Mittagsseminar 14.07.2023

A lower bound on the mixing time of Glauber dynamics

Largely based on (Hayes & Sinclair, 2007)

Talk by Sandro Roch

Glauber dynamics

- In a *spin system* we have:
  - $G = (V, E)$ graph, V *sites*
  - $Q = \{1, \cdots, q\}$ possible *spins*
  - *configuration*: assignemnt $\sigma : V \to Q$
  - *feasible configurations:* $\Omega \subseteq Q^V$
  - *hard constraints* iff $\Omega \neq Q^V$

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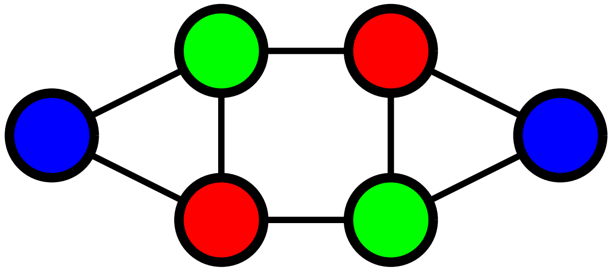
- **Example:** Proper colorings

$$\Omega = \{\sigma \in [q]^V : \forall \{v, w\} \in E : \sigma(v) \neq \sigma(w)\}$$



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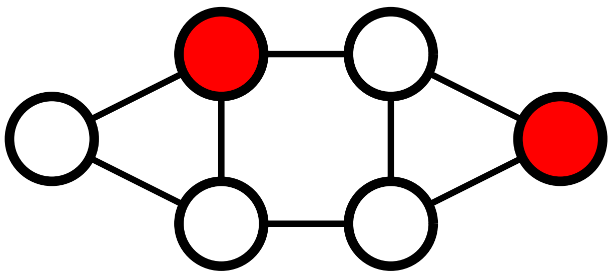
- **Example:** Independent sets

$$\Omega = \left\{\sigma \in \{\texttt{True}, \texttt{False}\}^V : \forall\{v, w\} \in E : \neg(\sigma(v) \land \sigma(w))\right\}$$



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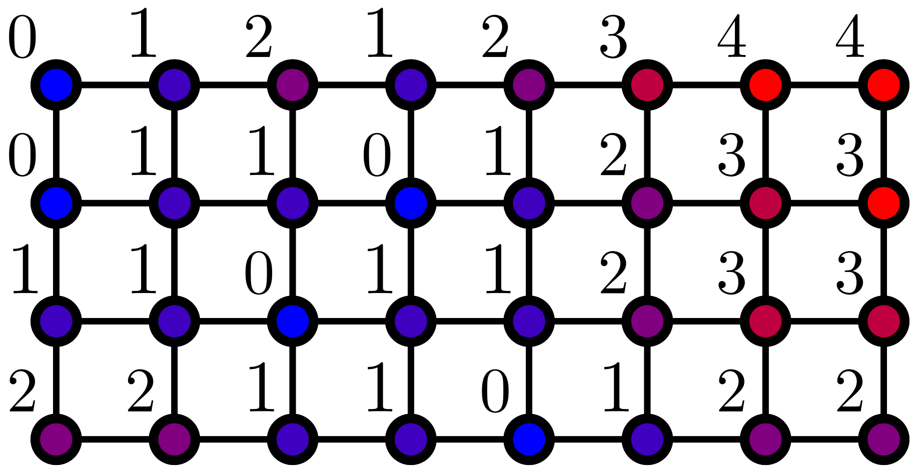
- **Example:** k-heights

$$\Omega = \{\sigma : V \to \{0, \cdots, k\} \mid \forall \{v, w\} \in E : |\sigma(v) - \sigma(w)| \leq 1\}$$



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- **Problem:** Given prob. distribution π on Q^V with

$$\pi(\sigma) > 0 \Leftrightarrow \sigma \in \Omega,$$

  how can we sample from π?
- Fast sampling from π often yields FPRAS for $|\Omega|$.

Glauber dynamics

- *Glauber dynamics*: Markov chain $(\sigma_t) \subset \Omega$
- Transition rule $\sigma \to \sigma'$:
  1) Pick $v \in V$ u.a.r.
  2) Update $\sigma'(v) \sim \kappa_{\sigma,v}$

- Update distribution $\kappa_{\sigma,v} : Q \times Q \to [0,1]$ satisfies:
  - $\kappa_{\sigma,v}$ is *local*, i.e. depends only on σ on $\mathcal{N}(v) \cup \{v\}$
  - $\kappa_{\sigma,v}$ is *reversible* w.r.t. distribution π on Ω, i.e.

$$\pi(\sigma) \cdot \kappa_{\sigma,v}(s, s') = \pi(\sigma') \cdot \kappa_{\sigma',v}(s', s)$$

- Assume (σ_t) is *irreducible* and *aperiodic*, hence *ergodic*.
- Implies $\sigma_t \to \pi$ as $t \to \infty$.

Update rules

Examples of reversible update rules:

- Usual choice: *heatbath* update rule:

$$\kappa_{\sigma,v}(s,s') := \mathbb{P}_\pi[\sigma'(v) = s' \mid \sigma'(w) = \sigma(w), w \neq v]$$

- *metropolis* update rule:

$$\kappa_{\sigma,v}(s,s') := \begin{cases} \frac{1}{|Q|} \cdot \min\left\{ \frac{\mathbb{P}_\pi[\sigma'(v)=s' \mid \sigma'(w)=\sigma(w), w\neq v]}{\mathbb{P}_\pi[\sigma'(v)=s \mid \sigma'(w)=\sigma(w), w\neq v]}, 1 \right\} & : s' \neq s \\ 1 - \sum_{s'\neq s} \kappa_{\sigma,v}(s,s') & : s' = s \end{cases}$$

- *up/down* update rule, if $\pi = \mathrm{Unif}(\Omega)$:
  - with prob. $\frac{1}{2}$, $\sigma'(v) := \sigma(v) + 1$ (if $\sigma'(v) \in Q$ and $\sigma' \in \Omega$)
  - otherwise, $\sigma'(v) := \sigma(v) - 1$ (if $\sigma'(v) \in Q$ and $\sigma' \in \Omega$)

