Mittagsseminar 05.05.2023

Block coupling on k-heights

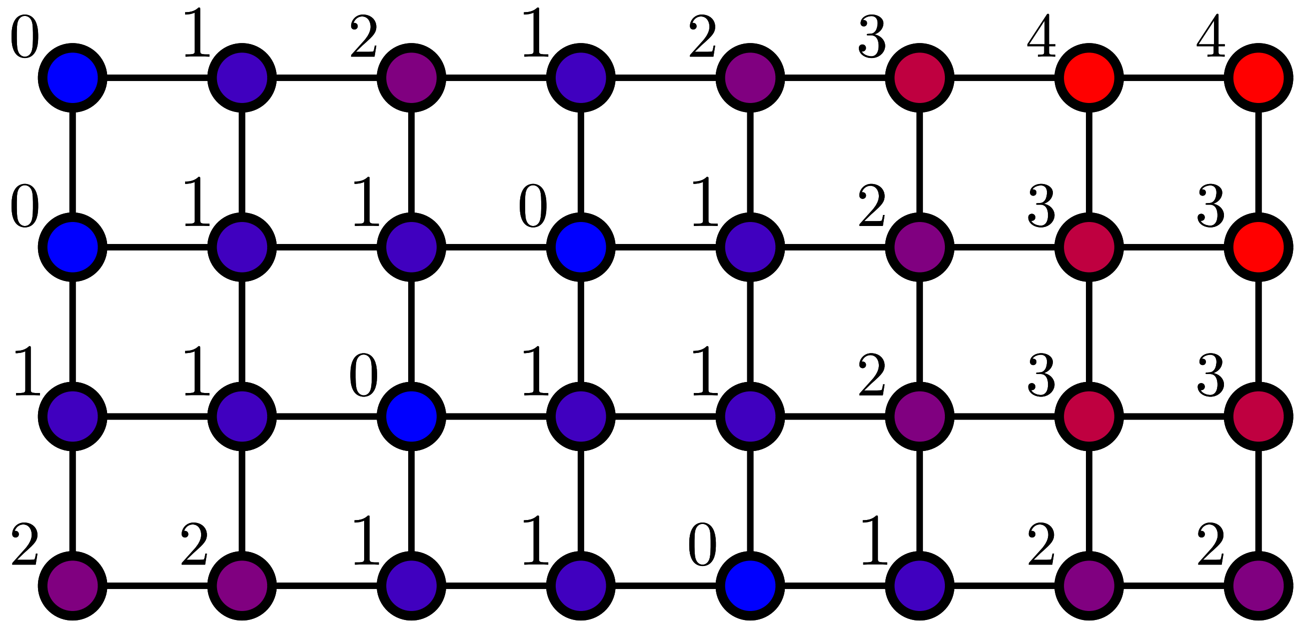
Largely based on the thesis by Daniel Heldt

Talk by Sandro Roch

k-heights

- Undirected graph $G = (V, E)$
- **k-height** of G: Assignment $f : V \to \{0, \cdots, k\}$ s.t.

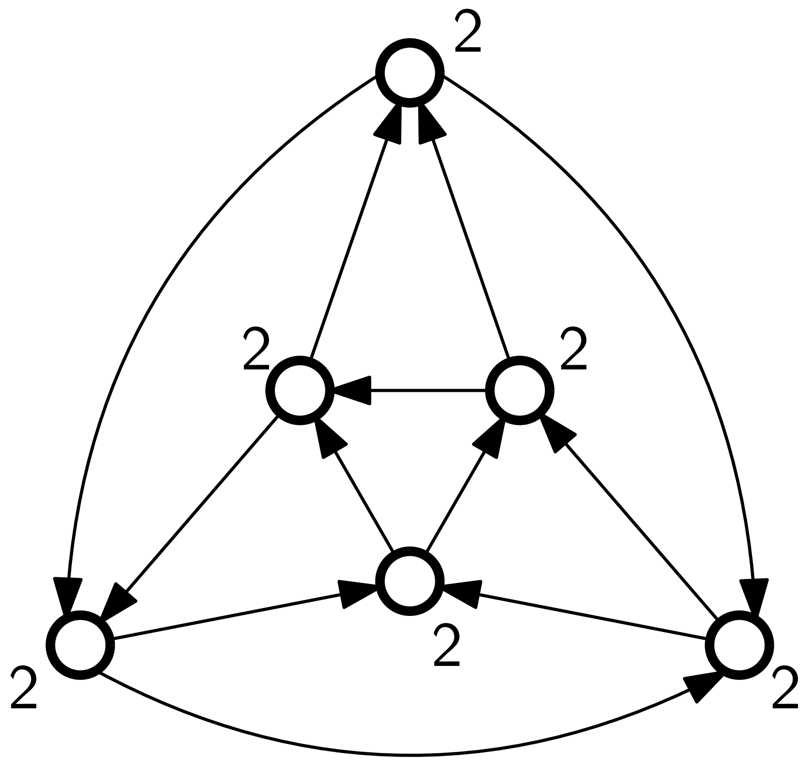
$$|f(v) - f(w)| \leq 1 \qquad \text{f.a. } (v, w) \in E$$



- Set Ω_G of k-heights is distributive lattice.
- Distance on pairs $X \leq Y$: $d(X, Y) := \sum_v (Y(v) - X(v))$

Motivation: α-orientations

- α-*orientation* of plane graph $G = (V, E)$:
  orientation of E with prescribed outdegrees $\alpha : V \to \mathbb{N}$



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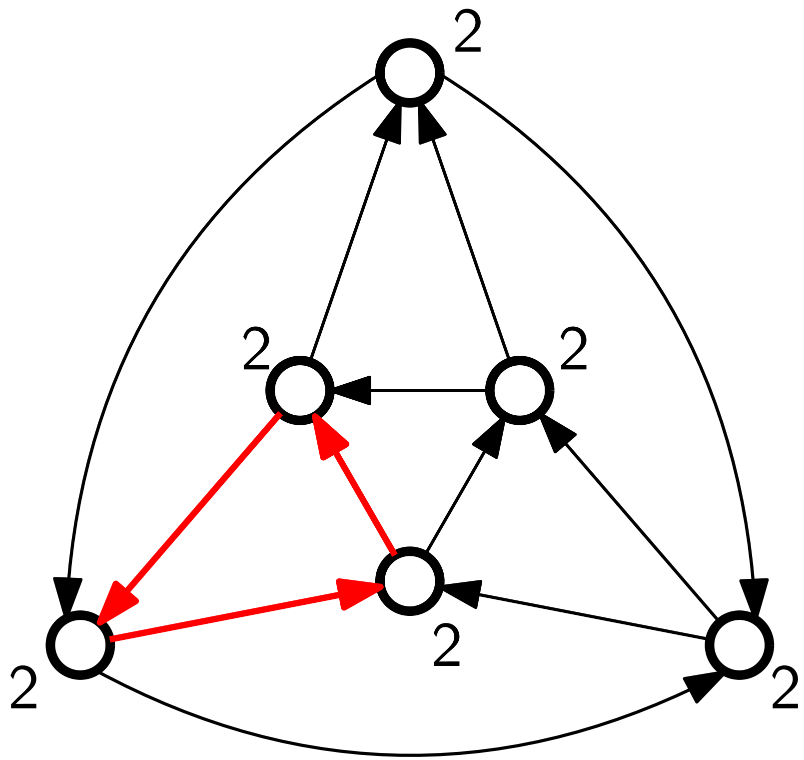

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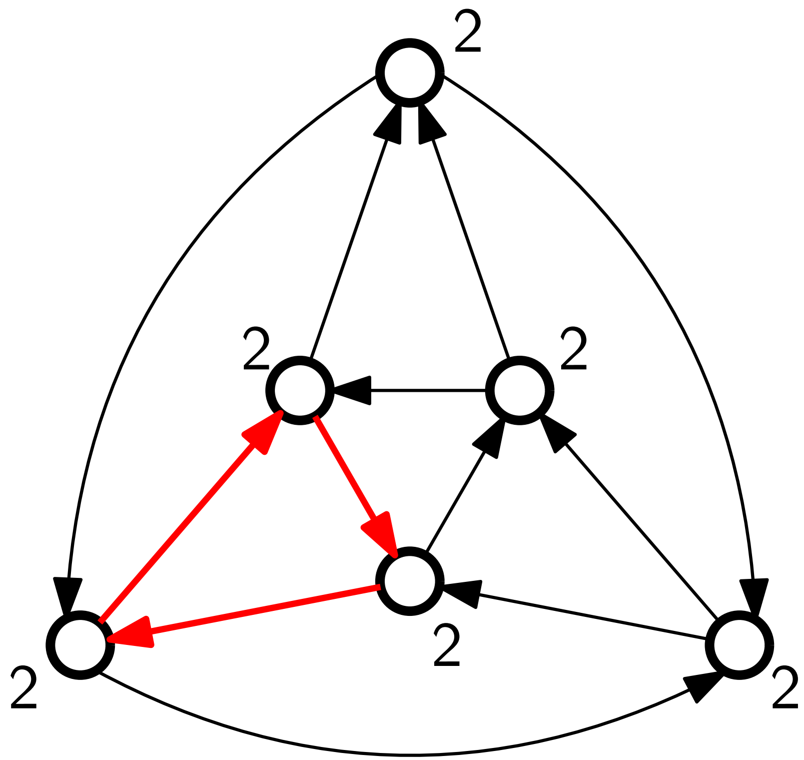

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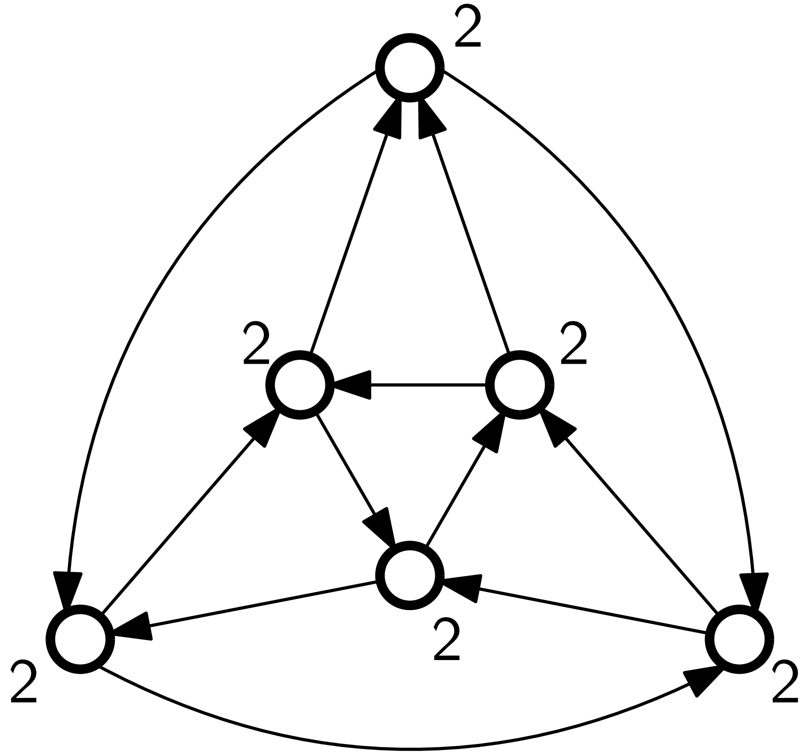

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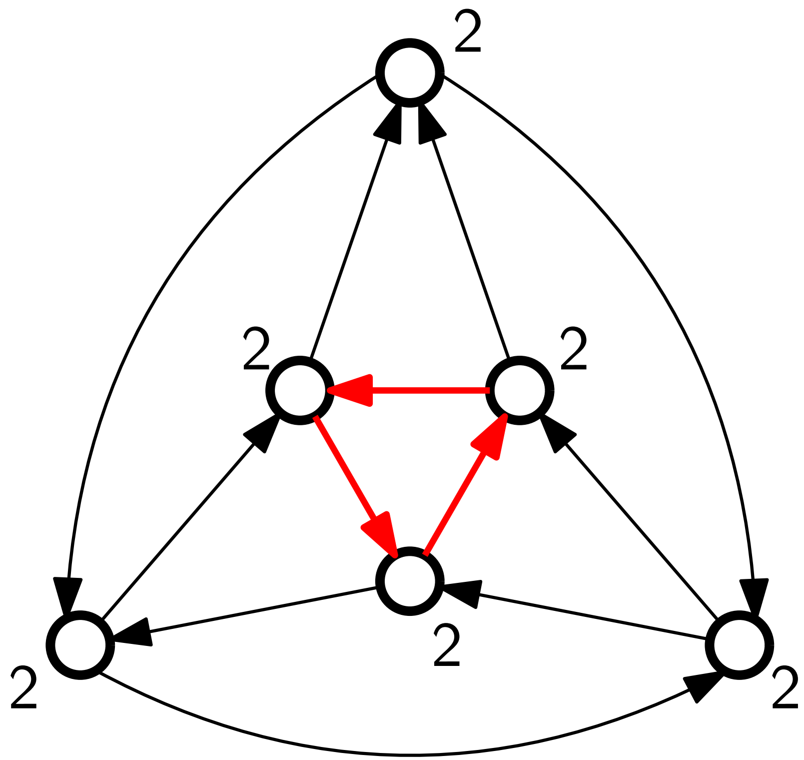

