuss the maximum number of digons or touching points. Grünbaum conjectured that every arrangement of n pairwise intersecting pseudocircles has at most $2n-2$ digons or equivalently at most $2n-2$ touchings. Using a result from Agarwal et al. (2004), who proved the conjecture for cylindrical arrangements, we show that the conjecture holds for any arrangement, where a triple of pseudocircles is pairwise touching. Even though the general conjecture remains open, this substantially narrows the options for potential counter-examples.

Second, we discuss the minimum number of triangular cells (triangles) in an arrangement of n pairwise intersecting pseudocircles without digons and touchings. While Snoeyink and Hershberger (1991) showed that there are at least $p_3 \geq \frac{4}{3}n$ triangles, Felsner and Scheucher (2017) showed that there exist arrangements on $n \geq 6$ pseudocircles with $p_3 < \lceil \frac{16}{11}n \rceil$ triangles, which disproved a long-standing conjecture of Grünbaum. Here we provide a construction for $n \geq 6$ with only $p_3 = \lceil \frac{4}{3}n \rceil$ triangles, showing that the lower bound of Snoeyink and Hershberger is tight.

1 Introduction

An *intersecting arrangement of pseudocircles* is a collection of simple closed curves on the sphere or plane such that any two of the curves either touch in a single point or intersect in exactly two points where they cross. Throughout this article, we consider all arrangements to be *simple*, that is, no three pseudocircles meet in a common point. An arrangement $\mathcal{A}$ partitions the plane into cells. Cells which have k crossings on their boundary are k*-cells* and we denote their number by $p_k(\mathcal{A})$. We also call 2-cells *digons* and 3-cells *triangles.*

The investigation of cells in arrangements started about 100 years ago with the study of *arrangements of (pairwise intersecting) pseudolines* by Levi [7], who showed that in the projective plane every pseudoline is incident to at least 3 triangles and proved the famous extension lemma. In the 1970's, Grünbaum [6] intensively investigated arrangements of pseudolines and initiated the study of arrangements of pseudocircles.

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1.1 Digons and touchings

Concerning digons in intersecting arrangements of pseudocircles, Grünbaum [6, Conjecture 3.6][1] posed the following conjecture:

▶ **Conjecture 1.1** (Grünbaum's digon conjecture [6]). *Every intersecting arrangement of n pseudocircles has at most $2n-2$ digons.*

An intersecting arrangement of pseudocircles is called *cylindrical*, if there is a pair of cells which are separated by each pseudocircle of the arrangement. It was shown by Agarwal et al. [1, Corollary 2.12] that Conjecture 1.1 holds for simple cylindrical arrangements.

Moreover, Agarwal et al. show for intersecting arrangements of pseudocircles that the number of digons is at most linear in n. The proof of this linear bound is based on the fact that every arrangement of intersecting pseudocircles can be stabbed by constantly many points. That is, there exists an absolute constant k, called the *stabbing number*, such that, for every arrangement of n pseudocircles in the plane, there exists a set of k points with the property that each pseudocircle contains at least one such point in its interior. In the literature, the stabbing number is also often referred to as piercing number or transversal number. Hence the arrangement can be decomposed into constantly many cylindrical subarrangements. The multiplicative constant of the linear term however remains unknown. In [5] we verified the conjecture for up to $n=7$ pseudocircles.

Here we show that Grünbaum's digon conjecture (Conjecture 1.1) holds for simple arrangements with three pseudocircles that pairwise form a digon; see Section 2. Before we state the result, let us introduce some notation which will be used extensively. Any arrangement $\mathcal{A}$ of pseudocircles can be perturbed so that any selection of its digons become touching points. Figure 1 gives an illustration. It is therefore sufficient to find an upper bound on the number of touchings. The *touching graph* $T(\mathcal{A})$ consists of the pseudocircles as vertices, and two of them share an edge if they have a touching.

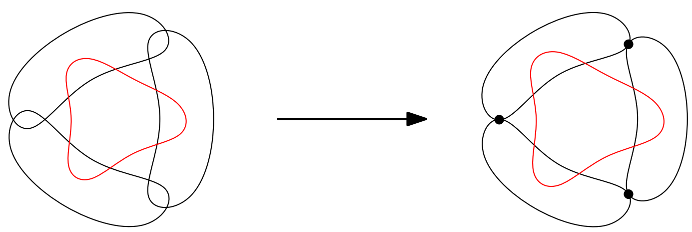


Figure 1 Contracting some of the digons to touchings.

▶ **Theorem 1.2.** *Let $\mathcal{A}$ be an arrangement of n pairwise intersecting pseudocircles. If the touching graph $T(\mathcal{A})$ contains a triangle, then there are at most $2n-2$ touchings.*

1.2 Triangles in digon- and touching-free arrangements

The study of triangles in arrangements goes back to Levi [7], who showed that every arrangement of n pseudolines in the projective plane contains at least n triangles. Since pseudoline arrangements are in correspondence with arrangements of *great-pseudocircles* (see

[1] Originally the conjecture extends to non-simple arrangements which are *non-trivial*, i.e., arrangements with at least 3 crossing points.

e.g. [4, Section 4]), it directly follows that an arrangement of n great-pseudocircles contains at least $p_3 \geq 2n$ triangles.

Grünbaum conjectured that every digon- and touching-free intersecting arrangement on n pseudocircles contains at least $p_3 \geq 2n-4$ triangles [6, Conjecture 3.7]. Snoeyink and Hershberger [10] proved a sweeping lemma for arrangements of pseudocircles. Using this powerful tool, they concluded that in every digon- and touching-free intersecting arrangement every pseudocircle has two triangles on each of its two sides (interior and exterior) and derived the lower bound $p_3(\mathcal{A}) \geq 4n/3$; see Section 4.2 in [10].

