Facetten der Komplexität
Facets of Complexity

Sandro M. Roch: Sampling of random pseudoline arrangements

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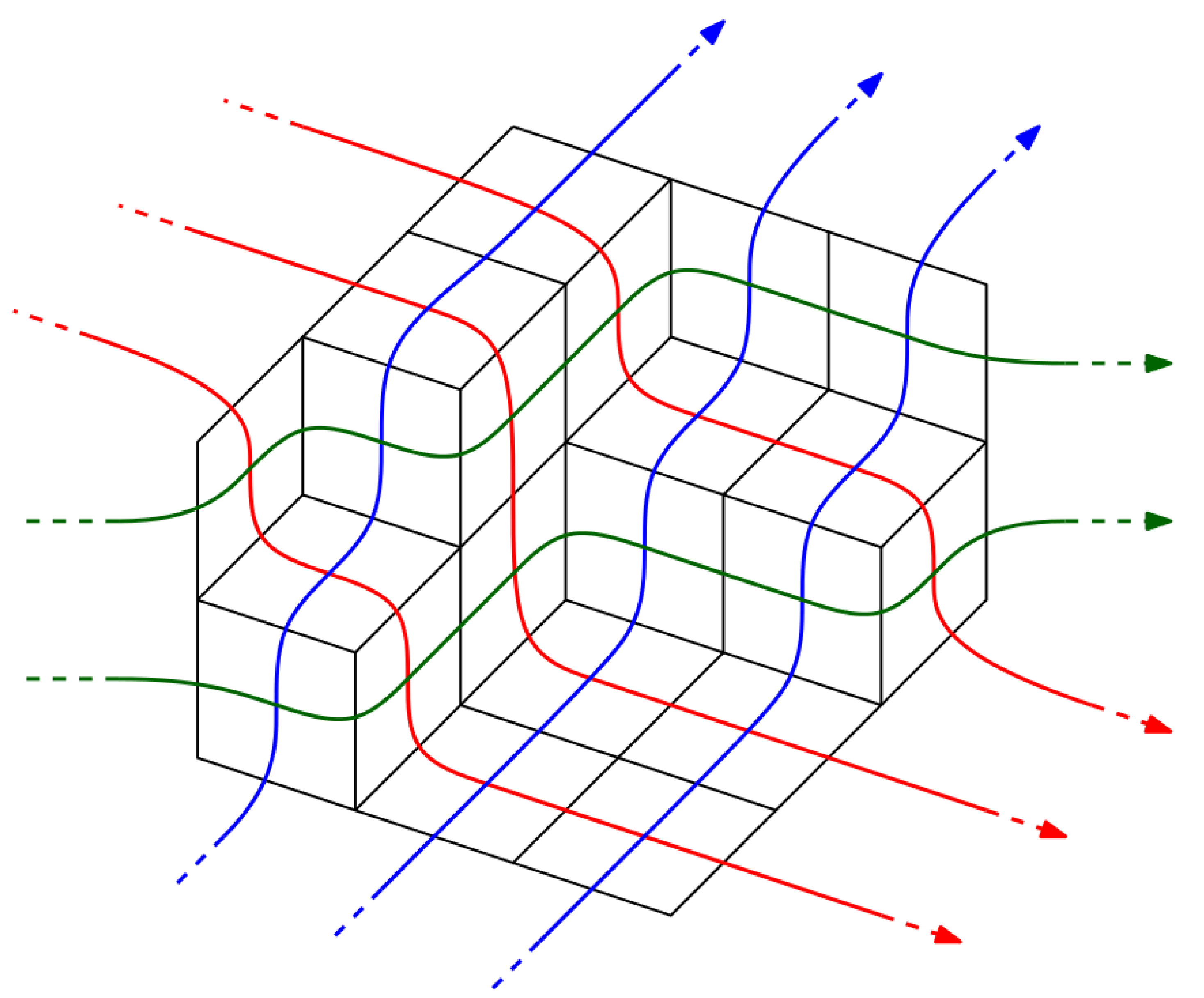
Sampling of random pseudoline arrangements

- Pseudoline arrangements and zonotopal tilings (A)
- Random sampling via...
  a. Rapidly mixing Markov chains (B)
  b. Random standard Young tableaux (C)

(A) Pseudoline arrangements and zonotopal tilings

- **Def.:** *Pseudoline arrangement (PLA)*: Collection of continuous, x-monotone curves $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}^2$, pairwise intersecting at a single point where they cross
- Assume *simple* PLAs: At any point, at most two curves intersect.
- *Generalized* PLA: r classes of $n_1, \dots, n_r$ curves, only curves of different classes cross
- **Example:** Red, green and blue curves:



- **Def.:** *2D-Zonotope*: For pw. indep. vectors $v_1, \dots, v_r \in \mathbb{R}^2$:

$$Z(v_1, \dots, v_r) := \left\{ \sum \lambda_i v_i : \lambda_i \in [-1, 1] \right\}$$

- **Def.:** *Rhombic tiling of shape* $(n_1, \dots, n_r)$: Tiling of $Z(v_1, \dots, v_r)$ with rhombi of type $Z(\frac{v_i}{n_i}, \frac{v_j}{n_j})$

Theorem [1]

Simple, generalized PLAs with $n_1, \dots, n_r$ curves are in **bijection** with rhombic tilings of shape $(n_1, \dots, n_r)$.

- **Example for Theorem**: See figure above.

