Matroids and Related Structures
Submodular functions and polymatroids

[Schrijver 2003, ch. 44]

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Submodular functions

Definition

S finite set. $f : \mathcal{P}(S) \to \mathbb{R}$ is *submodular*, if for $X, Y \subseteq S$:

$$f(X \cap Y) + f(X \cup Y) \leq f(X) + f(Y)$$

Examples:

- Cardinality $|X|$ for $X \subseteq S$
- $f(X) = \sum_{x \in X} w_x$ with weights $w_v \in \mathbb{R}$ for $v \in S$
- Rank function $\mathrm{rk}(X)$ for $X \subseteq S$ and matroid $(S, \mathcal{I})$
- $\delta_w^+(X)$ for $X \subseteq S$ and directed graph (S, E) with weights $w \in \mathbb{R}^E$

Lemma

Set function $f : \mathcal{P}(S) \to \mathbb{R}$ is submodular iff

$$\forall X \subseteq Y \subseteq S \;\; \forall s \in S \setminus Y : \quad f(Y \cup \{s\}) - f(Y) \leq f(X \cup \{s\}) - f(X)$$

Proof: Without. □

Polymatroids

Definition

$f : \mathcal{P}(S) \to \mathbb{R}$ submodular.

- The *polymatroid of* f:

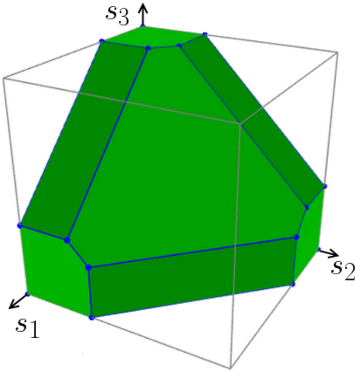
$$P_f := \left\{ x \in \mathbb{R}^S \mid x \geq \mathbf{0} \text{ and } \forall T \subseteq S : \sum_{s \in T} x(s) \leq f(T) \right\}$$

- The *extended polymatroid of* f:

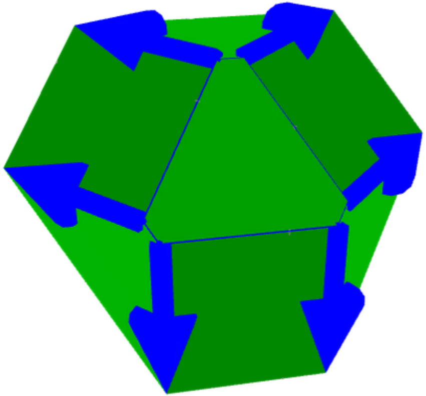
$$EP_f := \left\{ x \in \mathbb{R}^S \mid \forall T \subseteq S : \sum_{s \in T} x(s) \leq f(T) \right\}$$

- P_f is bounded polyhedron (polytope), $P_f \neq \emptyset \iff f \geq \mathbf{0}$
- EP_f is unbounded polyhedron (if $\neq \emptyset$), $EP_f \neq \emptyset \iff f(\emptyset) \geq \mathbf{0}$

Example: $X \subseteq \{s_1, s_2, s_3\}, f(X) = \begin{cases} 0 & X = \emptyset \\ 1 & |X| = 1 \\ 1.35 & |X| = 2 \\ 1.6 & X = S \end{cases}$



Polymatroid P_f



Extended polymatroid EP_f

Submodular functions and matroids

Two connections to chapter 40:

Matroid intersection problem

- For matroids $(S, \mathcal{I}_1), (S, \mathcal{I}_2)$, rank functions rk_1, rk_2, calculate:

$$\max\{|X| : X \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{\underbrace{\mathrm{rk}_1(X) + \mathrm{rk}_2(S \setminus X)}_{\text{submodular function in } X} : X \subseteq S\}$$

- Minimization of submodular functions solves matroid intersection.

Independent set polytope

- Matroid $M = (S, \mathcal{I})$, rank function $\mathrm{rk}(X)$, polymatroid

$$P_{\mathrm{rk}}(M) = \left\{x \in \mathbb{R}^S \mid x \geq \mathbf{0} \text{ and } x(T) \leq \mathrm{rk}(T)\right\}$$

  is called *independent set polytope*.