In [5] we constructed an infinite family of arrangements with $p_3 < \frac{16}{11}n$ which shows that Grünbaum's conjecture is wrong and verified that the lower bound $p_3 \geq 4n/3$ by Snoeyink and Hershberger is tight for $6 \leq n \leq 14$. We now have:

▶ **Theorem 1.3.** *For every $n \geq 6$, there exists a digon- and touching-free arrangement $\mathcal{A}_n$ of n pairwise intersecting pseudocircles with $p_3 = \lceil \frac{4}{3}n \rceil$ triangles.*

All arrangements constructed in Section 3 contain $\mathcal{A}_6$ (depicted on the left of Figure 7) as a subarrangement. This remarkable arrangement has been studied as the arrangement $\mathcal{N}_6^\Delta$ in [4] where it was shown that $\mathcal{N}_6^\Delta$ is *non-circularizable*, i.e., $\mathcal{N}_6^\Delta$ cannot be represented by an arrangement of proper circles. As a consequence, all arrangements constructed in Section 3 are as well non-circularizable. In fact, all known counter-examples to Grünbaum's triangle conjecture contain $\mathcal{N}_6^\Delta$ and are therefore non-circularizable. Hence, Grünbaum's conjecture may still be true when restricted to arrangements of proper circles.

▶ **Conjecture 1.4** (Weak Grünbaum triangle conjecture, [5, Conjecture 2.2])**.** *Every intersecting digon- and touching-free arrangement of n circles has at least $2n-4$ triangles.*

1.3 Discussion

For intersecting arrangements of unit-circles, Pinchasi showed an upper bound of $p_2 \leq n+3$ [8, Lemma 3.4 and Corollary 3.10]. For arrangements of unit circles there is a classical construction of Erdős [3] with n not necessarily pairwise intersecting circles and $\Omega(n^{1+c/\log\log n})$ touchings. An upper bound of $O(n^{3/2+\epsilon})$ on the number of digons in circle arrangements was shown by Aronov and Sharir [2]. We are not aware of upper bounds on the number of digons in the case of not necessarily intersecting pseudocircles.

Concerning intersecting arrangements with digons, the number of triangles behaves slightly different. While our best lower bound so far is $p_3 \geq 2n/3$, we have used computer assistance to verify that $p_3 \geq n-1$ is a tight lower bound for $3 \leq n \leq 7$ [5]. It remains open, whether $p_3 \geq n-1$ is a tight lower bound for every $n \geq 3$ [5, Conjecture 2.10]. For the maximum number of triangles in intersecting arrangements in [5], we have shown an upper bound $p_3 \leq \frac{4}{3}\binom{n}{2} + O(n)$ which is optimal up to a linear error term.

2 Sketch of the proof of Theorem 1.2

We outline the proof of Theorem 1.2. A complete proof is given in Appendix A.

Since the touching graph $T(\mathcal{A})$ contains a triangle, there are three pseudocircles in $\mathcal{A}$ that pairwise touch. Let $\mathcal{K}$ be the subarrangement induced by these three pseudocircles and let $\triangle$ and $\triangle'$ denote the two triangle cells in $\mathcal{K}$. We label the three touching points, which are also the corners of $\triangle$ and $\triangle'$, as a, b, c. Furthermore, we label the three boundary arcs of $\triangle$ (resp. $\triangle'$) as α, β, γ (resp. α', β', γ'), as shown in Figure 2(a).

Assume that all digons in $\mathcal{A}$ are contracted to touchings.

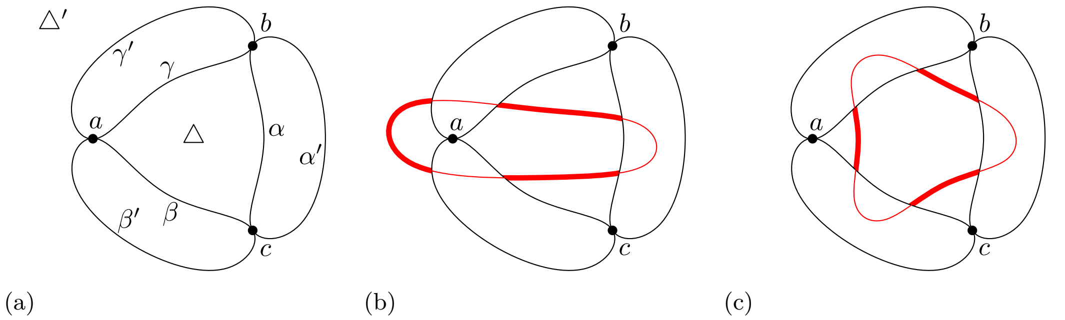


Figure 2 (a) An illustration of the subarrangement $\mathcal{K}$. (b) and (c) illustrate an additional pseudocircle C (red). The pc-arcs inside both $\triangle$ and $\triangle'$ are highlighted.

The intersection of a pseudocircle $C \in \mathcal{A} \setminus \mathcal{K}$ with $\triangle \cup \triangle'$ results in three connected segments, which we denote as the three *pc-arcs* of C, see Figures 2(b) and 2(c). Note that each pc-arc in $\triangle$ connects two of α, β or γ while a pc-arc in $\triangle'$ connects two of α', β' and γ'. Depending on the boundary arcs on which they start and end, they belong to one of the types $\alpha\beta$, $\beta\gamma$, $\alpha\gamma$, $\alpha'\beta'$, $\beta'\gamma'$ or $\alpha'\gamma'$.

