Mittagsseminar 3.11.2023

Coloring problems on arrangements of pseudolines

Sandro Roch

(Joint work with Rimma Hämäläinen)

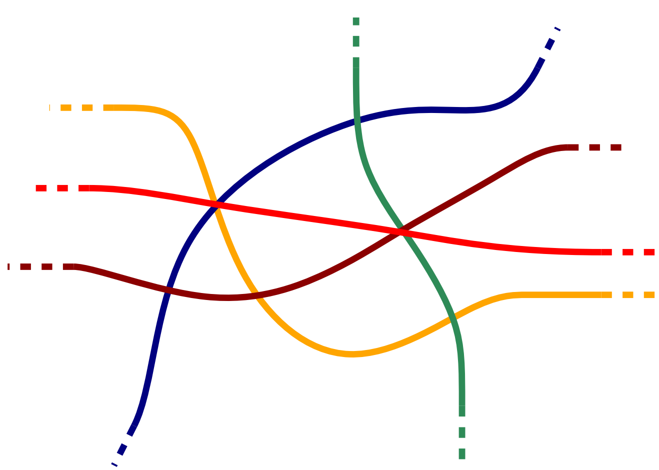
Pseudoline arrangements

Pseudoline arrangement: Finite family of continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}^2$ with

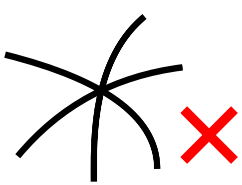
$$\lim_{t \to \infty} \|f_i(t)\| = \lim_{t \to -\infty} \|f_i(t)\| = \infty,$$

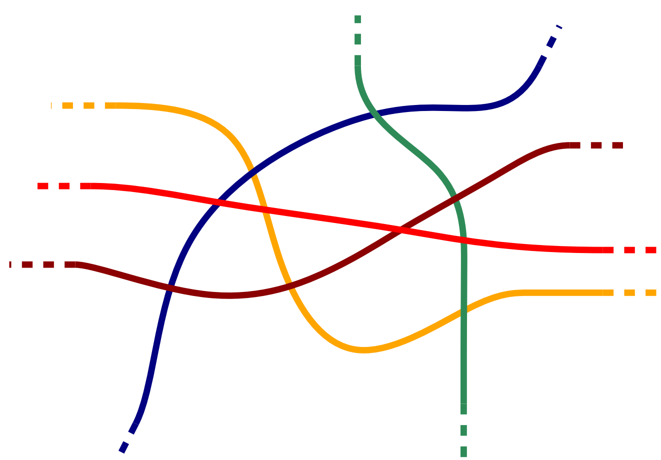
each two of which intersect in exactly one point.

Ex: *nonsimple* arrangement:



Ex: *simple* arrangement:



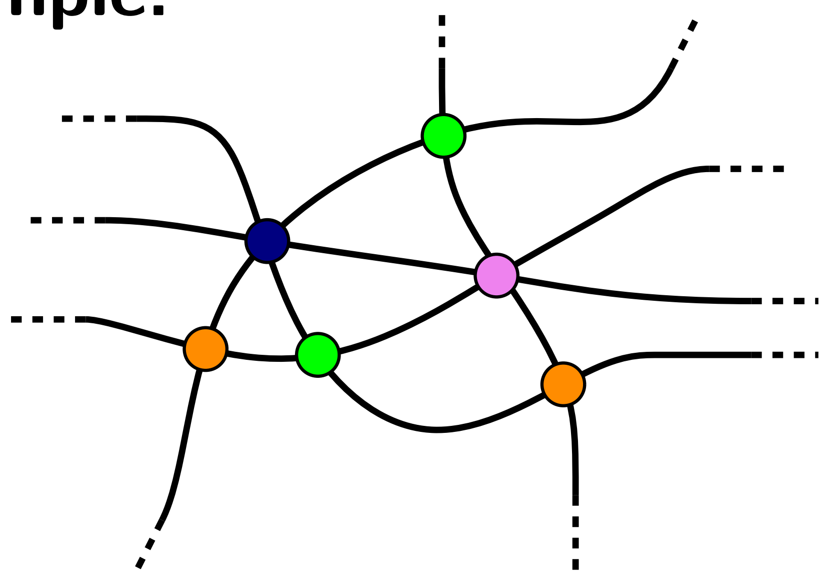


Crossing colorings

Def: *Crossing coloring* of arrangement $\mathscr{A}$:

- Coloring of the crossings of $\mathscr{A}$
- Avoiding twice the same color along any pseudoline

Example:



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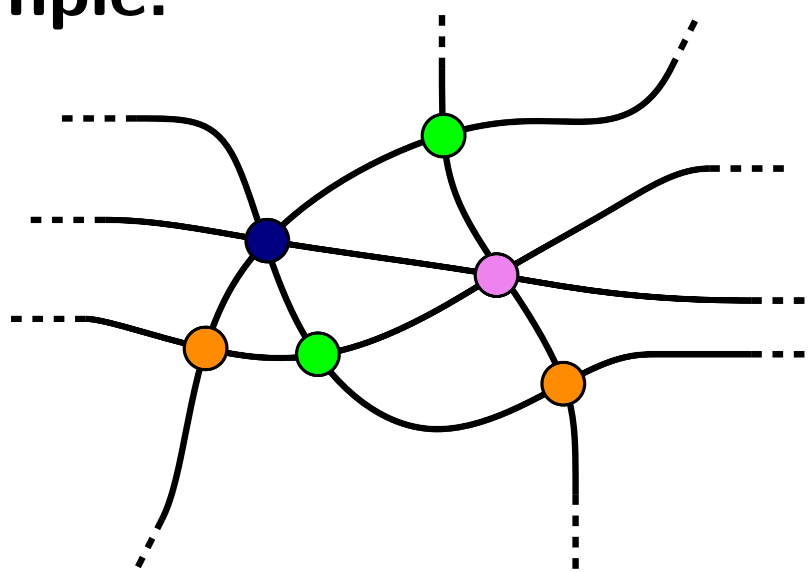

Question: How many colors are required, depending on n?