- Vertices of $P_{\mathrm{rk}}(M)$ are incidence vectors of sets $X \in \mathcal{I}$.

Integral monotonic submodular functions and matroid ranks

Definition

$f : \mathcal{P}(S) \to \mathbb{R}$ is *monotonic*, if $f(X) \leq f(Y)$ for all $X \subseteq Y \subseteq S$.

- Matroid $(E, \mathcal{I})$ with $E = \dot{\bigcup}_{s \in S} E_s$, $|S| < \infty$ index set. Then

$$f(U) := \text{rk}\left(\dot{\bigcup_{s \in U}} E_s \right) \qquad \text{for } U \subseteq S$$

  is integral monotonic submodular function with $f(\emptyset) = 0$.

- **Claim:** Each integral monotonic submodular $f : \mathcal{P}(S) \to \mathbb{R}_+$ with $f(\emptyset) = 0$ is of this kind.

Construction for claim:

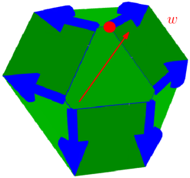
- $f : \mathcal{P}(S) \to \mathbb{R}_+$ integral monotonic submodular, $f(\emptyset) = 0$
- Choose disjoint sets E_s with $|E_s| = f(\{s\})$, let $E := \dot{\bigcup}_{s \in S} E_s$
- Define $r_f(U) := \min_{T \subseteq S} \left(|U \setminus \bigcup_{s \in T} E_s| + f(T) \right)$ for $U \subseteq E$
- r_f is rank function of some matroid $M = (E, \mathcal{I})$:
  R1: $0 \le r_f(U) \le |U|$ **R2:** r_f monotonic **R3:** r_f is submodular
- Check:

$$r_f\left(\bigcup_{s \in U} E_s\right) = \min_{T \subseteq S} \left(|\bigcup_{s \in U} E_s \setminus \bigcup_{s \in T} E_s| + f(T) \right)$$

$$= \min_{T \subseteq U} \left(\underbrace{\sum_{s \in U \setminus T} f(\{s\})}_{\ge f(U \setminus T)} \right) + f(T) = f(U)$$

Linear Optimization over Polymatroids

Problem: $\max \; w^\intercal x$
s.t. $x \in EP_f$



- Assume $f(\emptyset) = 0$ (Otherwise set $f(\emptyset) := 0$)
- Assume $w \geq \mathbf{0}$ (Otherwise unbounded)
- Assume $S = \{s_1, \ldots, s_n\}$ with $w_1 \geq \ldots \geq w_n$ (Otherwise reorder S)

Greedy algorithm for polymatroids

Input: (S, f, w) as above
Output: $x \in EP_f$ maximizing $w^\intercal x$

for $i = 1, \ldots, n$ **do**
  $x(s_i) \leftarrow f(\{s_1, \ldots, s_i\}) - f(\{s_1, \ldots, s_{i-1}\})$
end

Interpretation of greedy algorithm

In i-th iteration (when setting $x(s_i)$) have following bound:

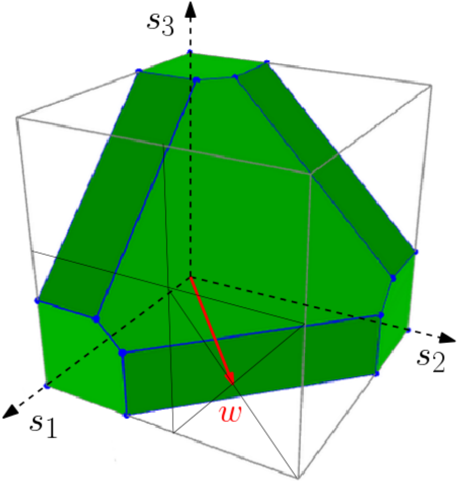
$$
\begin{aligned}
x(s_i) &= \left(\sum_{j=1}^{i} x(s_i) \right) - \left(\sum_{j=1}^{i-1} x(s_j) \right) \\
&\leq f\left(\{s_1, ..., s_i\}\right) - \left(\sum_{j=1}^{i-1} f\left(\{s_1, ..., s_j\}\right) - f\left(\{s_1, ..., s_{j-1}\}\right) \right) \\
&= f\left(\{s_1, ..., s_i\}\right) - \left(f\left(\{s_1, ..., s_{i-1}\}\right) - f(\emptyset)\right) \\
&= f\left(\{s_1, ..., s_i\}\right) - f\left(\{s_1, ..., s_{i-1}\}\right)
\end{aligned}
$$

Interpretation:

- Start with $x = \mathbf{0}$.
- Set x coordinate by coordinate to maximal possible value.

