Mittagsseminar 13.01.2023

Introduction to hyperplane arrangements

Based on selected topics from

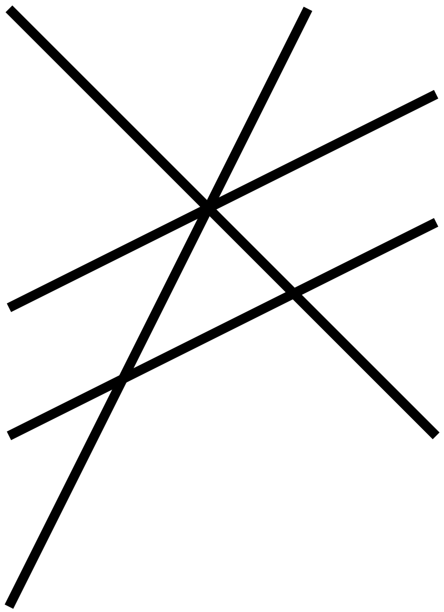
(Stanley, 2006)

Talk by Sandro Roch

Basic notions

Basic notions

- *hyperplane arrangement:*
  finite set of hyperplanes in $\mathbb{R}^n$



Basic notions

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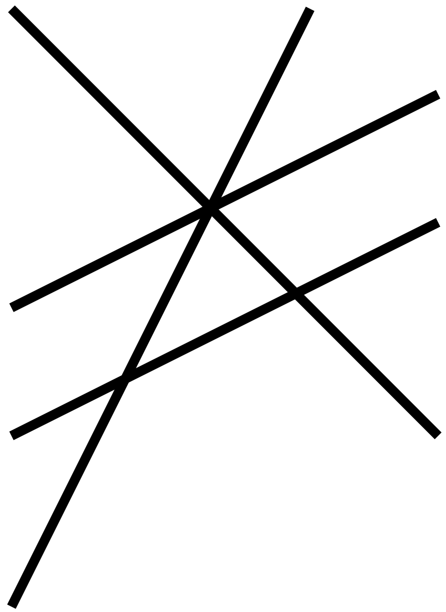
  if $\bigcap_{H\in\mathscr{A}} H \neq \emptyset$



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- *hyperplane arrangement:* finite set of hyperplanes in $\mathbb{R}^n$
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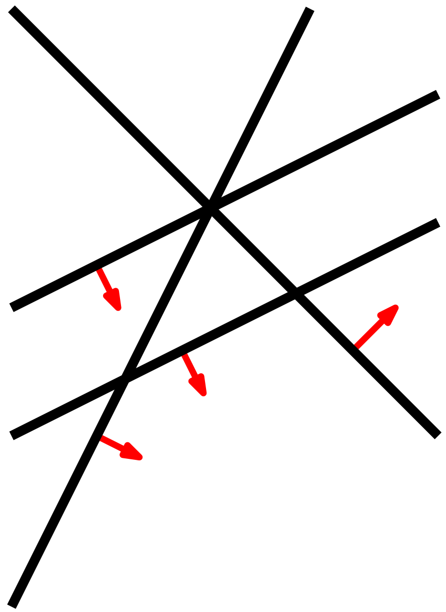

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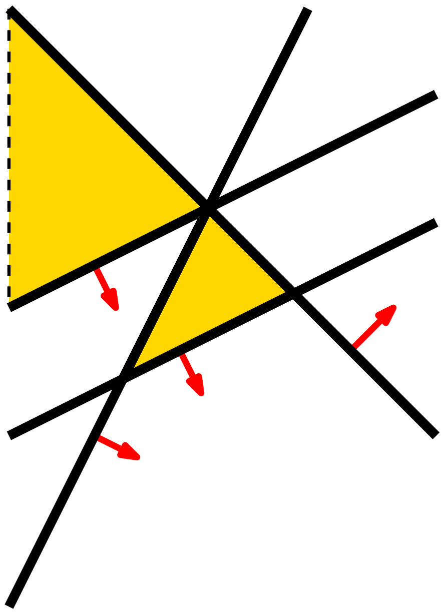
- *chamber in $\mathscr{A}$:* max. connected component of $\mathbb{R}^n - \bigcup_{H \in \mathscr{A}} H$



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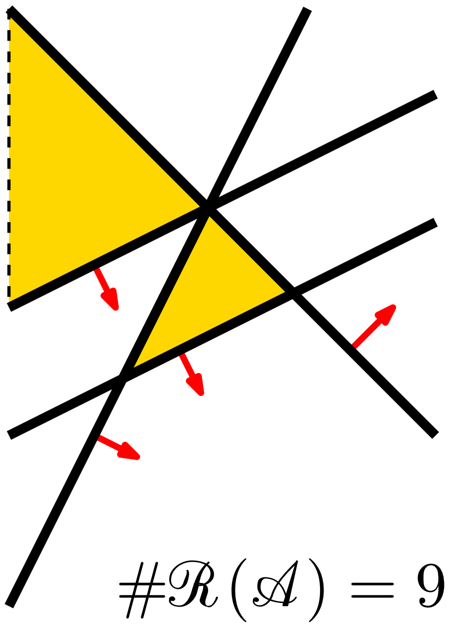
- *chamber in $\mathscr{A}$:* max. connected component of $\mathbb{R}^n - \bigcup_{H \in \mathscr{A}} H$
- $\mathscr{R}(\mathscr{A})$: set of all chambers in $\mathscr{A}$

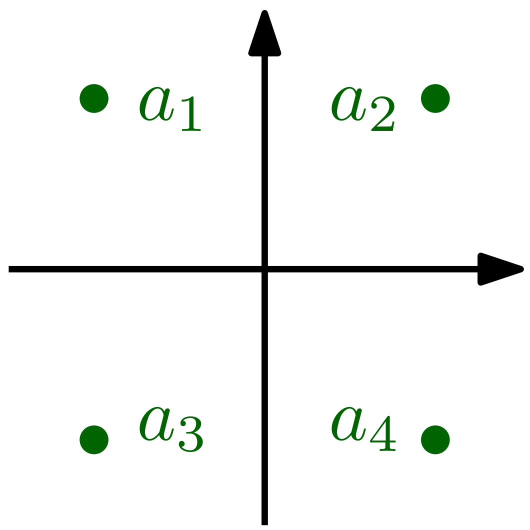


$\#\mathscr{R}(\mathscr{A}) = 9$

Chambers as sweep orders

Motivation

Chambers as sweep orders



point set $A \subset \mathbb{R}^n$

Motivation

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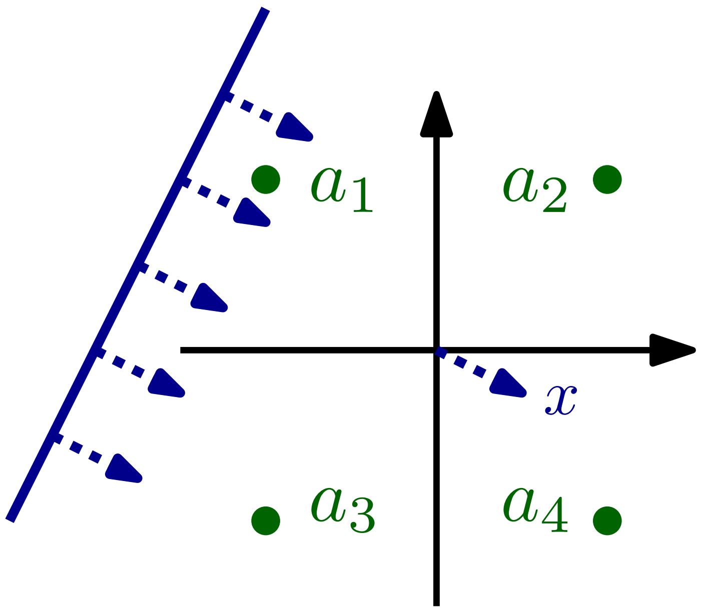

