Finding Domino Tilings with Maximal Monotonic Paths

using Wagner's Method

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Given $N, M \in \mathbb{N}$ and a $N \times M$ grid, what is the domino tiling that maximizes the number of unique monotonic paths from the top left corner point of the grid to the bottom right corner point.

Definitions

- A **Domino** is a 1×2 (horizontal) or a 2×1 (vertical) tile.
- A **Domino Tiling** is a full covering of a grid using dominos, such that all dominos are within the grid and disjoint.
- A **Monotonic Path** is a path along the edges of dominos that only traverses the grid rightwards and downwards.

Example of Domino Tiling ($N = 4, M = 4$)

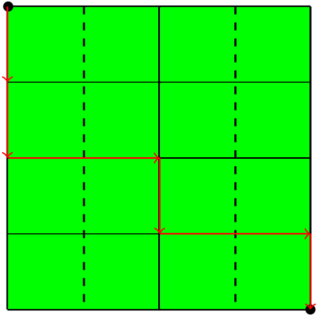


Figure 1: Standard Horizontal tiling with valid path

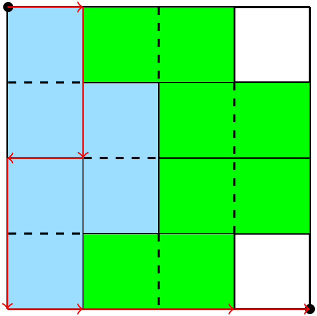


Figure 2: Invalid tiling with invalid path

A few Remarks

Corollary (Existence of Domino Tilings)

A domino tiling exists for a $N \times M$ grid if and only if $N \cdot M$ is even.

Proof.

$\Longrightarrow$: A domino has area 2 and thus a tiling can only cover a grid with even area.
$\Longleftarrow$: Assume w.l.o.g. N is even, then the standard vertical tiling is a valid tiling. □

Remark (Number of Dominos in Tiling)

Every domino tiling consists of $\frac{N \cdot M}{2}$ dominos.

Current Hypothesis for $N = M$

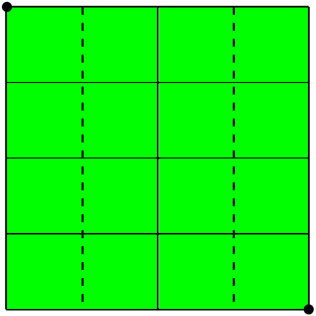


Figure 3: Standard Horizontal tiling

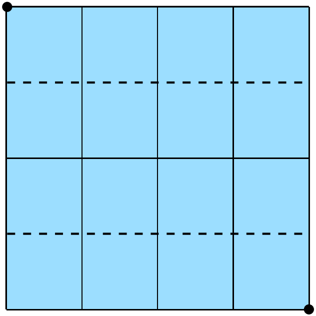


Figure 4: Standard Vertical tiling

Hypothesis

The standard horizontal and vertical tilings have the most montonic paths with $\frac{(\frac{3N}{2})!}{N!(\frac{N}{2})!}$ unique paths.

Known: Number of Domino Tilings in $N \times M$ Grid

Theorem (Kastelyn, 1961)

The number of domino tilings $D_{N,M}$ in a $N \times M$ grid with $N \cdot M \mod 2 = 0$ is given by

$$D_{N,M} = \prod_{j=1}^{\lceil \frac{M}{2} \rceil} \prod_{k=1}^{N} \left(2\sqrt{\cos^2 \frac{\pi j}{M+1} + \cos^2 \frac{\pi k}{N+1}} \right).$$

Naive Encoding

Idea

Using the observation that each tiling has $\frac{N \cdot M}{2}$ dominos, encode a tiling with a binary string of length $\frac{N \cdot M}{2}$, where 0 represents a horizontal tile and 1 a vertical tile. Induce a numbering on the grid, which uniquely maps each valid tiling to a binary string.

