Mittagsseminar 3.11.2023

Coloring problems on arrangements of pseudolines

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(Joint work with Rimma Hämäläinen)

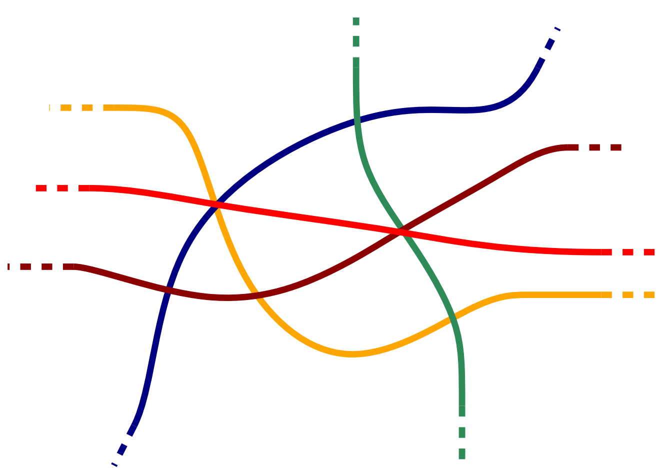
Pseudoline arrangements

Pseudoline arrangement: Finite family of continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}^2$ with

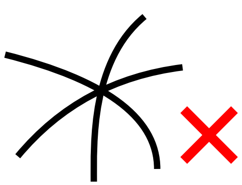
$$\lim_{t\to\infty} \|f_i(t)\| = \lim_{t\to-\infty} \|f_i(t)\| = \infty,$$

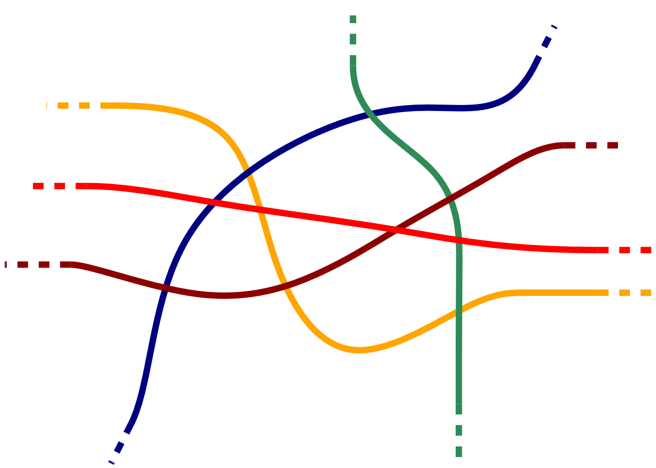
each two of which intersect in exactly one point.

Ex: *nonsimple* arrangement:



Ex: *simple* arrangement:



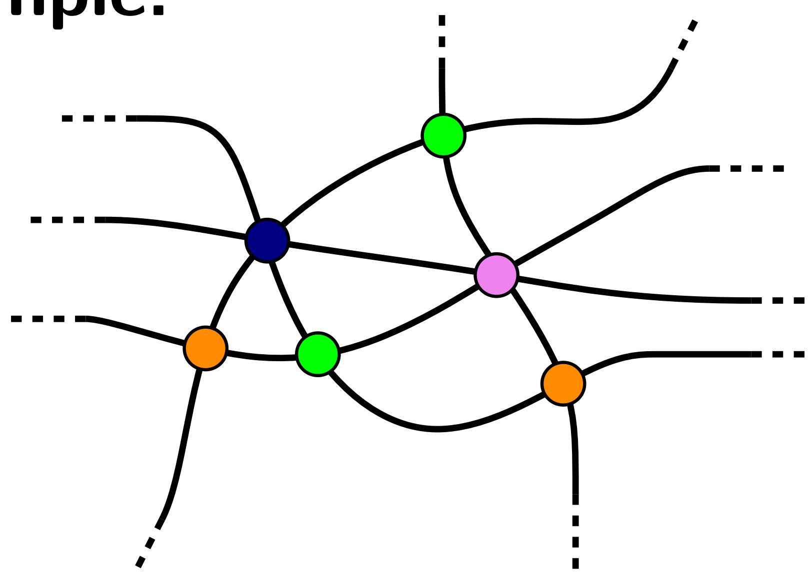


Crossing colorings

Def: *Crossing coloring* of arrangement $\mathscr{A}$:

- Coloring of the crossings of $\mathscr{A}$
- Avoiding twice the same color along any pseudoline

Example:



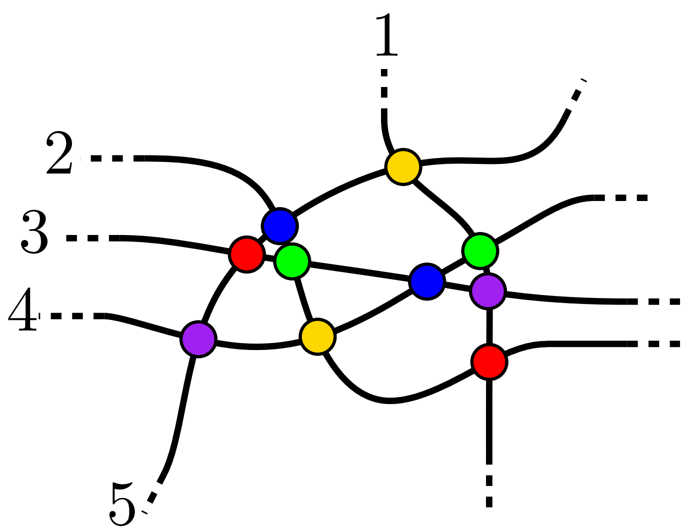
Question: How many colors are required, depending on n?