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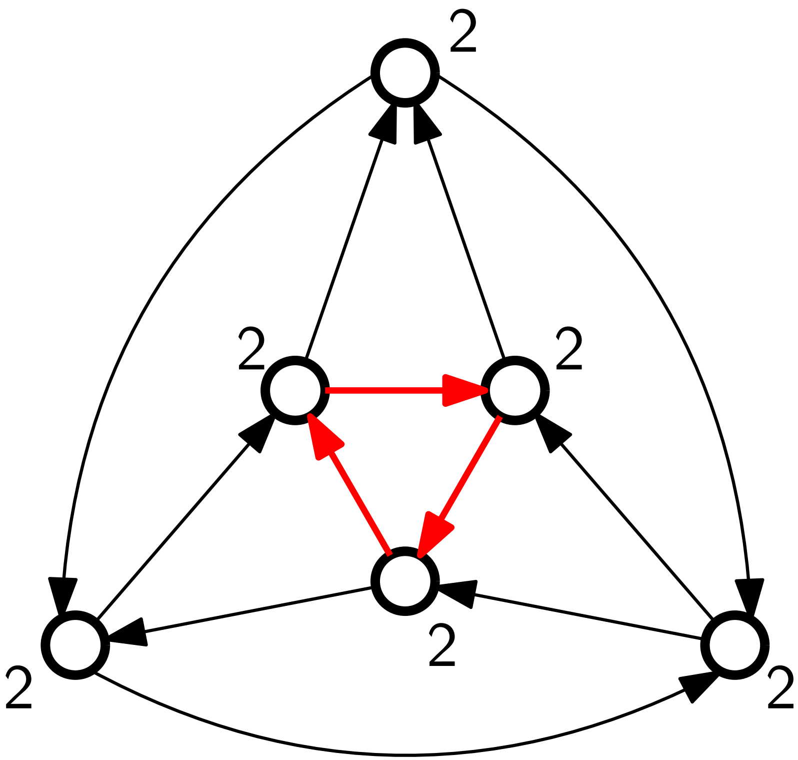

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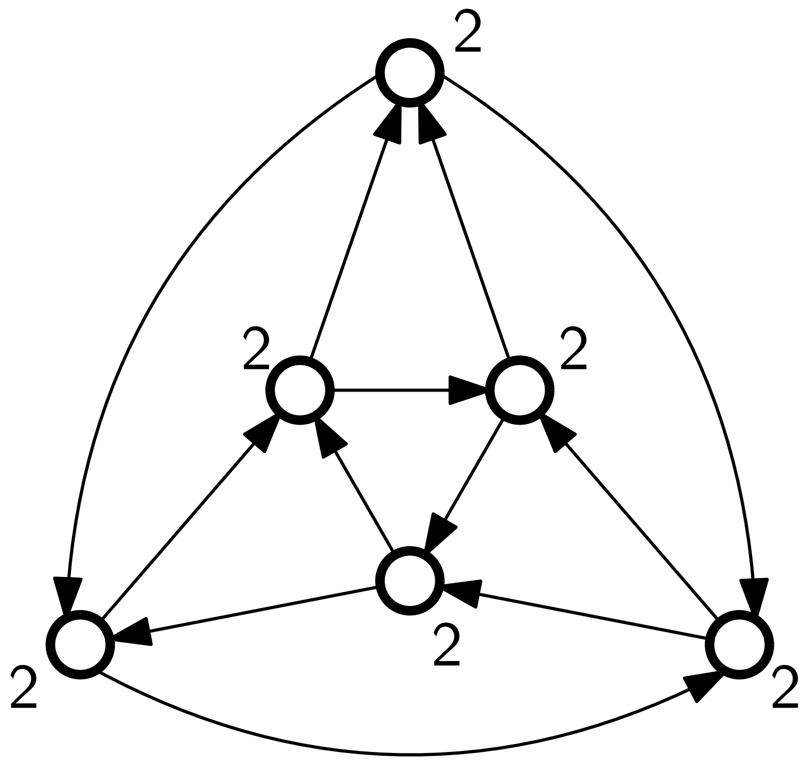

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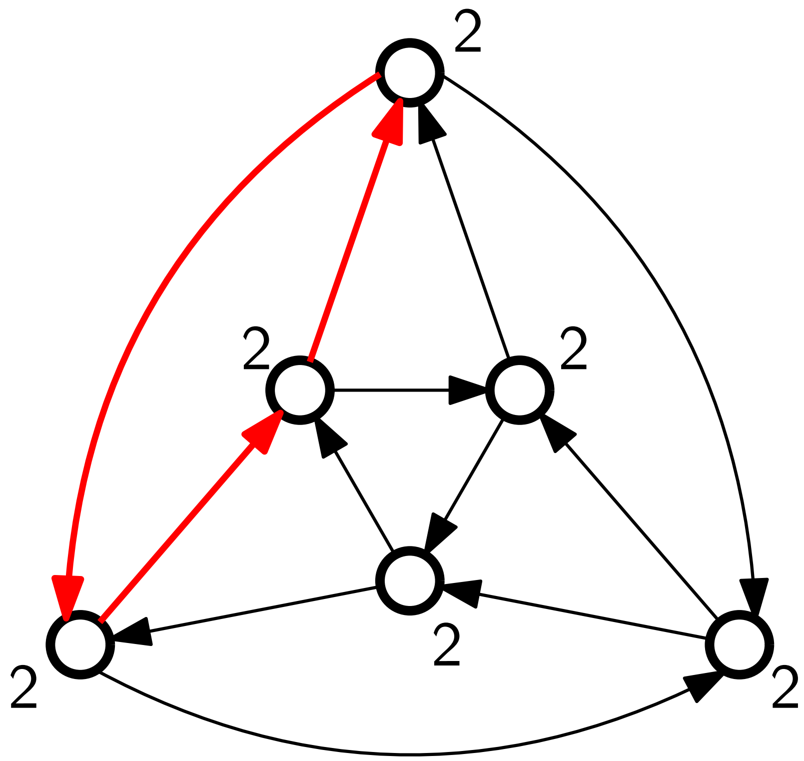

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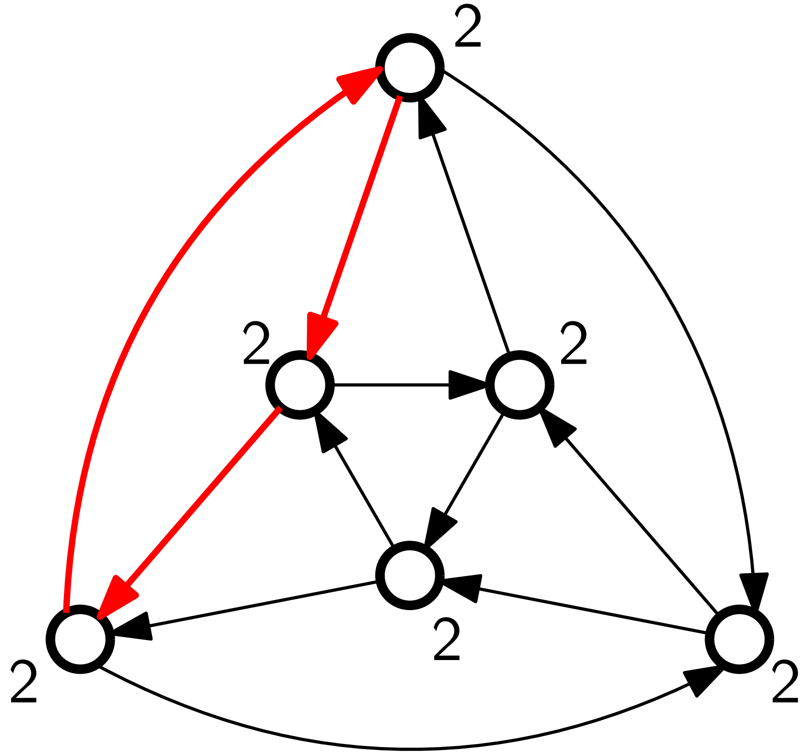

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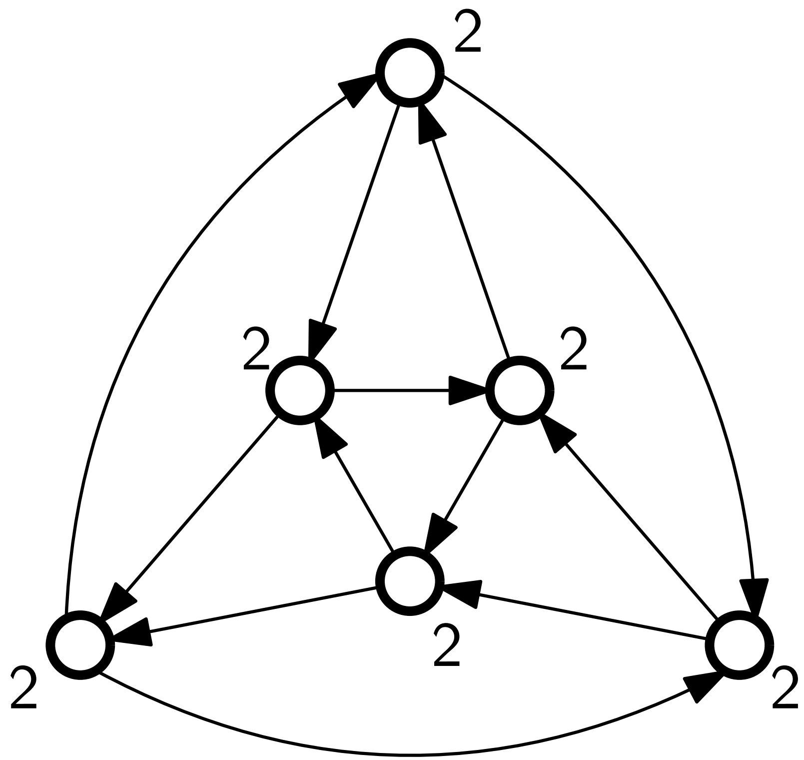

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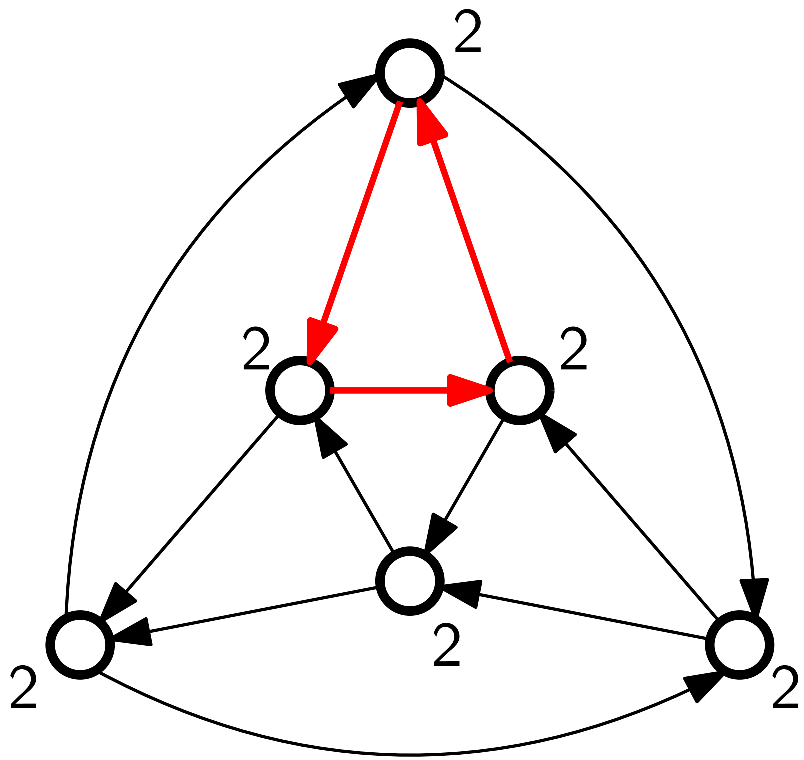