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

Figure 5: Numbering in Grid

Advantages and Disadvantages of Naive Encoding

Pros	Cons
• Only linear space required for encoding and tiling easily constructed from binary string.	• Mapping from valid domino tilings to binary string is not surjective, i.e. a lot of binary strings do not correspond to a valid tiling. (Big problem for Wagner's Method!) • $D_{N,M} << 2^{\frac{N \cdot M}{2}}$ for large $N, M \in \mathbb{N}$

Alternative Encoding (1)

- Domino tilings are equivalent to perfect matchings in an appropriately defined graph.

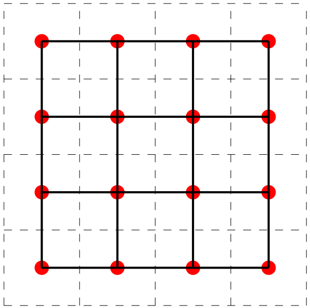


Figure 6: Graph

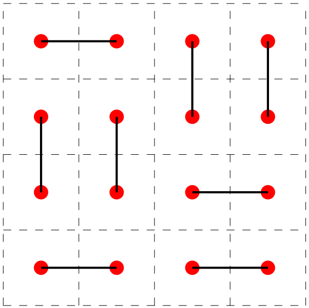
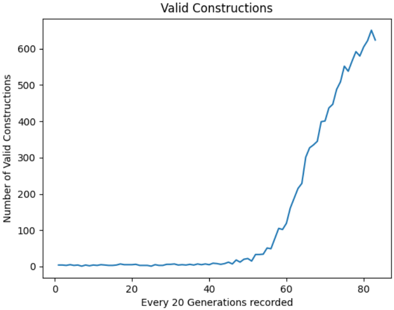


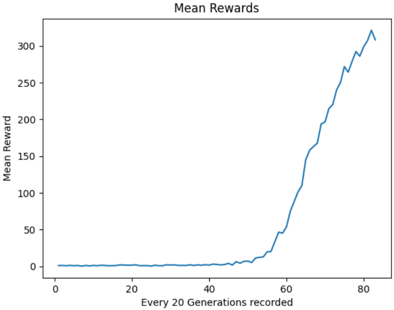
Figure 7: Perfect Matching

Alternative Encoding (2)

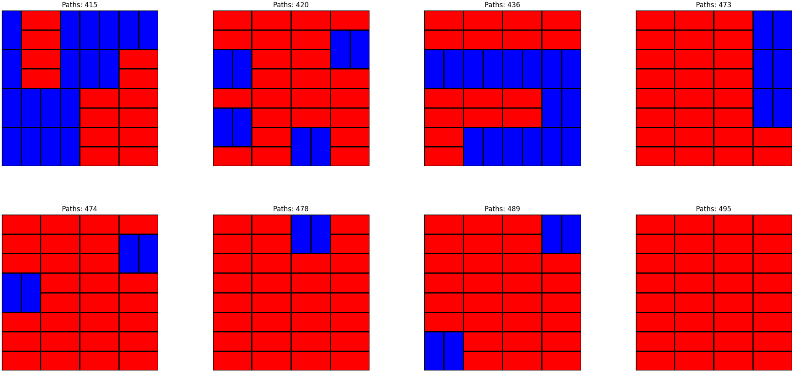
- Give each edge a weight of 1 or -1 and compute a maximum weighted perfect matching.
- Use binary string of length $N \cdot M - (N - 1) - (M - 1)$ to encode the weighting for each edge.
- Reward function is the number of unique paths, which can be computed using dynamic programming.