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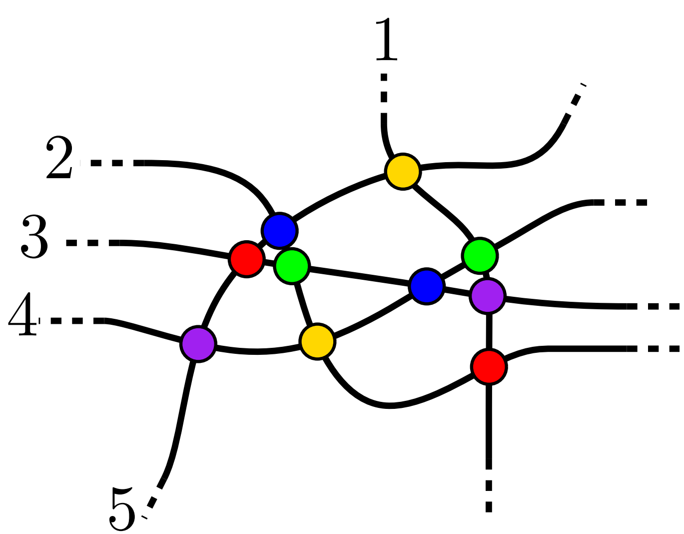
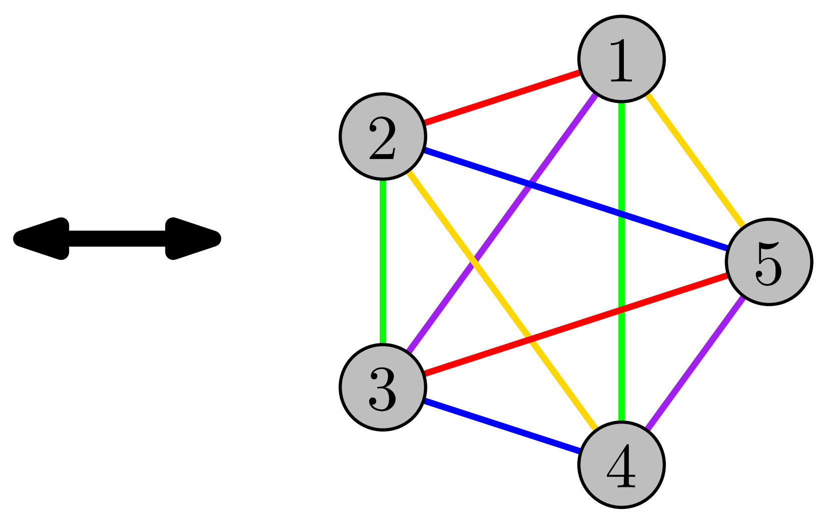
If $\mathscr{A}$ is simple: Equivalent to edge coloring of K_n:

- For $n \in 2\mathbb{Z}$: Need exactly $n-1$ colors.
- For $n \in 2\mathbb{Z}+1$: Need exactly n colors.





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Erdős-Faber-Lovász conjecture

Problem:
Let $|A_i| = n$ for $1 \le i \le n$, $|A_i \cap A_j| \le 1$ for all $i < j$.
Can one color the elements $\bigcup_i A_i$ using n colors such that each A_i contains all colors?

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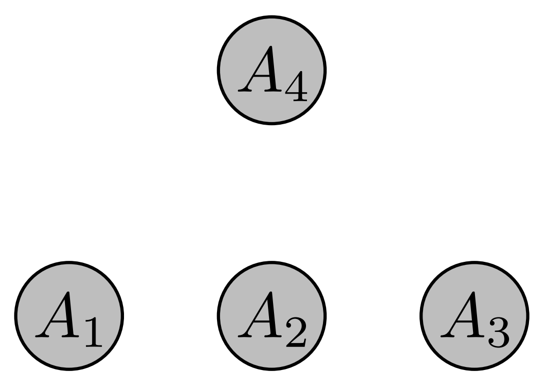

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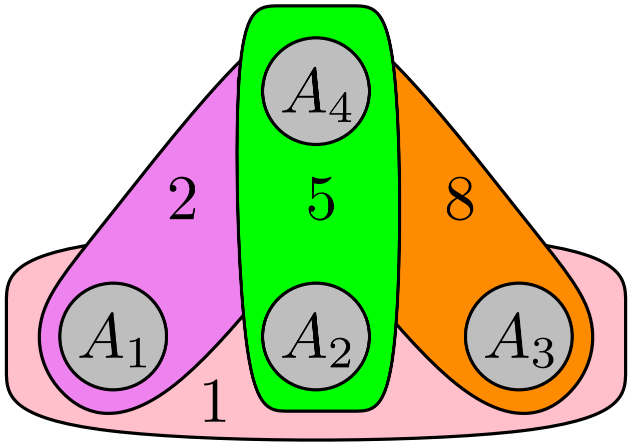

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Proven in 2021!
By D. Y. Kang, T. Kelly, D. Kühn, A. Methuku & D. Osthus

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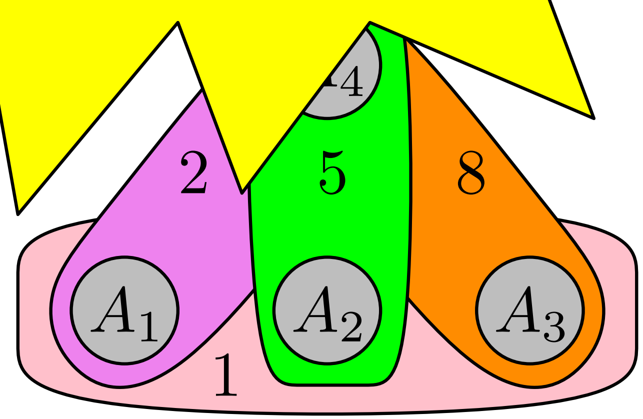

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Algorithm by Chang & Lawler (1987) uses $1.5n - 2$ colors.

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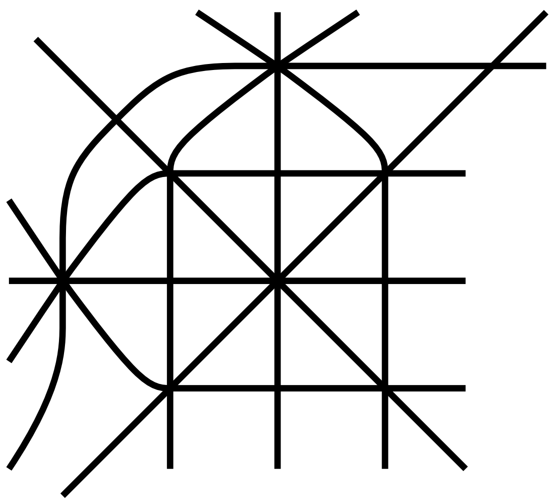