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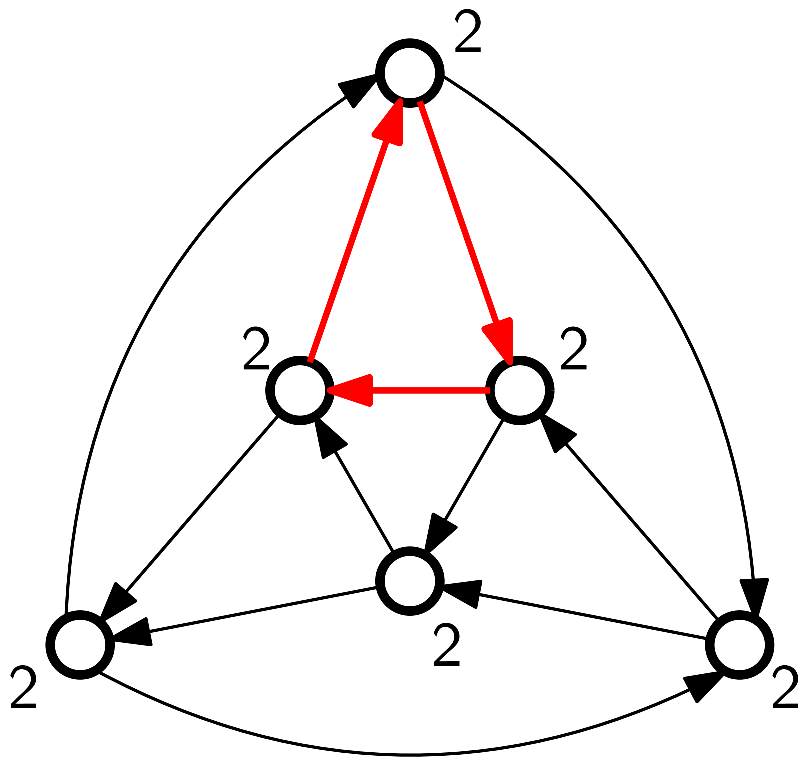

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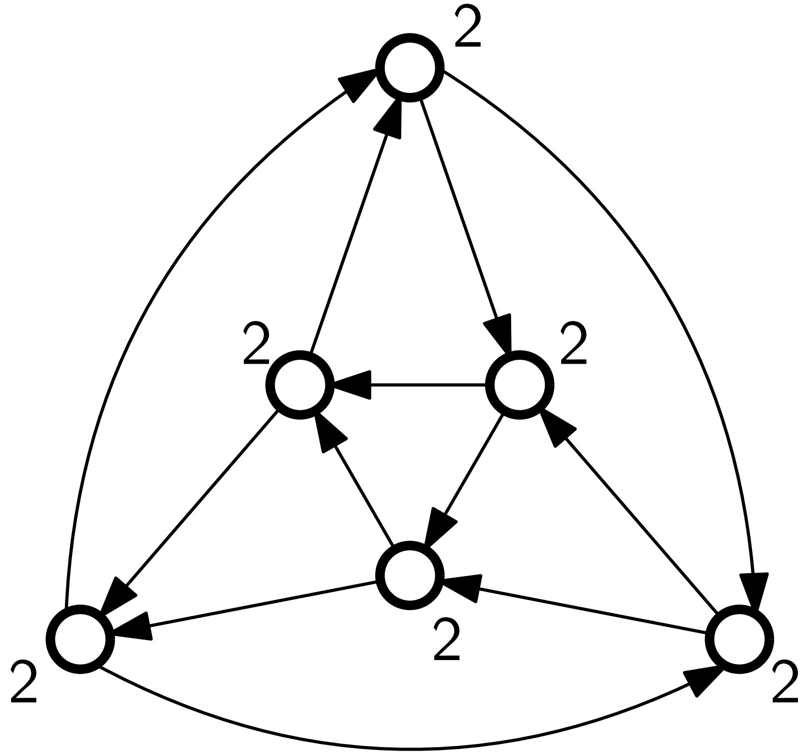

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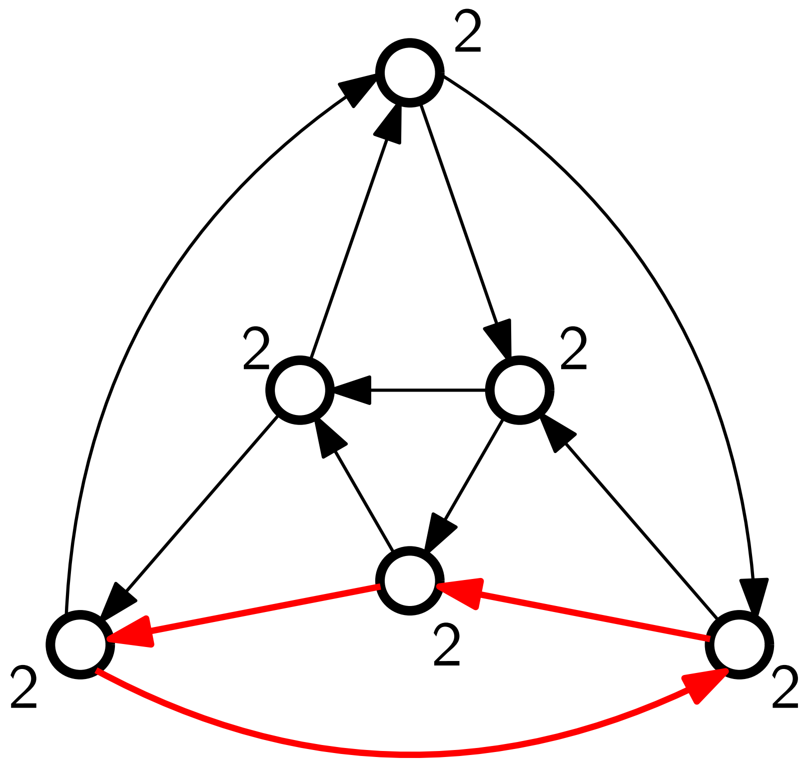

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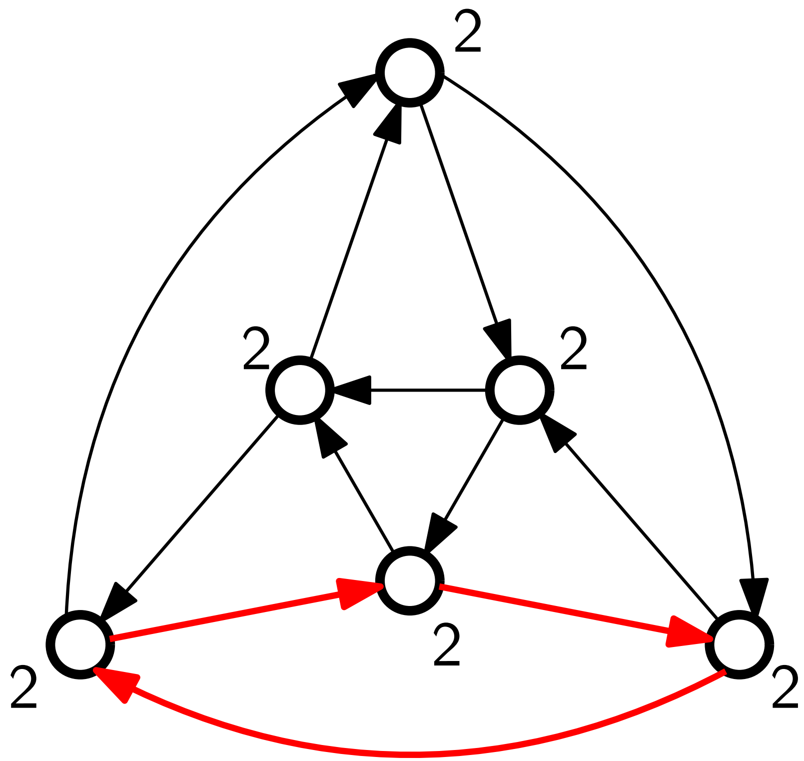

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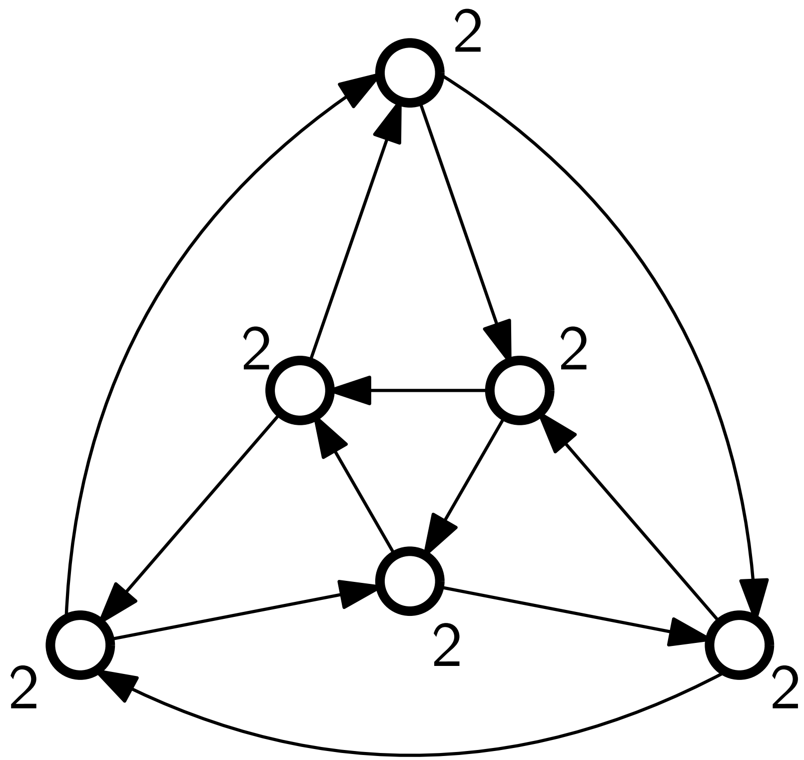

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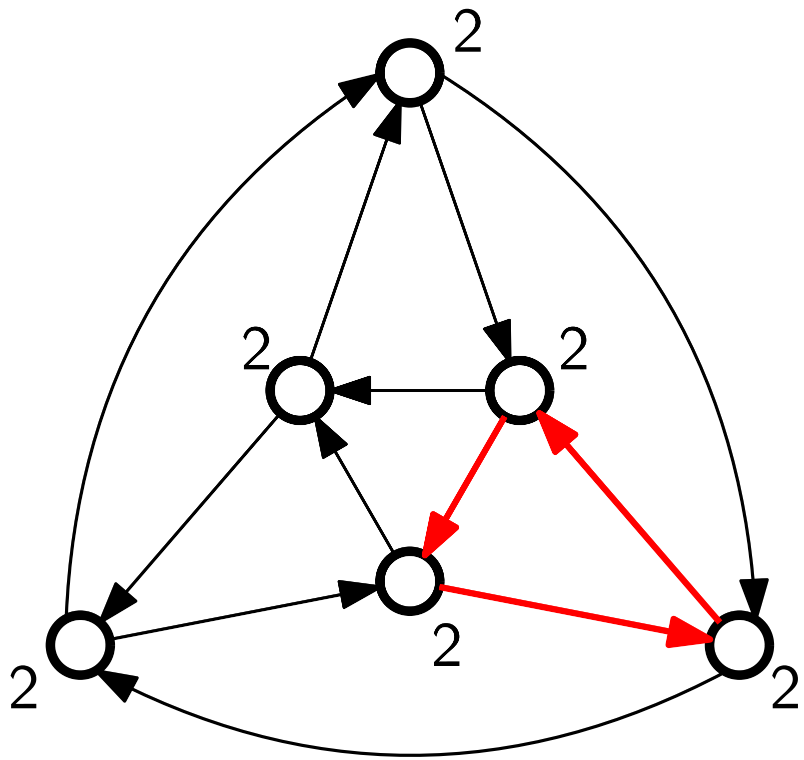

