Seminar on Graphs, Algorithms and Optimization: **A Fast Parametric Maximum Flow Algorithm and Applications**

(Gallo, Grigoriadis, Tarjan, 1989)

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02.07.2020

Parametric Maximum Flow Problem

Given: Directed Graph (V, E)
Source $s \in V$, Sink $t \in V$.
Capacities $c_\lambda(v, w) \geq 0$ for all $(v, w) \in E$,
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Problem for $\lambda \in \mathbb{R}$:

$$\kappa(\lambda) = \max \sum_{v \in V} f(v, t)$$

$$\begin{aligned}
\text{s.t.} \quad & f(v, w) = -f(w, v) && \text{for all } (v, w) \in V \times V && \text{(antisymmetry)} \\
& f(v, w) \leq c_\lambda(v, w) && \text{for all } (v, w) \in V \times V && \text{(capacity)} \\
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Problem class too general to expect interesting parametric algorithm
$\Rightarrow$ Will restrict to subclass

Restrictions on parametric capacities:

- $\lambda \mapsto c_\lambda(s, v)$ non-decreasing for all $v \in V$
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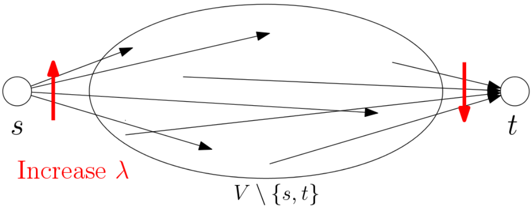
- $c_\lambda(v, w)$ constant for all $v, w \in V \setminus \{s, t\}$



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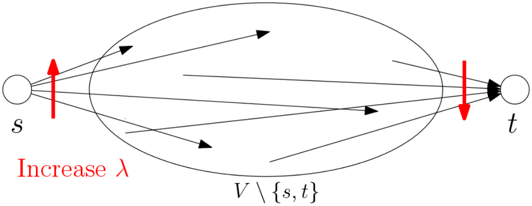

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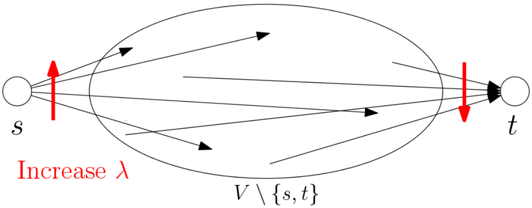

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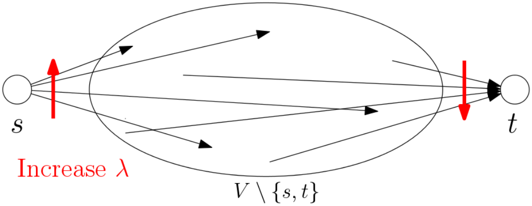

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- **Outline:** Modify Goldberg-Tarjan to obtain parametric algorithm that calculates $\kappa(\lambda)$ for $\lambda_1 \leq ... \leq \lambda_l$ at once.

Definition

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Remark: Flows are the feasible solutions of the LP (*conservation* instead of *leaky conservation*)

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- Given preflow f. If $f(v,w) < c_\lambda(v,w)$, then (v,w) is called *residual arc* with *residual capacity* $c_\lambda(v,w) - f(v,w)$.
- $d_f(v,w)$: Number of arcs of shortest residual path from v to w or ∞.

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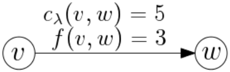
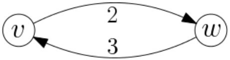


Residual arcs and capacities:



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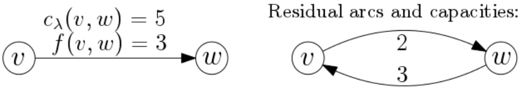

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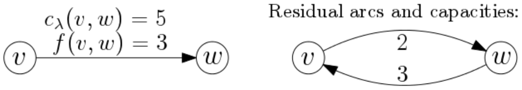

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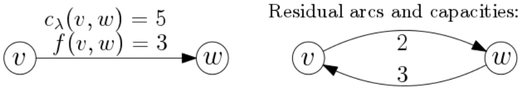

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- No sharp descents of valid labels d on residual arcs
- For all nodes v: $d(v) \leq \min\{d_f(v,t), d_f(v,s) + n\}$
- For all active nodes v: $\min\{d_f(v,t), d_f(v,s) + n\} \leq 2n - 1$

Claim: Flow f has valid labeling $d \Rightarrow f$ is maximal flow.