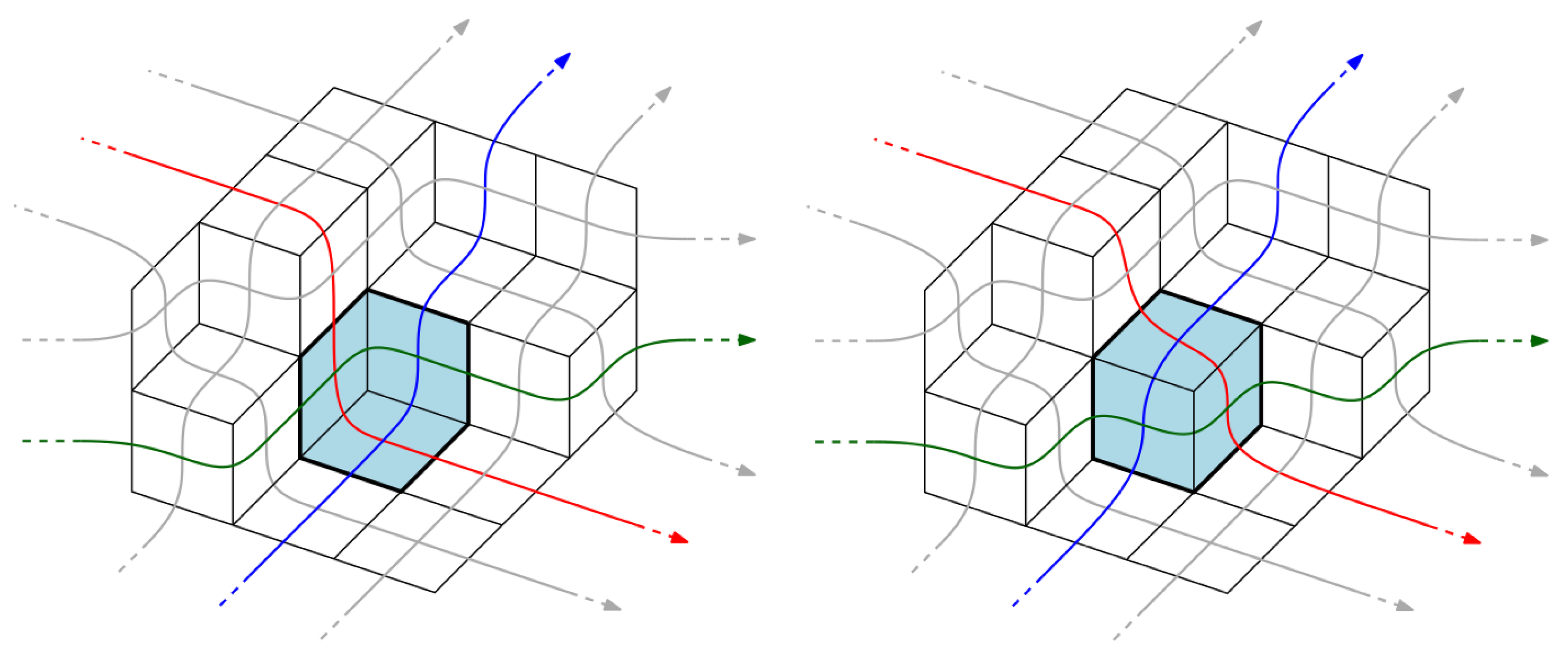
Guiding question

For fixed $n_1, \dots, n_r$, how can we efficiently sample pseudoline arrangements (or equivalently: rhombic tilings) by random (uniformly distributed)?

[1] S. Felsner, H. Weil, *Sweeps, arrangements and signotopes*, Disc. Appl. Math. 109 (2001).

(B) Sampling via rapidly mixing Markov chains

- **Idea:** Transitions are random triangle / hexagon flips:



(a) Before (b) After

- Stationary distribution equals uniform distribution
- **Def.:** Markov chain *rapidly mixing*, if, after polynomial number of iterations, chain is "ε-close" to stationary dist.

State of the art

- For $r = 3$ classes: Markov chain is rapidly mixing [2-4]
- For $r > 3$ classes: Unknown whether Markov chain is rapidly mixing, but suggested in [5]

- We prove following result for a "restricted version":

Theorem

The Markov chain that only flips random triangles, in which a curve of a certain class is involved, is rapidly mixing if and only if $r = 3$.

- We show how to efficiently insert a new class of curves randomly. Gives rise to other promising Markov chain:

Open question

Is the Markov chain, which in each iteration removes a class of curves and reinserts it unif. by random, rapidly mixing?

[2] D. Randall, P. Tetali, *Analyzing Glauber dynamics by comparison of Markov chains*, J. Math. Phys. 41 (2000).
[3] M. Luby, D. Randall, A. Sinclair, *Markov chain algorithms for planar lattice structures*, SIAM J. Comput. 31 (2001).
[4] D. B. Wilson, *Mixing times of lozenge tiling and card shuffling Markov chains*, Ann. Appl. Probab. 14 (2004).
[5] N. Destainville, *Mixing times of plane rhombus tilings*, DMTCS Proc. AA, DM-CCG (2001).

(C) Sampling via random standard Young tableaux

- *Sorting networks* are PLAs with a top. order on crossings
- Sorting networks in **bijection** with *standard Young tableaux* (SYT) of staircase shape [6,7]
- Via SYTs efficient sampling of sorting networks is possible [8], but PLAs are equivalence classes of these.
- In each equivalence class, we define a unique, efficiently recognizable element and employ rejection sampling.

[6] S. Felsner, *The skeleton of a reduced word and a correspondence of Edelman and Greene*, Electron. J. Comb. 8 (2001).
[7] Angel et al., *Random sorting networks*, Adv. Math. 215 (2007).
[8] C. Greene, A. Nijenhuis, H. Wilf, *A probabilistic proof of a formula for the number of Young tableaux of a given shape*, Adv. Math. 31 (1979)