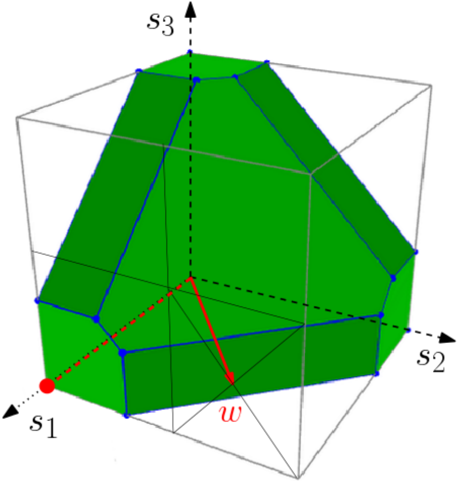
Example: Interpretation of greedy algorithm

Submodular function f as before, $w^\intercal = (1, \frac{3}{4}, \frac{1}{4})$.



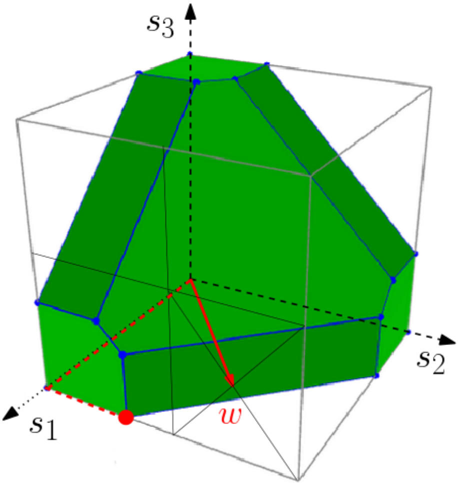
Example: Interpretation of greedy algorithm

Submodular function f as before, $w^\intercal = (1, \frac{3}{4}, \frac{1}{4})$.



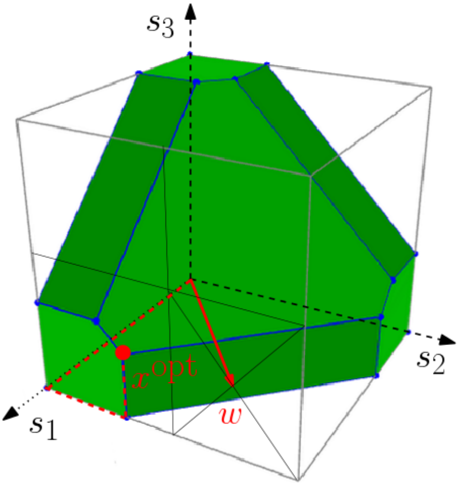
Example: Interpretation of greedy algorithm

Submodular function f as before, $w^{\mathsf{T}} = (1, \frac{3}{4}, \frac{1}{4})$.



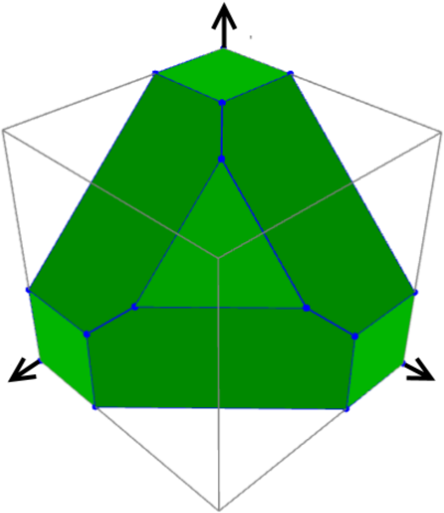
Example: Interpretation of greedy algorithm

Submodular function f as before, $w^\intercal = (1, \frac{3}{4}, \frac{1}{4})$.