These rules are local in *Markov random fields*

Excursion: Ising model

- Typically $G = (V, E)$ sublattice of $\mathbb{Z}^d$
- Spins $Q = \{-1, 1\}$, configurations $\Omega = Q^V$
- Hamiltonian $H : Q^V \to \mathbb{R}$:

$$H(\sigma) := \sum_{\{u,v\}\in E} \sigma(u)\sigma(v) + h \sum_{v\in V} \sigma(v)$$

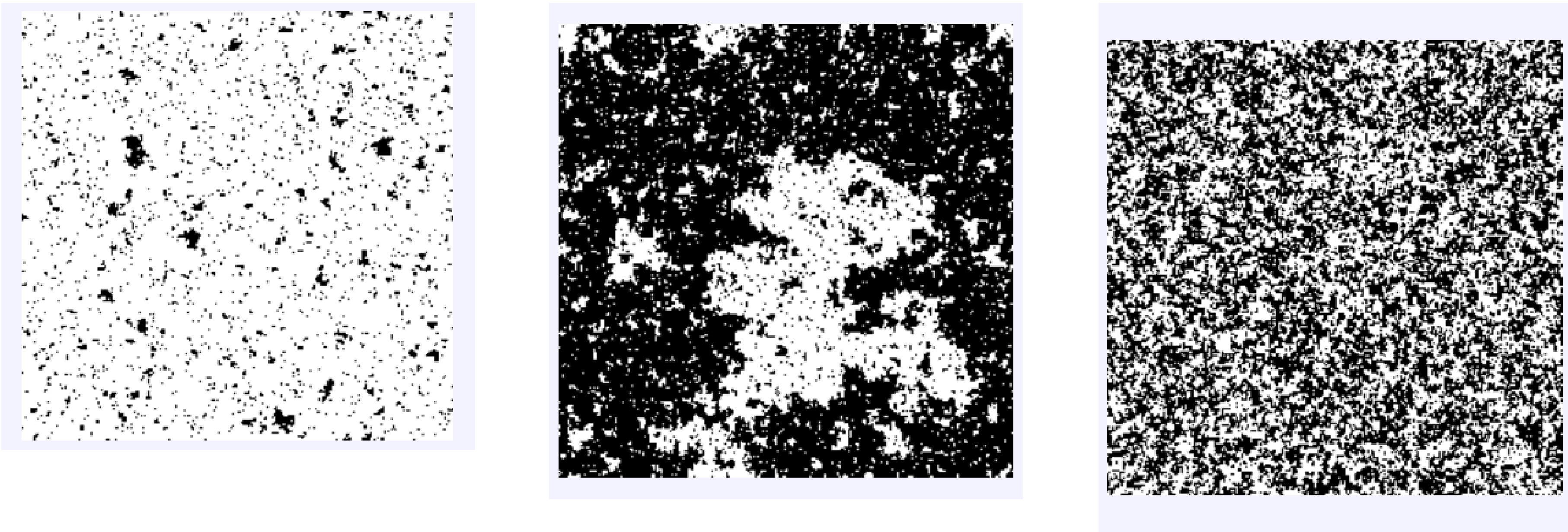
- *Boltzmann distribution* / *Gibbs measure* π on Ω:

$$\pi(\sigma) = \frac{e^{\beta H(\sigma)}}{Z_\beta(\sigma)} \qquad \text{where } Z_\beta = \sum_{\sigma\in\Omega} e^{\beta H(\sigma)}$$

at *inverse temperature* $\beta \geq 0$, $\beta \propto T^{-1}$

Excursion: Ising model

- Explains phase transition of *ferromagnetism* at critical temperature T_c.



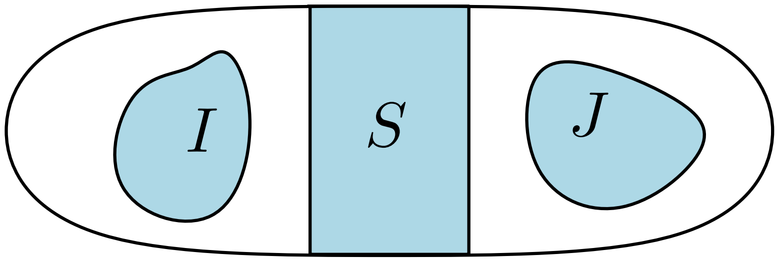
Ising model on 250×250 torus at low, critical, and high temperature, respectively. Figure taken from (Levin, Peres & Wilmer, 2017).

Excursion: Markov random fields

- *Markov random field*: graph $G = (V, E)$ with random variables $(X_v)_{v \in V}$ satisfying *global Markov property*:

$$X_I \perp\!\!\!\perp X_J \mid X_S$$

for all $I, J \subset V$ separated by $S \subset V$.



$G = (V, E)$

- In discrete case this means:
$$\mathbb{P}[X_i = x_i, i \in I \mid X_S, X_J] = \mathbb{P}[X_i = x_i, i \in I \mid X_S]$$

Excursion: Markov random fields

- Heatbath update rule becomes local:

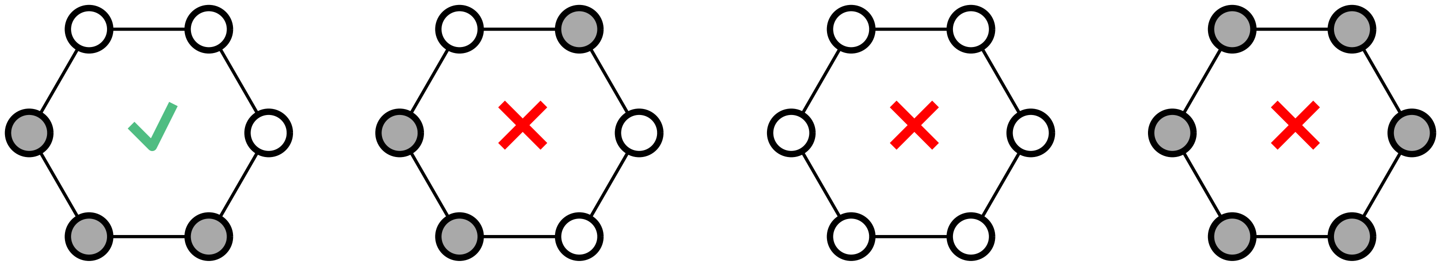
$$\kappa_{\sigma,v}(s,s') := \mathbb{P}_\pi[\sigma'(v) = s' \mid \sigma'(w) = \sigma(w), w \neq v]$$
$$= \mathbb{P}_\pi[\sigma'(v) = s' \mid \sigma'(w) = \sigma(w), w \in \mathcal{N}(v)]$$

- **Examples:**
  - Ising model with Gibbs measure
  - Unif. distrib. on colorings; indp. sets; k-heights; etc.