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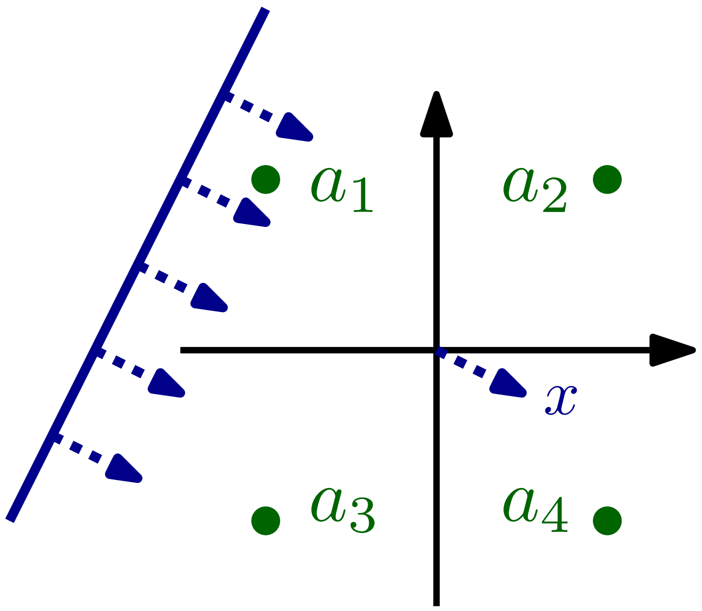

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$\pi_x = [a_1, a_3, a_2, a_4]$

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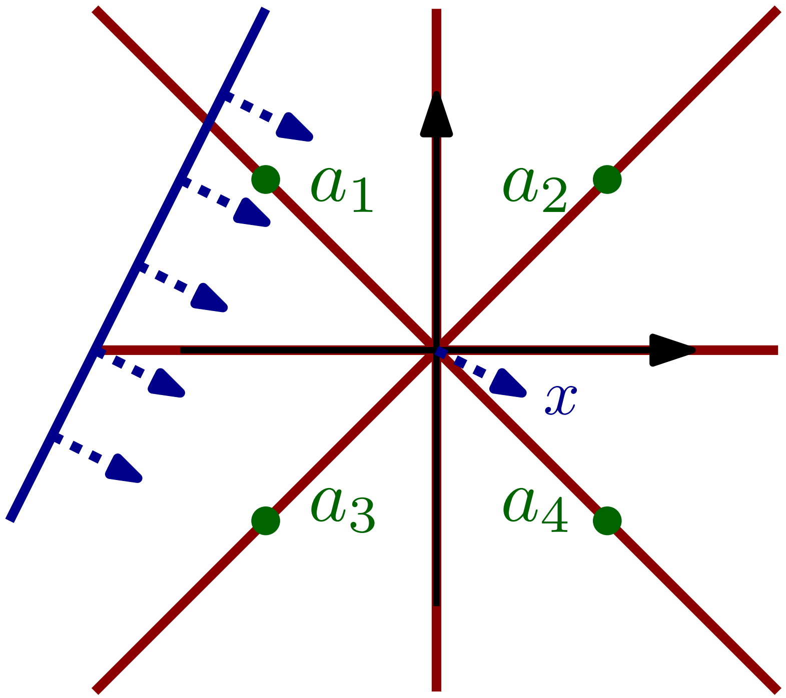

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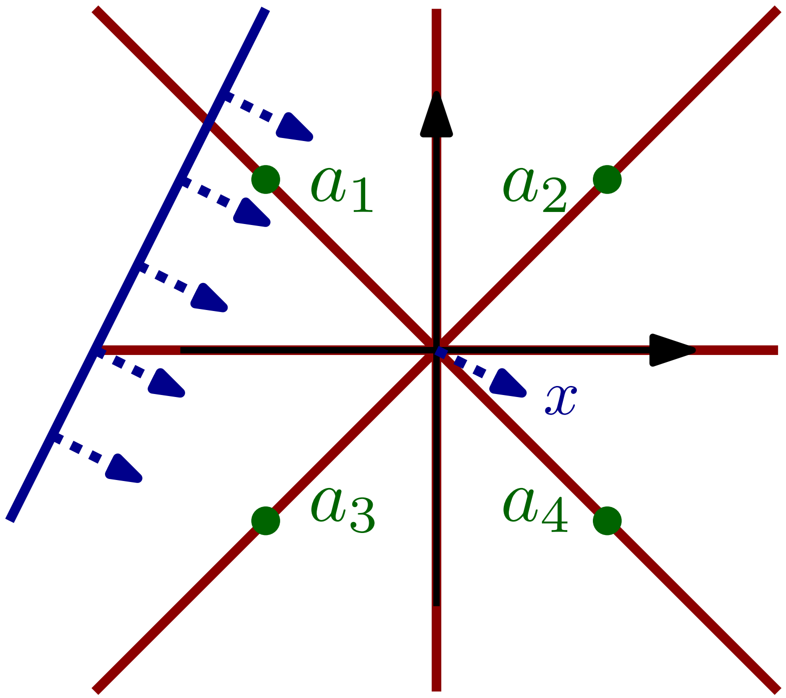