▶ **Claim 2.1.** *If two pc-arcs inside $\triangle$ or $\triangle'$ have a touching or cross twice, then they are of the same type.*

Proof of Claim 2.1. Suppose towards a contradiction that two distinct pseudocircles C, C' from $\mathcal{A} \setminus \mathcal{K}$ contain pc-arcs $A \subset C \cap \triangle$ and $A' \subset C' \cap \triangle$ of different types that have a touching or cross twice. One needs to check the four cases depicted in Figure 3. In none of these cases, pc-arc A' can be completed to a pseudocircle extending the intersecting arrangement of the four given pseudocircles. This is a contradiction. $\triangle$

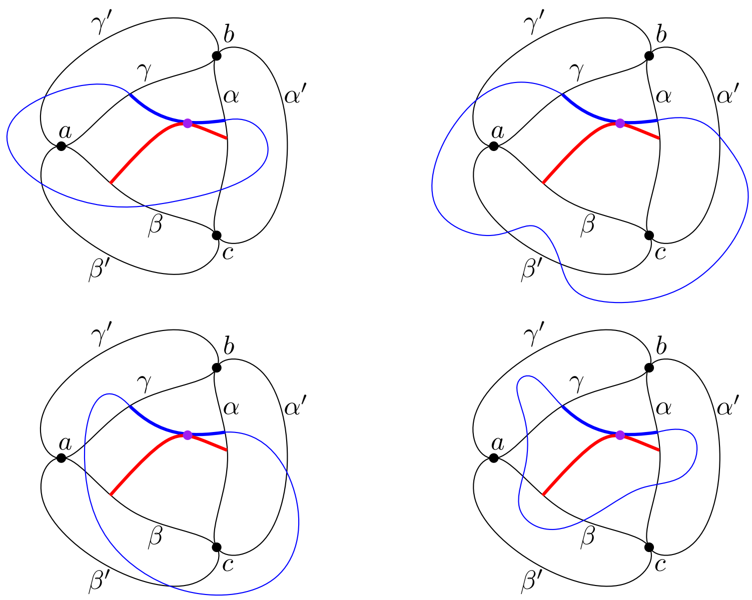


Figure 3 An illustration of the proof of Claim 2.1. The pseudocircles C and C' are highlighted blue and red, respectively. The pc-arcs A and A' are emphasized.

Next we explain how to transform $\mathcal{A}$ into another intersecting arrangement $\mathcal{A}'$ by changing the intersection pattern of pc-arcs within $\triangle$ and $\triangle'$. This transformation will ensure that the touching graphs of $\mathcal{A}$ and $\mathcal{A}'$ are identical and the arrangement $\mathcal{A}' \setminus \mathcal{K}$ will turn out to be cylindrical.

In both triangles, $\triangle$ and $\triangle'$, we concentrate all crossings and touchings of each arc type in a narrow region as depicted in Figure 4. For example, all the crossings of $\alpha\beta$ pc-arcs are in a region close to c and none of the crossings or touchings of these arcs is separated from c by an arc of type $\alpha\gamma$ or $\beta\gamma$. This is done in a way such that for each type of pc-arcs the arrangement of these arcs stays the same and all the endpoints of all pc-arcs stay at their original position.

By applying Claim 2.1, one can check that this transformation preserves the crossing and touching relations between any pair of pseudocircles. Hence we obtain again a valid intersecting pseudocircle arrangement $\mathcal{A}'$ with the same number of touchings.

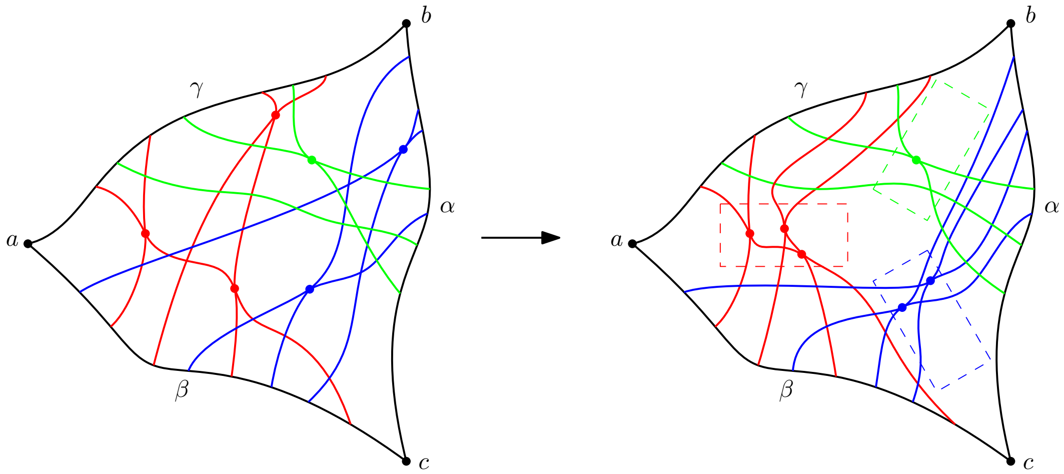


Figure 4 Concentrate all crossings and touchings of one arc type in a narrow region. The narrow regions are indicated by dashed rectangles.