- If f not maximal: Exists residual path from s to t of length $\leq n-1$
- Then d is no valid labeling, since $d(s) = n$ and $d(t) = 0$ cannot hold.

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- **Initialize** preflow f and valid labeling d.
- As long as exist active nodes:
  Eliminate active node by push-relabel-operation
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- Throughout execution: Labeling d remains valid and never decreases.

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- Termination is due to $d(v) \leq 2n-1$ for active nodes v.

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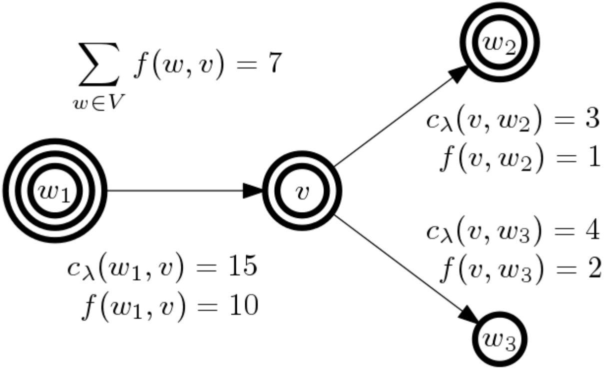

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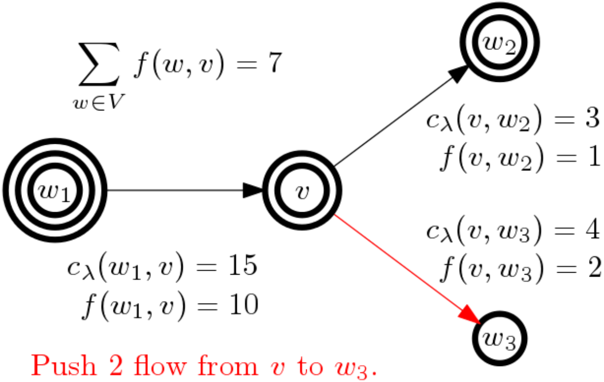

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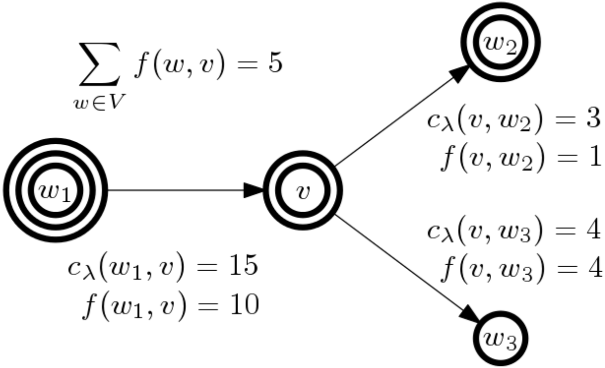

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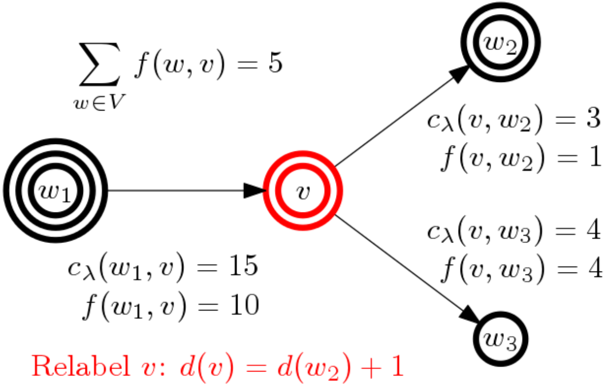

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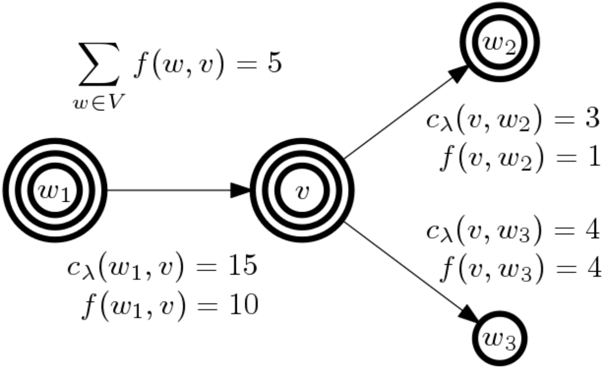