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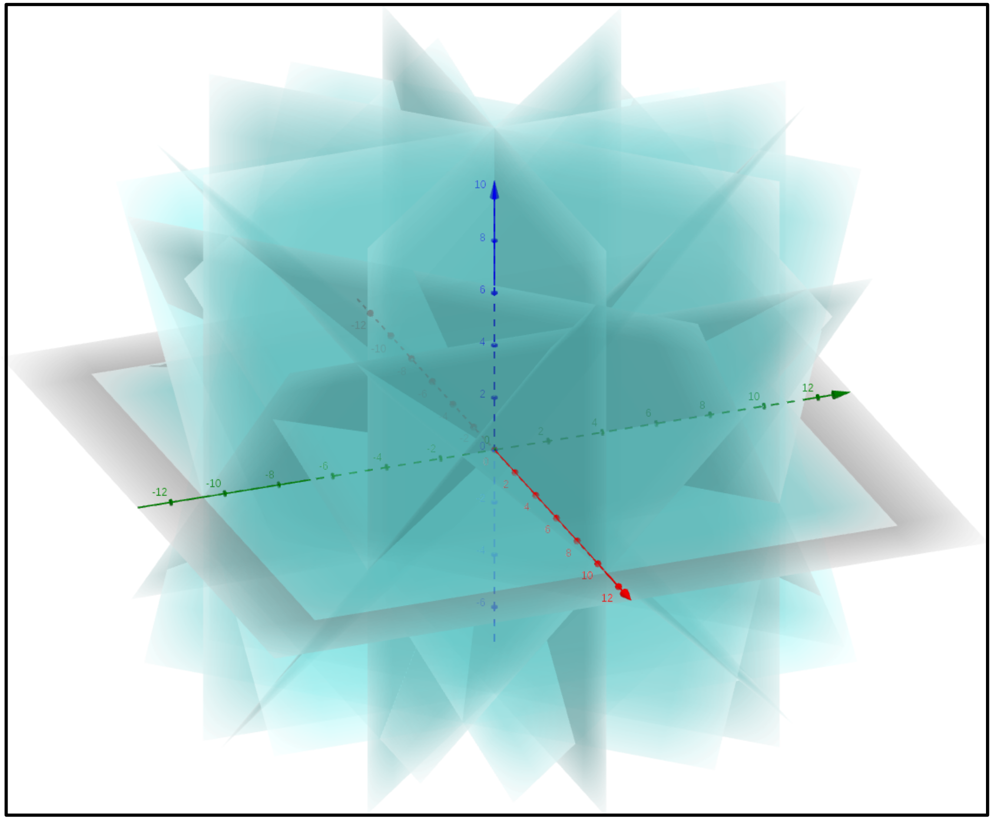
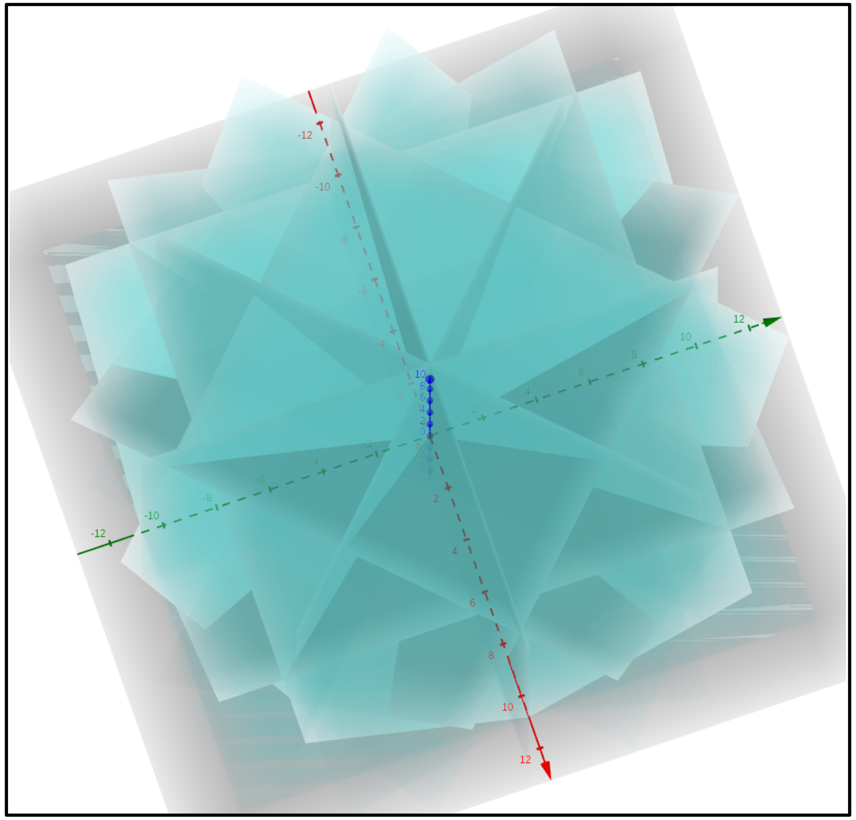
Correspondence: $\mathscr{R}(\mathscr{A}_A) \longleftrightarrow$ sweep orders of A

Motivation

Chambers as sweep orders

Example: sweep arrangement $\mathscr{A}_A$, where $A = \{-1, 1\}^3$





Motivation

Chambers as preference orders

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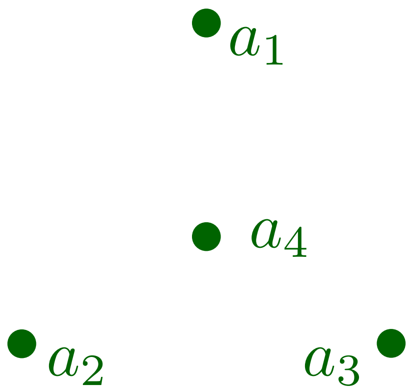
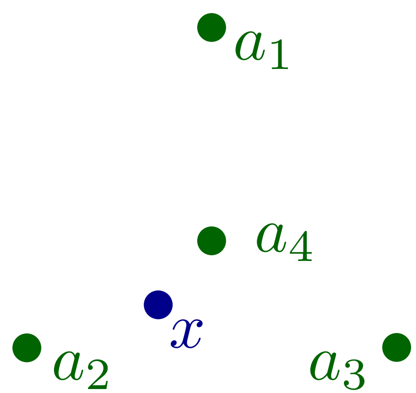
products $A \subset \mathbb{R}^n$



Chambers as preference orders



products $A \subset \mathbb{R}^n$

consumer's ideal product $x \in \mathbb{R}^n$

Motivation

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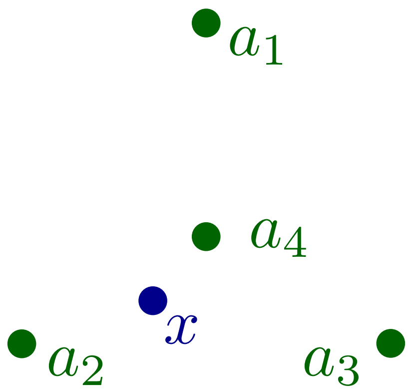

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consumer prefers a_i over a_j iff

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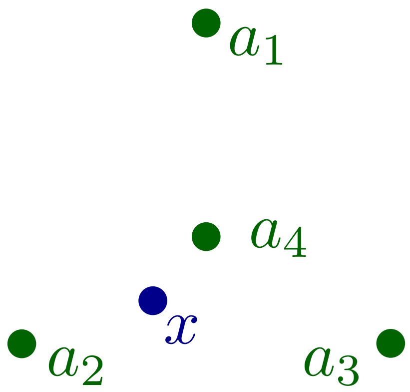