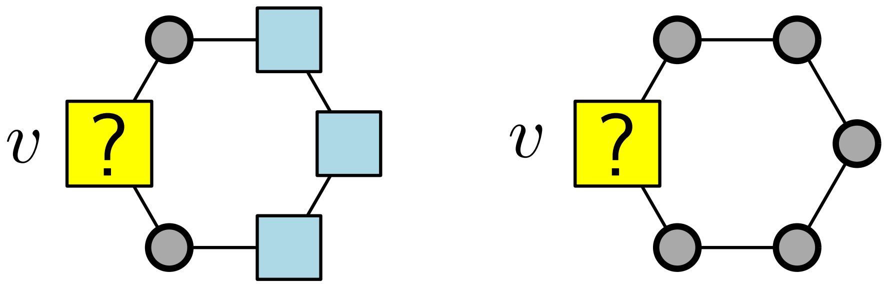
Theorem: (Hammersley & Clifford, 1971)
Let $p(x)$ be the joint density function of a family of random variables $(X_v)_{v \in V}$ with $p(x) > 0$. Then, (X_v) is a Markov random field if and only if $p(x)$ is a Gibbs distribution.

Non-markovian example

- $G = C_n$, $Q = \{0,1\}$,
  $\Omega = \left\{\sigma \in Q^V \mid 1\text{'s form block of length } 1 \le l \le n-1\right\}$



- Local update rule at v_i: flip, iff $\sigma(v_{i-1}) \neq \sigma(v_{i+1})$
- Dynamics reversible w.r.t. $\pi = \mathrm{Unif}(\Omega)$



$$\mathbb{P}_\pi[\sigma(v) = 1 \mid \mathcal{N}(v)] \neq \mathbb{P}_\pi[\sigma(v) = 1 \mid V \setminus \{v\}]$$

Mixing time

- *Total variation distance* between prob. dist. μ, μ' on Ω:

$$\begin{aligned} \|\mu - \mu'\|_{TV} &:= \max_{A \subset \Omega} |\mu(A) - \mu(A')| \\ &= \frac{1}{2} \sum_{\sigma \in \Omega} |\mu(\sigma) - \mu(\sigma')| \\ &= \inf\{\mathbb{P}[X \neq Y] \mid \text{cpl. } X \sim \mu, Y \sim \mu'\} \end{aligned}$$

- *Mixing time* of ergodic Markov chain $X_t \to \pi$:

$$\tau := \max_{X_0 \in \Omega} \min \left\{ t \ : \ \|X_t - \pi\|_{TV} < \frac{1}{2e} \right\}$$

Main result

Theorem A (Hayes & Sinclair, 2007)

Let $\Delta \geq 2$ fixed, and let G be any graph on n vertices with maximum degree at most Δ. Any *nonredundant* Glauber dyamics on G has mixing time $\Omega(n \log n)$, where the constant in the $\Omega(\cdot)$ depends only on Δ.

- *nonredundant* means:
  For all v, there exist $\sigma, \sigma' \in \Omega$ with $\sigma(v) \neq \sigma(v')$

Intuition: Coupon collector's problem

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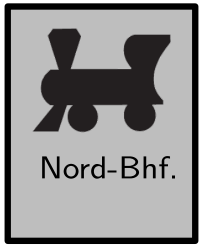

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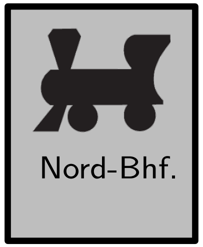

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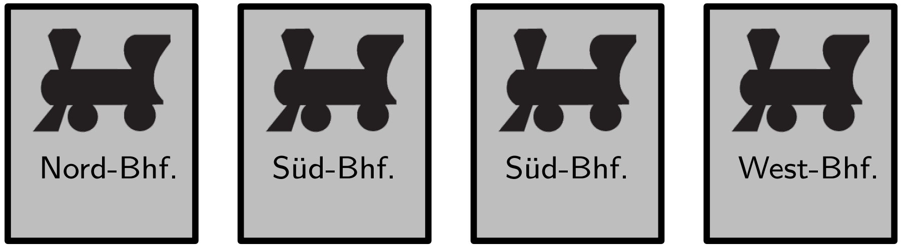

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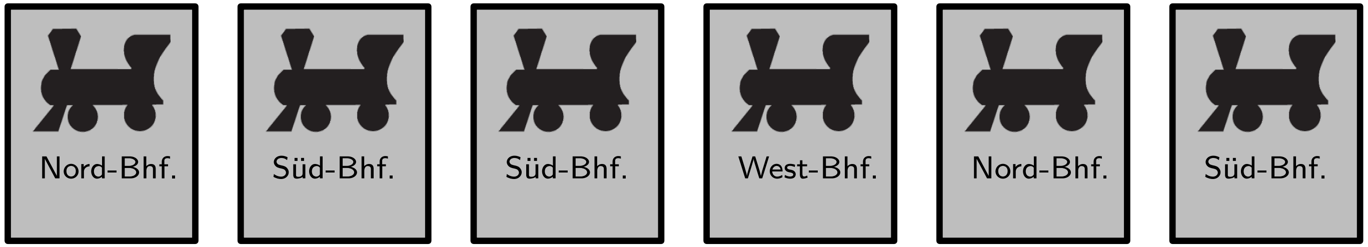