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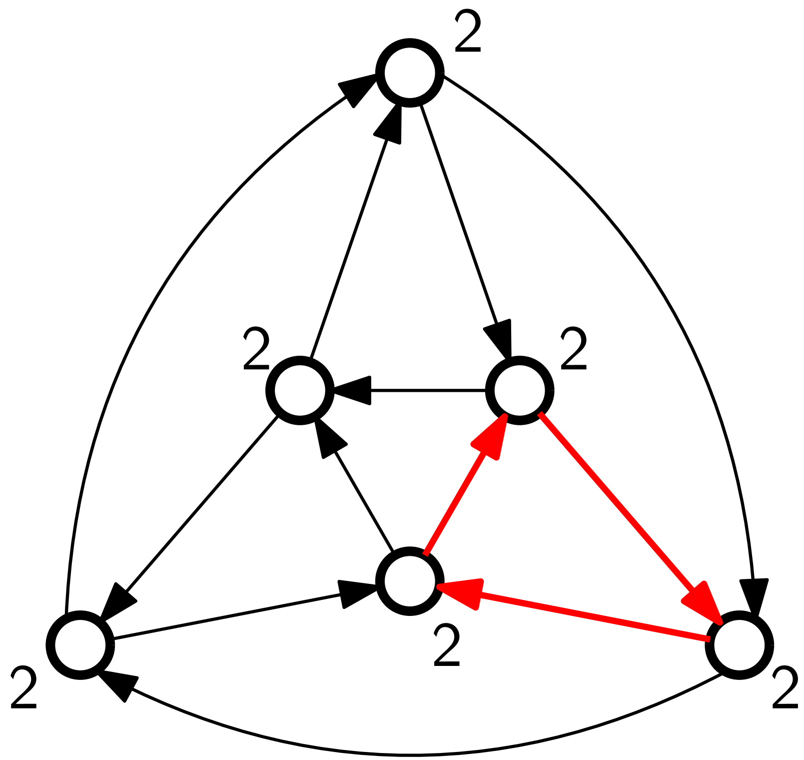

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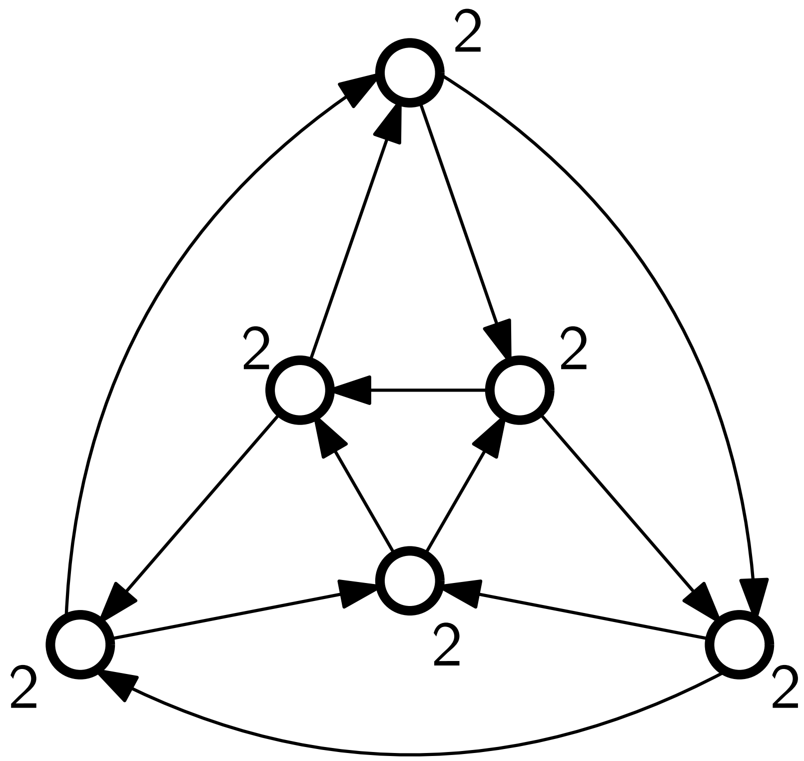

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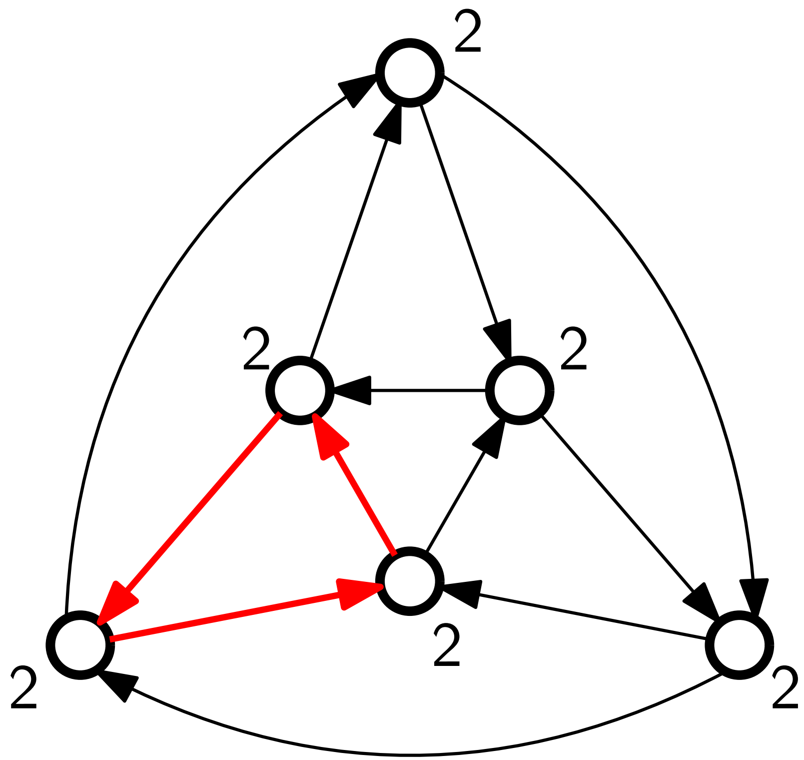

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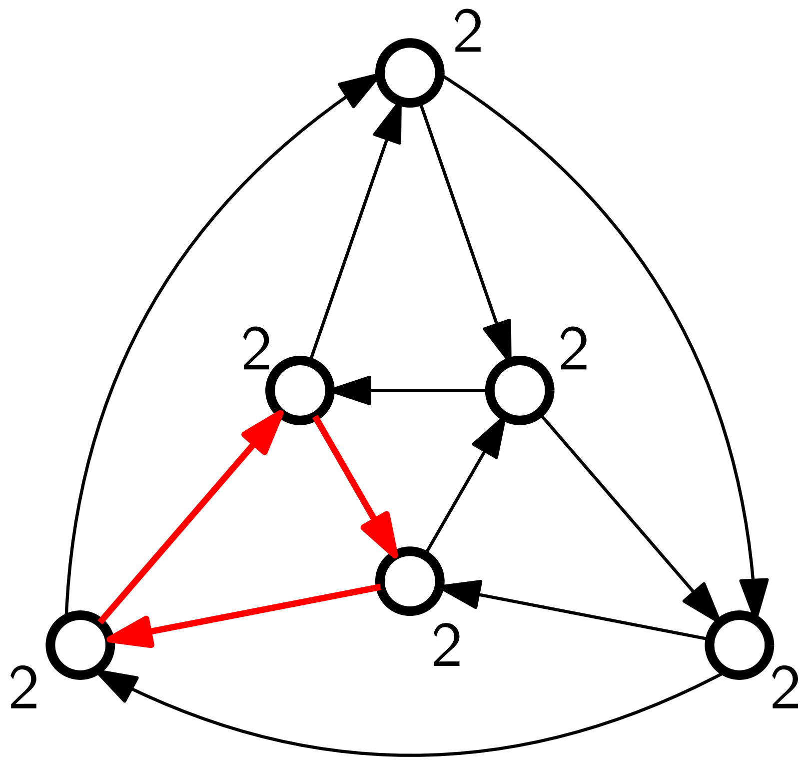

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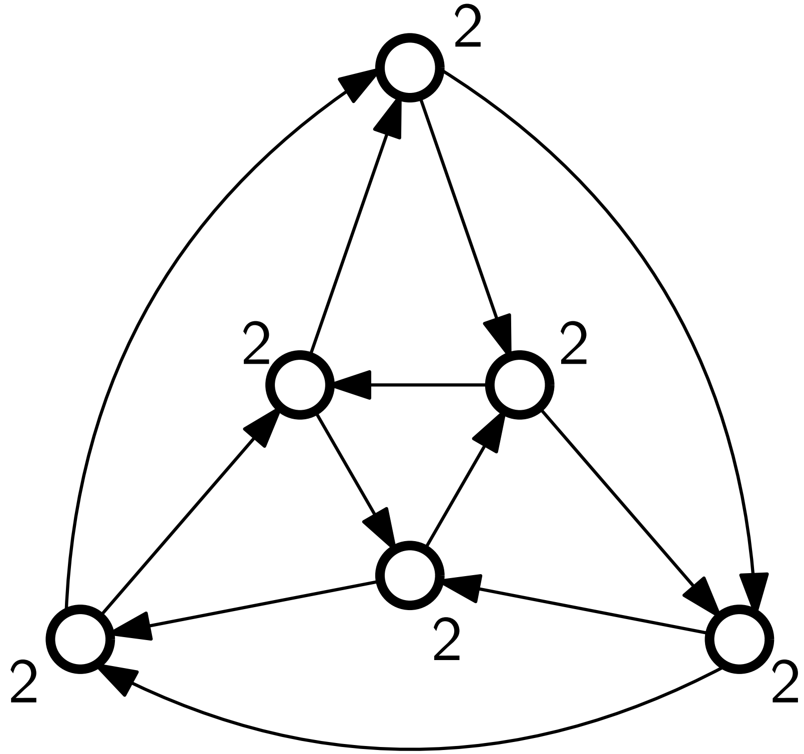

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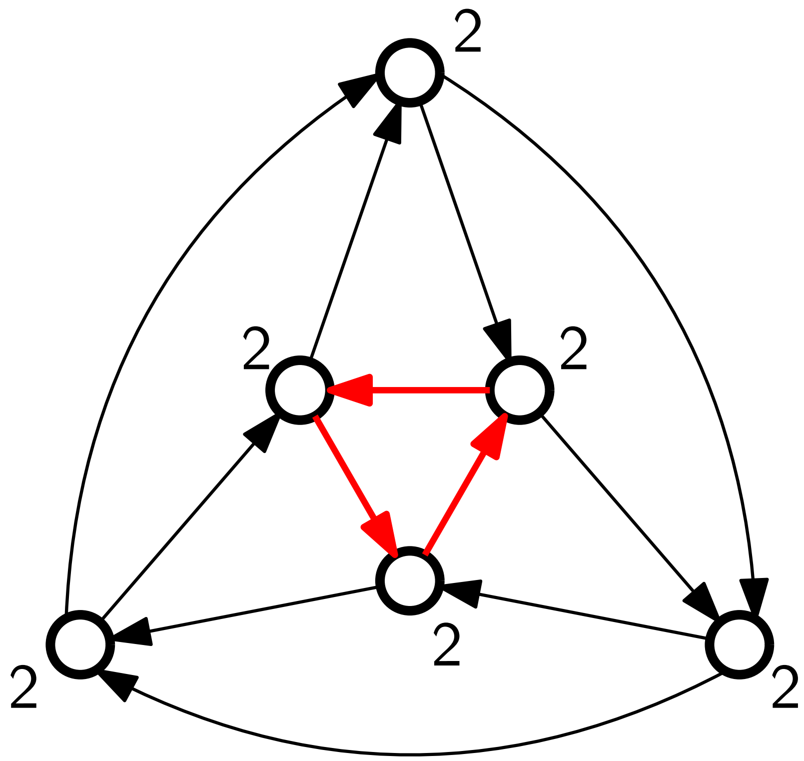