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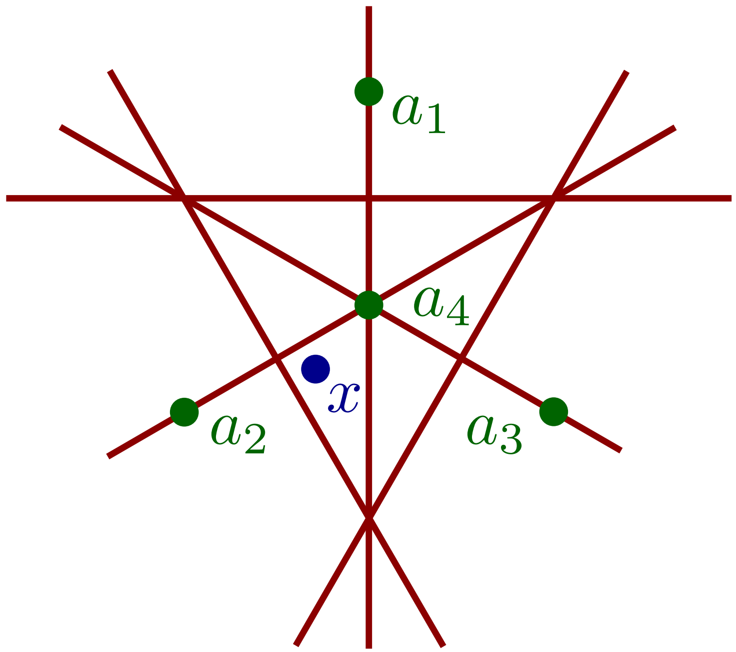

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arrangement $\mathscr{P}_A$:
H_{ij} = perp. bisector of a_i and a_j

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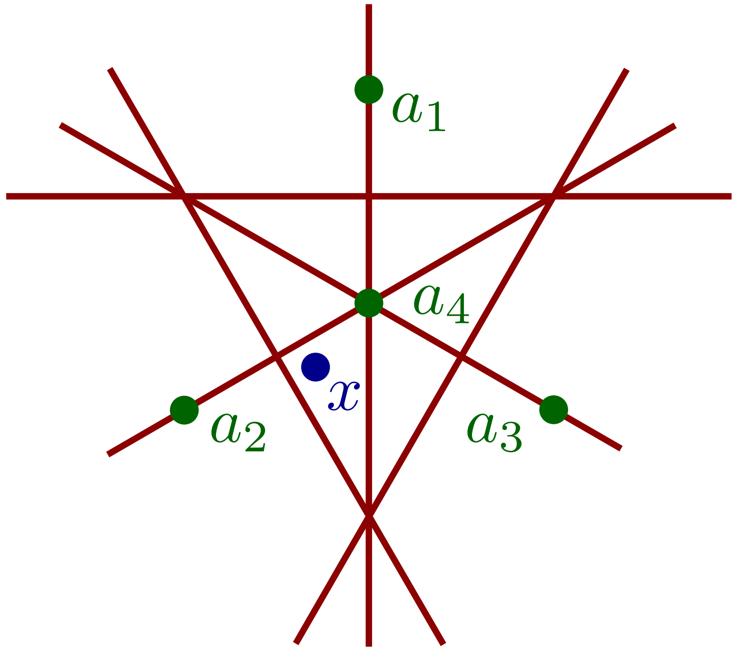

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- *Graphical arrangement* $\mathscr{A}_G$ consisting of hyperplanes
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- **Correspondence:**
$$\mathscr{R}(\mathscr{A}_G) \longleftrightarrow \text{ acyclic orientations of } G$$
$$x \in \mathbb{R}^n \longmapsto \text{ Total order } \pi \text{ on } V = [n] \text{ s.t.}$$
$$x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}$$
induces acyclic orientation of G.

Chambers as interval orders

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- **Def.:** Partial order P on $[n]$ is called *interval order of prescribed lengths* $(l_1, \cdots, l_n)$, if there exist intervals $I_i \subset \mathbb{R}$, $\operatorname{len}(I_i) = l_i$, with

$$i < j \iff L_i \text{ lies entirely to the left of } L_j$$

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$x \in \mathbb{R}^n \longmapsto$ order induced by intervals $I_i = [x_i, x_i + l_i]$

(because then $L_i < L_j \Leftrightarrow x_i - x_j > l_j$)

Chambers as facet visibilities

Motivation

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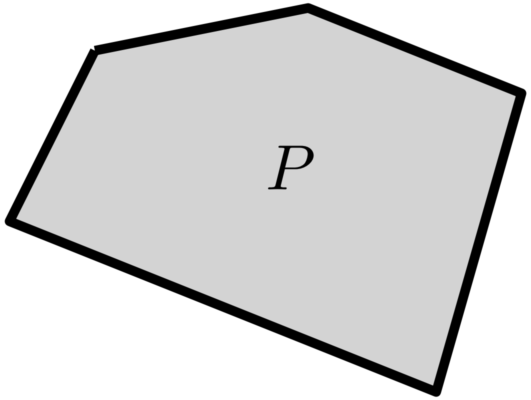
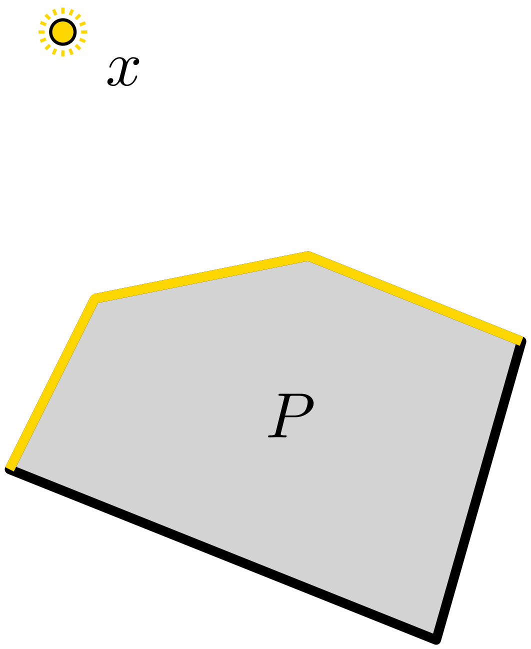


- Polytope $P \subset \mathbb{R}^n$

Motivation

Chambers as facet visibilities



- Polytope $P \subset \mathbb{R}^n$
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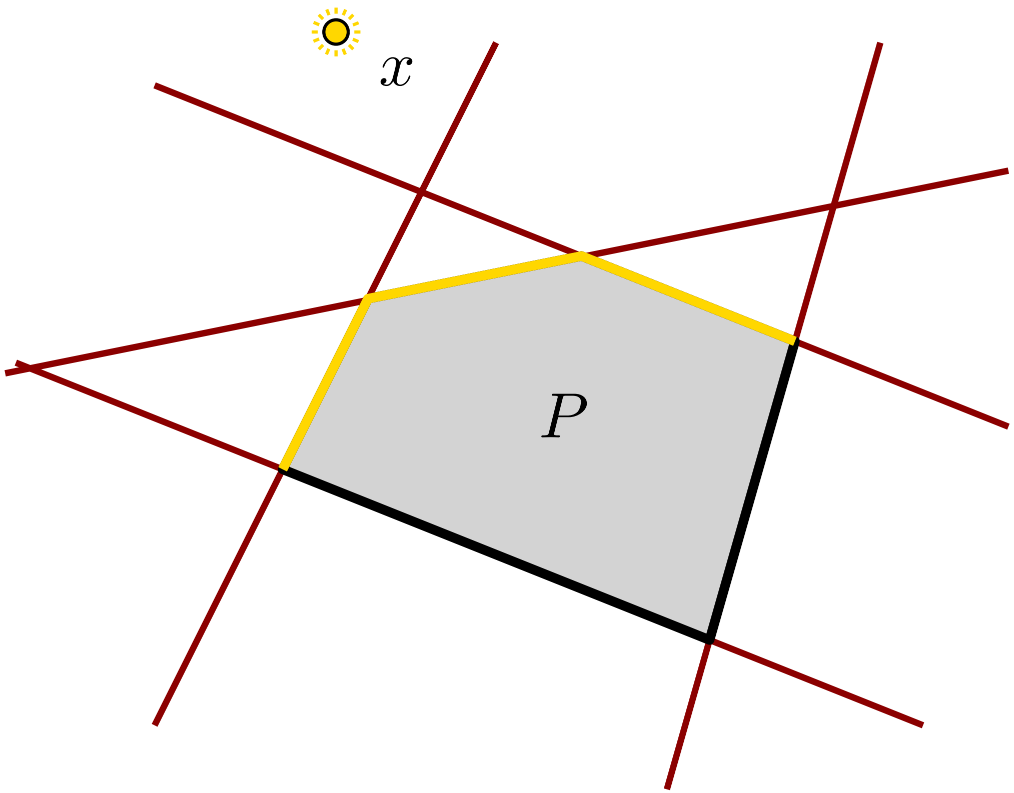