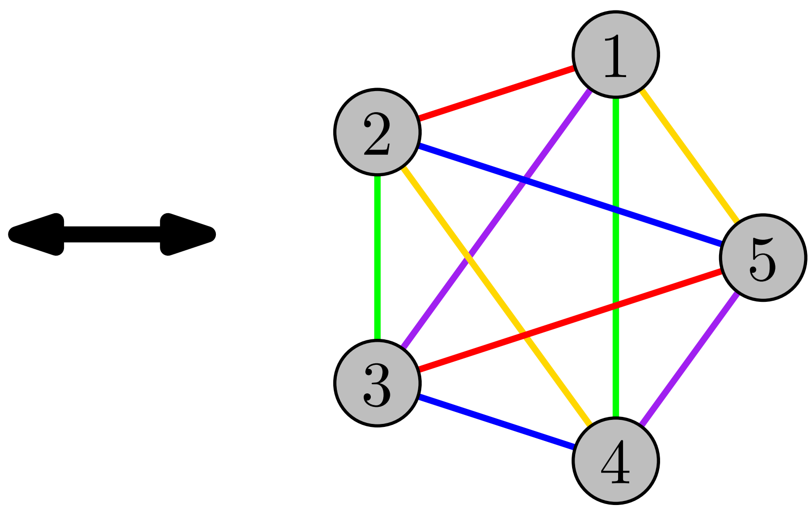
Crossing colorings

Proposition: For every arrangement $\mathscr{A}$ there is a crossing coloring using n colors.

If $\mathscr{A}$ is simple: Equivalent to edge coloring of K_n:

- For $n \in 2\mathbb{Z}$: Need exactly $n - 1$ colors.
- For $n \in 2\mathbb{Z} + 1$: Need exactly n colors.





Excursion: hypergraph coloring

Convention:

- *vertex coloring*: coloring of the vertices, avoiding monochromatic hyperedges
- *edge coloring*: coloring of the hyperedges, avoiding twice the same color at any vertex

Terminology for hypergraph $\mathscr{H} = (V, \mathscr{E})$:

- *simple*: For all $E_1, E_2 \in \mathscr{E} : |E_1 \cap E_2| \leq 1$ and $|E| \geq 2$ f.a. $E \in \mathscr{E}$.
- *codegree*: $\max_{u,v \in V} \# \{E \in E : \{u, v\} \subset E\}$
- k*-uniform*: For all $E \in \mathscr{E} : |E| = k$.
- k*-bounded*: For all $E \in \mathscr{E}$: $|E| \leq k$.

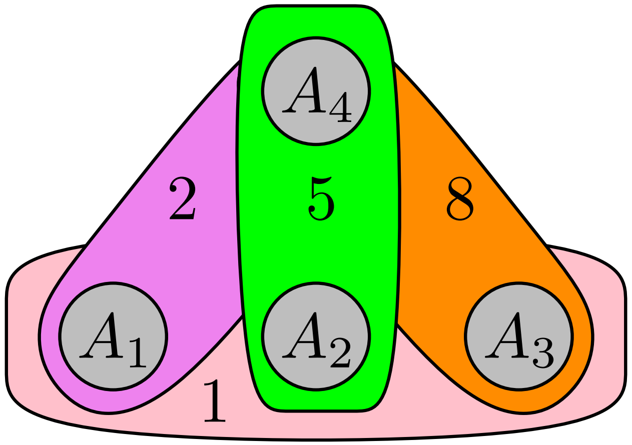
Erdős-Faber-Lovász conjecture

Problem:
Let $|A_i| = n$ for $1 \leq i \leq n$, $|A_i \cap A_j| \leq 1$ for all $i < j$.
Can one color the elements $\bigcup_i A_i$ using n colors such that each A_i contains all colors?

Equivalent: Every simple hypergraph on n vertices can be edge-colored using n colors.

Example:

$$A_1 = \{1, 2, 3, 4\}$$
$$A_2 = \{1, 5, 6, 7\}$$
$$A_3 = \{1, 8, 9, 10\}$$
$$A_4 = \{2, 5, 8, 11\}$$



Erdős-Faber-Lovász conjecture

Problem:

Let $|A_i| = n$ for $1 \le i$ … $\le n$, $|A$ … $\cap A_j| \le 1$ for all $i < j$.