Moreover, one can verify that $\mathcal{A}'$ can always be drawn as in Figure 5 on a cylinder, so that all pseudocircles except the three pseudocircles of $\mathcal{K}$ wrap around the cylinder. This means that the following claim holds:

▶ **Claim 2.2.** *The arrangement induced by $\mathcal{A}' \setminus \mathcal{K}$ is cylindrical.*

Next we replace the three pseudocircles of $\mathcal{K}$ by six pseudocircles as illustrated in Figure 6, so that the resulting arrangement $\mathcal{A}''$ is cylindrical. Each of the three touching points a, b, c in $\mathcal{K}$ is replaced by two new touching points and altogether we obtain touchings $a', a'', b', b'', c', c''$. Hence, when transforming $\mathcal{A}$ into $\mathcal{A}''$, the number of pseudocircles is increased by 3 and the number of touchings is also increased by 3.

An *intersecting arrangement of pseudoparabolas* is a collection of infinite x-monotone curves, called *pseudoparabolas*, where each two of them either have a single touching or intersect in exactly two points where they cross. As every cylindrical pseudocircle arrangement can be represented as an arrangement of pseudoparabolas and vice versa, Agarwal et al. [1] proved the $p_2(\mathcal{A}) \le 2n-2$ upper bound on the number of touchings in arrangements of cylindrical intersecting arrangements by bounding the number of touchings in an intersecting arrangement of pseudoparabolas. They show that their touching graph is planar and bipartite [1, Theorem 2.4]. In fact, the drawing of $\mathcal{A}''$ in Figure 6 can be seen as an

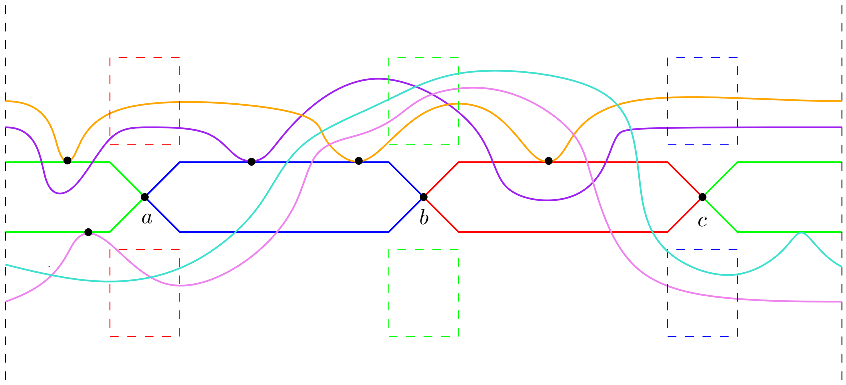


Figure 5 A cylindrical drawing of $\mathcal{A}' \setminus \mathcal{K}$.

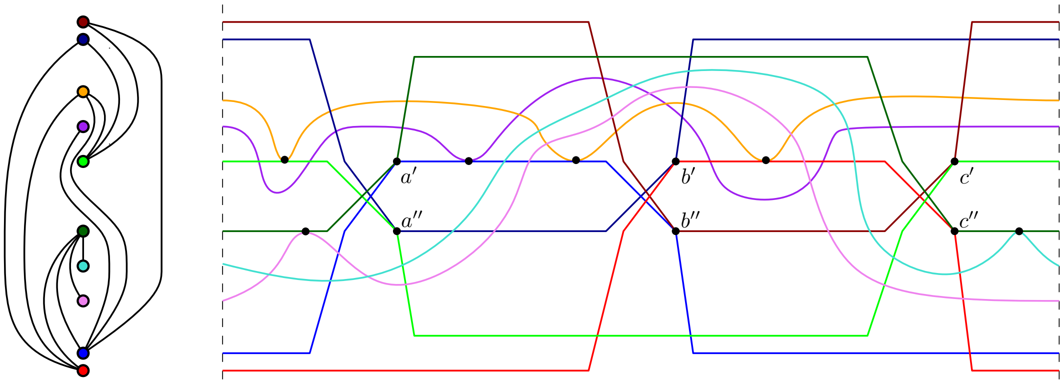


Figure 6 Replace each of the three pseudocircles of $\mathcal{K}$ by two new pseudocircles so that the entire arrangement is now cylindrical. On the left: the touching graph $T(\mathcal{A}'')$ of the arrangement.

intersecting arrangement of pseudoparabolas. We review their proof to prove the following claim.

▶ **Claim 2.3.** *$T(\mathcal{A}'')$ remains planar and bipartite after adding a certain edge.*

Since $T(\mathcal{A}'')$ remains planar and bipartite after adding an edge, and since planar bipartite n-vertex graphs have at most $2n-4$ edges, we obtain

$$p_2(\mathcal{A}) + 3 = p_2(\mathcal{A}'') \leq 2(n+3) - 5 \quad \Longrightarrow \quad p_2(\mathcal{A}) \leq 2n-2.$$

This completes the sketch of the proof of Theorem 1.2.

3 Proof of Theorem 1.3

We denote by $\mathcal{A}_6$, $\mathcal{A}_7$, and $\mathcal{A}_8$ the three arrangements shown in Figure 7. These three arrangements on 6, 7, and 8 pseudocircles, respectively, are digon- and touching-free and contain 8, 10, and 11 triangles, respectively. In each of the three arrangements, there is a pseudocircle C and four incident triangles which are alternatingly inside and outside of C in the cyclic order around C. In fact, this *alternation property* holds for all pseudocircles of these three arrangements.