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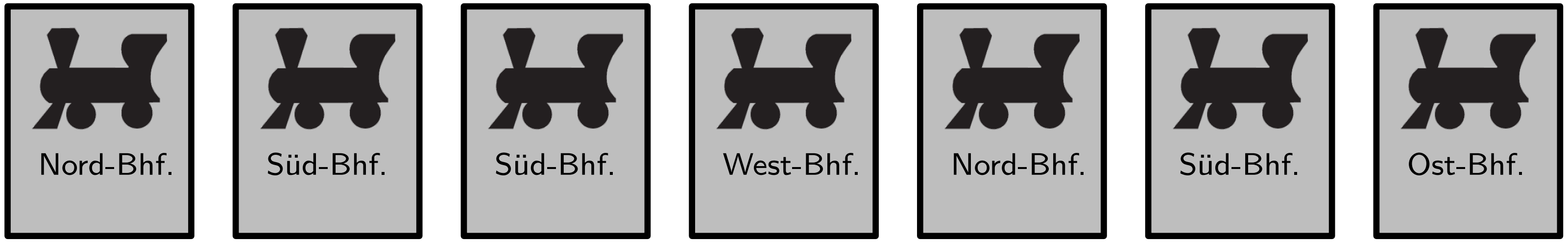

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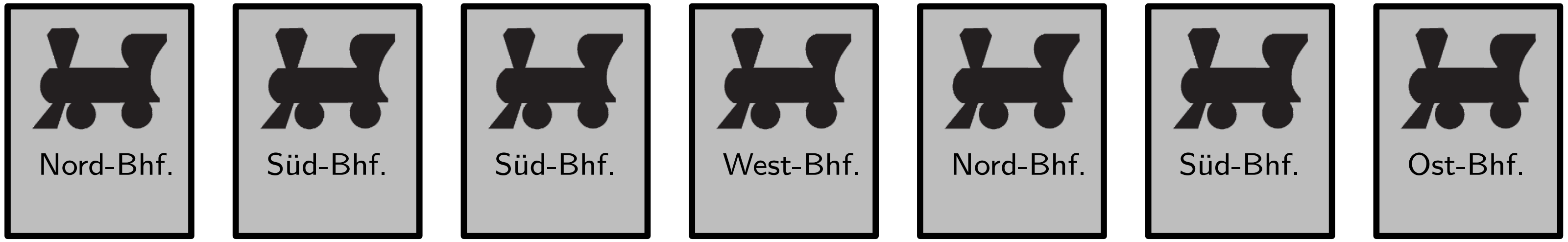

- **Problem:** How many coupons do you have to sample u.a.r. in order to observe all?

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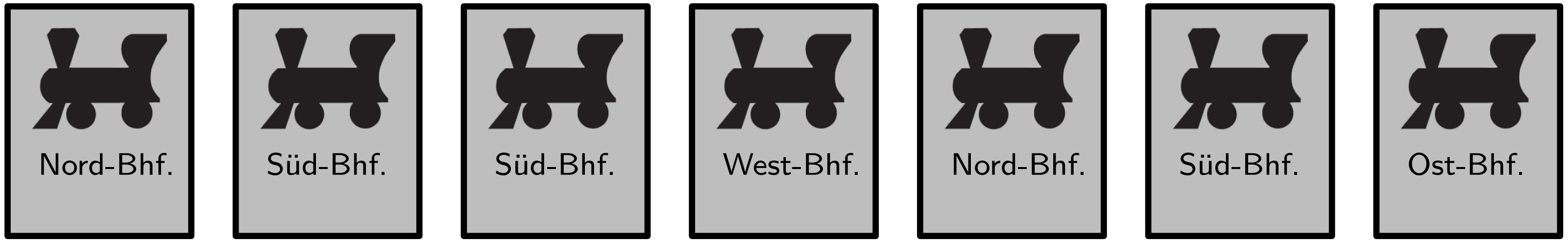


- **Problem:** How many coupons do you have to sample u.a.r. in order to observe all?
- **Answer:**
$$\mathbb{P}\left[|T - nH_n| \geq cn\right] \leq \frac{\pi^2}{6c^2}$$
where $nH_n = n \log n + \gamma n + O(n^{-1})$
- In less than $\Omega(n \log n)$ steps, some sites have never been updated (with high prob.)
- This does **not** imply $\|\sigma_t - \pi\|_{TV} > \frac{1}{2e}$!

Complementary result

Bounding $\Delta(G)$ is necessary for a lower bound of $\tau \in \Omega(n \log n)$ on the mixing time.

Theorem B (Hayes & Sinclair, 2007)

For each n, let $\Delta(n)$ be any natural number satisfying $2 \leq \Delta(n) < n$. Then there exists a family of graphs G_n with n vertices and maximum degree $\Delta(n)$, and an associated Glauber dynamics on G_n with mixing time $O\left(\frac{n \log n}{\log \Delta(n)}\right)$.

Reduction to continuous time

- $(X_t^{\mathscr{D}})_{t\in\mathbb{N}}$ discrete-time Glauber dynamics as before
- $(X_t^{\mathscr{C}})_{t\geq 0}$ *continuous-time Glauber dynamics:*
  - Each vertex v has independent rate-1 *Poisson clock*
  - When clock at v rings: update v
  - Number of updates till time t is $\mathrm{Poi}(nt)$-distributed.
- Express $X_t^{\mathscr{C}}$ in terms of $X_t^{\mathscr{D}}$:

$$\mathbb{P}[X_t^{\mathscr{C}} = \sigma] = \sum_{s=0}^{\infty} e^{-nt}\frac{(nt)^s}{s!} \cdot \mathbb{P}[X_s^{\mathscr{D}} = \sigma]$$

- One verifies: $\tau^{\mathscr{D}} \geq \frac{n}{6} \cdot \tau^{\mathscr{C}}$
- Remains to show: $\tau^{\mathscr{C}} \in \Omega(\log n)$