Can one color the elen … ts $\bigcup$ … usi … n colors such that each A_i contai … ll co … ?

Proven in 2021!

By D. Y. Kang, T. Kelly, D. Kühn, A. Methuku & D. Osthus

Equivalent: Every … ertices can be edge-colored u …

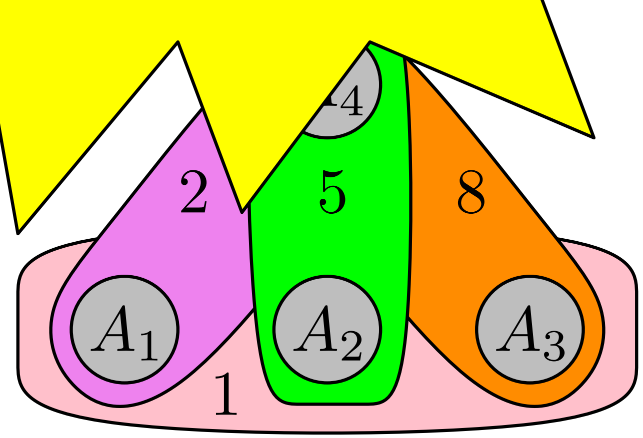
Example:

$A_1 = \{1, 2, \cancel{3}, \cancel{4}\}$

$A_2 = \{1, 5, \cancel{6}, \cancel{7}\}$

$A_3 = \{1, 8, \cancel{9}, \cancel{10}\}$

$A_4 = \{2, 5, 8, \cancel{11}\}$



Crossing colorings

Proposition: For every arrangement $\mathscr{A}$ there is a crossing coloring using n colors.

Proof: Define simple hypergraph $\mathscr{H}(\mathscr{A})$:

- vertices $\sim$ n pseudolines
- hyperedges $\sim$ crossings

Then: edge coloring of $\mathscr{H}(\mathscr{A}) \Leftrightarrow$ crossing coloring of $\mathscr{A}$ $\square$

Open problem:
Find a deterministic algorithm for crossing colorings using n colors.

Algorithm by Chang & Lawler (1987) uses $1.5n - 2$ colors.

Crossing colorings

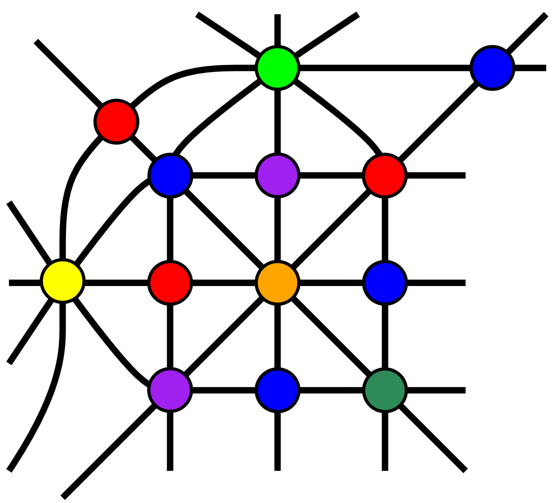
- Let

$$\mathrm{mx}(\mathscr{A}) := \max_{l \in \mathscr{A}} \#\text{crossings on } l$$

- Observe: $\mathscr{A}$ simple $\Rightarrow \mathrm{mx}(\mathscr{A}) = n - 1$

Conjecture:
For every arrangement $\mathscr{A}$ there is a crossing coloring using $\mathrm{mx}(\mathscr{A}) + c$ colors, for some constant c.

Example:



$\mathrm{mx}(\mathscr{A}) = 4$

Need $\mathrm{mx}(\mathscr{A}) + 3 = 7$ colors.

Crossing colorings

Theorem: (Pippenger & Spencer, 1989)
For every $k, \varepsilon > 0$, there exists $\delta > 0$ such that the following holds:

If $\mathscr{H}$ is a k-uniform D-regular hypergraph of codegree at most δD, then $\mathscr{H}$ can be edge colored using $(1+\varepsilon) \cdot D$ colors.

Theorem: (Kahn, 1996)
For every $k, \varepsilon > 0$, there exists $\delta > 0$ such that the following holds:

If $\mathscr{H}$ is a k-bounded hypergraph of max-degree at most D and codegree at most δD, then $\mathscr{H}$ can be list edge colored using $(1+\varepsilon) \cdot D$ colors.