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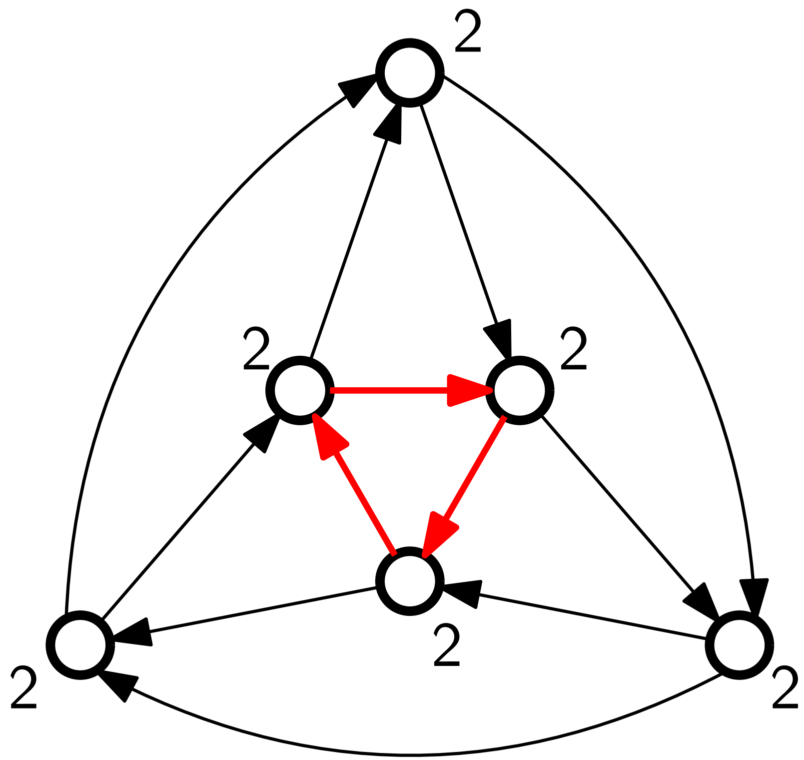

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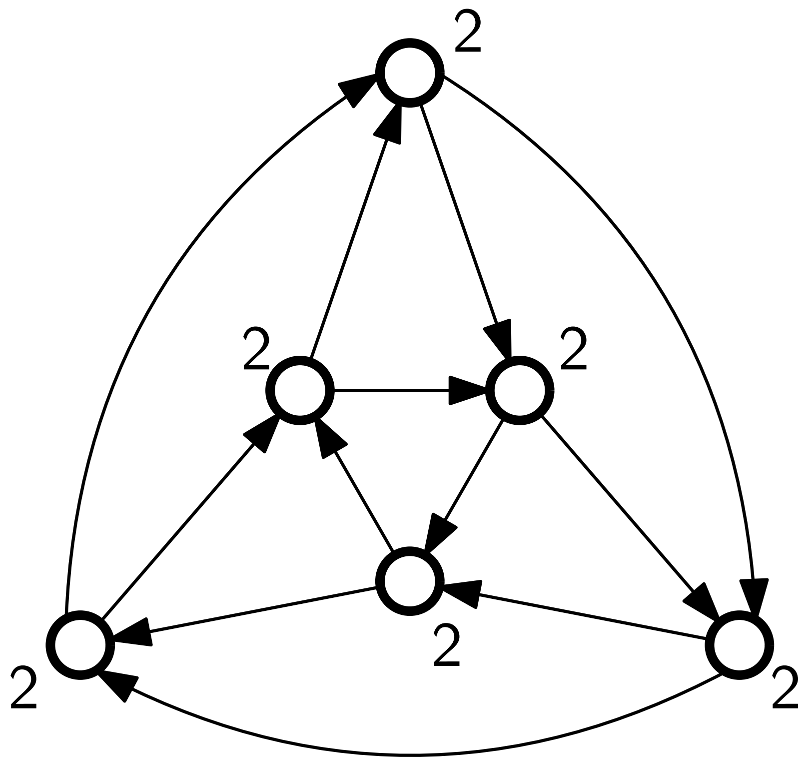

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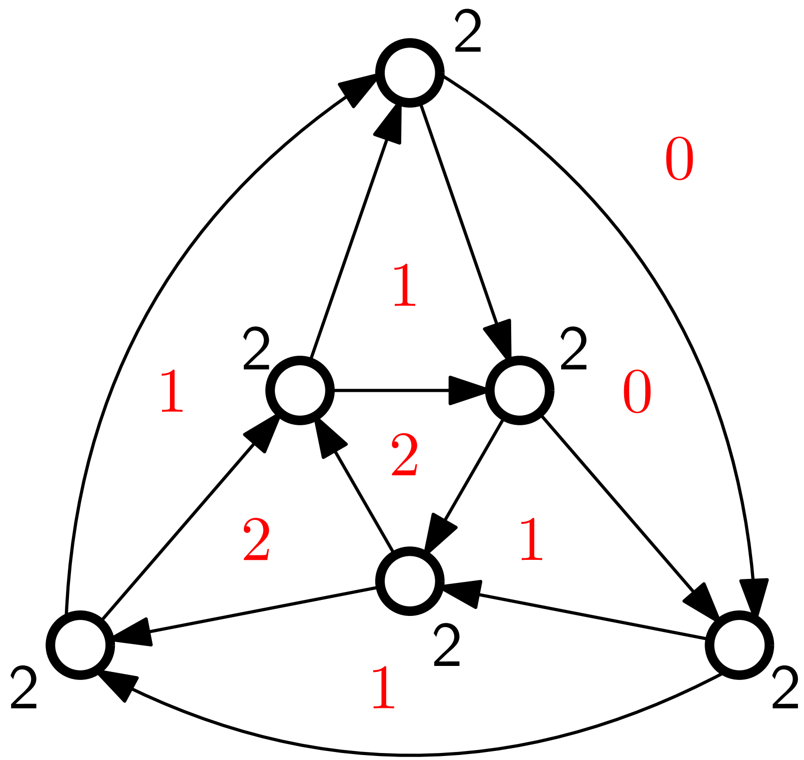

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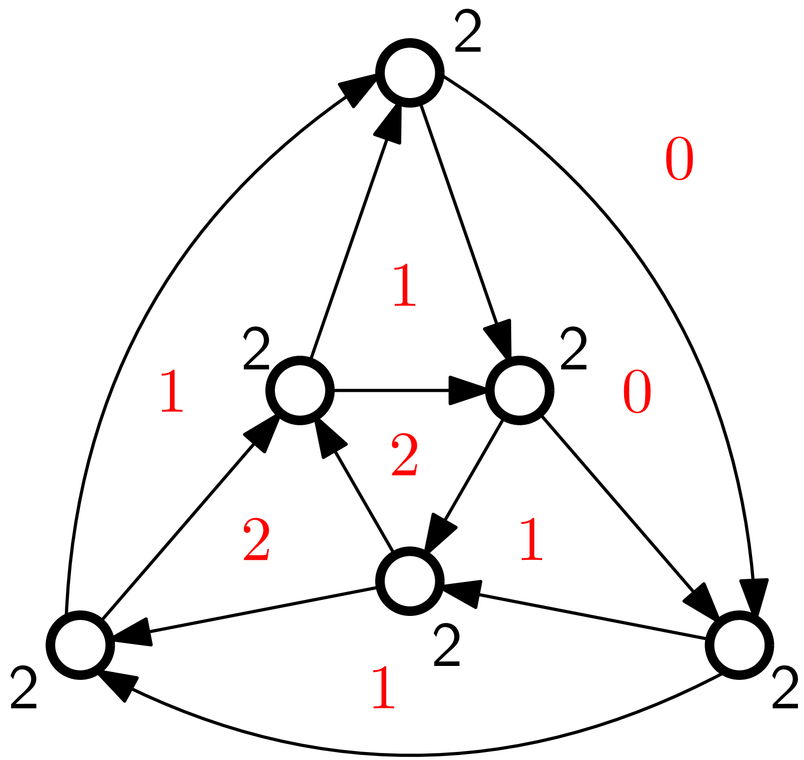


k-height on dual graph!

Markov chain on k-heights

Chain $\mathcal{M}$ on k-heights:

Transition $X_t \to X_{t+1}$:

- with probability $\frac{1}{2}$: $X_{t+1} \leftarrow X_t$
- otherwise:
  - choose $\tilde{v} \in V$ uniformly at random
  - choose $\triangle \in \{-1, +1\}$ uniformly at random
  - define

$$f(v) := \begin{cases} X_t(v) + \triangle & \text{if } v = \tilde{v} \\ X_t(v) & \text{sonst} \end{cases}$$

  - if $f \in \Omega_G$: $X_{t+1} \leftarrow f$
  - otherwise: $X_{t+1} \leftarrow X_t$

Markov chain on k-heights

- **Problem:** Bound mixing time

$$\tau(\varepsilon) := \min\{t > 0 : \|X_t - U_\Omega\|_{TV} < \varepsilon\}$$

- Typical approach: Coupling $(X_t, Y_t) \in \Omega \times \Omega$
  - Two copies starting with $X_0 \equiv 0, Y_0 \equiv k$
  - In transitions $X_t \to X_{t+1}$ and $Y_t \to Y_{t+1}$ use the same $(\tilde{v}, \triangle)$ chosen at random.
  - Invariant: $X_t \le Y_t$ for all t (*monotone coupling*)

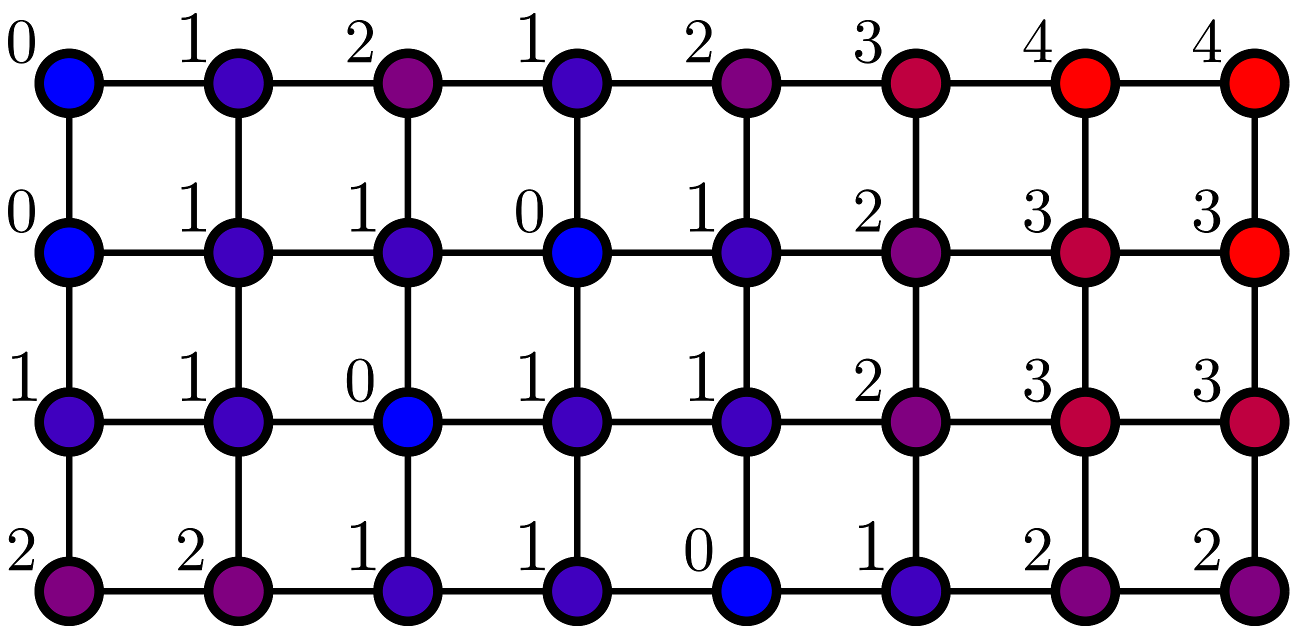
- **Theorem** (Dyer & Greenhill): If there is $\beta < 1$ s.t.

$$\mathbb{E}[d(X_{t+1}, Y_{t+1})] \le \beta \cdot d(X_t, Y_t) ,$$

then $\tau(\varepsilon) \le \frac{\log(d_{\mathsf{max}} \cdot \frac{1}{\varepsilon})}{1-\beta}$.