Greedy coupling

- Two copies (X_t), (Y_t) of same dynamics
- (X_t) and (Y_t) use identical clocks on vertices.
- When clock on v rings, coupling $(X,Y) \to (X',Y')$:
  - Choose $(X'(v), Y'(v))$ by *greedy coupling* of $\mu := \kappa_{(X,v)}(X(v),\cdot)$ and $\mu' := \kappa_{(Y,v)}(Y(v),\cdot)$.
  - *greedy couling* means

$$\mathbb{P}[X'(v) \neq Y'(v)] = \|\mu - \mu'\|_{TV} \ .$$

  - If $X = Y$ on $\mathcal{N} \cup \{v\}$, then

$$\mathbb{P}[X'(v) = Y'(v)] = 1$$

Disagreement percolation

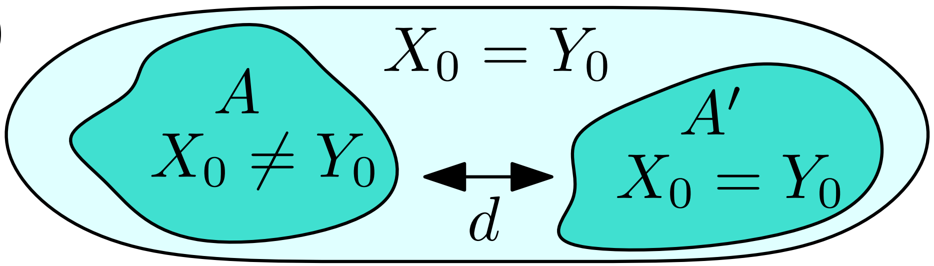
Lemma: (Percolation-Lemma)

Let (X_t) and (Y_t) be continuous-time Glauber dynamics on G with max. deg. at most Δ. Suppose $X_0 = Y_0$ on all sites in $V \setminus A$. Let $A' \subset V$ with $d := \mathrm{dist}(A', A) > 0$. Then, the greedy coupling of (X_t) and (Y_t) satisfies

$$\mathbb{P}[X_t = Y_t \text{ on } A'] \geq 1 - \min\{|\delta A|, |\delta A'|\} \left(\frac{et\Delta}{d}\right)^d$$

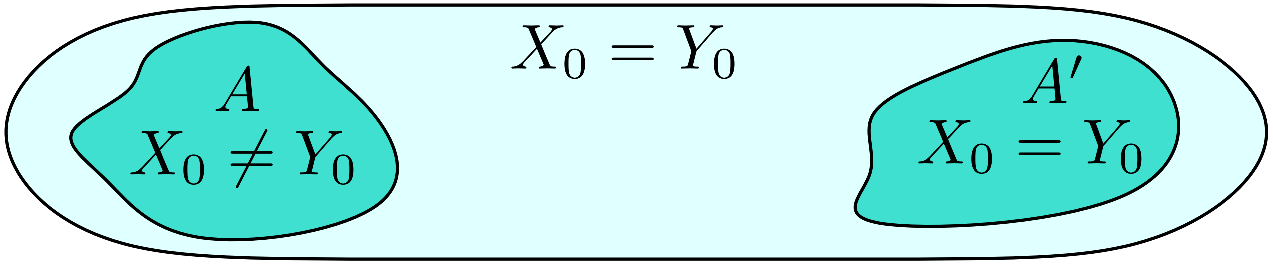
The same holds even if the update probabilities of (X_t) and (Y_t) differ at sites in A.

$G = (V, E)$



Disagreement percolation

Proof:



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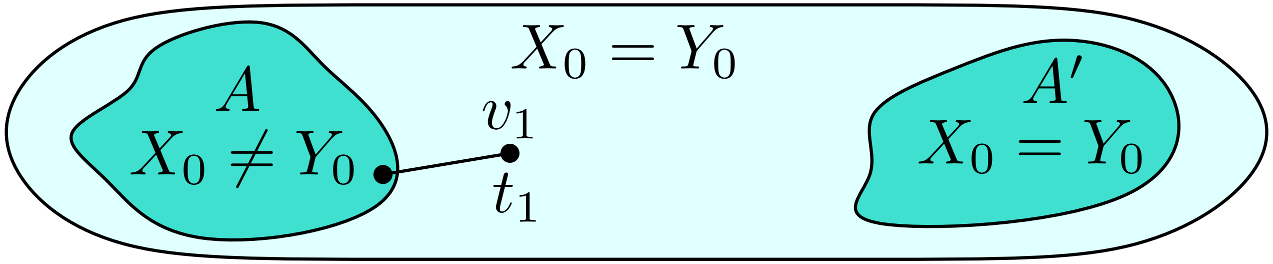

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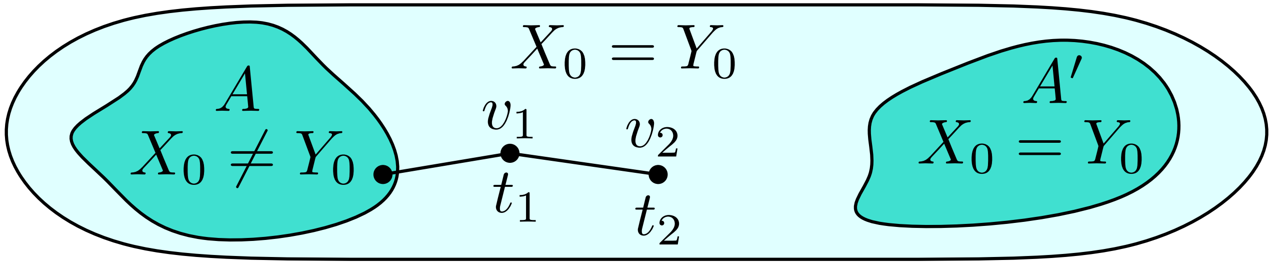

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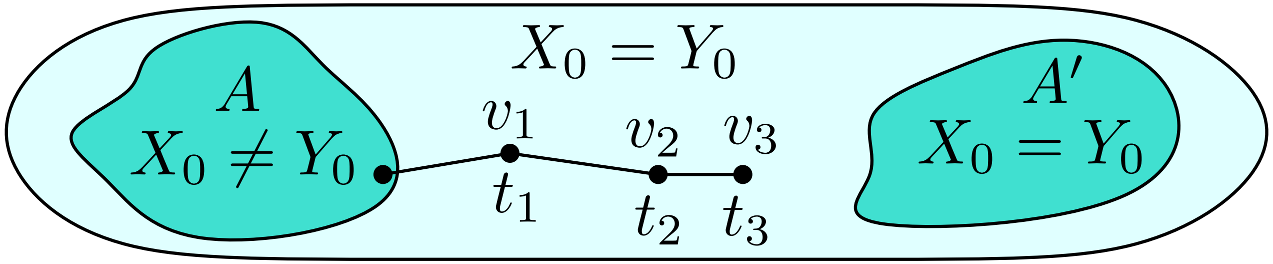