Crossing colorings

Consequence: For every $k, \varepsilon > 0$ there is a mx_0 so that for every arrangement $\mathscr{A}$ with $\mathrm{mx}(\mathscr{A}) \geq \mathsf{mx}_0$ and only having crossings of degree at most k there exists a valid crossing coloring using

$$\chi(\mathscr{A}) \leq (1 + \varepsilon) \cdot \mathrm{mx}(\mathscr{A})$$

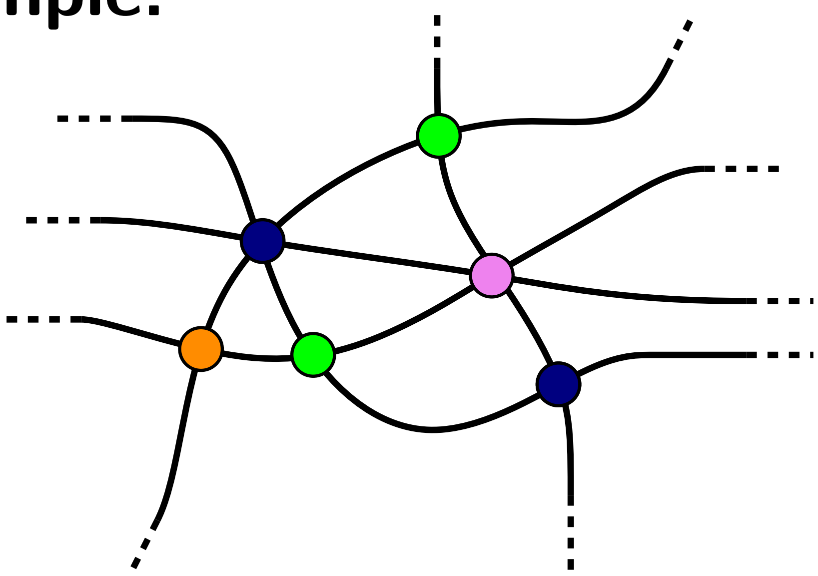
colors.

Face respecting colorings

Def: *Face respecting coloring* of arrangement $\mathscr{A}$:

- Coloring of the crossings of $\mathscr{A}$
- Avoiding twice the same color on any face

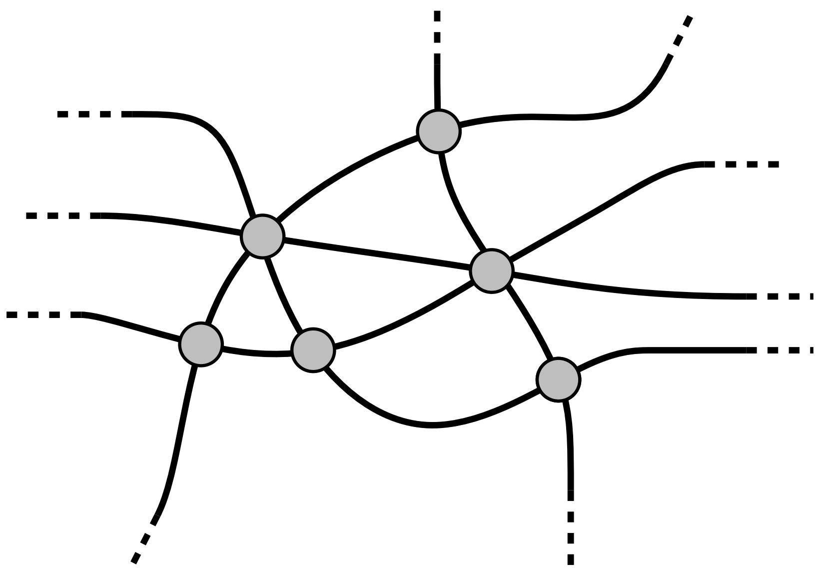
Example:



Face respecting colorings

Proposition: For every arrangement $\mathscr{A}$ there is a face respecting coloring using n colors.

Proof:



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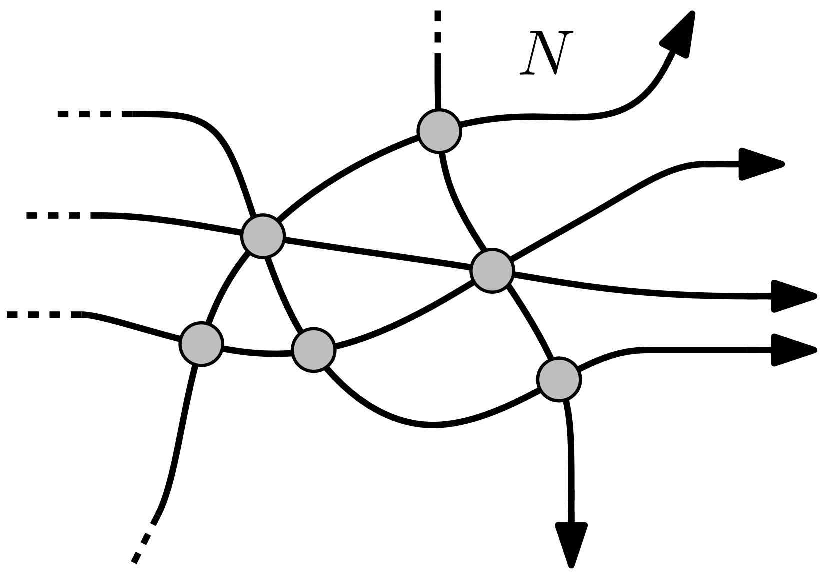