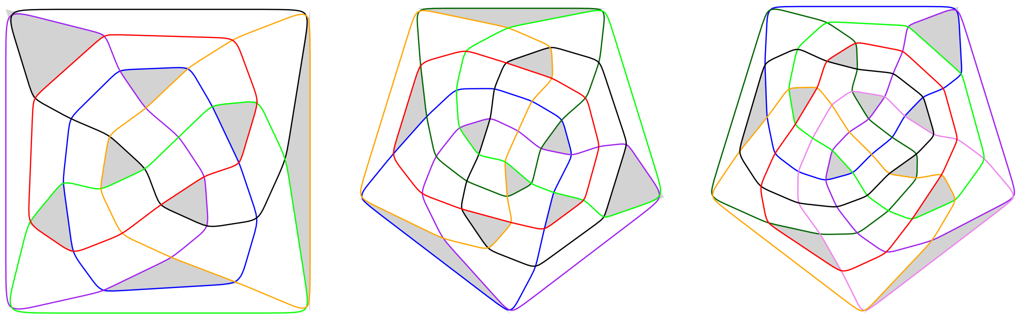
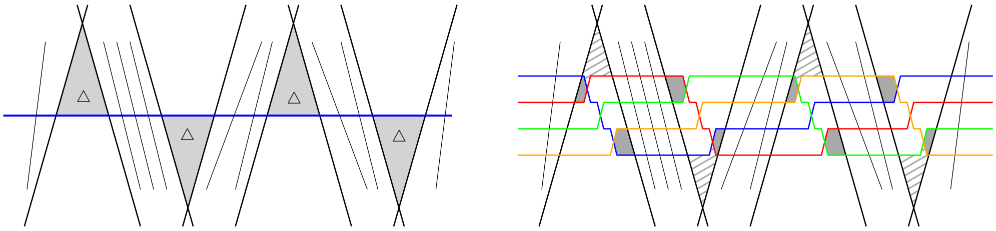


Figure 7 Digon- and touching-free intersecting arrangements of $n = 6, 7, 8$ pseudocircles with 8, 10, 11 triangles, respectively. Triangular cells are highlighted gray. [5, Fig. 2]



(a) (b)

Figure 8 Replacing one pseudocircle with the alternation property (i.e., four triangles on alternating sides) by a particular arrangement of four pseudocircles.

To recursively construct $\mathcal{A}_n$ for $n \geq 9$, we replace a pseudocircle C with the alternation property from $\mathcal{A}_{n-3}$ by a particular arrangement of four pseudocircles as depicted in Figure 8.

With this replacement we destroy 4 triangles incident to C in the original arrangement, and in total the four new pseudocircles are incident to eight new triangles. Hence, we have $p_3(\mathcal{A}_n) = p_3(\mathcal{A}_{n-3}) + 4 = \lceil \frac{4}{3}(n-3) \rceil + 4 = \lceil \frac{4}{3} n \rceil$.

Moreover, for each of the four new pseudocircles, there are four new triangles (among the eight new triangles) that lie on alternating sides. This allow us to recurse by using one of the four new pseudocircles in the role of C for the next iteration. This completes the proof.

It is worth noting that $\mathcal{A}_6$ can be created as illustrated in Figure 9 by extending the Krupp arrangement of three pseudocircles, in which all cells are triangles.

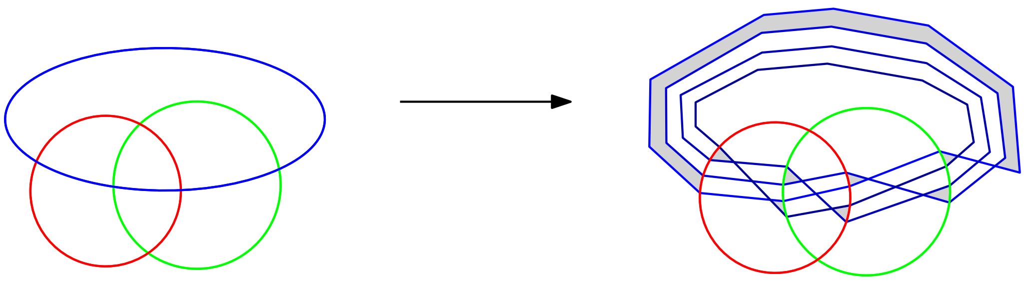


Figure 9 Extending the Krupp arrangement (left) to the arrangement $\mathcal{A}_6$ (right).

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A Complete proof of Theorem 1.2

Since the touching graph $T(\mathcal{A})$ contains a triangle, there are three pseudocircles in $\mathcal{A}$ that pairwise touch. Let $\mathcal{K}$ be the subarrangement induced by these three pseudocircles and let $\triangle$ and $\triangle'$ denote the two triangle cells in $\mathcal{K}$. We label the three touching points, which are also the corners of $\triangle$ and $\triangle'$, as a, b, c. Furthermore, we label the three boundary arcs of $\triangle$ (resp. $\triangle'$) as α, β, γ (resp. α', β', γ'), as shown in Figure 10(a).

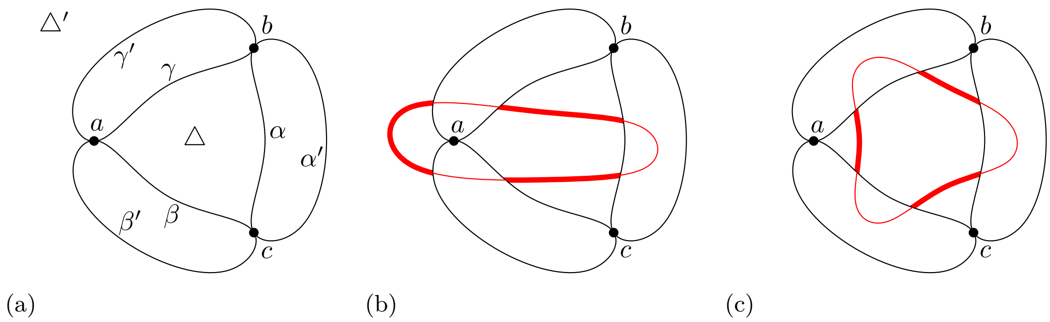

Figure 10 (a) An illustration of the subarrangement $\mathcal{K}$. (b) and (c), respectively, illustrate an additional pseudocircle C (red). The pc-arcs inside $\triangle$ and $\triangle'$, respectively, are highlighted.