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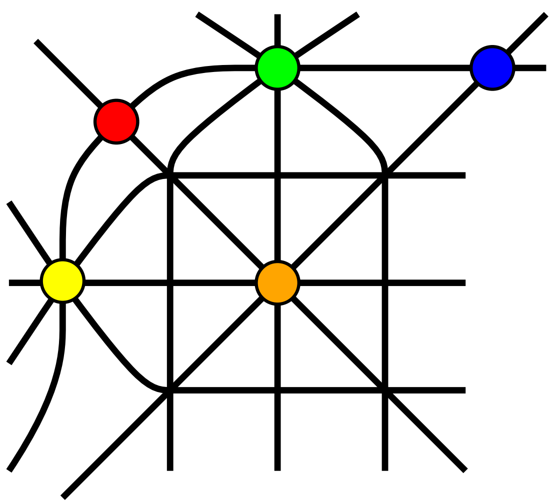

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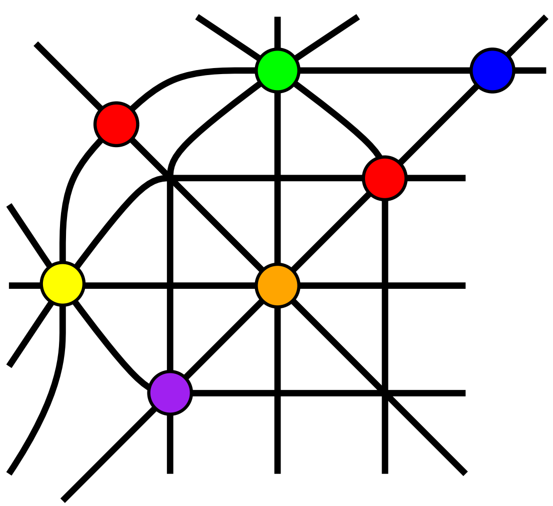

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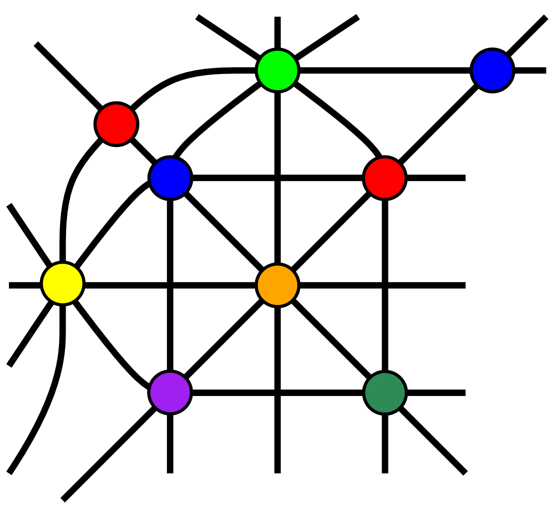

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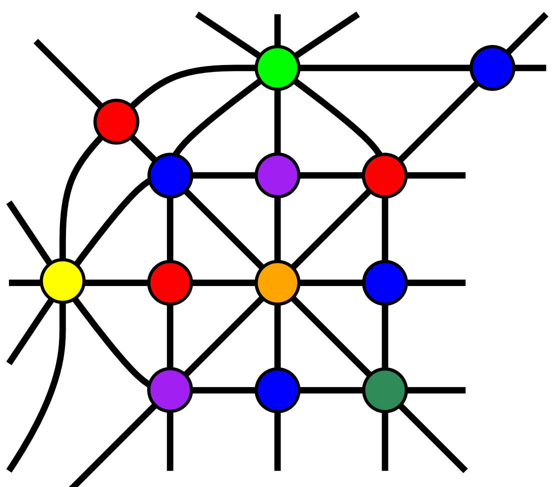

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Need $\mathrm{mx}(\mathscr{A}) + 3 = 7$ colors.

Crossing colorings

Theorem: (Pippenger & Spencer, 1989)
For every $k, \varepsilon > 0$, there exists $\delta > 0$ such that the following holds:

If $\mathscr{H}$ is a k-uniform D-regular hypergraph of codegree at most δD, then $\mathscr{H}$ can be edge colored using $(1+\varepsilon) \cdot D$ colors.

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Theorem: (Kahn, 1996)
For every $k, \varepsilon > 0$, there exists $\delta > 0$ such that the following holds:

If $\mathscr{H}$ is a k-bounded hypergraph of max-degree at most D and codegree at most δD, then $\mathscr{H}$ can be list edge colored using $(1 + \varepsilon) \cdot D$ colors.

Crossing colorings

Consequence: For every $k, \varepsilon > 0$ there is a mx_0 so that for every arrangement $\mathscr{A}$ with $\mathrm{mx}(\mathscr{A}) \geq \text{mx}_0$ and only having crossings of degree at most k there exists a valid crossing coloring using

$$\chi(\mathscr{A}) \leq (1 + \varepsilon) \cdot \mathrm{mx}(\mathscr{A})$$

colors.

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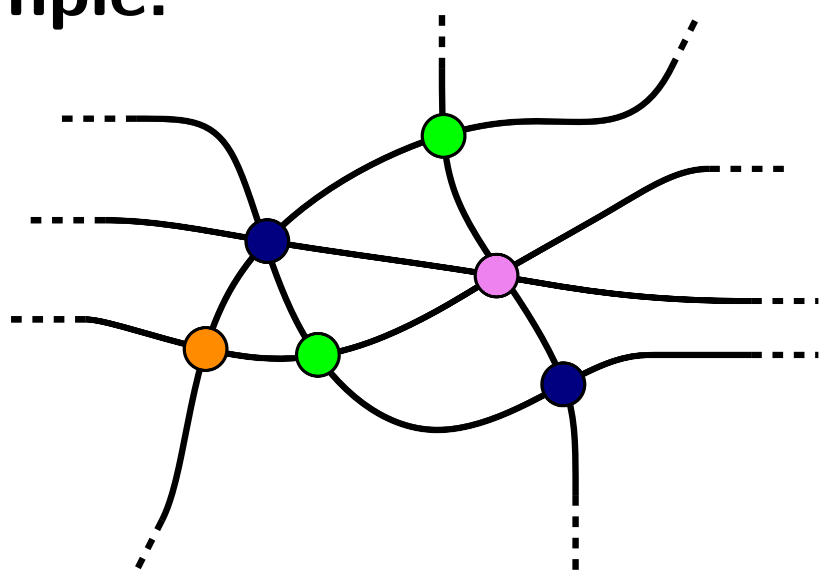