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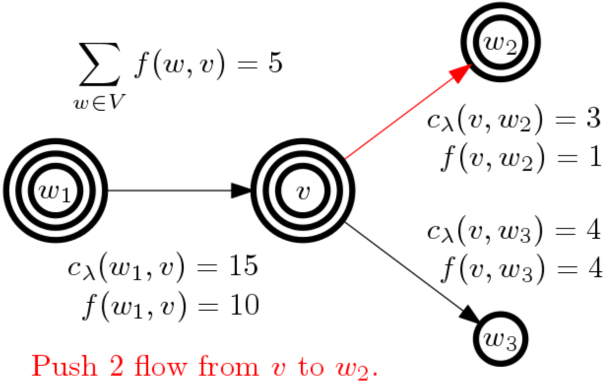

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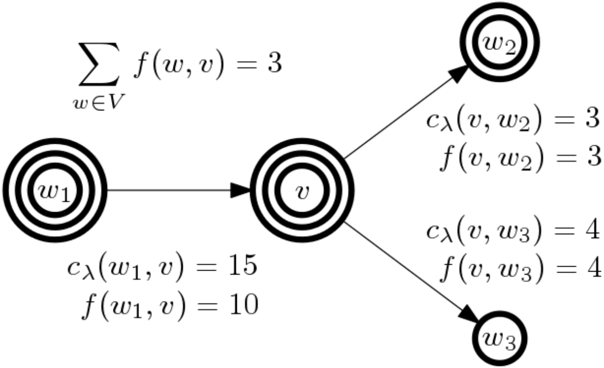

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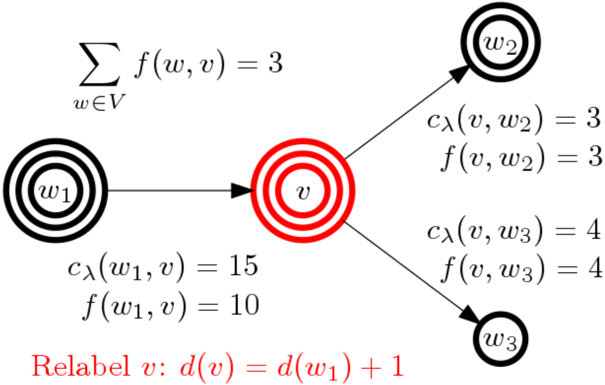

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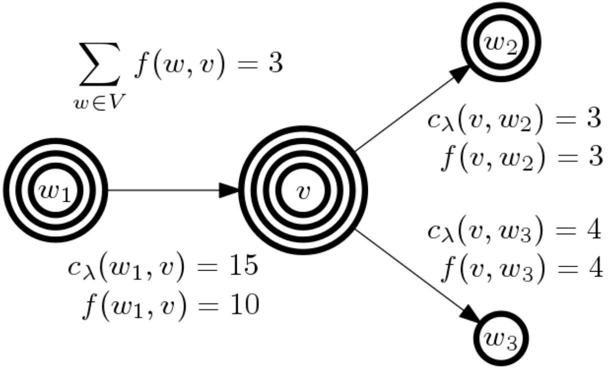

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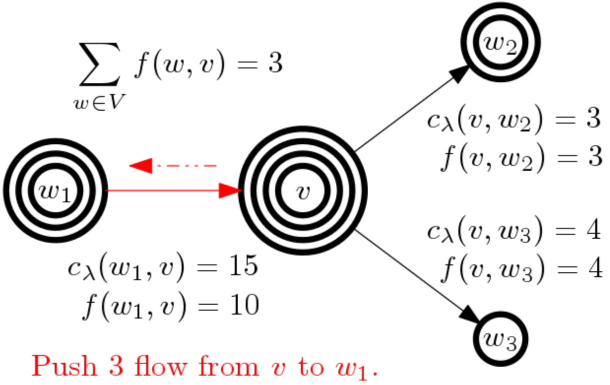

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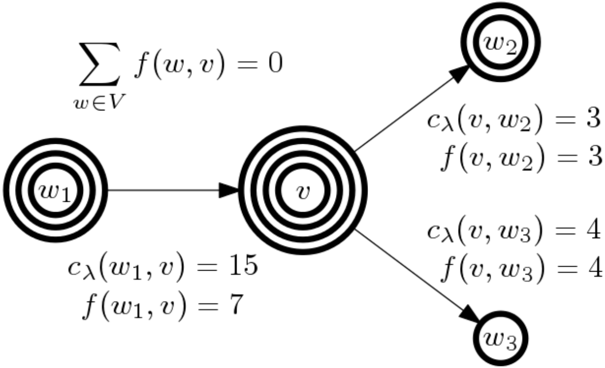