Assume that all digons in $\mathcal{A}$ are contracted to touchings.

The intersection of a pseudocircle $C \in \mathcal{A} \setminus \mathcal{K}$ with $\triangle \cup \triangle'$ results in three connected segments, which we denote as the three *pc-arcs* of C, see Figures 10(b) and 10(c). Note that each pc-arc in $\triangle$ connects two of α, β or γ while a pc-arc in $\triangle'$ connects two of α', β' and γ'. Depending on the boundary arcs on which they start and end, they belong to one of the types $\alpha\beta$, $\beta\gamma$, $\alpha\gamma$, $\alpha'\beta'$, $\beta'\gamma'$ or $\alpha'\gamma'$.

▶ **Claim 1.1.** *If two pc-arcs inside $\triangle$ have a touching or cross twice, then they are of the same type.*

Proof of Claim 1.1. Suppose towards a contradiction that two distinct pseudocircles C, C' from $\mathcal{A} \setminus \mathcal{K}$ contain pc-arcs $A \subset C \cap \triangle$ and $A' \subset C' \cap \triangle$ of different types that have a touching or cross twice. For simplicity, consider only the arrangement induced by the five pseudocircles $\mathcal{K} \cup \{C, C'\}$. By symmetry we may assume that A is of type $\alpha\gamma$ and A' is of type $\alpha\beta$. We may further assume that A and A' have a touching, since otherwise, if they cross twice, they form a digon and we can contract it. This allows us to distinguish four cases which are depicted in Figure 11 (up to further possible contractions of digons formed between C and the pseudocircles of $\mathcal{K}$).

Case 1: C separates a from b and c.
Case 2: C separates b from a and c.
Case 3: C separates c from a and b.
Case 4: C does not separate a, b, c.

In the next paragraph we show that in neither case, it is possible to extend the arc A' to a pseudocircle C' intersecting the three pseudocircles of $\mathcal{K}$. This is a contradiction.

Extend A' starting from its endpoint on α. The only way to reach γ or γ', avoiding an invalid, additional intersection with C, is via the pseudocircle $\beta \cup \beta'$. But the other endpoint of A' already lies on β, so either the pseudocircle extending A' has at lest 3 intersections with $\beta \cup \beta'$ or it misses $\gamma \cup \gamma'$. Both is prohibited in an intersecting arrangement extending $\mathcal{K}$. △

Clearly a statement analog to Claim 1.1 holds with $\triangle'$ replacing $\triangle$.

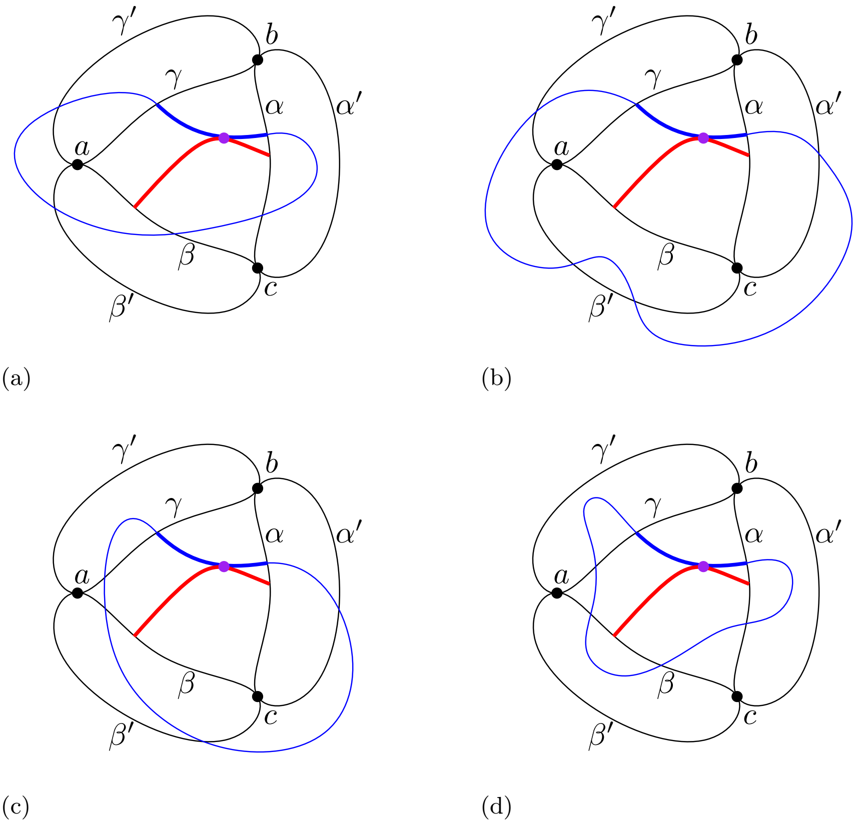

Figure 11 (a)–(d) illustrate Case 1–4 from the proof of Claim 1.1. The pseudocircles C and C' are highlighted blue and red, respectively. The pc-arcs A and A' are emphasized.

Next we explain how to transform $\mathcal{A}$ into another intersecting arrangement $\mathcal{A}'$ by changing the intersection pattern of pc-arcs within $\triangle$ and $\triangle'$. This transformation will ensure that the touching graphs of $\mathcal{A}$ and $\mathcal{A}'$ are identical and the arrangement $\mathcal{A}' \setminus \mathcal{K}$ will turn out to be cylindrical.