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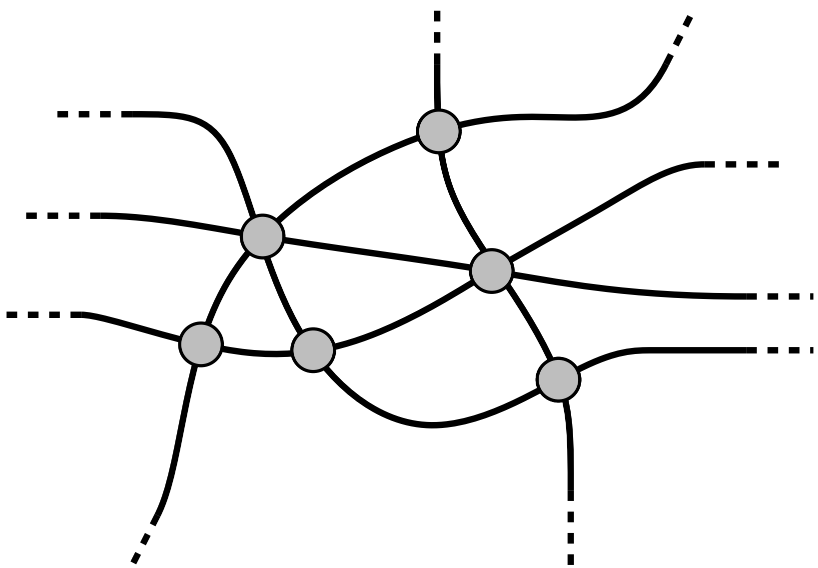

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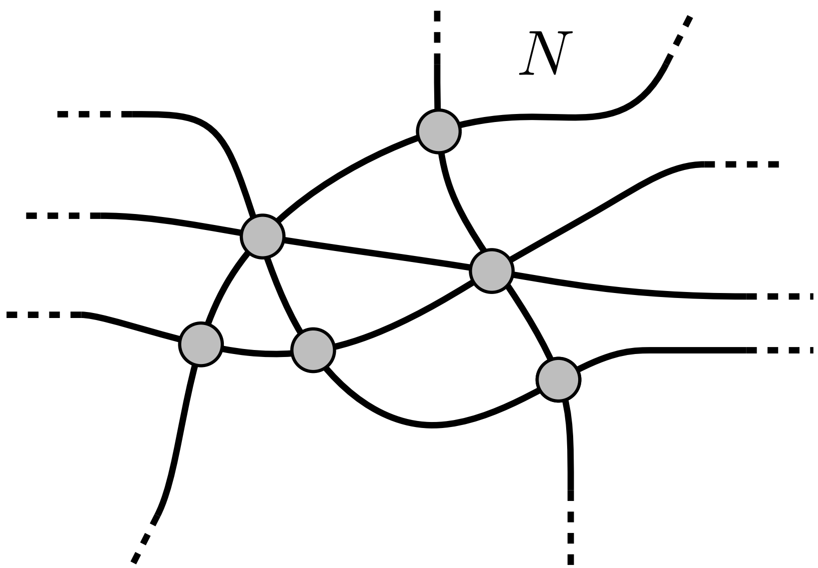

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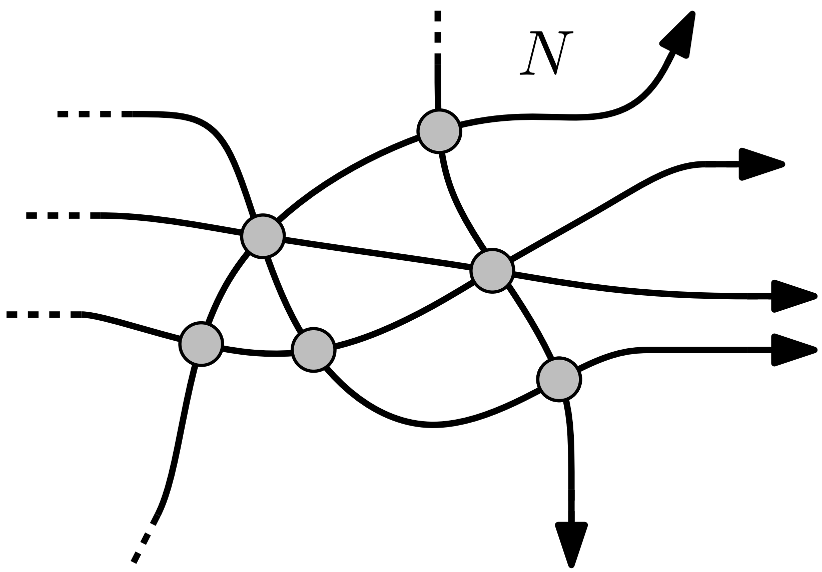

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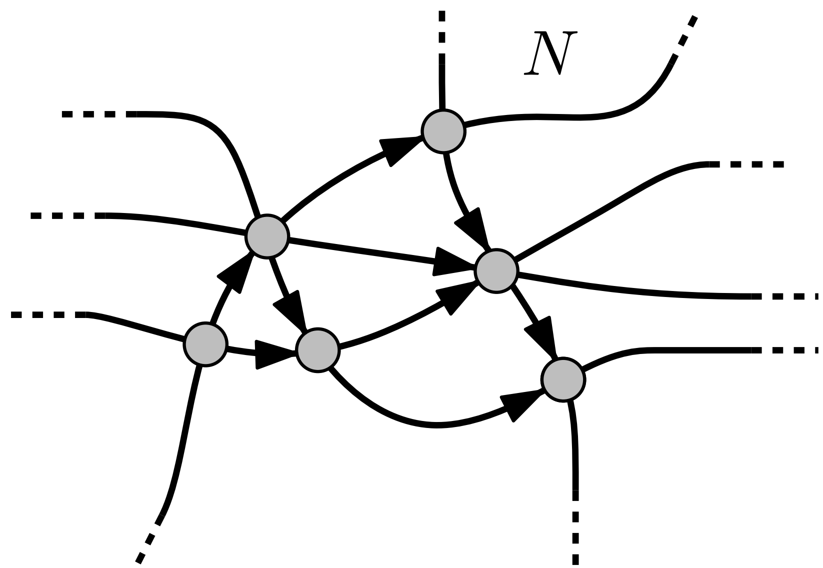