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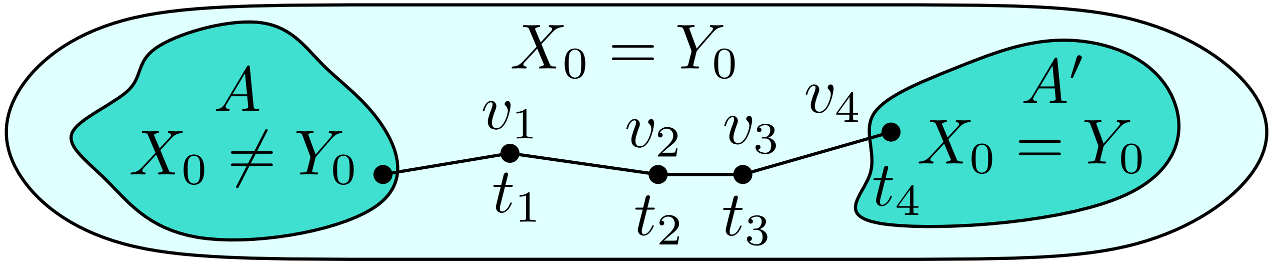

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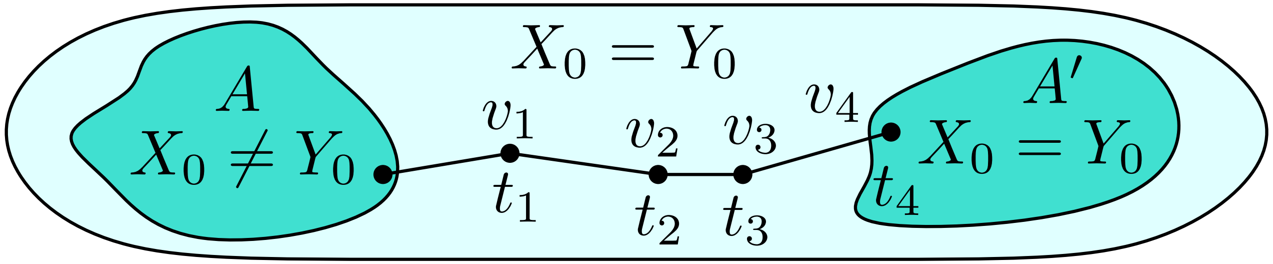


- For affecting A' from A, need updates at times $t_1 < \cdots < t_r$ along some connecting path $v_1, \cdots, v_r$.
- Waiting time for t_{i+1} after t_i is $\mathrm{Exp}(1)$ distributed.
- Prob. p of observing update sequence $t_1 < \cdots < t_d < t$ equals prob. of $\geq d$ rings within time t of rate-1 clock.

$$p = \sum_{i=d}^{\infty} \frac{t^i}{i!} e^{-t} < \left(\frac{et}{d}\right)^d$$

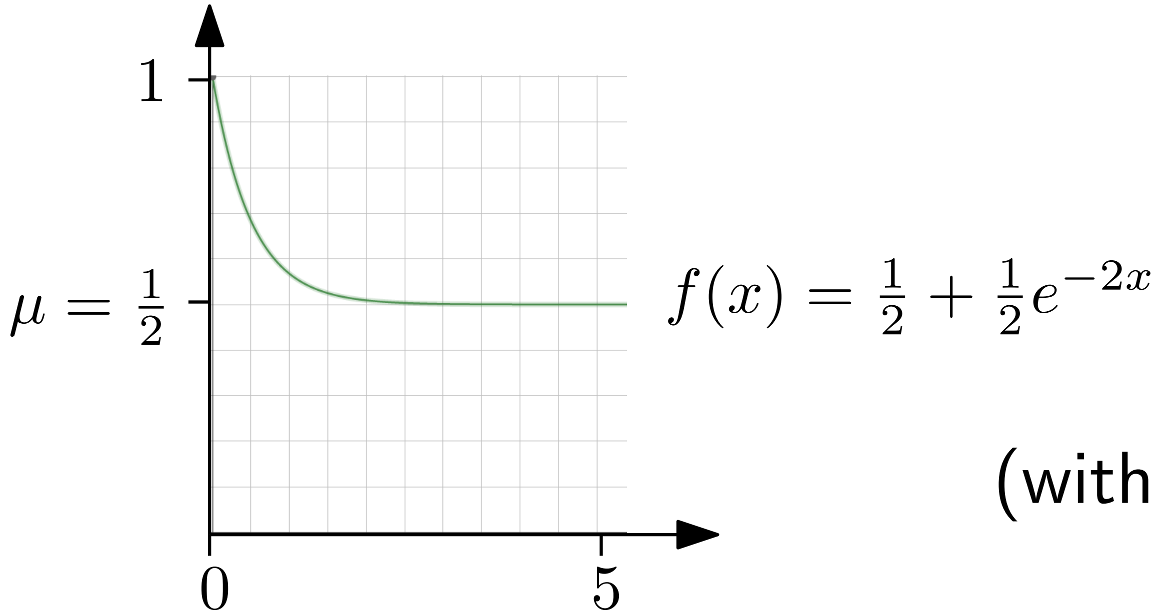
- Union bound on at most $\min\{|\delta A|, |\delta A'|\}\Delta^d$ paths. $\square$

Monotonicity property

Lemma: (Monotonicity property)

Fix $v \in V$. Let $Q_v \subset Q$, set $\mu := \mathbb{P}_\pi[\sigma(v) \in Q_v]$, and suppose $0 < \mu < 1$. Sample $X_0 \sim (\pi \,|\, X_0(v) \in Q_v)$. Then, for every $t \geq 0$,

$$\mathbb{P}[X_t(v) \in Q_v] \geq \mu + (1-\mu) \cdot \exp\left(\frac{-t}{1-\mu}\right) .$$



(without proof)

Proof sketch of main result

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Let $\Delta \geq 2$ fixed, and let G be any graph on n vertices with maximum degree at most Δ. Any continuous-time Glauber dyamics on G has mixing time $\Omega(\log n)$, where the constant in the $\Omega(\cdot)$ depends only on Δ.