In both triangles, $\triangle$ and $\triangle'$, we concentrate all crossings and touchings of each arc type in a narrow region as depicted in Figure 12. For example, all the crossings of $\alpha\beta$ pc-arcs are in a region close to c and none of the crossings or touchings of these arcs is separated from c by an arc of type $\alpha\gamma$ or $\beta\gamma$. This is done in a way such that for each type of pc-arcs the arrangement of these arcs stays the same and all the endpoints of all pc-arcs stay at their original position.

The transformation preserves the crossing and touching relations between any pair of pc-arcs. For two pc-arcs $\phi, \psi \subset \triangle$ of the same type this is clear by construction. With Claim 1.1 the only case that remains to show is when two pc-arcs ϕ and ψ are of different types and cross exactly once. By symmetry we may assume that ϕ is of type $\alpha\beta$ and ψ is of type $\alpha\gamma$. Since both endpoints of ϕ and ψ on α remain unchanged by the transformation, ϕ and ψ cross exactly once in $\mathcal{A}'$ if and only if they cross exactly once in $\mathcal{A}$. For two pc-arcs $\phi, \psi \subset \triangle'$ the argument is analogous.

It follows that the transformation preserves the crossing and touching relations between any pair of pseudocircles. Hence, $\mathcal{A}'$ is again a valid intersecting pseudocircle arrangement and the touching graphs $T(\mathcal{A})$ and $T(\mathcal{A}')$ are identical. In particular, $\mathcal{A}'$ has the same number of touchings as $\mathcal{A}$.

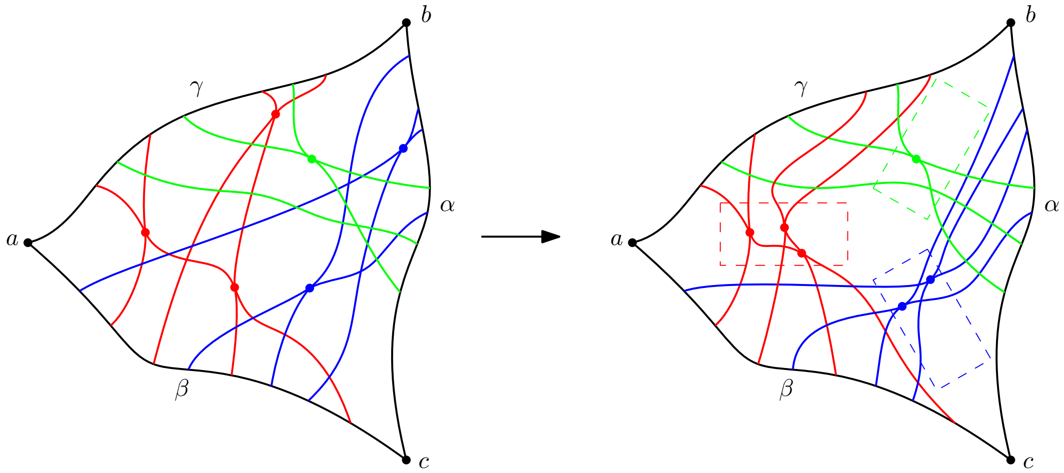

Figure 12 Concentrate all crossings and touchings of one arc type in a narrow region. The narrow regions are indicated by dashed rectangles.

▶ **Claim 1.2.** *The arrangement induced by $\mathcal{A}' \setminus \mathcal{K}$ is cylindrical.*

Proof of Claim 1.2. For each pseudocircle $C \in \mathcal{A}' \setminus \mathcal{K}$, the intersection

$$C \cap (\triangle \cup \triangle') = (C \cap \triangle) \cup (C \cap \triangle')$$

consists of three pc-arcs, and each of these three pc-arcs is of a different type. The first arc is of type $\alpha\beta$ or $\alpha'\beta'$ (depending whether it is inside $\triangle$ or $\triangle'$), the second is of type $\beta\gamma$ or $\beta'\gamma'$, and the third is of type $\alpha\gamma$ or $\alpha'\gamma'$.

Now we redraw $\mathcal{A}'$ on a cylinder as illustrated in Figure 13. Since all crossings and touchings of the arc type are within a small region, all pseudocircles from $\mathcal{A}' \setminus \mathcal{K}$ wrap around the cylinder, and hence the arrangement induced by $\mathcal{A}' \setminus \mathcal{K}$ is cylindrical. $\triangle$

Next we replace the three pseudocircles of $\mathcal{K}$ by six pseudocircles as illustrated in Figure 14, so that the resulting arrangement $\mathcal{A}''$ is cylindrical. Each of the three touching points a, b, c in $\mathcal{K}$ is replaced by two new touching points and altogether we obtain touchings $a', a'', b', b'', c', c''$. Hence, when transforming $\mathcal{A}$ into $\mathcal{A}''$, the number of pseudocircles is increased by 3 and the number of touchings is also increased by 3.

An *intersecting arrangement of pseudoparabolas* is a collection of infinite x-monotone curves, called *pseudoparabolas*, where each two of them either have a single touching or intersect in exactly two points where they cross. As every cylindrical pseudocircle arrangement can be represented as an arrangement of pseudoparabolas and vice versa, Agarwal et al. [1] proved the $p_2(\mathcal{A}) \leq 2n - 2$ upper bound on the number of touchings in arrangements of cylindrical intersecting arrangements by bounding the number of touchings in an intersecting arrangement of pseudoparabolas. They show that their touching graph is planar and bipartite [1, Theorem 2.4]. In fact, the drawing of $\mathcal{A}''$ in Figure 14 can be seen as an intersecting arrangement of pseudoparabolas. We review their proof to prove the following claim.