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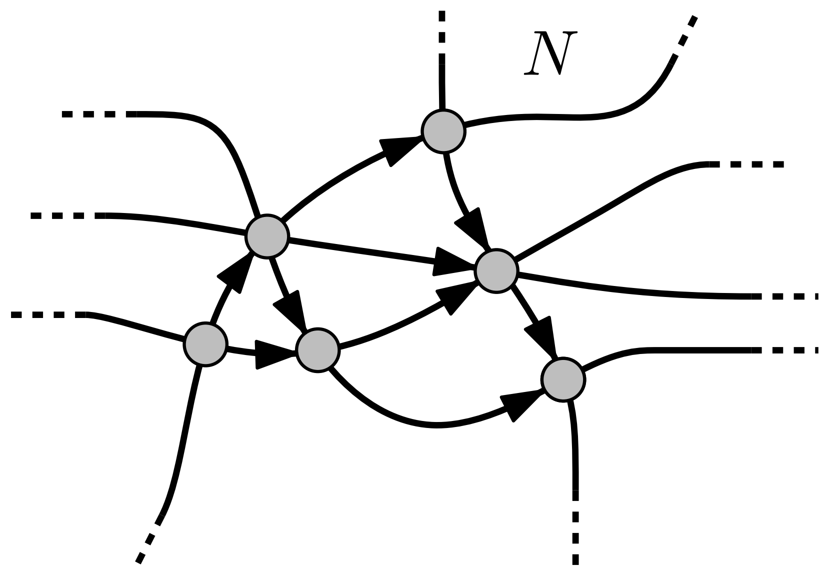

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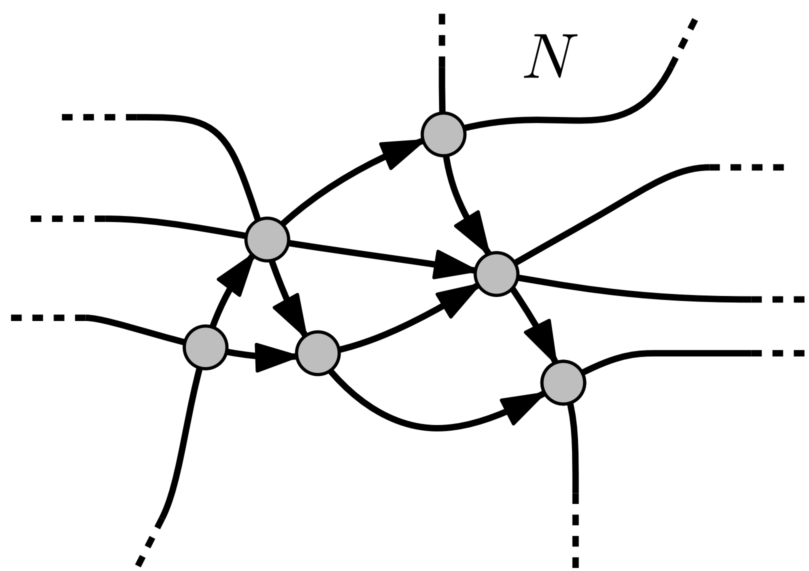


Two facts:

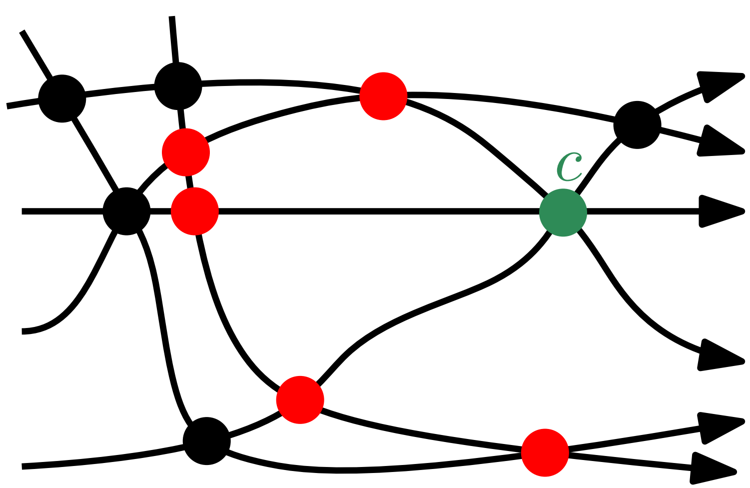
- This oriented graph is acyclic
- A pseudoline touches a face at most once

Face respecting colorings

Proposition: For every arrangement $\mathscr{A}$ there is a face respecting coloring using n colors.

Color greedily in order of top. sort.

Potential conflict ancestors of a crossing c:



Claim: Every crossing has at most $n-1$ conflict ancestors.