Block Markov chain

- **Boosted chain $\mathscr{M}_{\mathscr{B}}$ on k-heights:**
  - Assign vertices to blocks $\mathfrak{B} \subset \mathscr{P}(V)$



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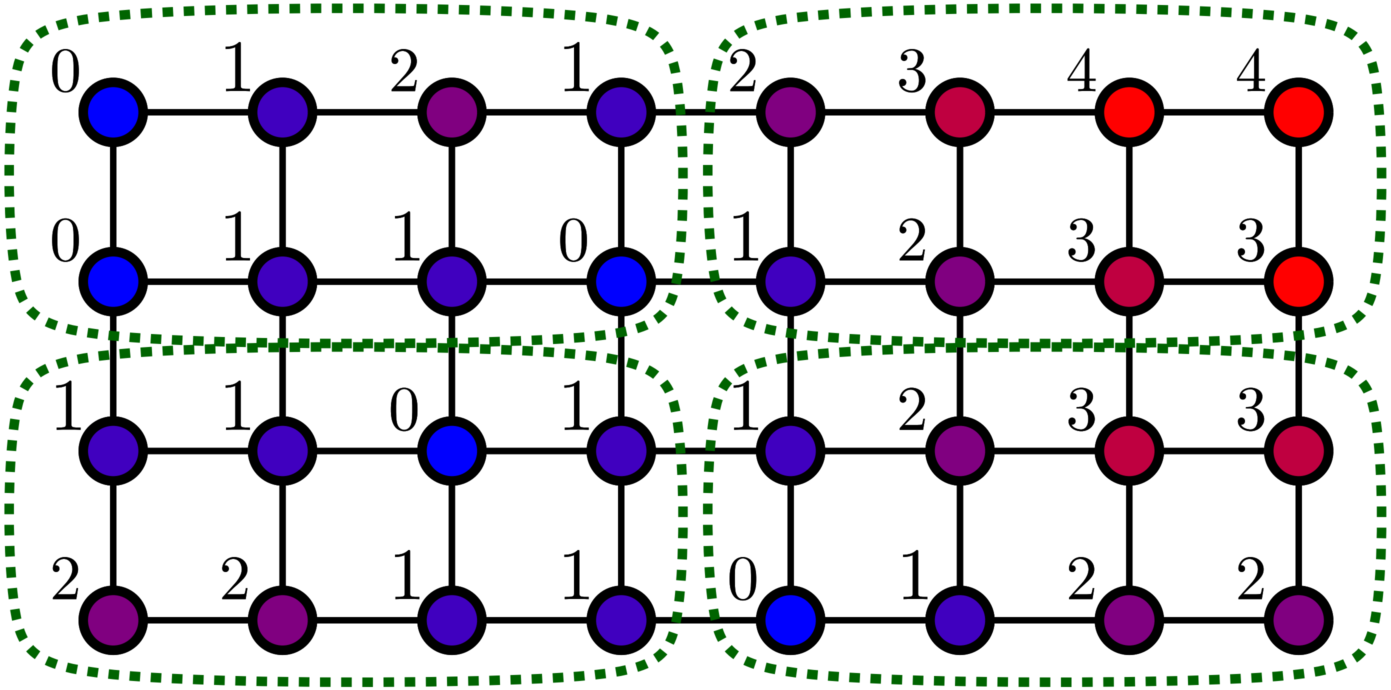

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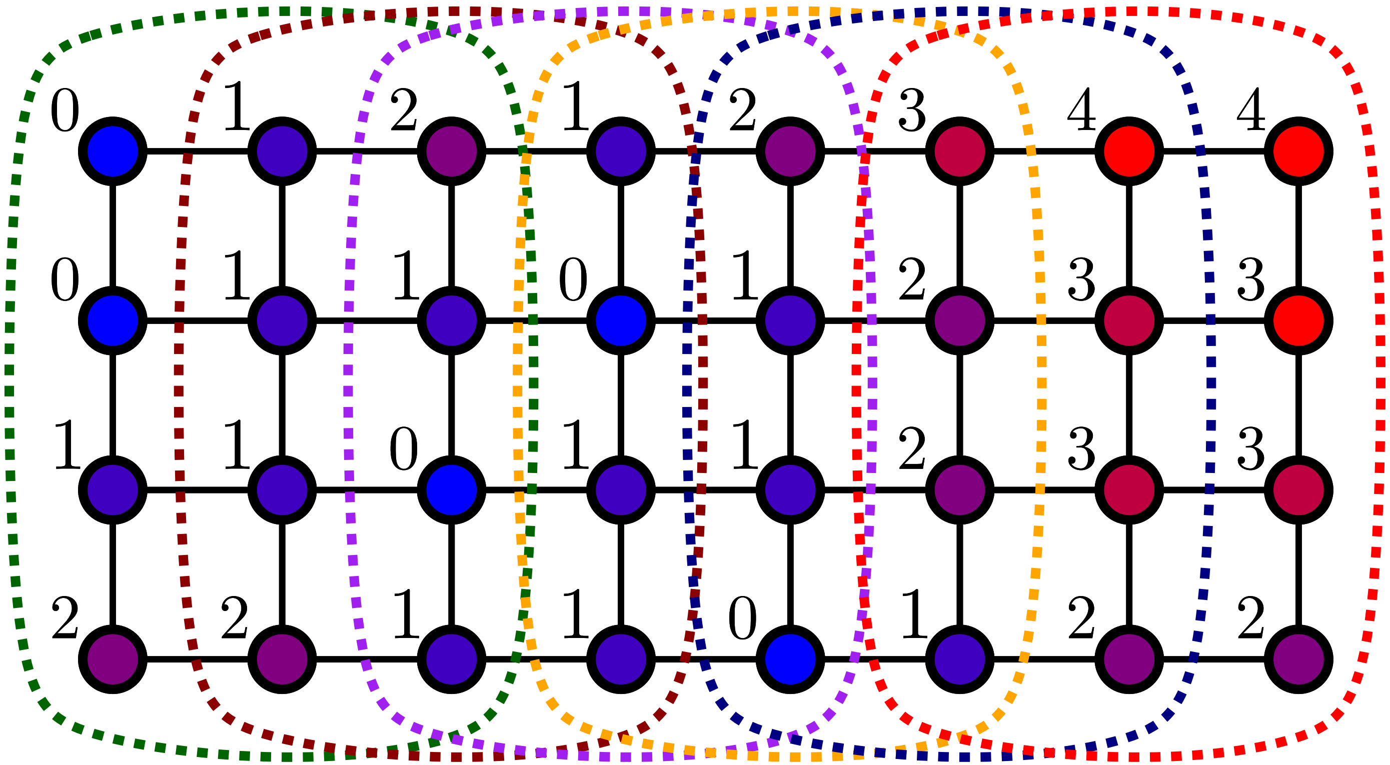
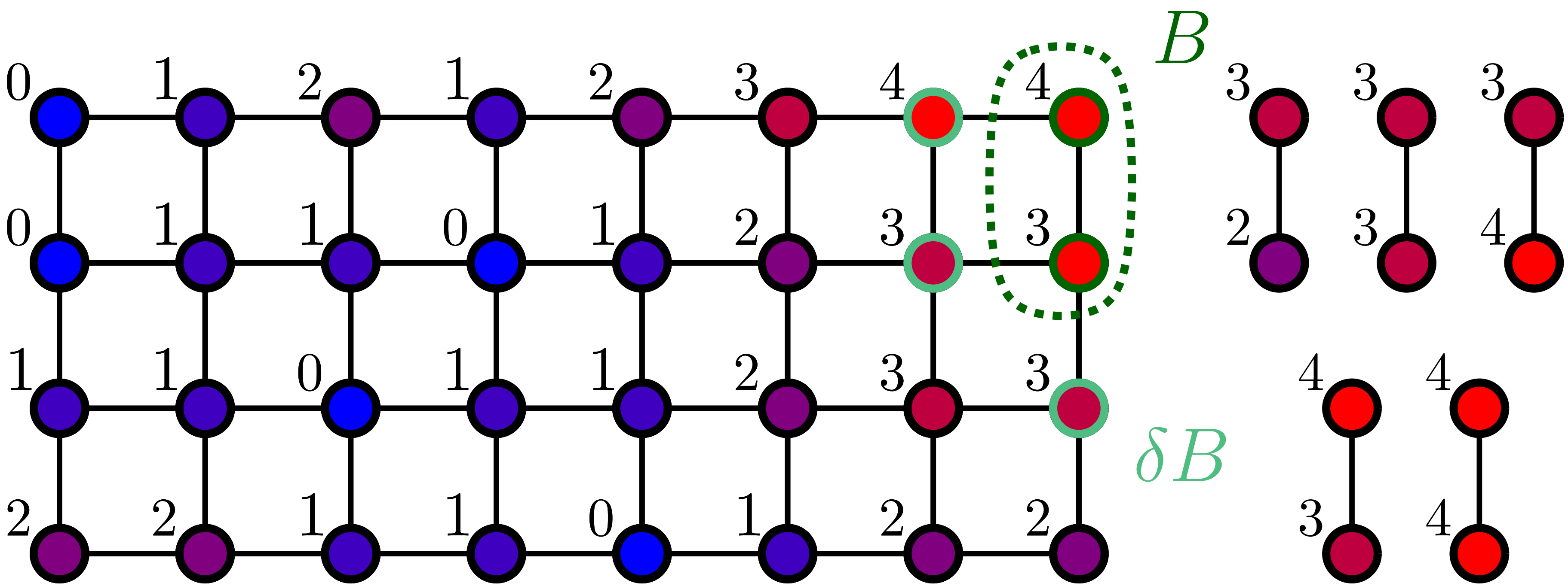


Block Markov chain

- **Boosted chain $\mathscr{M}_{\mathfrak{B}}$ on k-heights:**
  - Assign vertices to blocks $\mathfrak{B}$
  - In each transition, choose block $B \in \mathfrak{B}$ at random and sample among admissible k-heights on B



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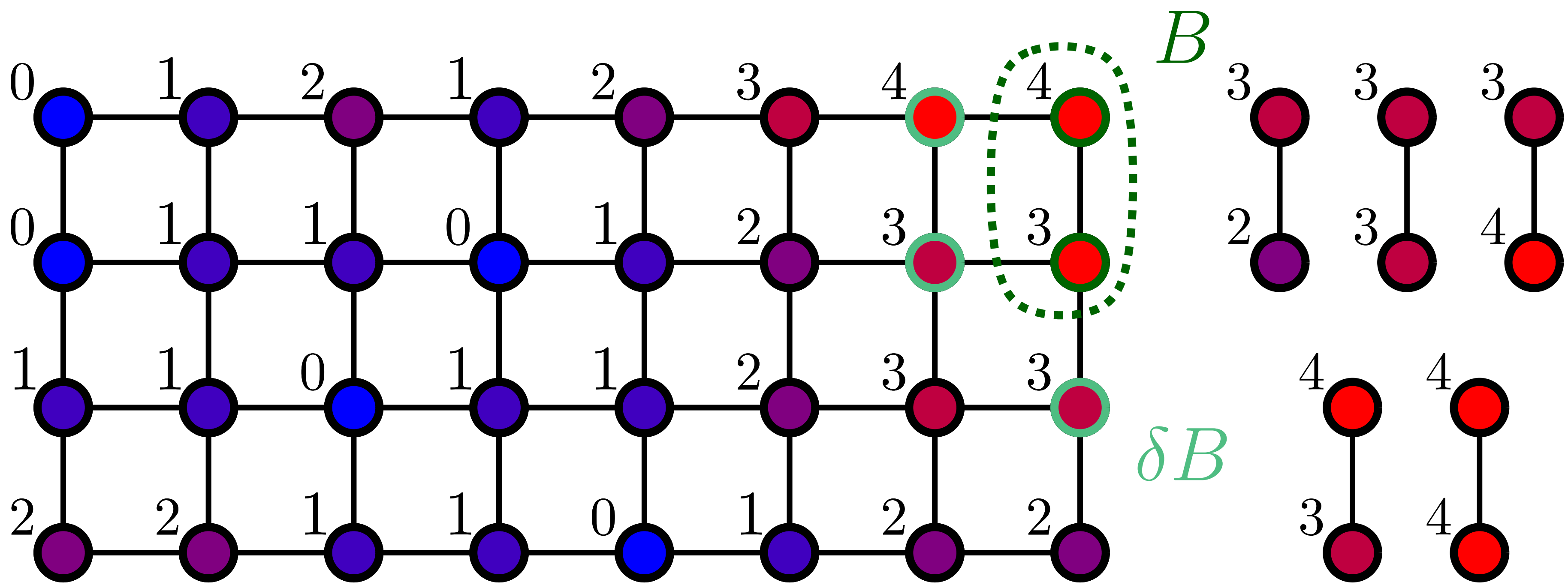
- **Problem:** Monotone coupling $X_t \leq Y_t$???