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- *Visibility arrangement:*

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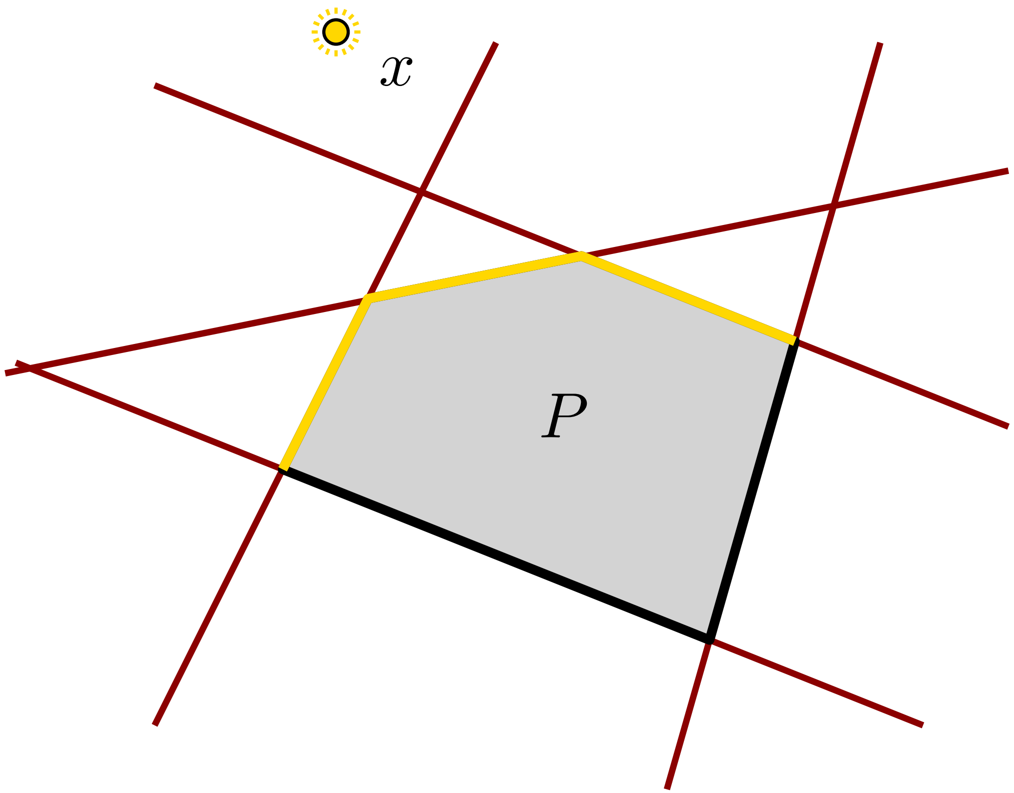

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- **Correspondence:**

$\mathscr{R}\,(\mathrm{vis}(P)) \longleftrightarrow$ possible visibilities of facets of P

Characteristic polynomial

Characteristic polynomial

- **Def.:** *Intersection poset* of arrangement $\mathscr{A}$:

$$L(\mathscr{A}) := \left\{ \bigcap_{H \in I} I \neq \emptyset : I \subseteq \mathscr{A} \right\}$$

  - ordered by **reverse** inclusion: $x \leq y \iff y \subseteq x$
  - graded by $\operatorname{codim}(x) := n - \dim(x)$
  - has minimum element $\hat{0} := \mathbb{R}^n$

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- If $\mathscr{A}$ central, then
  - $L(\mathscr{A})$ has maximum element $\hat{1} := \cap_{H \in \mathscr{A}} H$.
  - $L(\mathscr{A})$ is a lattice.

Characteristic polynomial

- **Def.:** *Characteristic polynomial* of arrangement $\mathscr{A}$:

$$\chi_{\mathscr{A}}(t) := \sum_{x \in L(\mathscr{A})} \mu(\hat{0}, x)\, t^{\dim(x)}$$

where $\mu : \{(x, y) : x \leq y\} \to \mathbb{Z}$ *Möbius function*:

$$\mu(x, y) := \begin{cases} 1 & \text{if } x = y \\ -\sum_{x<z<y} \mu(x, y) & \text{if } x < y \end{cases}$$

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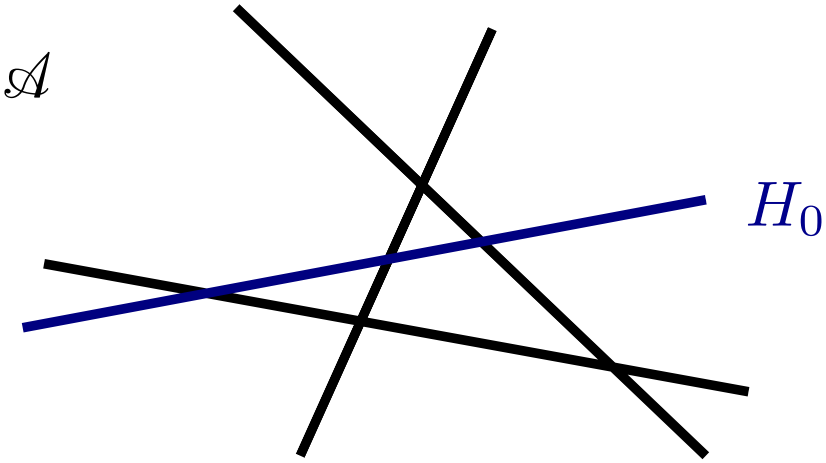
Theorem (Whitney)

$$\chi_{\mathscr{A}}(t) = \sum_{\mathscr{B} \subseteq \mathscr{A} \text{ central}} (-1)^{\#\mathscr{B}}\, t^{n - \operatorname{rank}(\mathscr{B})}$$