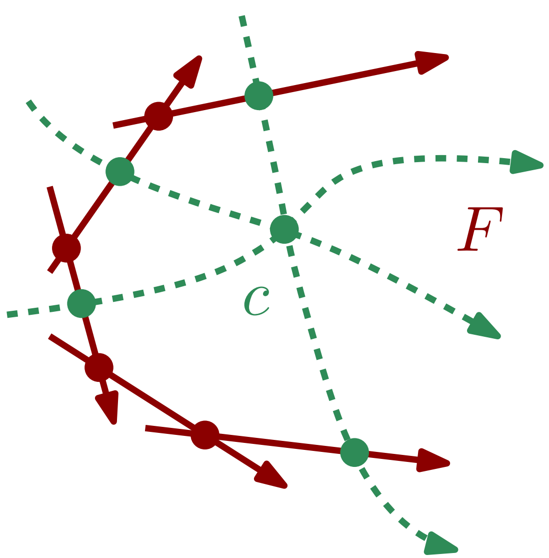
Face respecting colorings

Proposition: For every arrangement $\mathscr{A}$ there is a face respecting coloring using n colors.

Pseudolines crossing in c: $p_1, \cdots, p_k$

Let $\mathscr{A}' := \mathscr{A} - \{p_1, \cdots, p_k\}$

Case I: c lies in *unbounded* face F of $\mathscr{A}'$.



- At most $n - k - 1$ crossings on boundary of F
- $p_1, \cdots, p_k$ add at most k conflict ancestors

Face respecting colorings

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Pseudolines crossing in c: $p_1, \cdots, p_k$

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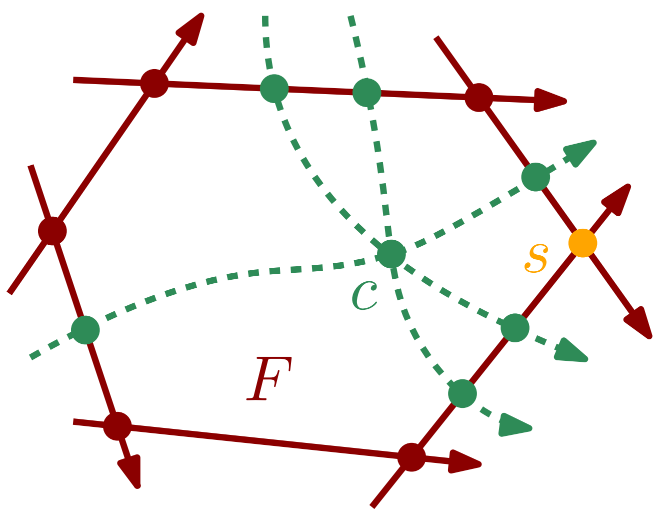


- At most $n-k$ crossings on boundary of F
- Crossing s cannot be conflict ancestor
- $p_1, \cdots, p_k$ add at most k conflict ancestors

□

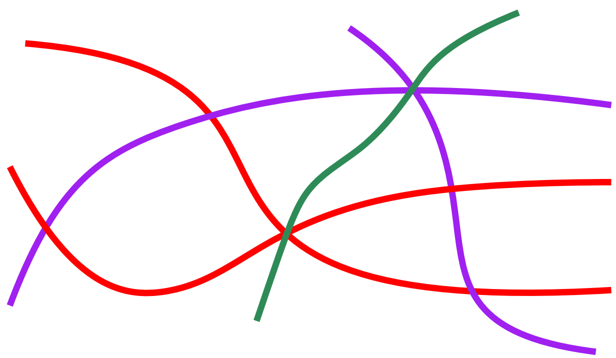
Pseudoline colorings

Def: *Pseudoline coloring* of arrangement $\mathscr{A}$:

- Coloring of the pseudolines of $\mathscr{A}$
- Avoiding monochromatic crossings

$\chi_{pl}(\mathscr{A})$: minimal number of colors in pseudoline coloring

Example:



$\chi_{pl}(\mathscr{A}) = 3$

First observations:

- $2 \leq \chi_{pl}(\mathscr{A}) \leq n$ (unless $n < 2$)
- $\mathscr{A}$ simple $\Leftrightarrow \chi_{pl}(\mathscr{A}) = n$

Pseudoline colorings

- *Ordinary point:* Crossing of degree 2.
- Set $\sigma_k(n) := \max\{\#\text{ord. points of } \mathscr{A} : \chi_{pl}(\mathscr{A}) = k\}$

Proposition: We have $\sigma_k(n) \in \Theta(n^2)$. Precisely,

$$t_k(n) - n \leq \sigma_k(n) \leq t_k(n),$$

where $t_k(n)$ is the Turan number.

Proof upper bound:

- *„ordinary graph“* $G_o(\mathscr{A})$:
  vertices $\sim$ pseudolines; edges $\sim$ ordinary points
- If more then $t_k(n)$ ord. points in $\mathscr{A}$,
  then $G_o(\mathscr{A})$ has $(k+1)$-clique (Turán theorem).
- Then these pseudolines need pw. different colors, $\chi_{pl}(\mathscr{A}) > k$.