Results with Naive Encoding for $N = M = 8$

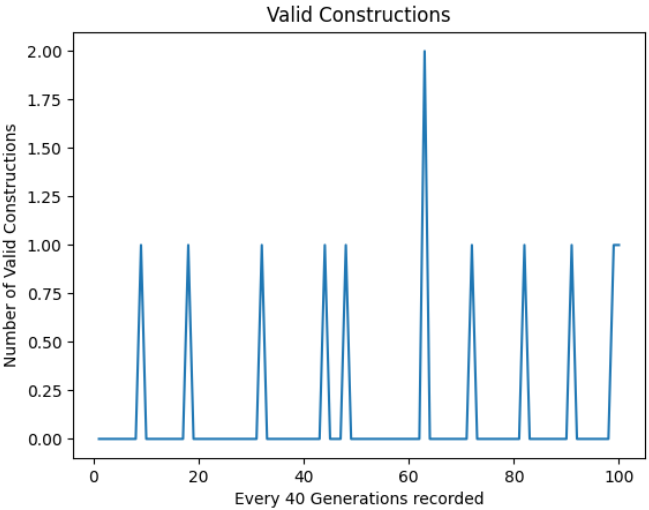




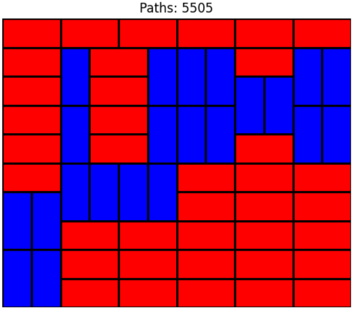
Progression of Constructed Tilings (1)

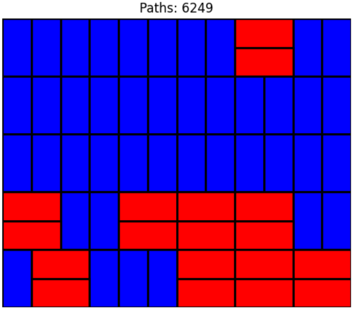


Results with Naive Encoding for $N = 10, M = 12$

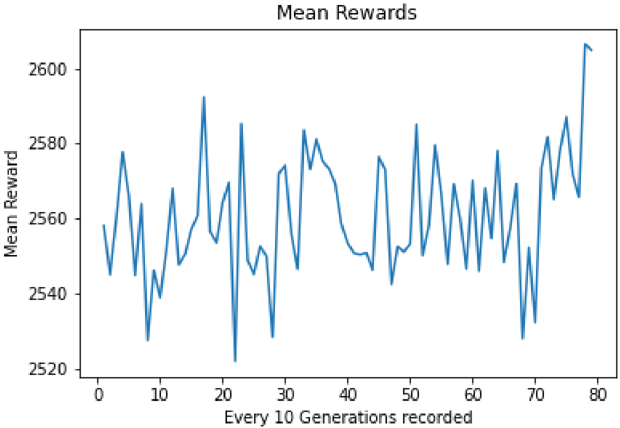


Progression of Constructed Tilings (2)



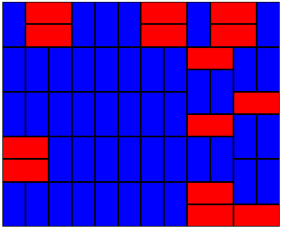


Results with Alternative Encoding for $N = 10, M = 12$

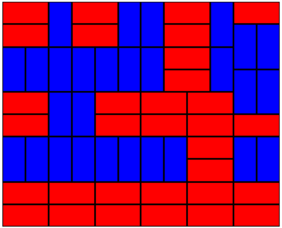


Progression of Tilings with Alternative Encoding (3)

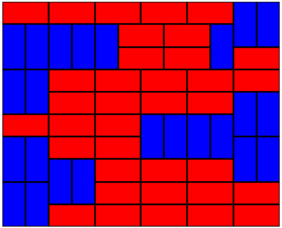
Paths: 4495



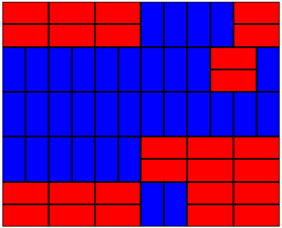
Paths: 5065



Paths: 5325



Paths: 5932



References

- Wagner, A. Z. (2021, April 29). Constructions in Combinatorics via neural networks. arXiv.org. Retrieved January 26, 2023, from https://arxiv.org/abs/2104.14516
- Borys, K. (2015). Domino Tiling. University of Chicago http://math.uchicago.edu/ may/REU2015/REUPapers/Borys.pdf