▶ **Claim 1.3.** *$T(\mathcal{A}'')$ remains planar and bipartite after adding a certain edge.*

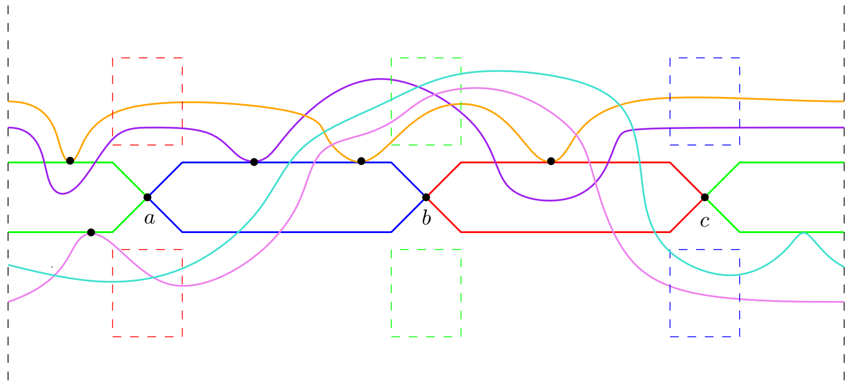

Figure 13 A cylindrical drawing of $\mathcal{A}' \setminus \mathcal{K}$.

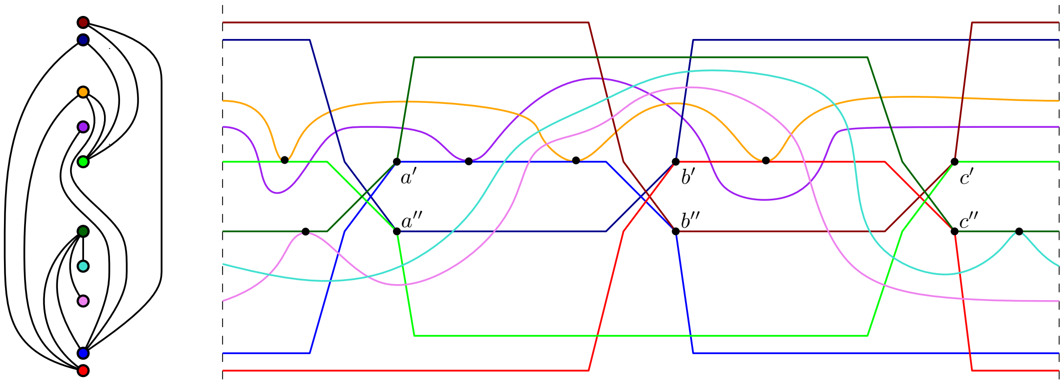

Figure 14 Replace each of the three pseudocircles of $\mathcal{K}$ by two new pseudocircles so that the entire arrangement is now cylindrical. On the left: the touching graph $T(\mathcal{A}'')$ of the arrangement.

Proof of Claim 1.3. We label the pseudoparabolas with starting segments sorted from top to bottom as $P_1, \ldots, P_n$. In the touching graph $T(\mathcal{A}'')$, we label the corresponding vertices as $1, \ldots, n$.

Bipartiteness: the bipartition comes from the fact that the digons incident to one pseudoparabola are either all from below or all from above. Suppose towards a contradiction that a pseudoparabola P_j has a touching from above with P_i and from below with P_k ($i < j < k$). Now, P_i and P_k cannot not intersect because P_j separates them – a contradiction since the arrangement is intersecting.

We now further observe that the uppermost pseudoparabola P_1 and the lowermost pseudoparabola P_n belong to distinct parts of the bipartition, because P_1 has all touchings below (i.e. with parabolas of greater index); P_n has all touchings above (i.e. with parabolas of smaller index). Hence, the touching graph remains bipartite after adding the edge $\{1, n\}$.

Planarity: For the planarity of $T(\mathcal{A}'')$, Agarwal et al. [1] create a particular drawing: the vertices are drawn on a vertical line and each edge $e = \{u, v\}$ is drawn as y-monotone curve according to the following *drawing rule*: For each w with $u < w < v$, we route e to the left of w if the pseudoparabola P_w intersects P_u before P_v, and to right otherwise. It is then shown that in the so-obtained drawing $\mathcal{D}$, each pair of non-incident edges has an even number of intersections. Hence, the Hanani–Tutte theorem (cf. Section 3 in [9]) asserts that $T(\mathcal{A}'')$ is planar.

Notice that $\{1, n\}$ is not an edge in $T(\mathcal{A}'')$, since by construction, the lowermost and uppermost pseudocircles do not touch. We further observe that, since all edges in $\mathcal{D}$ are drawn as y-monotone curves, the entire drawing lies in a box which is bounded from above by vertex 1 and from below by vertex n. Hence, we can draw an additional edge from 1 to n which is routed entirely outside of the box and does not intersect any other edge. Again, by the Hanani–Tutte theorem, we have planarity. $\triangle$

Since $T(\mathcal{A}'')$ remains planar and bipartite after adding an edge, and since planar bipartite n-vertex graphs have at most $2n-4$ edges, we obtain

$$p_2(\mathcal{A}) + 3 = p_2(\mathcal{A}'') \leq 2(n+3) - 5 \quad \Longrightarrow \quad p_2(\mathcal{A}) \leq 2n - 2.$$

This completes the proof of Theorem 1.2.