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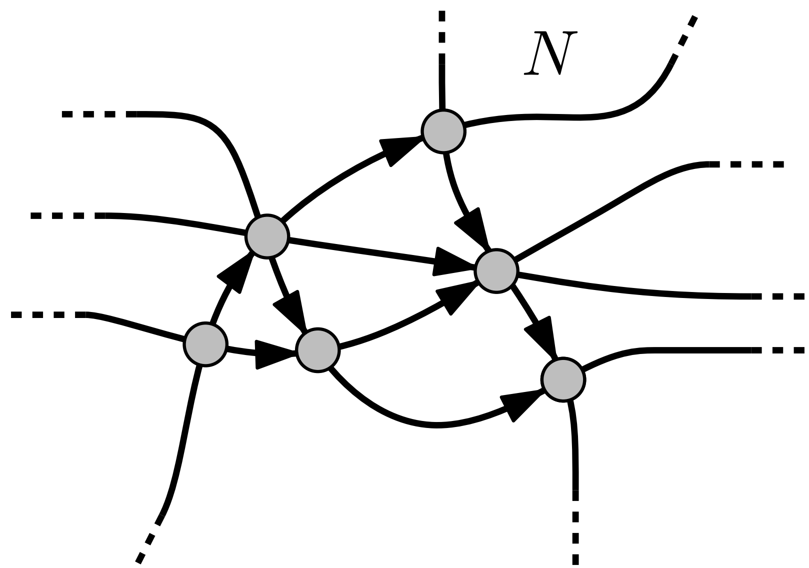

Two facts:

- This oriented graph is acyclic
- A pseudoline touches a face at most once

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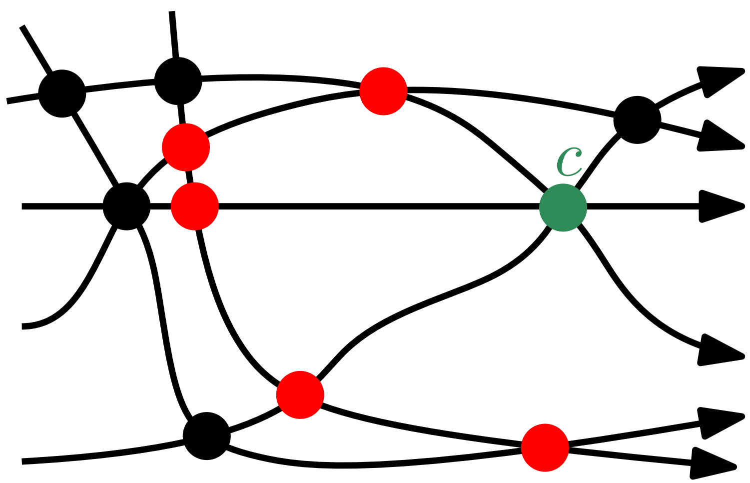
Potential conflict ancestors of a crossing c:



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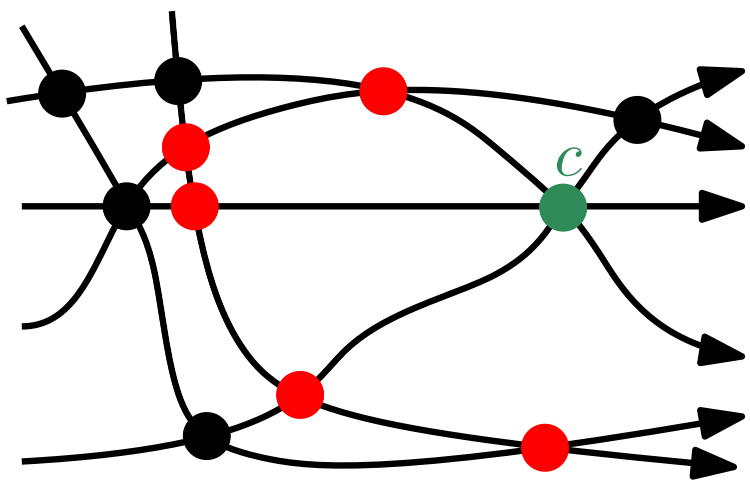

Claim: Every crossing has at most $n-1$ conflict ancestors.

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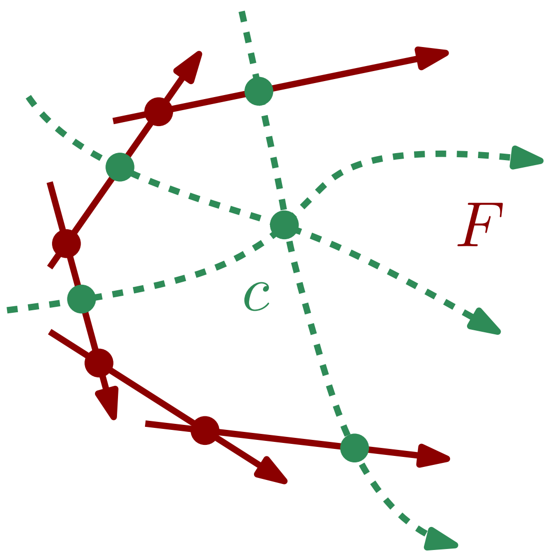

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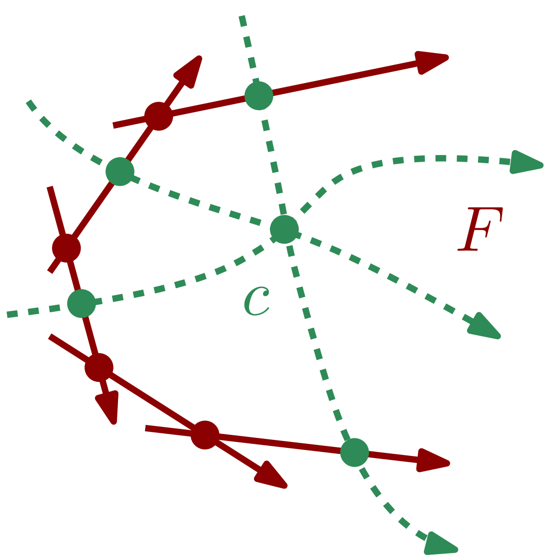

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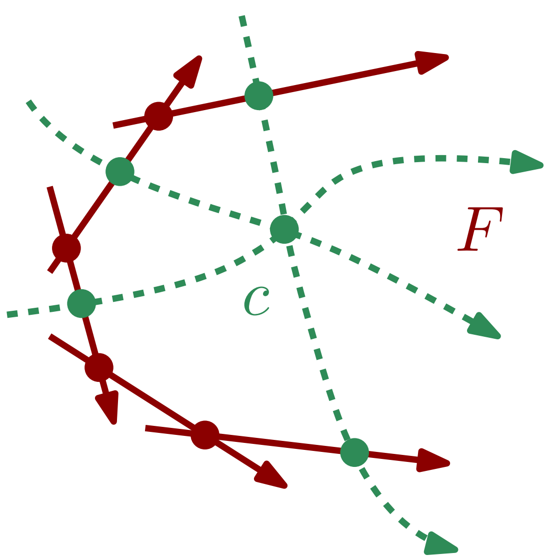