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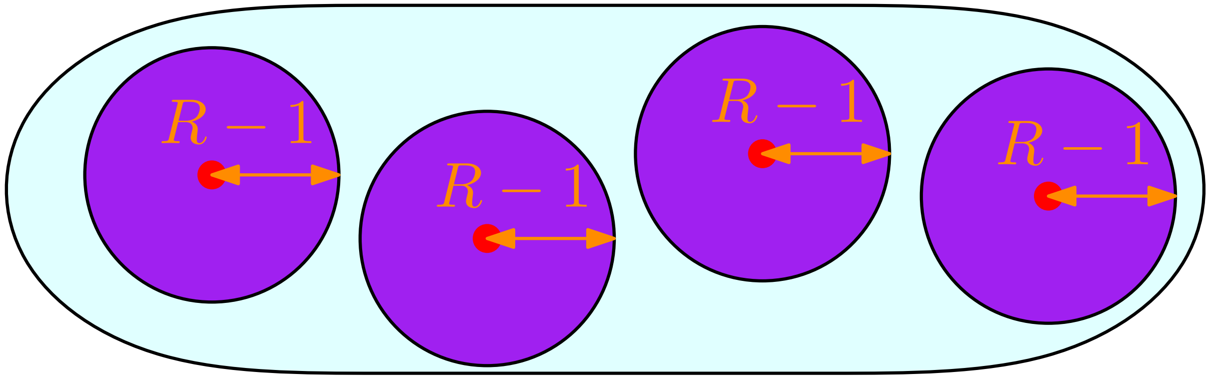
Set $R := \lceil \frac{\log n}{4 \log \Delta} \rceil$. Choose $\lceil \frac{n}{\Delta^{2R}} \rceil$ pw. disjoint and non-adjacent balls of radius $R-1$ and with centers $C \subset V$.



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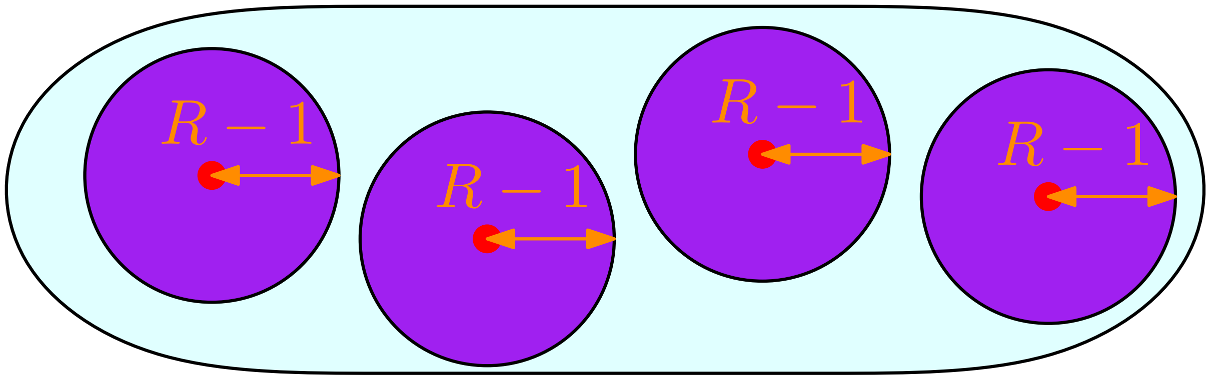

For each $v \in C$, choose arbitrary $\emptyset \neq Q_v \subsetneq Q$ set of „good spins".

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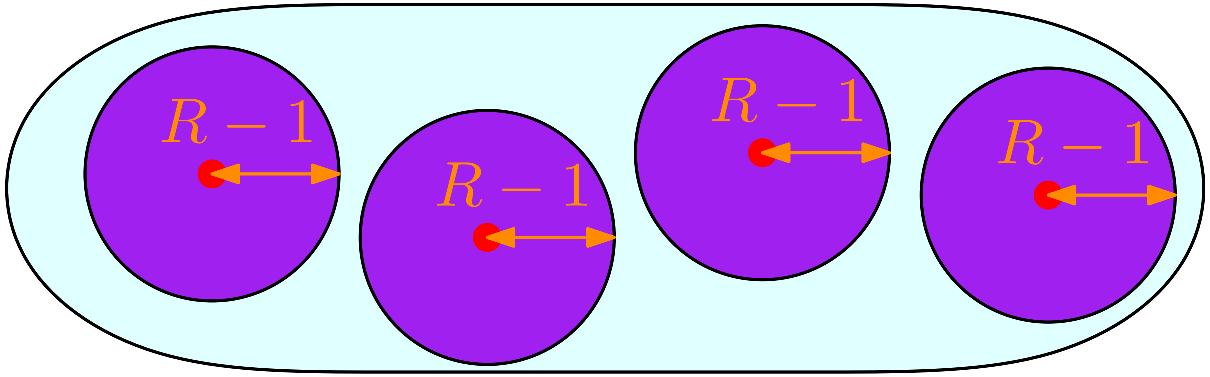

For each $v \in C$, choose arbitrary $\emptyset \neq Q_v \subsetneq Q$ set of „good spins“.

$$f(X) := \frac{\#\{v \in C : X(v) \in Q_v\}}{|C|} \quad \text{for } X \in \Omega$$