Example for NOT being a polymatroid

Because some vertices „cannot be reached“ by greedy algorithm.



Theorem

The greedy algorithm for extended polymatroids works correctly.

Proof: Use duality theory: Value primal LP = Value dual LP

Primal LP:

$$\max \; w^\intercal x$$

$$\text{s.t. } x(T) \leq f(T) \;\; \text{f.a. } T \subseteq S$$

$$x \in \mathbb{R}^S$$

Dual LP:

$$\min \sum_{T \subseteq S} y(T) f(T)$$

$$\text{s.t.} \sum_{T \subseteq S,\; s \in T} y(T) = w(s) \quad \text{f.a. } s \in S$$

$$y \in {\mathbb{R}_{\geq 0}}^{\mathcal{P}(S)}$$

Proof in 3 steps:

- Step I: $x \in \mathbb{R}^S$ by greedy is feasible for primal LP.
- Step II: Construct feasible dual solution y.
- Step III: Show that x and y have same value $\rightarrow$ Optimality.

Primal LP:

max $w^\intercal x$

s.t. $x(T) \leq f(T)$ f.a. $T \subseteq S$

$x \in \mathbb{R}^S$

Dual LP:

min $\sum_{T \subseteq S} y(T) f(T)$

s.t. $\sum_{T \subseteq S,\, s \in T} y(T) = w(s)$ f.a. $s \in S$

$y \in \mathbb{R}_{\geq 0}{}^{\mathcal{P}(S)}$

Step I: $x \in \mathbb{R}^S$ by greedy is feasible for primal LP.

By induction over $|T|$: $x(T) \leq f(T)$ for all $T \subseteq S$.

IA ($|T| = 0$): $x(\emptyset) = 0 = f(\emptyset)$

IS ($|T| - 1 \to |T|$): Let $k := \max\{i : s_i \in T\}$.

$$\begin{aligned} x(T) &= x(T \setminus \{s_k\}) + x(s_k) \leq^{\text{I.H.}} f(T \setminus \{s_k\}) + x(s_k) \\ &= f(T \setminus \{s_k\}) + f(\{s_1, ..., s_k\}) - f(\{s_1, ..., s_{k-1}\}) \\ &\leq f(T) \quad \text{(By submodularity)} \end{aligned}$$

Primal LP:

$\textbf{max} \;\; w^\intercal x$

s.t. $x(T) \leq f(T)$ f.a. $T \subseteq S$

$x \in \mathbb{R}^S$

Dual LP:

$\textbf{min} \;\; \sum_{T \subseteq S} y(T) f(T)$

s.t. $\sum_{T \subseteq S,\; s \in T} y(T) = w(s)$ f.a. $s \in S$

$y \in \mathbb{R}_{\geq 0}{}^{\mathcal{P}(S)}$

Step II: Construct feasible dual solution y:

$$y(T) := \begin{cases} w(s_i) - w(s_{i+1}) & : T = \{s_1, ..., s_i\} \text{ for some } i = 1, ..., n-1 \\ w(s_n) & : T = S \\ 0 & : \text{otherwise} \end{cases}$$

- $y \geq \mathbf{0}$ because $w(s_i) \geq w(s_{i+1})$
- $\forall s_i \in S : \sum_{T \subseteq S, s_i \in T} y(T) = \sum_{j \geq i} y(\{s_1, ..., s_j\}) = w(s_i)$

Primal LP:

max $w^{\intercal} x$

s.t. $x(T) \leq f(T)$ f.a. $T \subseteq S$

$x \in \mathbb{R}^S$

Dual LP:

min $\sum_{T \subseteq S} y(T) f(T)$

s.t. $\sum_{T \subseteq S,\ s \in T} y(T) = w(s)$ f.a. $s \in S$

$y \in \mathbb{R}_{\geq 0}{}^{\mathcal{P}(S)}$

Step III: Show that x and y have same value.