Characteristic polynomial

Characteristic polynomial

Arrangement $\mathscr{A}$



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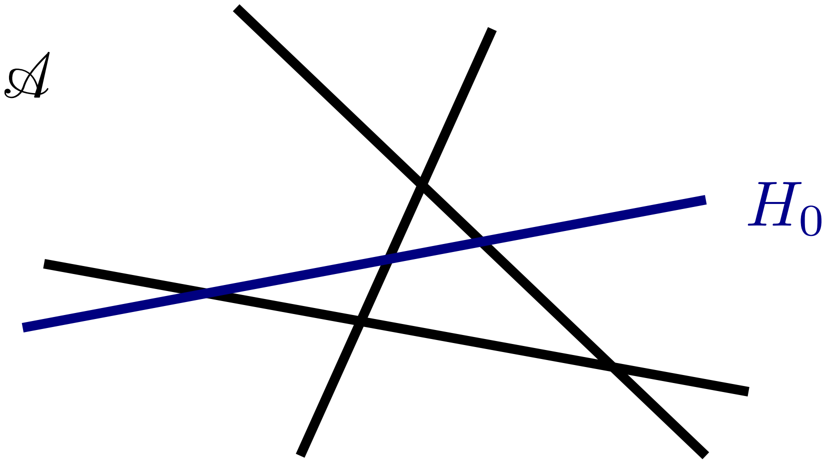
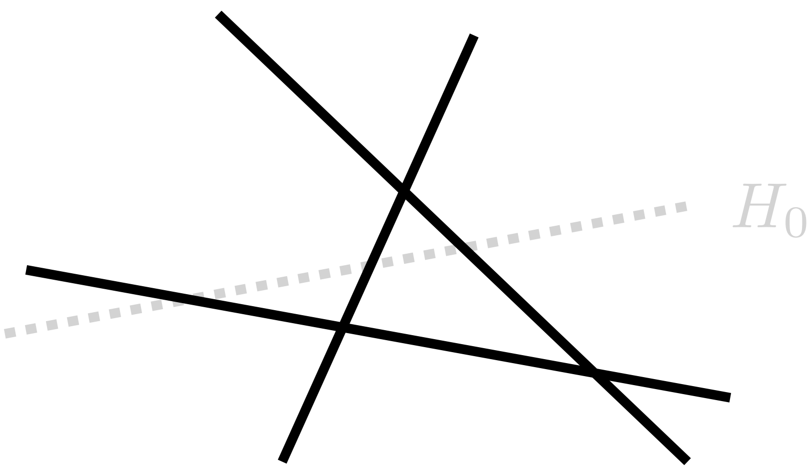
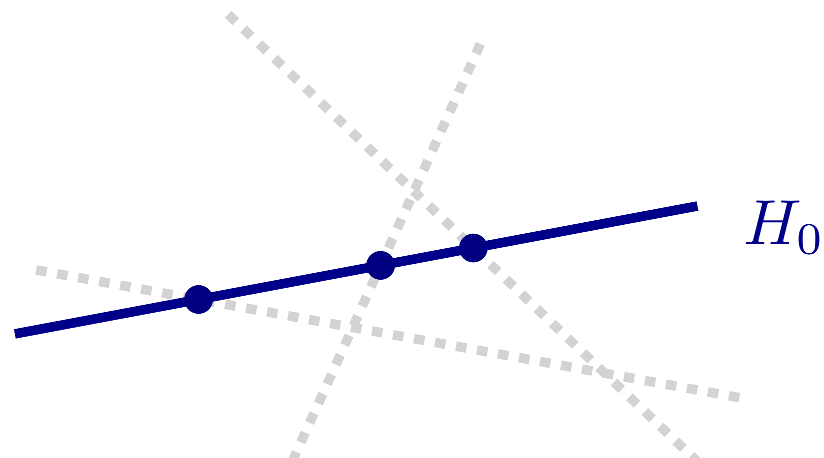




Arrangement $\mathscr{A} \setminus H_0$

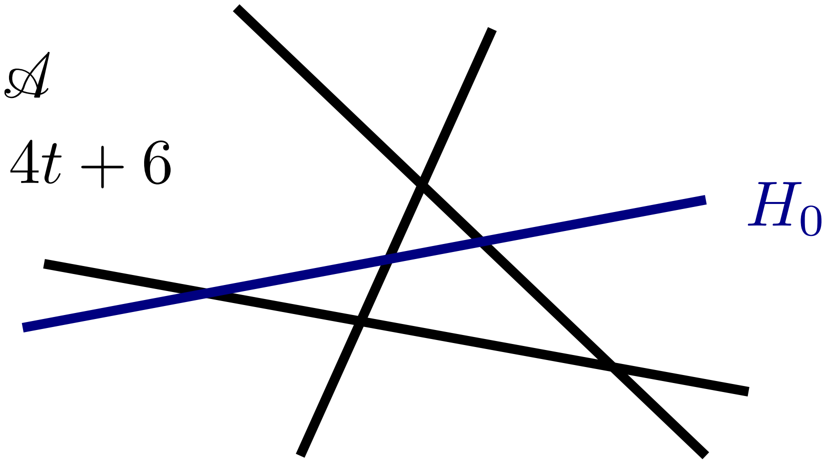


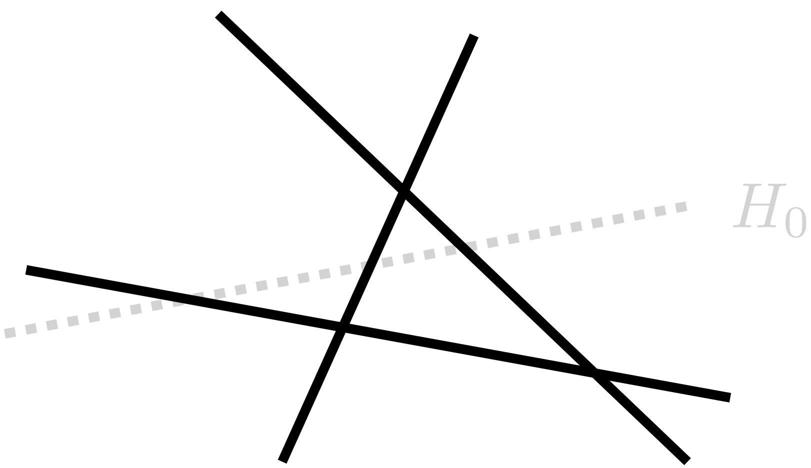
Arrangement $\mathscr{A}/H_0$

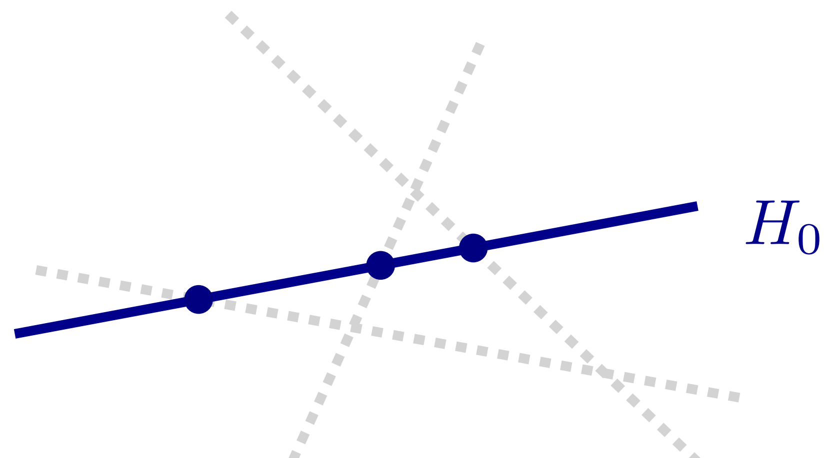
Characteristic polynomial

Arrangement $\mathscr{A}$

$\chi_{\mathscr{A}}(t) = t^2 - 4t + 6$







Arrangement $\mathscr{A} \setminus H_0$

$\chi_{\mathscr{A} \setminus H_0}(t) = t^2 - 3t + 3$

Arrangement $\mathscr{A} / H_0$

$\chi_{\mathscr{A} / H_0}(t) = t - 3$

Characteristic polynomial

Characteristic polynomial

Lemma: "Deletion & Restriction"

$$\chi_{\mathscr{A}}(t) = \chi_{\mathscr{A} \setminus H_0}(t) - \chi_{\mathscr{A} / H_0}(t)$$