Pseudoline colorings

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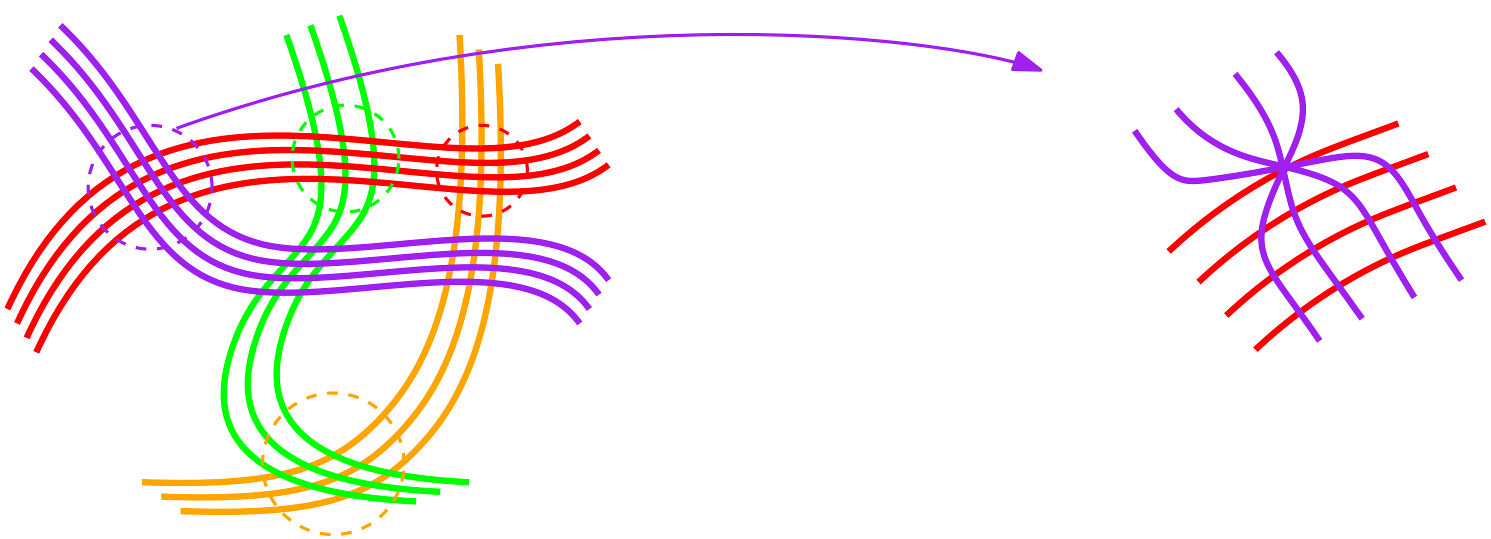
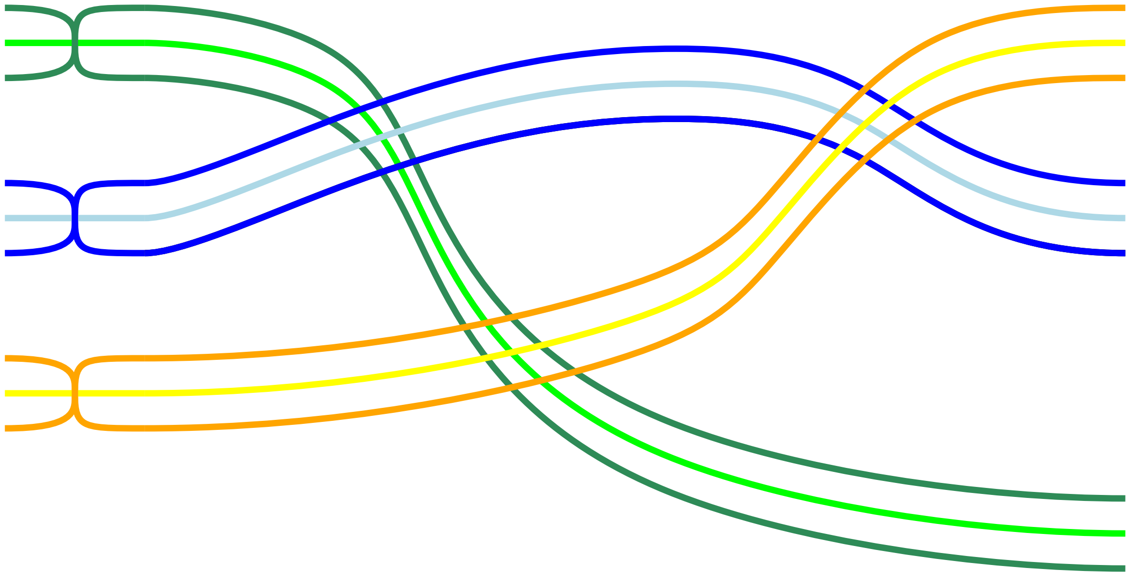
Proof lower bound: Construction for $n = 14$ and $k = 4$:



$\square$

Pseudoline colorings

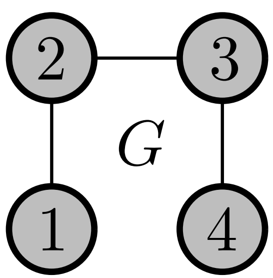
Obs.: There are arbitrary large arrangements $\mathscr{A}$ with $\chi_{pl}(\mathscr{A}) = 2 \cdot \chi(G_o(\mathscr{A}))$.