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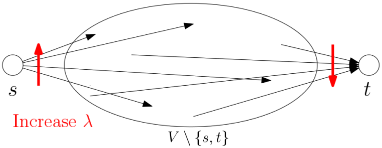
- Want to calculate MaxFlow for $\lambda_1 \leq \lambda_2$



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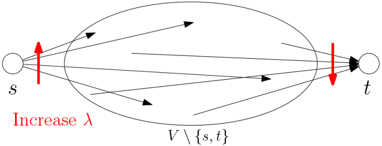

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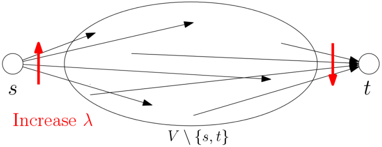

- **Idea** of Parametric Goldberg Tarjan algorithm:
  - Run Goldberg Tarjan first for λ_1: MaxFlow f, valid labeling d
  - Modify flow f for c_{λ_1} to obtain preflow f' for c_{λ_2}:

$$f'(v,w) := \begin{cases} \min\{c_{\lambda_2}(v,t), f(v,t)\} & \text{if } w = t \\ \max\{c_{\lambda_2}(s,w), f(s,w)\} & \text{if } v = s \text{ and } d(w) < n \\ f(v,w) & \text{otherwise} \end{cases}$$

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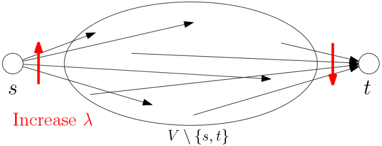

- **Idea** of Parametric Goldberg Tarjan algorithm:
  - Run Goldberg Tarjan first for λ_1: MaxFlow f, valid labeling d
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$$f'(v,w) := \begin{cases} \min\{c_{\lambda_2}(v,t), f(v,t)\} & \text{if } w = t \\ \max\{c_{\lambda_2}(s,w), f(s,w)\} & \text{if } v = s \text{ and } d(w) < n \\ f(v,w) & \text{otherwise} \end{cases}$$

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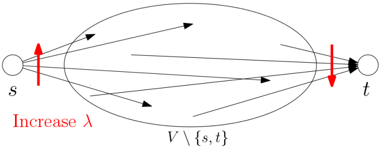

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  - Observe: f' is indeed preflow and d valid labeling for f'.
  - Run Goldberg Tarjan for λ_2, but start with f' and d.

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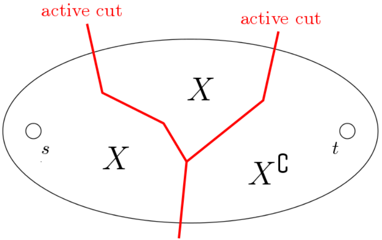

Parametric Goldberg Tarjan Algorithm

Input: Directed graph (V, E); $s, t \in V$; capacities c_λ; $\lambda_1 \leq ... \leq \lambda_l$
Output: MaxFlow f_i and MinCut $(X_i, X_i^\complement)$ for $i = 1, ..., l$

Initialize $f = 0$; $d(s) \leftarrow n$; $d(v) \leftarrow 0$ f.a. $v \neq s$
for $i = 1, ..., l$ **do**

Step 1: Update preflow

$$f(v,w) \leftarrow \begin{cases} \min\{c_{\lambda_i}(v,t), f(v,t)\} & \text{if } w = t \\ \max\{c_{\lambda_i}(s,w), f(s,w)\} & \text{if } v = s \text{ and } d(w) < n \\ f(v,w) & \text{otherwise} \end{cases}$$

Step 2: Run Goldberg Tarjan
$(f, d) \leftarrow \text{GoldbergTarjan}(f, d)$

Step 3: Find MinCut
$d(v) \leftarrow \min\{d_f(v,s) + n, d_f(v,t)\}$ for $v \in V$
$X \leftarrow \{v \in V \mid d(v) \geq n\}$
Output „$X_i = X, f_i = f$“

end

Analysis of Parametric Goldberg Tarjan

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Proof of 1):

- Correctness from discussion before
- Step 1 (Update preflow) requires $\mathcal{O}(m)$ per iteration
- Step 2 (Goldberg Tarjan) $\mathcal{O}(nm) + \#$*non-saturating pushs*
- Step 3 (Find MinCut) requires $\mathcal{O}(m)$ per iteration
- Σ: $\mathcal{O}((n+l)m) + \#$*non-saturating pushs*
- As in ADM I: $\#$*non-saturating pushs* $\in \mathcal{O}(n^2(l+m))$
- In total: Running time $\mathcal{O}(n^2(l+m))$

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Proof of 2):

- Throughout execution, d only increases.
- Cut is chosen as $X_i = \{v \in V \mid d(v) \geq n\}$ □

Corollary

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Remark:

- Improve both to $\mathcal{O}((n+l)m \log \frac{n^2}{m})$ by advanced strategies for push-relabel operations and advanced data structures.
- Parametric Goldberg Tarjan algorithm works **on-line** for sequence $\lambda_1 \leq ... \leq \lambda_l$.