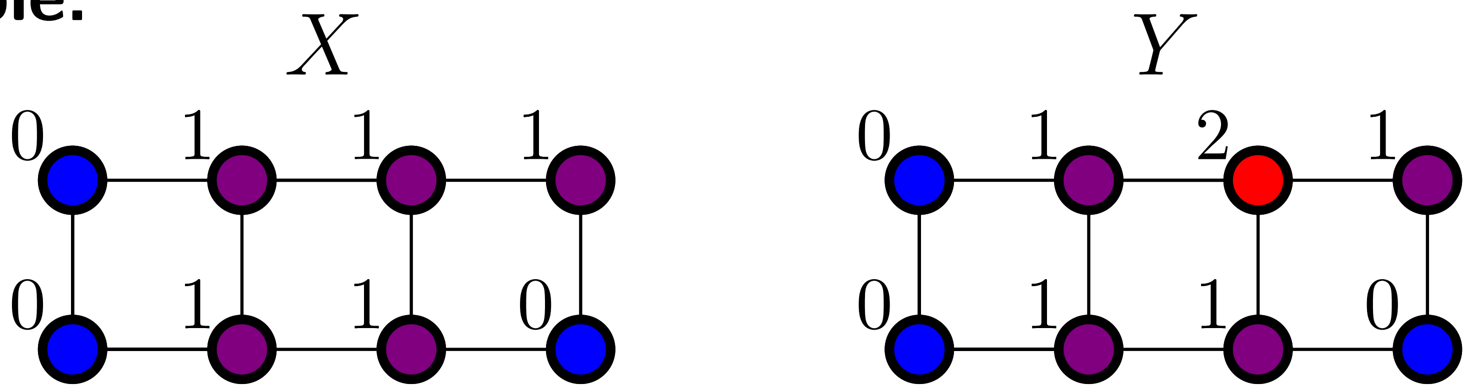


Coupling on cover relations

- **Def:** Tuple $(X, Y) \in \Omega \times \Omega$, $X \leq Y$ is *cover relation*, if

$$Y(v) = \begin{cases} X(v) + 1 & v = \tilde{v} \\ X(v) & v \neq \tilde{v} \end{cases}$$

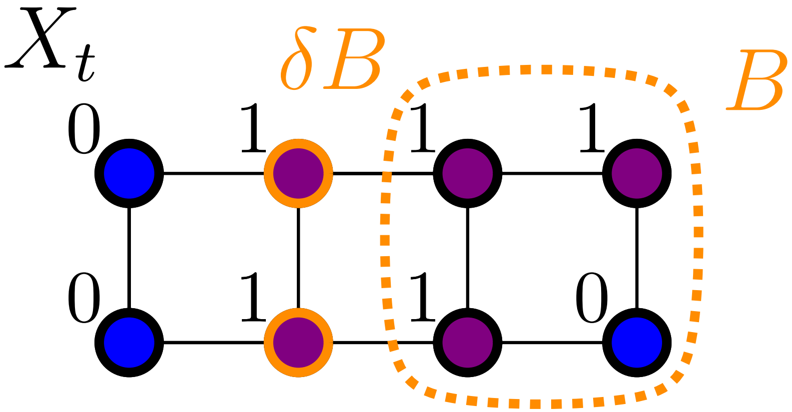
- **Example:**

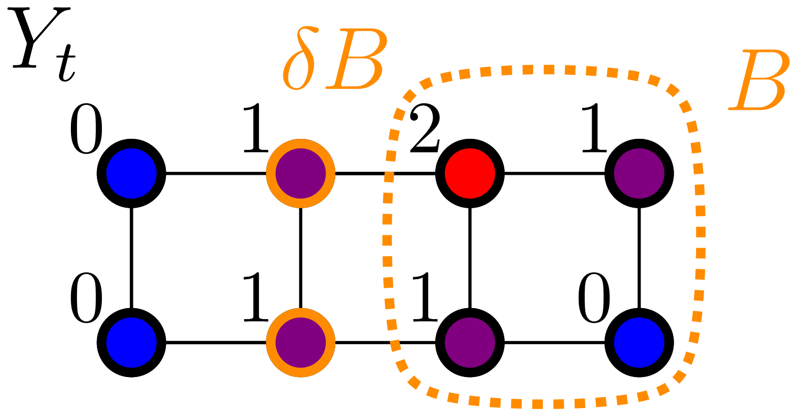


- **Goal:** Find coupling of block-MC on cover relations!

Coupling on cover relations

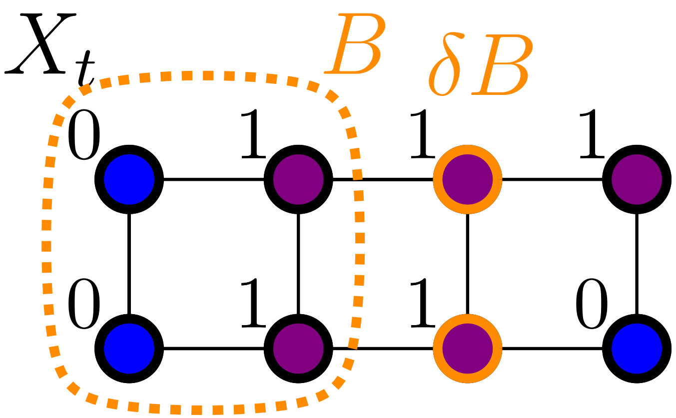
- Choose same block $B \in \mathfrak{B}$ for X_t and Y_t
- **Case I:** X_t and Y_t are equal on δB

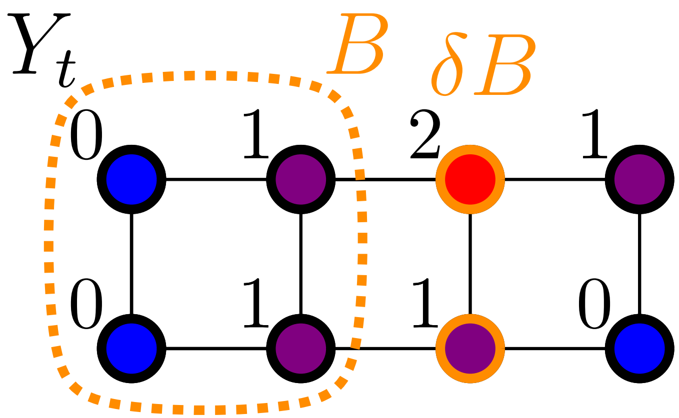




Sample admissible B-filling for X_t and Y_t identically!

- **Case II:** X_t and Y_t differ on $v \in \delta B$





Coupling on cover relations

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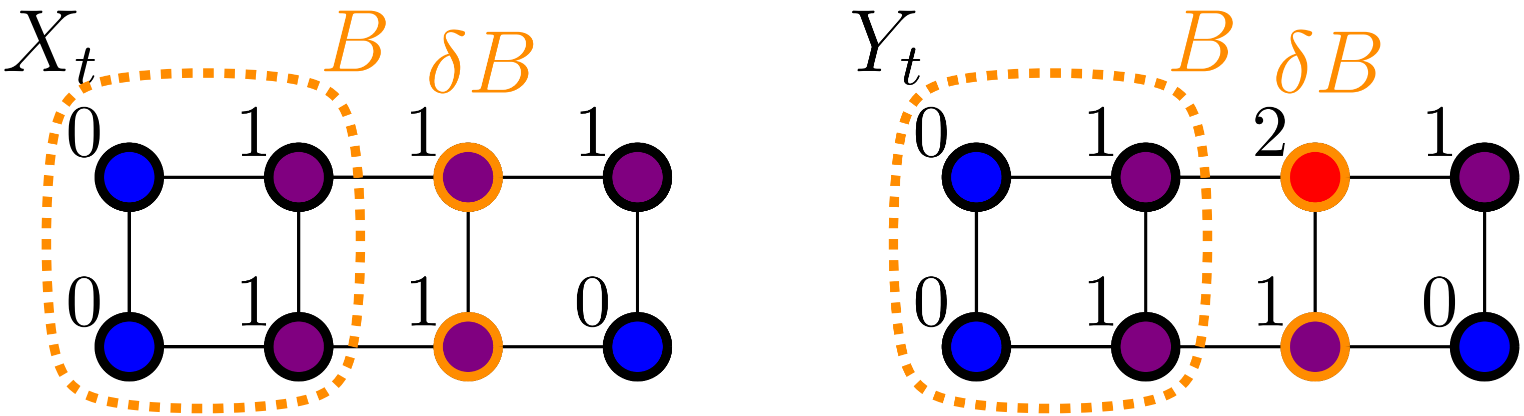

Two different probability distributions:

- $d^{\mathscr{B}}_{X_t}$: Unif. distrib. among adm. fillings on B wrt. $X_t \restriction_{\delta B}$
- $d^{\mathscr{B}}_{Y_t}$: Unif. distrib. among adm. fillings on B wrt. $Y_t \restriction_{\delta B}$

Claim: $d^{\mathscr{B}}_{X_t}$ is *stochastically dominated* by $d^{\mathscr{B}}_{Y_t}$,
i.e. $d^{\mathscr{B}}_{X_t}(U) \leq d^{\mathscr{B}}_{Y_t}(U)$ for all upsets U.

Coupling on cover relations

- **Case II:** X_t and Y_t differ on $v \in \delta B$

Theorem (Discrete version of Strassen's theorem)
Let d_1, d_2 be distributions on finite poset Ω. If d_1 is stoch. dom. by d_2, then there is a distribution q on $\Omega \times \Omega$ with

- $\sum_{y \in \Omega} q(x, y) = d_1(x)$ for all $x \in \Omega$
- $\sum_{x \in \Omega} q(x, y) = d_2(y)$ for all $y \in \Omega$
- $q(x, y) > 0$ implies $x \leq y$

- Apply theorem on $d_{X_t}^{\mathscr{B}}$ and $d_{Y_t}^{\mathscr{B}}$
- Transition $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$ from distribution q

$\Rightarrow$ Have monotone coupling on cover relations.

Aim for: $\mathbb{E}[d(X_{t+1}, Y_{t+1})] < 1 = d(X_t, Y_t)$

Path coupling

Idea: Extend coupling on cover relations to all $X_t \leq Y_t$.

Theorem (Dyer & Greenhill, 1997)

Given:

- Markov chain $\mathcal{M}$ on Ω
- Integral distance $d : \Omega \times \Omega \to \{0, \cdots, D\}$
- Subset $S \subset \Omega \times \Omega$ and for all $X, Y \in \Omega$ shortest path $\gamma_{X,Y} : X = X_0, \cdots, X_r = Y$ with $(X_i, X_{i+1}) \in S$
- Coupling $S \to \Omega \times \Omega$, $(X_t, Y_t) \mapsto (X_{t+1}, Y_{t+1})$ of $\mathcal{M}$ fulfilling $\mathbb{E}[d(X', Y')] \leq \beta \cdot d(X, Y)$

Then: Applying this coupling along the paths $\gamma_{X,Y}$ yields a coupling of $\mathcal{M}$ on $\Omega \times \Omega$ fulfilling

$$\mathbb{E}[d(X', Y')] \leq \beta \cdot d(X, Y)$$

Monotone coupling on $\mathscr{M}_{\mathscr{B}}$

Transition $(X_t \leq Y_t) \to (X_{t+1} \leq Y_{t+1})$:

- with probability $\frac{1}{2}$: $X_{t+1} \leftarrow X_t, \; Y_{t+1} \leftarrow Y_t$
- otherwise:
  - choose $B \in \mathscr{B}$ uniformly at random
  - **Case I:** X_t and Y_t are equal on δB:
    - sample adm. B-filling for X_{t+1} and Y_{t+1} identically
    - outside B, set $X_{t+1} := X_t$ and $Y_{t+1} := Y_t$
  - **Case II:** $X_t \leq Y_t$ differ on δB by one:
    - sample B adm. B-filling for X_{t+1} and Y_{t+1} according to Strassen's Theorem.
    - outside B, set $X_{t+1} := X_t$ and $Y_{t+1} := Y_t$
  - **Case III:** X_t and Y_t differ on δB by more than one:
    - (X_{t+1}, Y_{t+1}) determined by path coupling technique

Monotone coupling on $\mathscr{M}_{\mathscr{B}}$

- **Recall:** Need $\mathbb{E}[d(X', Y')] < 1 = d(X, Y)$ on all cover relations $(X, Y) \in \Omega \times \Omega$.
- Supp. $X \leq Y$ cover relation, differing only in $v \in V$, coupling choses block $B \in \mathscr{B}$ at random