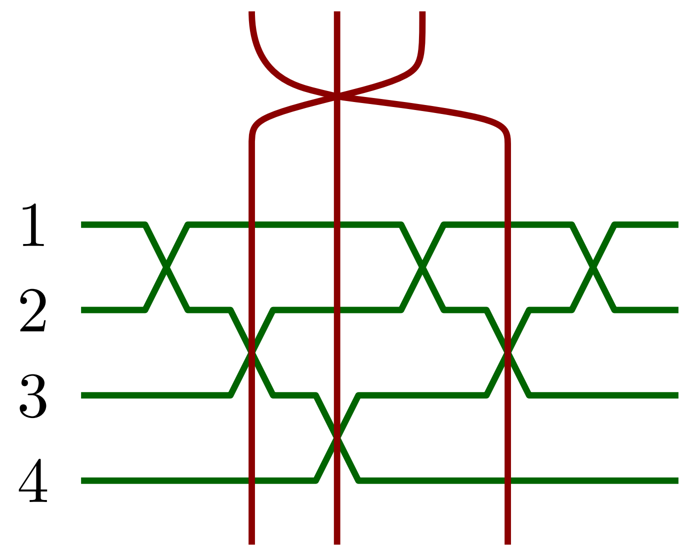


Pseudoline colorings

Computing $\chi_{pl}(\mathscr{A})$ is hard:

- $G = (V, E)$: arbitrary simple graph, $V = [n]$
- Start with arbitrary wiring diagram of pseudolines $p_1, \cdots, p_n$.
- $\forall \{i, j\} \notin E$: *„vertical antenna“* through crossing of p_i and p_j, all antennas meet in common crossing
- **Then:** $G \leq G_o(\mathscr{A})$ and $\chi(G) \leq \chi_{pl}(\mathscr{A}) \leq \chi(G) + 2$





Pseudoline colorings

Observation: Crossings of low degree, especially *ordinary points* (degree 2) play a crucial role!

Proposition:
Using $\mathcal{O}(n^{\frac{1}{l-1}})$ one can color the pseudolines avoiding monochromatic crossings of degree in $\{l, l+1, \cdots, l+r\}$, for any constant r.

Proposition:
Using $\mathcal{O}(\sqrt{n})$ colors one can color the pseudolines avoiding monochromatic crossings of degree at least 4.

Proof: Case $l = 2$ is trivial, assume $l > 2$.

- Assign each pseudoline uniform random color out of $\{1, \cdots, k\}$.
- For each crossing c with $l \leq \deg(c) \leq l + r$ have „*bad event*"

$$\mathcal{E}_c : \ c \text{ is monochromatic} \quad .$$

- We have

$$\mathbb{P}[\mathcal{E}_c] = \frac{1}{k^{\deg(c)-1}} \leq \frac{1}{k^{l-1}} =: p \quad .$$

- $\mathcal{E}_c$ is mut. indep. to all $\mathcal{E}_{c'}$ without pseudoline through c and c'.
- These are all $\mathcal{E}_{c'}$ but at most

$$\deg(c) \frac{n}{l-1} \leq \frac{l+r}{l-1} \cdot n =: d$$

  many.
- By Lovász Local Lemma, if $4pd \leq 1$, i.e.

$$k \geq \left(\frac{4(l+r)n}{l-1} \right)^{\frac{1}{l-1}}$$

  then with positive porbability none of the $\mathcal{E}_c$ happens. □

Pseudoline colorings

Theorem: (Frieze & Mubay, 2013)
Fix $k \geq 3$. Every simple k-uniform hypergraph with maximum degree Δ can be vertex colored using

$$c \cdot \left(\frac{\Delta}{\log \Delta} \right)^{\frac{1}{k-1}}$$

colors, where c depends only on k.

Pseudoline colorings

Idea: Encode arrangement $\mathscr{A}$ as hypergraph $\mathscr{H}$:

- vertices $\sim$ pseudolines
- hyperedges $\sim$ crossings

Consequence: Fix $l \geq 3$. Using

$$c \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{l-1}}$$

colors, one can find a pseudoline coloring avoiding monochromatic crossings of degree l.

Coloring lines of line arrangements

Theorem: (Ackermann, Pach, Pinchasi, Radoičić, Toth, 2014)
The lines of every arrangement of lines can be colored using $\mathcal{O}\left(\sqrt{n/\log n}\right)$ colors avoiding monochromatic faces.

Theorem: (Bose, Cardinal, Collette, Hurtado, Korman, Langerman, Taslaskian, 2013)

For every $n_0 \in \mathbb{N}$ there is an arrangement of $n \geq n_0$ lines that needs at least $\Omega(\log n/\log\log n)$ colors for coloring the lines avoiding monochromatic faces.

Questions.... ?