Maximization of $\kappa(\lambda)$

- **Goal:** Determine $\lambda \in \mathbb{R}$ so that $\lambda \mapsto \kappa(\lambda)$ maximal
- Assume $\lambda \mapsto c_\lambda(v, w)$ affine linear for all $(v, w) \in V \times V$

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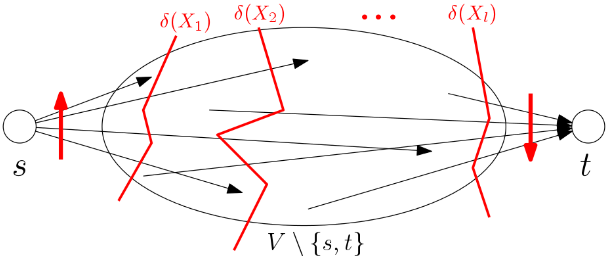
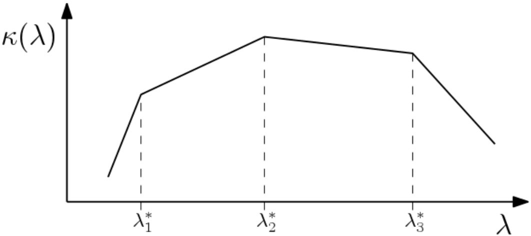
Proof: Follows directly from nested property $X_1 \subseteq X_2 \subseteq ... \subseteq X_l$.



Recall from ADM II (Global Dependence on the Cost Vector):

- $\kappa(\lambda)$ is concave and piecewise linear.
- Breakpoints correspond to basis changes.



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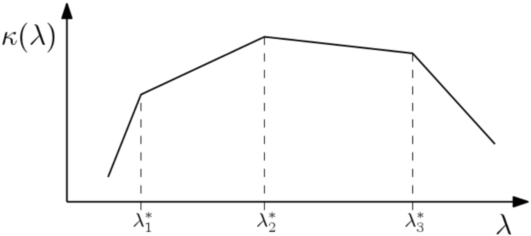

Consequence:

- Now basis change means change of MinCut $(X, X^{\complement})$.
- Therefore at most $n-1$ breakpoints.

Calculate the line segments of $\kappa(\lambda)$:

- Assume parametric capacities are given for $v \in V \setminus \{s, t\}$ by

$$c_\lambda(s, v) = a_0(v) + \lambda \cdot a_1(v)$$
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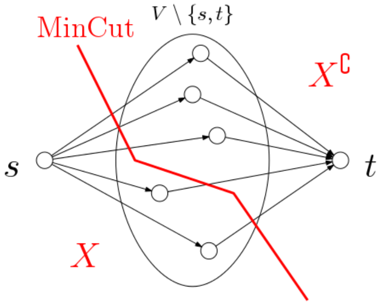
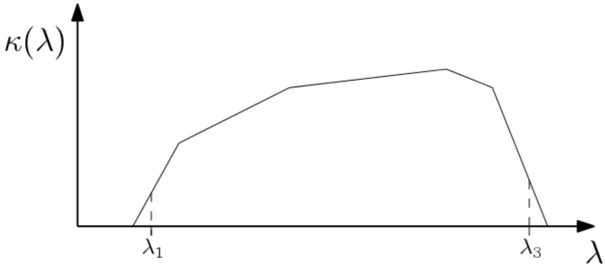
- Line segment of $\kappa(\lambda)$ around λ_0:
  $L(\lambda) = \kappa(\lambda_0) + (\lambda - \lambda_0) \cdot \left(\sum_{v \in X^{\complement} \setminus \{t\}} a_1(v) - \sum_{v \in X \setminus \{s\}} b_1(v)\right)$



Example: Assume $\lambda_{\max} \in [\lambda_1, \lambda_3]$:



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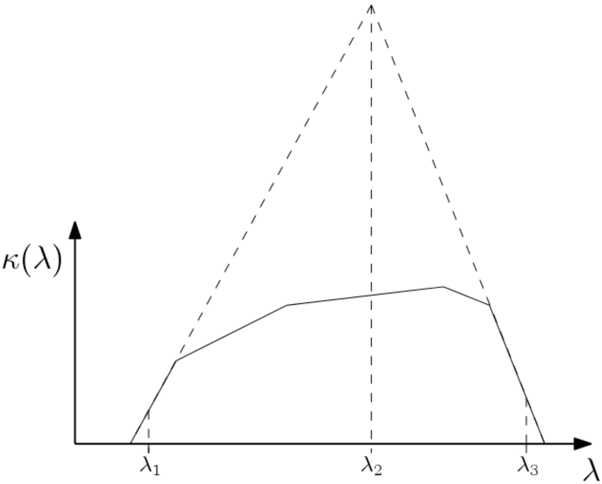

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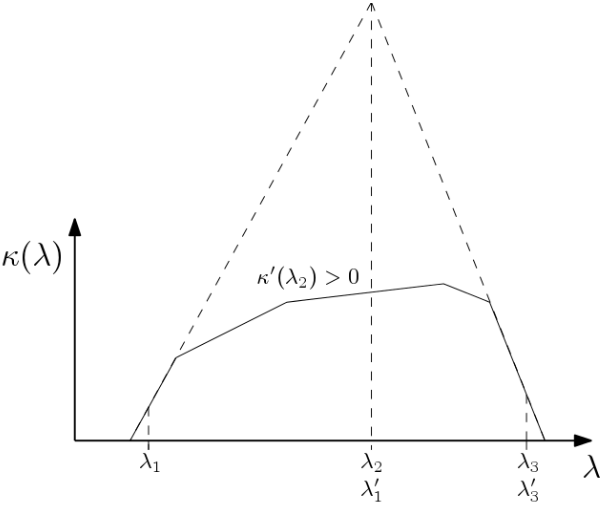

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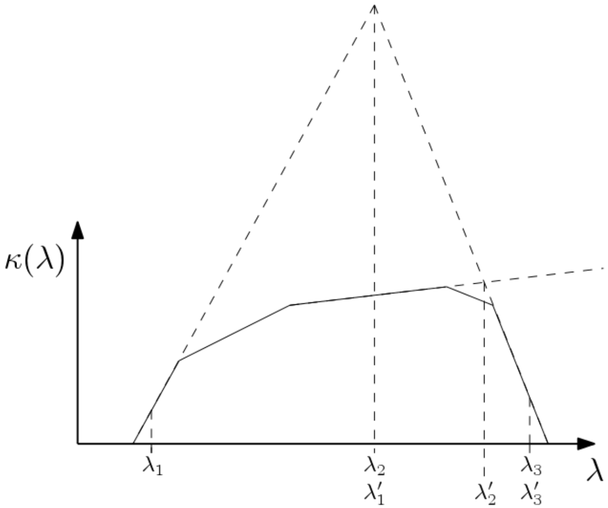

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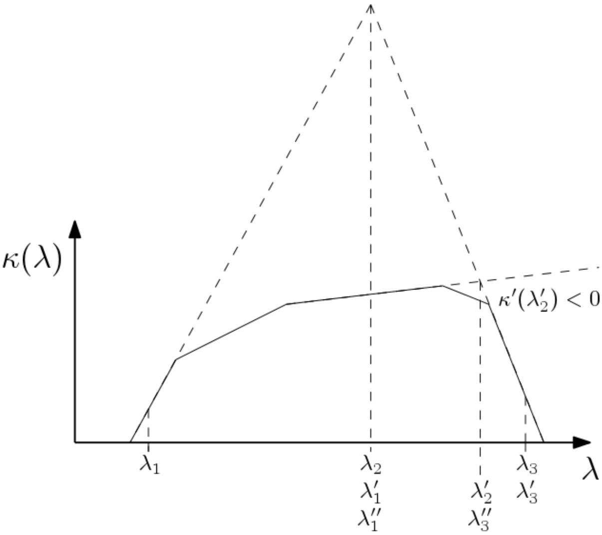