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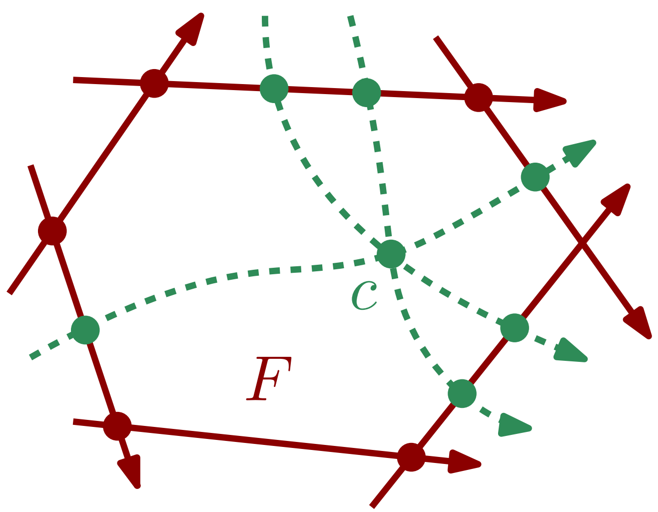

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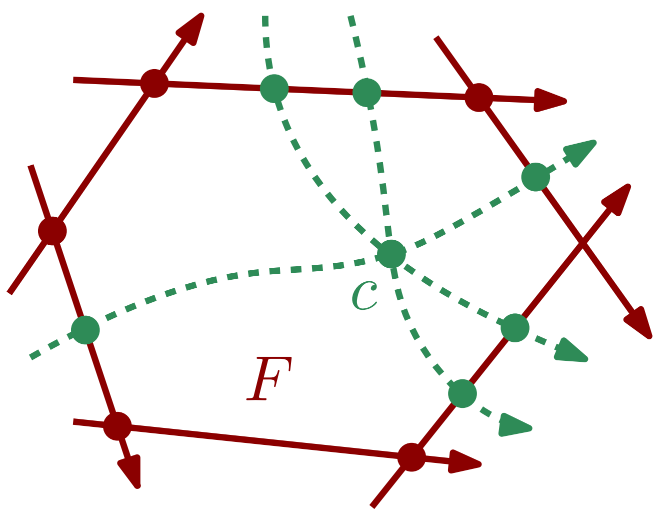

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Proposition: For every arrangement $\mathscr{A}$ there is a face respecting coloring using n colors.

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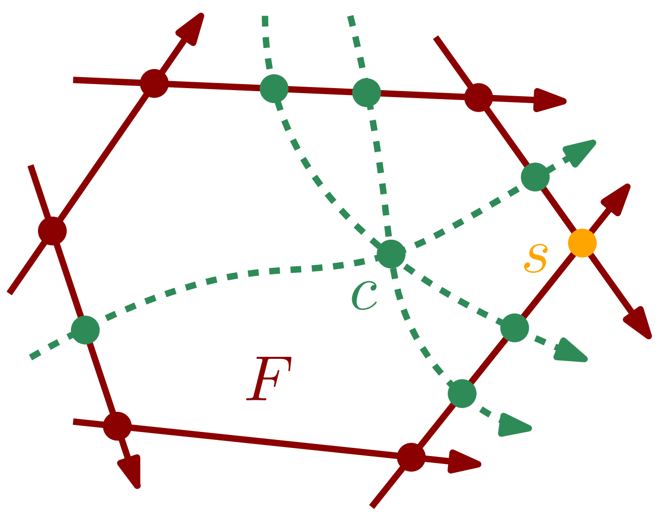
Let $\mathscr{A}' := \mathscr{A} - \{p_1, \cdots, p_k\}$

Case II: c lies in *bounded* face F of $\mathscr{A}'$.



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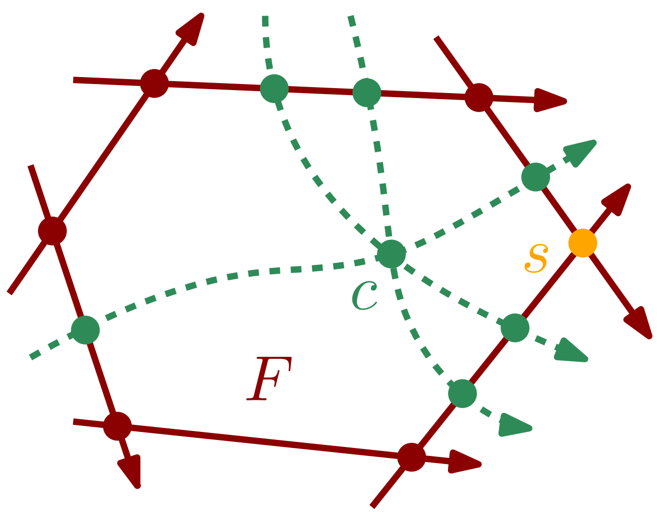

- At most $n - k$ crossings on boundary of F
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- $p_1, \cdots, p_k$ add at most k conflict ancestors

□

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Def: *Pseudoline coloring* of arrangement $\mathscr{A}$:

- Coloring of the pseudolines of $\mathscr{A}$
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$\chi_{pl}(\mathscr{A})$: minimal number of colors in pseudoline coloring

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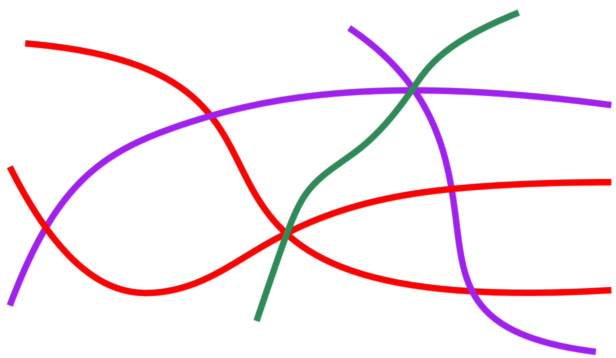

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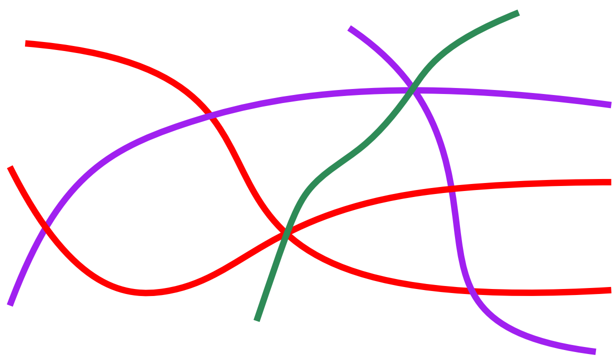