Characteristic polynomial

Lemma: "Deletion & Restriction"

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Proof:

Characteristic polynomial

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$$\chi_{\mathscr{A}}(t) = \chi_{\mathscr{A} \setminus H_0}(t) - \chi_{\mathscr{A} / H_0}(t)$$

Proof:

By Whitney's Theorem:

$$\chi_{\mathscr{A}}(t) = \sum_{\mathscr{B} \subseteq \mathscr{A} \text{ central}} (-1)^{\#\mathscr{B}} t^{n - \operatorname{rank}(\mathscr{B})}$$

$$= \sum_{\substack{\mathscr{B} \subseteq \mathscr{A} \\ \mathscr{B} \text{ central} \\ H_0 \notin \mathscr{B}}} (-1)^{\#\mathscr{B}} t^{n - \operatorname{rank}(\mathscr{B})} + \sum_{\substack{\mathscr{B} \subseteq \mathscr{A} \\ \mathscr{B} \text{ central} \\ H_0 \in \mathscr{B}}} (-1)^{\#\mathscr{B}} t^{n - \operatorname{rank}(\mathscr{B})}$$

Characteristic polynomial

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Characteristic polynomial

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$$\sum_{\substack{\mathscr{B} \subseteq \mathscr{A} \\ \mathscr{B} \text{ central} \\ H_0 \in \mathscr{B}}} (-1)^{\#\mathscr{B}} t^{n-\operatorname{rank}(\mathscr{B})}$$

Characteristic polynomial

$$\mathscr{B} \longmapsto \mathscr{B}' := \mathscr{B}/H_0$$
$$\#\mathscr{B}' = \#\mathscr{B} - 1$$
$$\mathrm{rank}(\mathscr{B}') = \mathrm{rank}(\mathscr{B}) - 1$$

$$\sum_{\substack{\mathscr{B} \subseteq \mathscr{A} \\ \mathscr{B} \text{ central} \\ H_0 \in \mathscr{B}}} (-1)^{\#\mathscr{B}} t^{n-\mathrm{rank}(\mathscr{B})}$$

Characteristic polynomial

$$\begin{aligned}\mathscr{B} \longmapsto\ & \mathscr{B}' := \mathscr{B}/H_0 \\ & \#\mathscr{B}' = \#\mathscr{B} - 1 \\ & \mathrm{rank}(\mathscr{B}') = \mathrm{rank}(\mathscr{B}) - 1\end{aligned}$$

$$\sum_{\substack{\mathscr{B} \subseteq \mathscr{A} \\ \mathscr{B} \text{ central} \\ H_0 \in \mathscr{B}}} (-1)^{\#\mathscr{B}} t^{n-\mathrm{rank}(\mathscr{B})} = \sum_{\substack{\mathscr{B}' \subseteq (\mathscr{A}/H_0) \\ \mathscr{B}' \text{ central}}} (-1)^{\#\mathscr{B}'+1}\, t^{n-1-\mathrm{rank}(\mathscr{B}')}$$

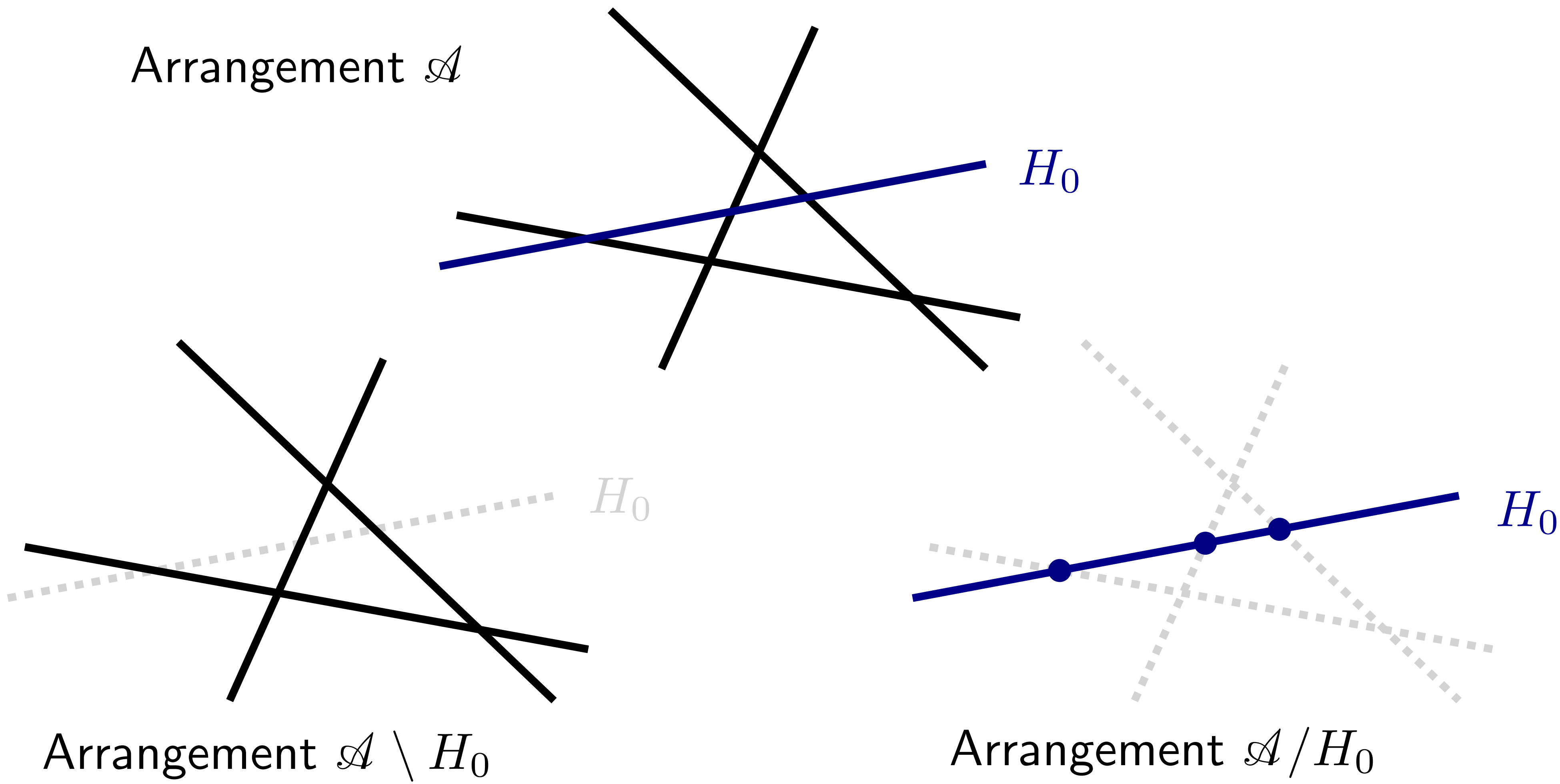
Characteristic polynomial

$$\mathscr{B} \longmapsto \mathscr{B}' := \mathscr{B}/H_0$$
$$\#\mathscr{B}' = \#\mathscr{B} - 1$$
$$\operatorname{rank}(\mathscr{B}') = \operatorname{rank}(\mathscr{B}) - 1$$