$$
\begin{aligned}
w^{\intercal} x &= \sum_{s \in S} w(s) x_s = \sum_{i=1}^{n} w(s_i) \Big(f(\{s_1, \ldots, s_i\}) - f(\{s_1, \ldots, s_{i-1}\}) \Big) \\
&= \sum_{i=1}^{n-1} f(\{s_1, \ldots, s_i\}) \Big(w(s_i) - w(s_{i+1}) \Big) + f(S) w(s_n) - \underbrace{f(\emptyset) w(s_1)}_{=0} \\
&= \sum_{T \subseteq S} y(T) f(T)
\end{aligned}
$$

□

Corollary

For $f : \mathcal{P}(S) \to \mathbb{R}$ submodular, $w \in {\mathbb{R}_{\geq 0}}^S$:

1. *If f is integral, there is integral $x \in \mathbb{Z}^S$ maximizing $w^\intercal x$ s.t. $x \in EP_f$.*
2. *If f is monotonic, the greedy algorithm maximizes $w^\intercal x$ s.t. $x \in P_f$.*

Proof: From greedy algorithm:

For 1):

$$x(s_i) = f(\{s_1, ..., s_i\}) - f(\{s_1, ..., s_{i-1}\}) \in \mathbb{Z}$$

For 2):

$$x(s_i) = f(\{s_1, ..., s_i\}) - f(\{s_1, ..., s_{i-1}\}) \in \mathbb{R}_{\geq 0}$$

$\square$

Greedy algorithm generalizes classical greedy algorithm

- Matroid $M = (S, \mathcal{I})$, rank function $\text{rk}(X)$, polymatroid $P_{\text{rk}}(M)$
- $S = \{s_1, ..., s_n\}$, weights $w_1 \geq ... \geq w_n > 0$
- Run greedy algorithm:
  - First iteration:

$$x(s_1) \leftarrow \text{rk}(\{s_1\}) - \text{rk}(\emptyset) = 1 - 0 = 1$$

  - i-th iteration:

$$x(s_i) \leftarrow \text{rk}(\{s_1, ..., s_i\}) - \text{rk}(\{s_1, ..., s_{i-1}\}) = \begin{cases} 1 & s_i \text{ inc. rank} \\ 0 & s_i \text{ doesn't inc. rank} \end{cases}$$

  - After iteration i the 1-entries of x form max. indep. set in $\{s_1, ..., s_i\}$
- $\{s \in S : x(s_i) = 1\}$ equals solution of classical greedy algorithm.

Submodular function from polymatroid

Theorem

For $f : \mathcal{P}(S) \to \mathbb{R}$ submodular with $f(\emptyset) = 0$:

$$\forall T \subseteq S : \qquad f(T) = \max\{x(T) \mid x \in EP_f\}$$

Remark: One to one correspondence:
Extended polymatroids $\leftrightarrow$ Submodular functions with $f(\emptyset) = 0$ (non-empty)

Proof: Let $T \subseteq S$, w.l.o.g. $T = \{s_1, ..., s_j\}$.
$w^\intercal := (\underbrace{1, ..., 1}_{j \text{ times}}, 0, ..., 0)$, x^* greedy solution of $w^\intercal x$ s.t. $x \in EP_f$.

$$\begin{aligned}\max\{x(T) \mid x \in EP_f\} = \max\{w^\intercal x \mid x \in EP_f\} = w^\intercal x^* &= \sum_{i=1}^{j} x^*(s_i) \\ &= \sum_{i=1}^{j} f(\{s_1, ..., s_i\}) - f(\{s_1, ..., s_{i-1}\}) = f(T)\end{aligned}$$

□

Definition

For submodular $f : \mathcal{P}(S) \to \mathbb{R}_+$ let $\bar{f} : \mathcal{P}(S) \to \mathbb{R}$ with

$$\bar{f}(\emptyset) := 0 \qquad \bar{f}(U) := \min_{T \supseteq U} f(T) \quad \text{for } U \neq \emptyset, U \subseteq S$$

Observations:

- $\bar{f}$ is monotonic submodular and $\bar{f}(\emptyset) = 0$.
- If f is monotonic submodular with $f(\emptyset) = 0$, then $\bar{f} = f$.
- f and $\bar{f}$ define the same polymatroid: $P_f = P_{\bar{f}}$

  Proof:
  $$x \in P_{\bar{f}} \implies \forall U \subseteq S : x(U) \leq \min_{T \supseteq U} f(T)$$
  $$\implies \forall U \subseteq S : x(U) \leq f(U) \implies x \in P_f$$
  $$x \in P_f \implies \forall U \subseteq T \subseteq S : x(U) \leq x(T) \leq f(T)$$
  $$\implies \forall U \subseteq S : x(U) \leq \bar{f}(U) \implies x \in P_{\bar{f}}$$

- To apply greedy algorithm on P_f, can replace f by $\bar{f}$.