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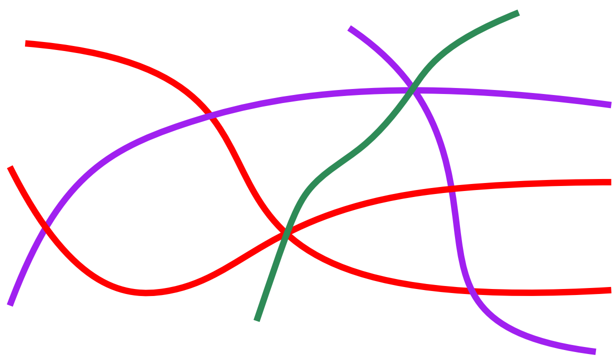

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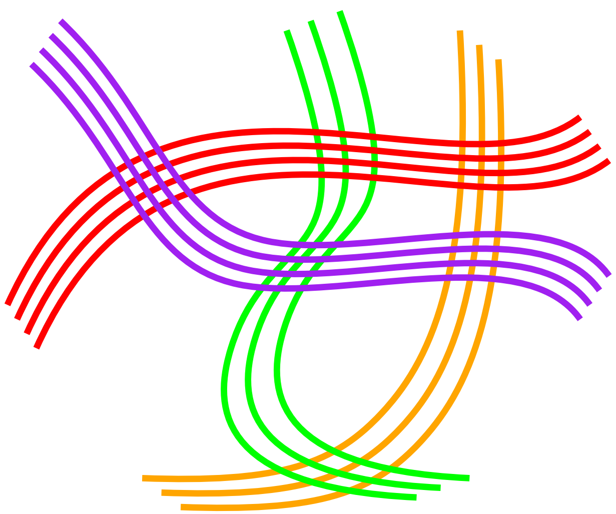

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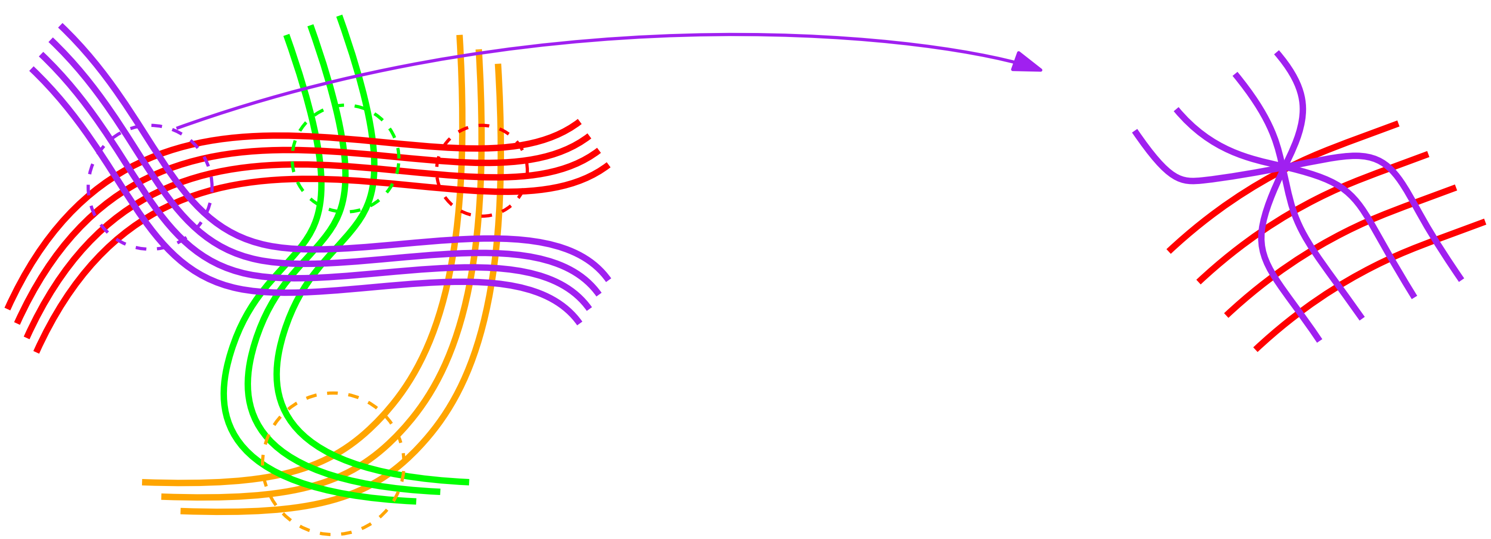

□

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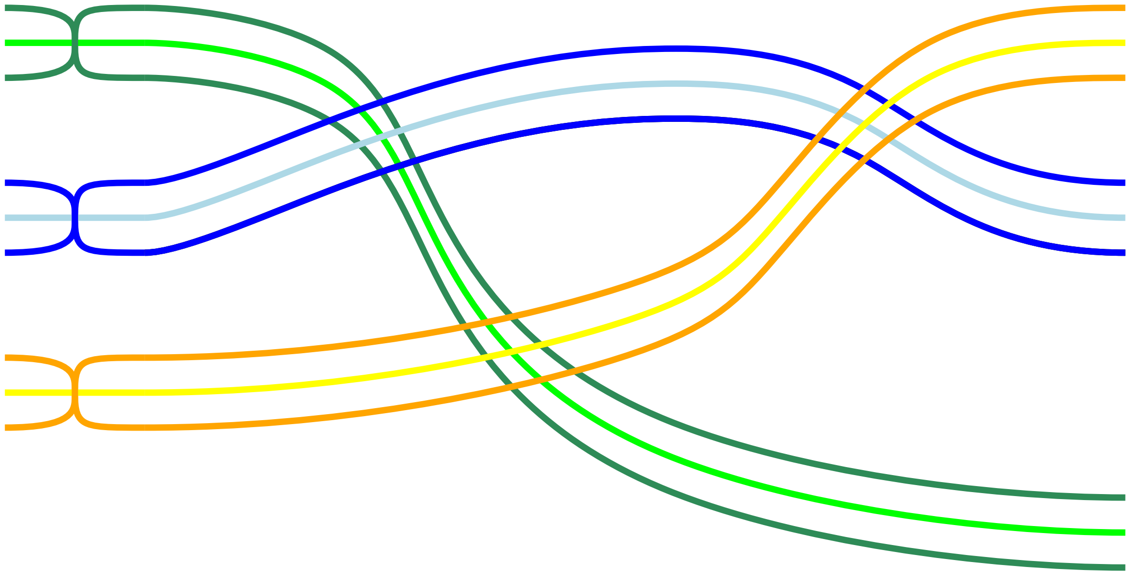
Obs.: There are arbitrary large arrangements $\mathscr{A}$ with $\chi_{pl}(\mathscr{A}) = 2 \cdot \chi(G_o(\mathscr{A}))$.