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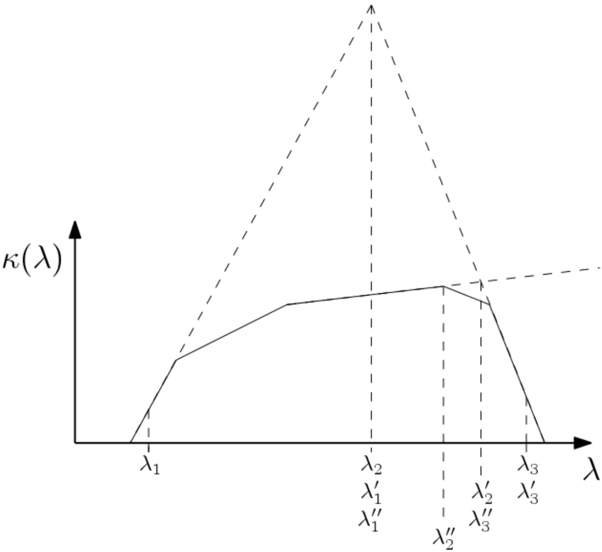

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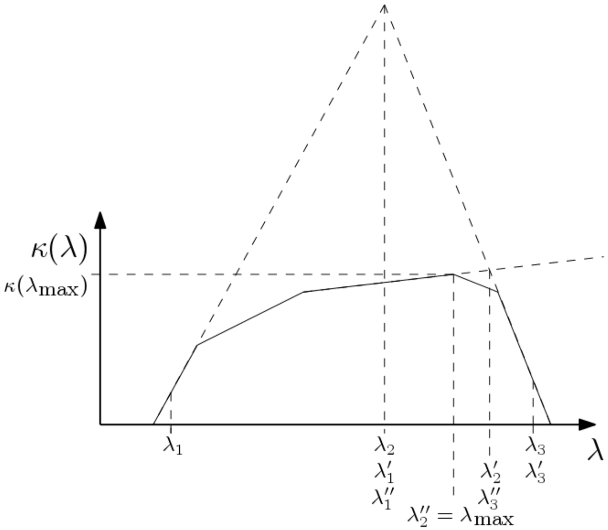

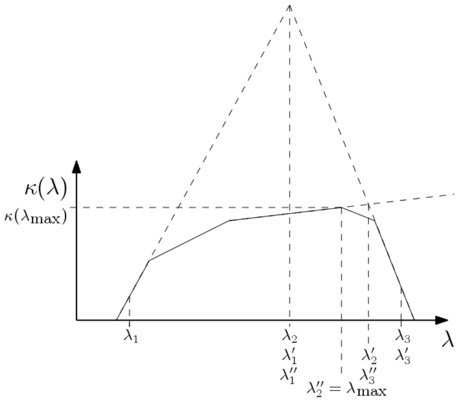

Basic idea:

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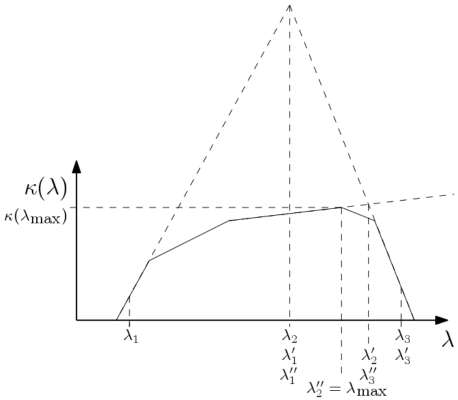

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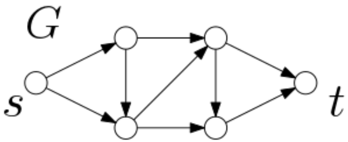
- For calculating $\kappa(\lambda)$ for $\lambda_1 \leq \lambda_1' \leq \lambda_1'' \leq \ldots$ and $\lambda_3 \geq \lambda_3' \geq \lambda_3'' \geq \ldots$ use two concurrent runs of Parametric Goldberg Tarjan.

Adapt Parametric Goldberg Tarjan to decreasing values of λ

- **Idea:** From G obtain equivalent problem G' by reversing all arcs and interchanging source and sink.



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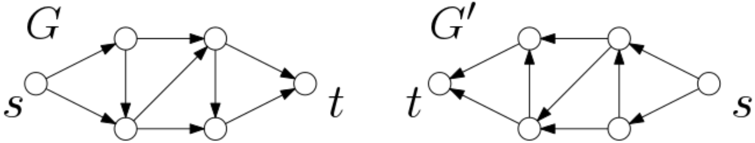

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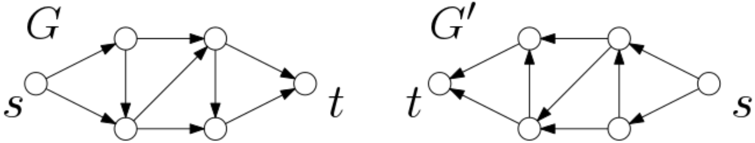