$$\sum_{\substack{\mathscr{B} \subseteq \mathscr{A} \\ \mathscr{B} \text{ central} \\ H_0 \in \mathscr{B}}} (-1)^{\#\mathscr{B}} t^{n-\operatorname{rank}(\mathscr{B})} = \underbrace{\sum_{\substack{\mathscr{B}' \subseteq (\mathscr{A}/H_0) \\ \mathscr{B}' \text{ central}}} (-1)^{\#\mathscr{B}'+1} \, t^{n-1-\operatorname{rank}(\mathscr{B}')}}_{= -\chi_{\mathscr{A}/H_0}(t)}$$

$\square$

Number of chambers $\#\mathscr{R}(\mathscr{A})$



Oberservation: $\#\mathscr{R}(\mathscr{A}) = \#\mathscr{R}(\mathscr{A} \setminus H_0) + \#\mathscr{R}(\mathscr{A} / H_0)$

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Proof: By induction using:

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- Let $G = (V, E)$ graph, χ_G *chromatic polynomial*
  - For any edge $e \in E$, χ_G also satisfies recurrence

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  - By induction one also shows: $\chi_{\mathscr{A}_G} = \chi_G$

Finite field method

- Assume: All hyperplanes of $\mathscr{A}$ have integral coefficients
- Let $q = p^r$ prime power
- Take coefficients modulo p
  $\Longrightarrow$ Yields arrangement $\mathscr{A}_q$ of hyperplanes in $\mathbb{F}_q^n$

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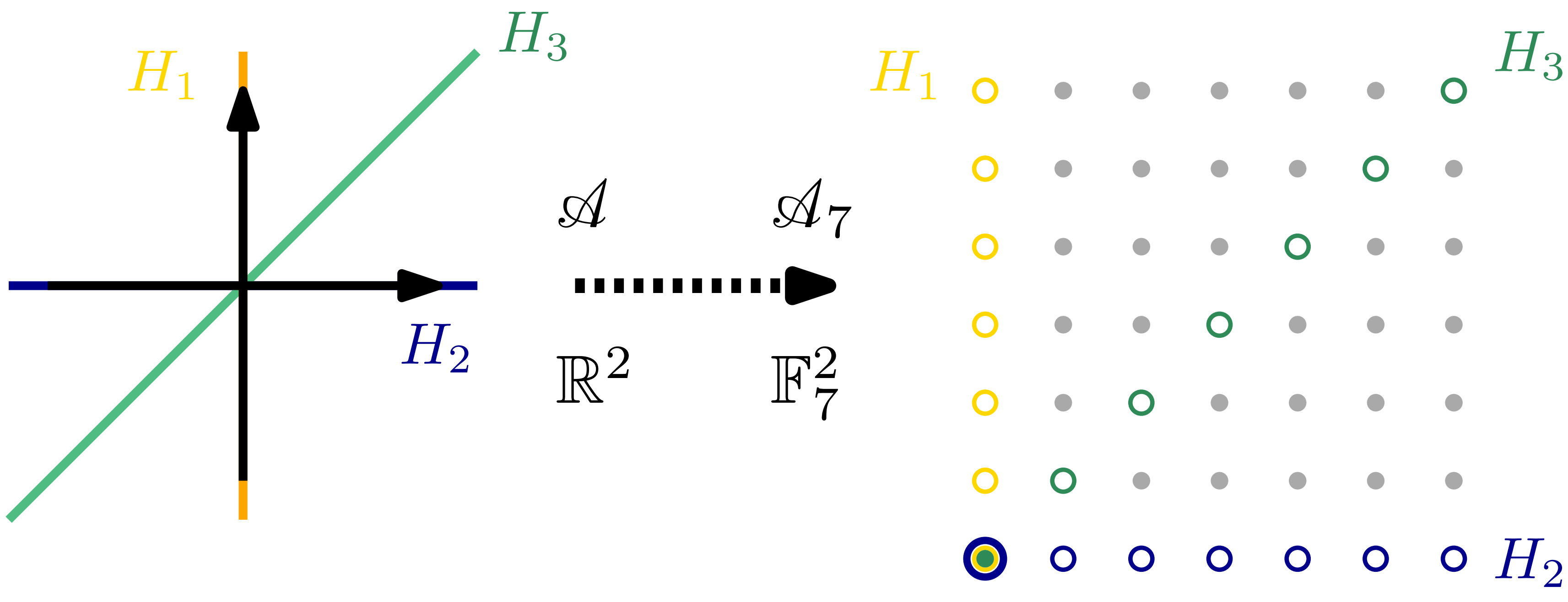
Example: $H_1 : x_1 = 0\ ;\quad H_2 : x_2 = 0\ ;\quad H_3 : x_1 = x_2$



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Theorem (Althanasiadis, 1996): "Finite field method"

Let $\mathscr{A}$ be an arrangement with integral hyperplanes and let $q = p^r$ be some prime power s.t. $L(\mathscr{A}) \cong L(\mathscr{A}_q)$.

Then:

$$\chi_{\mathscr{A}}(q) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathscr{A}} H\right) = q^n - \#\left(\bigcup_{H \in \mathscr{A}} H\right)$$

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Proof:

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Define: $f, g : L(\mathscr{A}_q) \to \mathbb{Z}$

$$f(x) := \#x = q^{\dim_{\mathbb{F}_q}(x)}$$

$$g(x) := \#\left(x - \bigcup_{y>x} y\right)$$

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Möbius inversion

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Möbius inversion

Then: $$g(\hat{0}) = g(\mathbb{F}_q^n) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathscr{A}_q} H\right)$$

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Observe: $$f(x) = \sum_{y \geq x} g(y) \quad \overset{\text{Möbius inversion}}{\Longrightarrow} \quad g(x) = \sum_{y \geq x} \mu(x,y) f(y)$$

Then: $$g(\hat{0}) = g(\mathbb{F}_q^n) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathscr{A}_q} H \right)$$ bec. $L(\mathscr{A}) \cong L(\mathscr{A}_q)$

$$g(\hat{0}) = \sum_{y \in L(\mathscr{A}_q)} \mu(\hat{0}, y) f(y) = \sum_{y \in L(\mathscr{A}_q)} \mu(\hat{0}, y) q^{\dim_{\mathbb{F}_q}(y)} = \chi_{\mathscr{A}}(q)$$

□

Questions?