Theorem

For $f : \mathcal{P}(S) \to \mathbb{R}_+$ *monotonic submodular with* $f(\emptyset) = 0$*:*

$$\forall T \subseteq S : \quad f(T) = \max\{x(T) \mid x \in P_f\}$$

Proof: Analogous to theorem before (apply greedy algorithm)

Remark: One to one correspondence:
Polymatroids $\leftrightarrow$ Monotonic submodular functions with $f(\emptyset) = 0$ (non-empty)

For **any** submodular $f : \mathcal{P}(S) \to \mathbb{R}_+$:

$$\max\{x(T) \mid x \in P_f\} = \bar{f}(T) = \min_{T' \supseteq T} f(T') \qquad \text{for } T \neq \emptyset$$

Calculate $\max\{x(T) \mid x \in P_f\}$ is
equivalent to calculate $\min_{T' \supseteq T} f(T')$.

Vertices of polymatroids and integrality

Theorem

Submodular $f : \mathcal{P}(S = \{s_1, ..., s_n\}) \to \mathbb{R}$ *with* $f(\emptyset) = 0$.

1. *All vertices of* EP_f *are given by*

$$x(s_{\pi(i)}) = f(\{s_{\pi(1)}, ..., s_{\pi(i)}\}) - f(\{s_{\pi(1)}, ..., s_{\pi(i-1)}\})$$

   for all $\pi \in \mathfrak{S}_n$.

2. *If* f *is integral, then* EP_f *is integral.*

Proof: From greedy algorithm. □

Theorem

Submodular $f : \mathcal{P}(S = \{s_1, \ldots, s_n\}) \to \mathbb{R}$ *with* $f(\emptyset) = 0$.

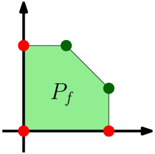
1. *All vertices of* P_f *have the form*

$$x(s_{\pi(i)}) = \begin{cases} f(\{s_{\pi(1)}, \ldots, s_{\pi(i)}\}) - f(\{s_{\pi(1)}, \ldots, s_{\pi(i-1)}\}) & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$$

   for some $\pi \in \mathfrak{S}_n$, $k \in \{0, \ldots, n\}$.

2. *If* f *is integral, then* P_f *is integral.*

Proof: From greedy algorithm.
Need case distinction for reaching vertices with zero components.



Theorem

Submodular $f : \mathcal{P}(S = \{s_1, ..., s_n\}) \to \mathbb{R}$ *with* $f(\emptyset) = 0$.

1. *All vertices of* P_f *have the form*

$$x(s_{\pi(i)}) = \begin{cases} f(\{s_{\pi(1)}, ..., s_{\pi(i)}\}) - f(\{s_{\pi(1)}, ..., s_{\pi(i-1)}\}) & \text{if } i \le k \\ 0 & \text{if } i > k \end{cases}$$

   for some $\pi \in \mathfrak{S}_n$, $k \in \{0, ..., n\}$.

2. *If* f *is integral, then* P_f *is integral.*

Remark: Assume f is monotonic.

- Iterating over $(\pi, k) \in \mathfrak{S}_n \times [n]$ gives **exactly** the vertices P_f.
- Alternatively: Iterate over totally ordered subsets $B \subseteq S$.
- Characterisation of totally ordered $B \subseteq S$ generating given vertex v by
  - prescribed subset and superset $A_v \subseteq B \subseteq C_v$
  - **and** prescribed partial order $\preceq_v$ on C_v

See: Bixby, Cunningham, Topkis (1985):
The Partial Order of a Polymatroid Extreme Point

Conclusion

- Polymatroids defined by submodular functions
- Optimization of (extended) polymatroids by greedy algorithm