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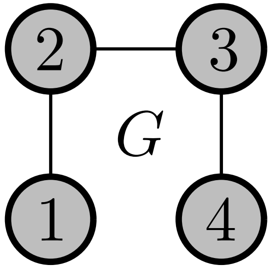

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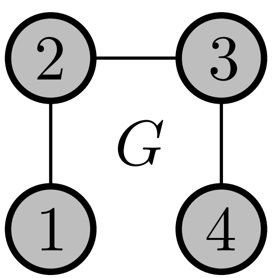

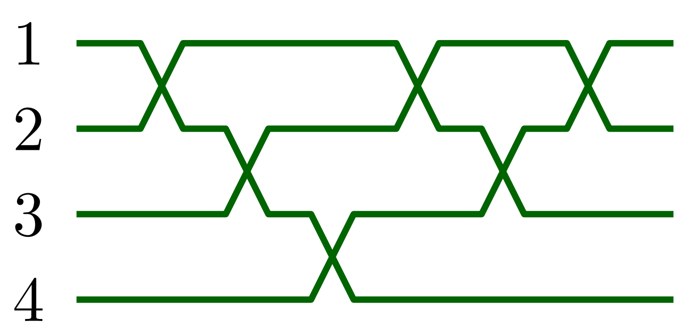

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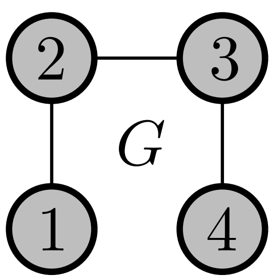

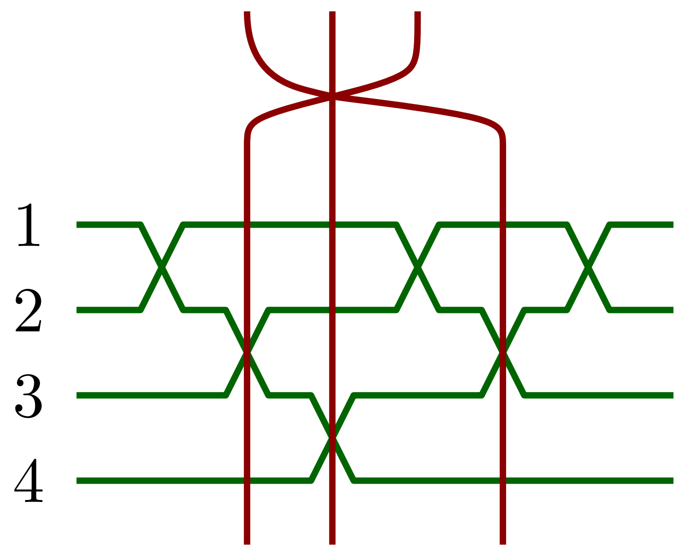

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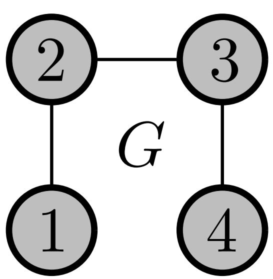
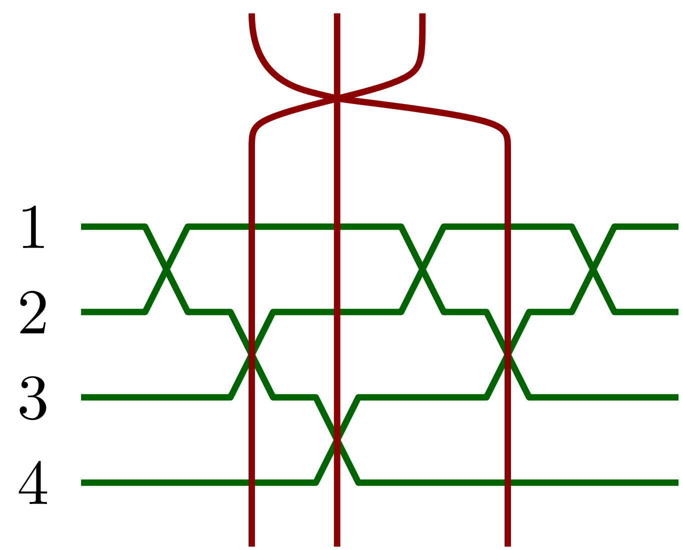
- **Then:** $G \leq G_o(\mathscr{A})$ and $\chi(G) \leq \chi_{pl}(\mathscr{A}) \leq \chi(G) + 2$





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Using $\mathcal{O}(\sqrt{n})$ colors one can color the pseudolines avoiding monochromatic crossings of degree at least 4.

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  many.
- By Lovász Local Lemma, if $4pd \leq 1$, i.e.

$$k \geq \left(\frac{4(l+r)n}{l-1} \right)^{\frac{1}{l-1}}$$

  then with positive porbability none of the $\mathcal{E}_c$ happens. $\square$

Pseudoline colorings

Theorem: (Frieze & Mubay, 2013)
Fix $k \geq 3$. Every simple k-uniform hypergraph with maximum degree Δ can be vertex colored using

$$c \cdot \left(\frac{\Delta}{\log \Delta} \right)^{\frac{1}{k-1}}$$

colors, where c depends only on k.

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Idea: Encode arrangement $\mathscr{A}$ as hypergraph $\mathscr{H}$:

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Consequence: Fix $l \geq 3$. Using

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colors, one can find a pseudoline coloring avoiding monochromatic crossings of degree l.

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Theorem: (Ackermann, Pach, Pinchasi, Radoičić, Toth, 2014)
The lines of every arrangement of lines can be colored using $\mathcal{O}\left(\sqrt{n/\log n}\right)$ colors avoiding monochromatic faces.

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Theorem: (Bose, Cardinal, Collette, Hurtado, Korman, Langerman, Taslaskian, 2013)

For every $n_0 \in \mathbb{N}$ there is an arrangement of $n \geq n_0$ lines that needs at least $\Omega(\log n / \log\log n)$ colors for coloring the lines avoiding monochromatic faces.

Questions.... ?