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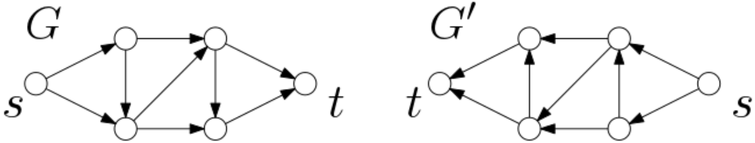

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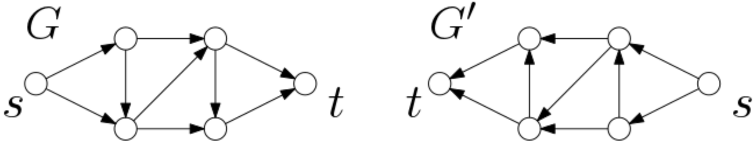

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- Apply Parametric Goldberg Tarjan on G'.

Application: Flow sharing

Given: Directed Graph (V, E) with capacities $c \in \mathbb{R}^E$
Multiple sources $s_1, ..., s_k \in V$, sink $t \in V$
Source weights $w_1, ..., w_k > 0$

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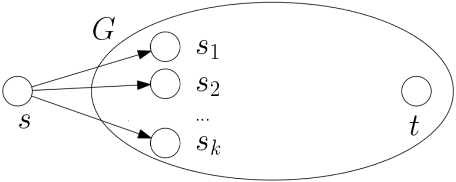
Perfect sharing:

- Restrict to flows with equal ratios $\frac{u_1}{w_1}, ..., \frac{u_k}{w_k}$.
- Find maximal flow under this restriction.

Solving perfect sharing:

- Add supersource s, edge (s, s_i) with $c_\lambda(s, s_i) = \lambda \cdot w_i$ for $i = 1, \ldots, k$.



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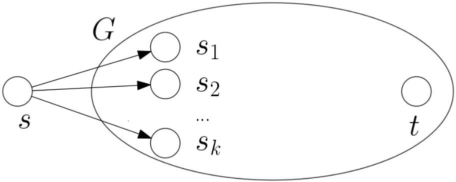

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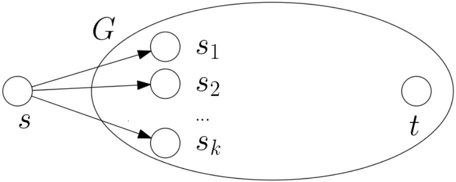

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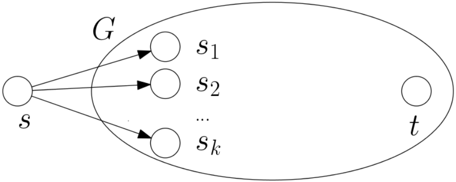

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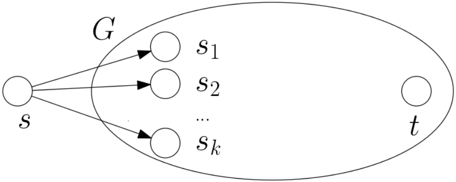

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- λ_0 maximizes MaxFlow under perfect sharing.

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  maximize $\min_{i=1,\ldots,k} \frac{u_i}{w_i}$ and minimize $\max_{i=1,\ldots,k} \frac{u_i}{w_i}$ simultaneously.
- Lexicographic sharing: Among maximum flows, minimize

  $$\frac{u_{i_1}}{w_{i_1}} < \ldots < \frac{u_{i_1}}{w_{i_1}}$$

  lexicograpically.

Other flow sharing problems

- MaxMin sharing: Among maximum flows, maximize $\min_{i=1,\dots,k} \frac{u_i}{w_i}$.
- MinMax sharing: Among maximum flows, minimize $\max_{i=1,\dots,k} \frac{u_i}{w_i}$.
- Optimal sharing: Among maximum flows, maximize $\min_{i=1,\dots,k} \frac{u_i}{w_i}$ and minimize $\max_{i=1,\dots,k} \frac{u_i}{w_i}$ simultaneously.
- Lexicographic sharing: Among maximum flows, minimize

$$\frac{u_{i_1}}{w_{i_1}} < \dots < \frac{u_{i_1}}{w_{i_1}}$$

lexicograpically.

Parametric Goldberg Tarjan can solve all of them!

References

- Gallo, Grigoriadis & Tarjan (1989). *A fast parametric maximum flow algorithm and applications.* SIAM J. Comput. 18 (1), pp. 30