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Case A: Boring case

- $\mathscr{M}_{\mathscr{B}}$ pauses for aperodicity
- $p = \frac{1}{2}$
- $d(X', Y') = 1$

Monotone coupling on $\mathscr{M}_{\mathscr{B}}$

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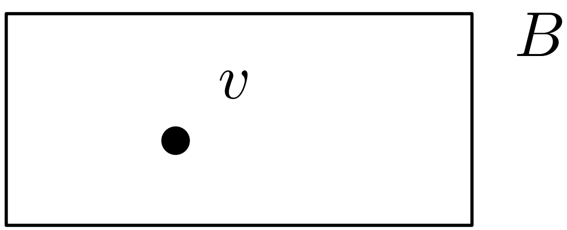
Case B: Good case

- $v \in B$



- $p = \frac{1}{2|\mathscr{B}|} \cdot \#\{B \in \mathscr{B} | v \in B\}$
- $d(X', Y') = 0$

Monotone coupling on $\mathscr{M}_{\mathscr{B}}$

- **Recall:** Need $\mathbb{E}[d(X', Y')] < 1 = d(X, Y)$ on all cover relations $(X, Y) \in \Omega \times \Omega$.
- Supp. $X \leq Y$ cover relation, differing only in $v \in V$, coupling choses block $B \in \mathscr{B}$ at random

Case C: Neutral case

- $v \notin (B \cup \delta B)$

- $d(X', Y') = 1$

Monotone coupling on $\mathscr{M}_{\mathscr{B}}$

- **Recall:** Need $\mathbb{E}[d(X', Y')] < 1 = d(X, Y)$ on all cover relations $(X, Y) \in \Omega \times \Omega$.
- Supp. $X \leq Y$ cover relation, differing only in $v \in V$, coupling choses block $B \in \mathscr{B}$ at random

Case D: Bad and complicated case

- $v \in \delta B$

- $p = \frac{1}{2|\mathscr{B}|}$ for each block $B \in \mathscr{B}$ with $v \in \delta B$
- Define worst case $\mathbb{E}[d(X', Y')]$ as $E_{B,v}$.

Main result

Theorem (Felsner, Heldt & Winkler, 2016)

Given:

- Finite graph $G = (V, E)$
- Family of blocks $\mathscr{B} \subset \mathscr{P}(V)$
- Number $\beta < 1$ s.t. for all $v \in V$:

$$1 - \frac{1}{2|\mathscr{B}|}\left(\#\{B \in \mathscr{B} \mid v \in B\} - \sum_{B \in \mathscr{B} \mid v \in B} E_{B,v}\right) \leq \beta$$

Then $\mathscr{M}_{\mathscr{B}}$ is rapidly mixing and so is $\mathscr{M}$.

Main result

Corollary (Felsner, Heldt & Winkler, 2016)

Given:

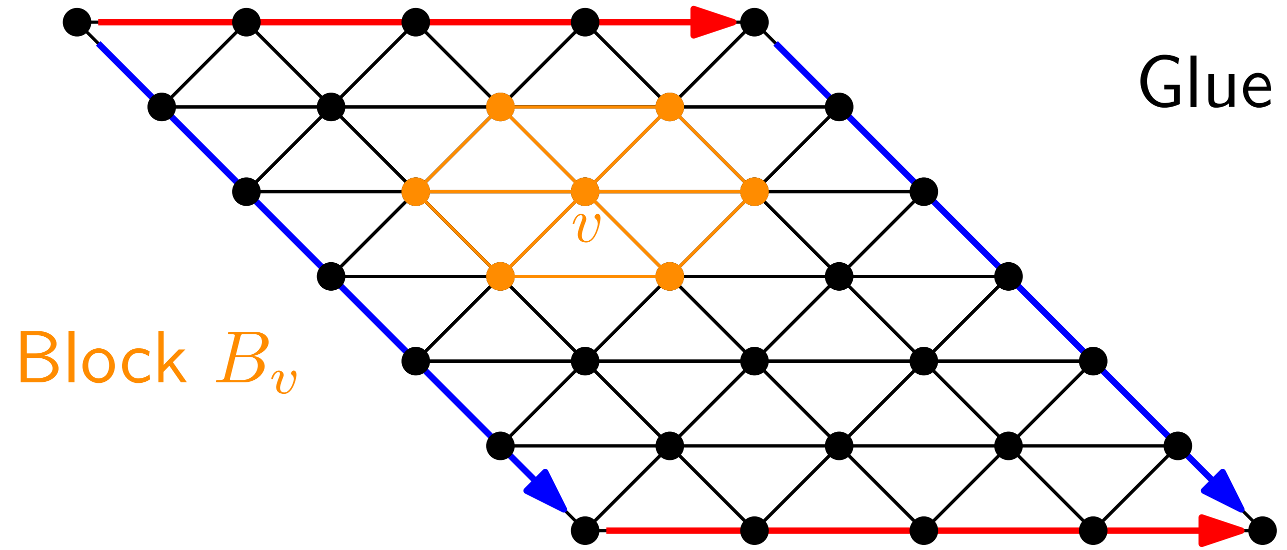
- Finite graph $G = (V, E)$
- Family of blocks $\mathscr{B} \subset \mathscr{P}(V)$
- Each $v \in V$ is contained in at least m blocks.
- Each $v \in V$ is contained in at most l borders of blocks.
- Value $E := \max_{B \in \mathscr{B}, v \in \delta B} E_{B,v}$ satisfies

$$1 + \frac{1}{|\mathscr{B}|}(l \cdot E - m) < 1$$

Then $\mathscr{M}_{\mathscr{B}}$ is rapidly mixing and so is $\mathscr{M}$.

Applications: Toroidal triangle grid graphs

Toroidal triangle grid graphs:



Glue opposite sites!

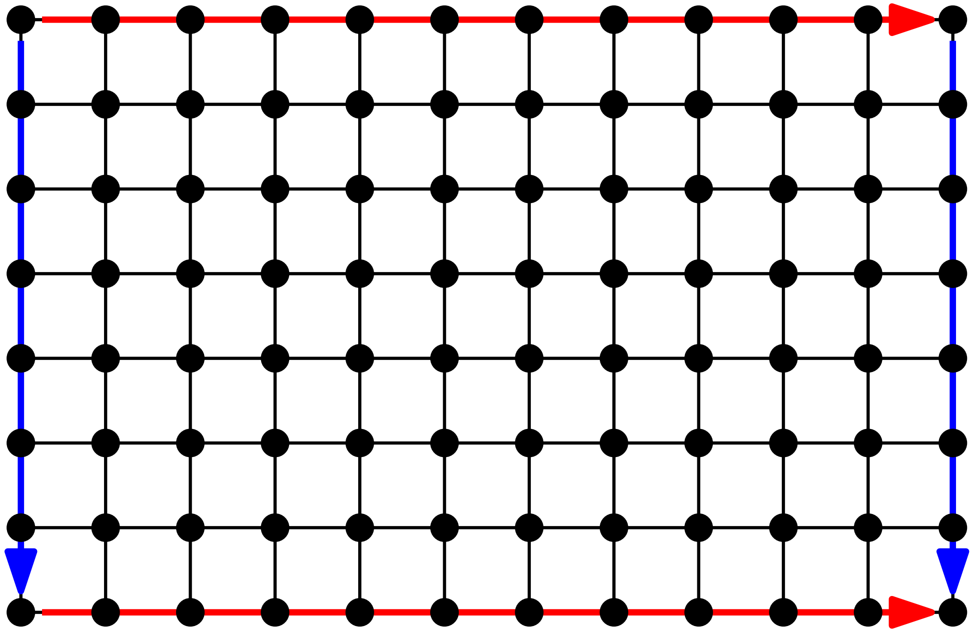
Family of blocks: $\mathscr{B} = \{B_v | v \in V\}$

Theorem (Heldt, Felsner & Winkler, 2016)

The Markov chain $\mathscr{M}$ is rapidly mixing on 2-heights of toroidal triangle grid graphs: $\tau_{\mathscr{M}}(\varepsilon) \in \mathcal{O}(|V|^3 \log |V|)$

Applications: Toroidal rectangular grid graphs

Toroidal rectangular grid graphs:



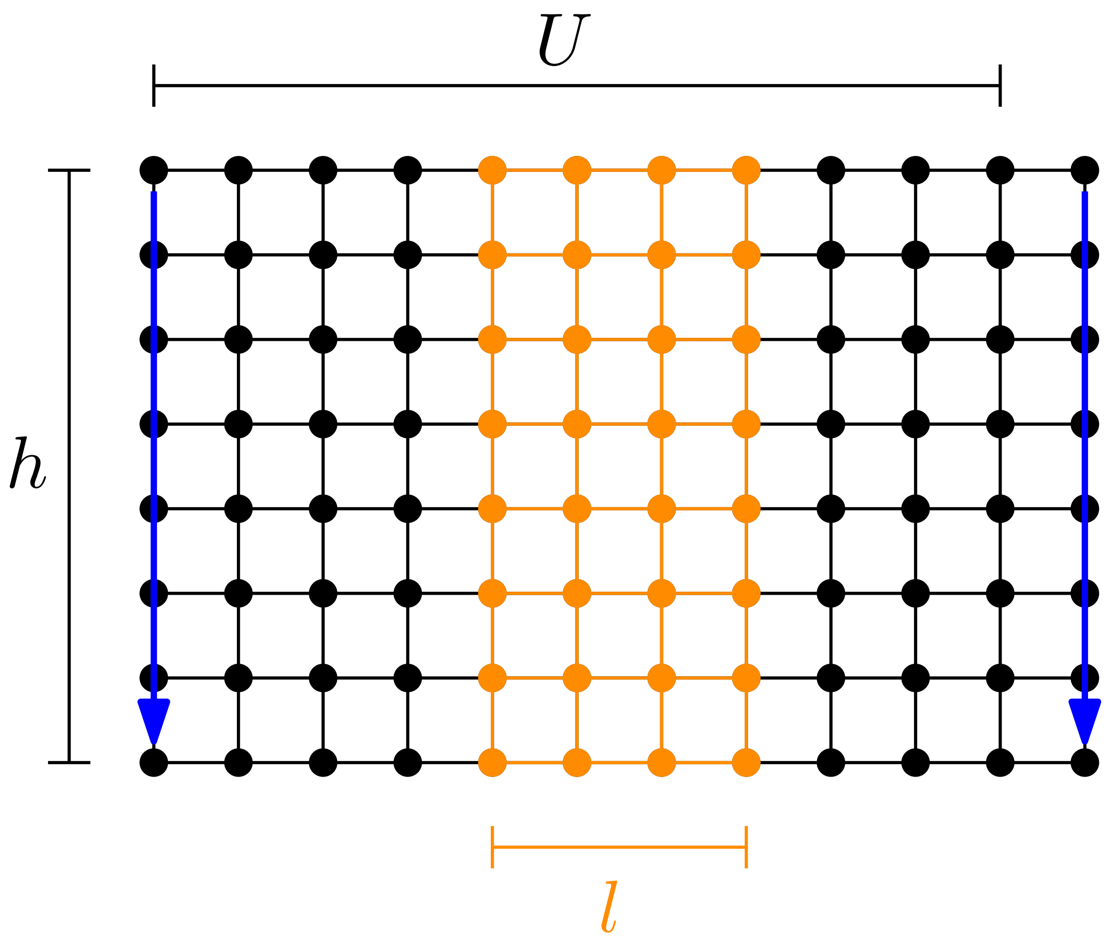
Glue opposite sides!

$\mathscr{B}$: blocks of size 6×6

Theorem (Heldt, Felsner & Winkler, 2016)
The Markov chain $\mathscr{M}$ is rapidly mixing on 2-heights of toroidal rectangular grid graphs: $\tau_{\mathscr{M}}(\varepsilon) \in \mathcal{O}(|V|^3 \log |V|)$

Applications: Cylindrical rectangular grid graphs

Cylindrical rectangular grid graphs:



Glue left & right side!

Use U blocks of shape $l \times h$ as $\mathscr{B}$!

Applications: Cylindrical rectangular grids

Main theorem applicaple?
Required block length l for different heights h:

k \ h	2	3	4	5	6	7
2	2	2	2	3	3	3
3	3	4	5	5	5	?
4	5	7	9	10	?	?
5	7	11	14	16	?	?

Conjecture: $\mathscr{M}$ rapidly mixing for all values of k and h.

Questions?