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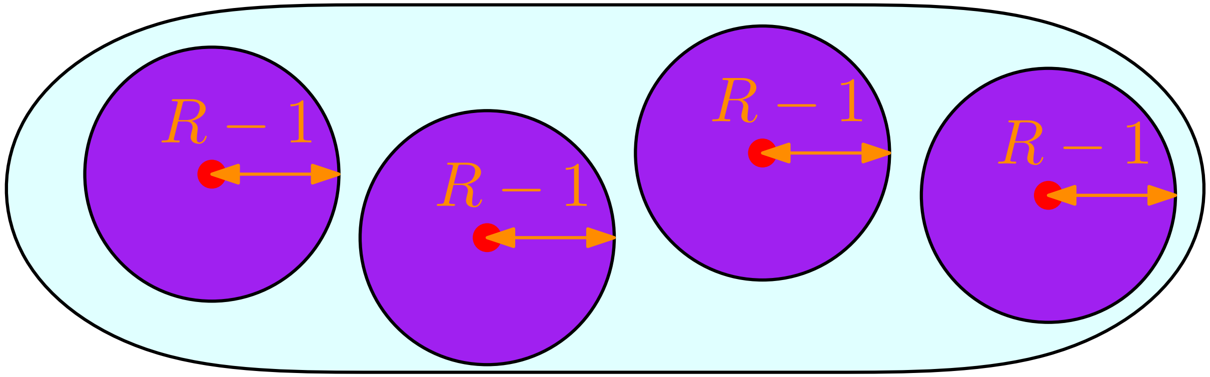

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$$f(X) := \frac{\#\{v \in C : X(v) \in Q_v\}}{|C|} \qquad \text{for } X \in \Omega$$

Goal: Specify distribution on X_0 and threshold $\hat{\mu} > 0$ s.t. with $T := \frac{\log n}{8e\Delta \log \Delta}$ we have

$$\left|\mathbb{P}[f(X_T) \geq \hat{\mu}] - \mathbb{P}_\pi[f(X) \geq \hat{\mu}]\right| > \frac{1}{2e}$$

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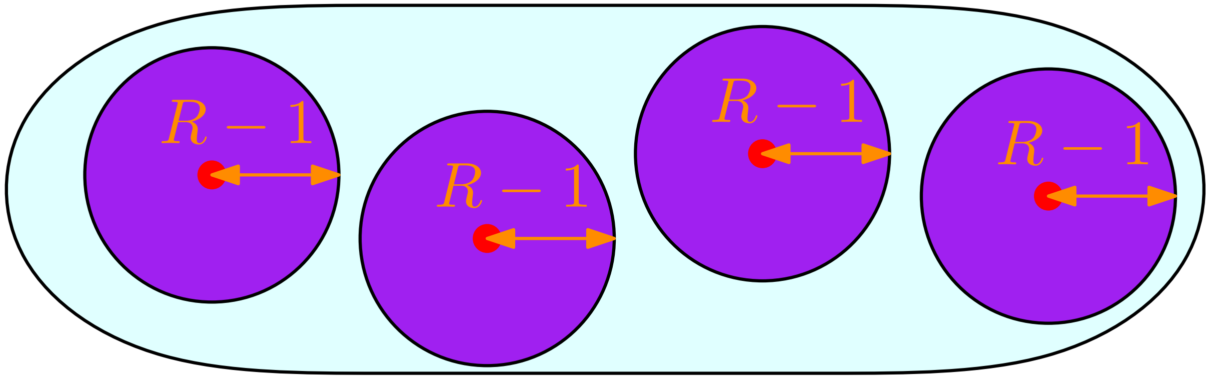

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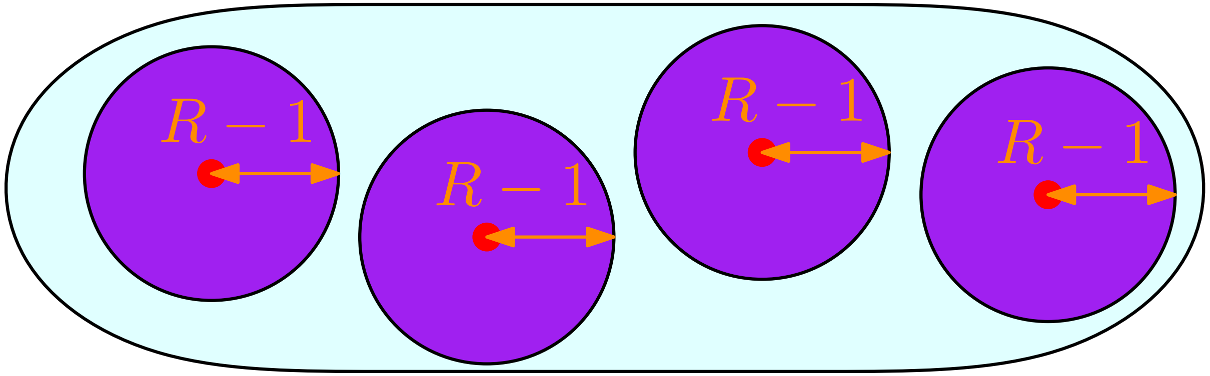
$$\left| \mathbb{P}[f(X_T) \geq \hat{\mu}] - \mathbb{P}_\pi[f(X) \geq \hat{\mu}] \right| > \frac{1}{2e} \implies \tau \in \Omega(\log n)$$



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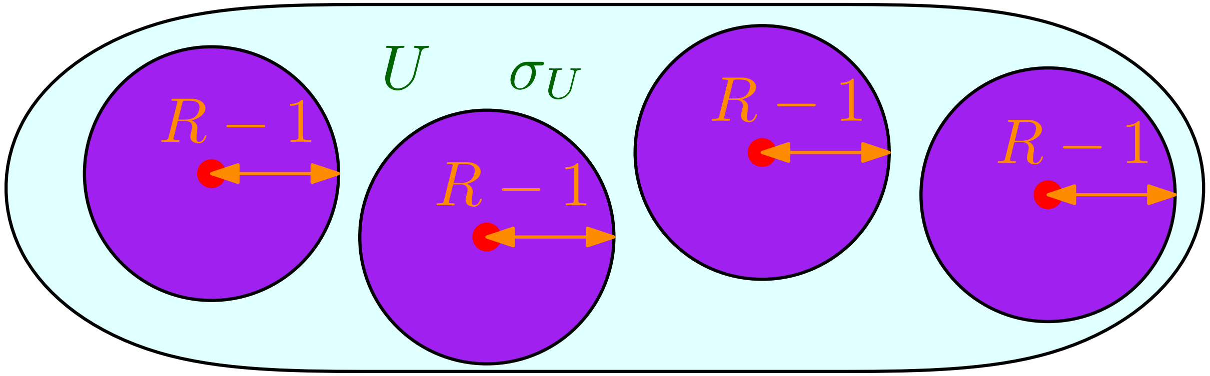
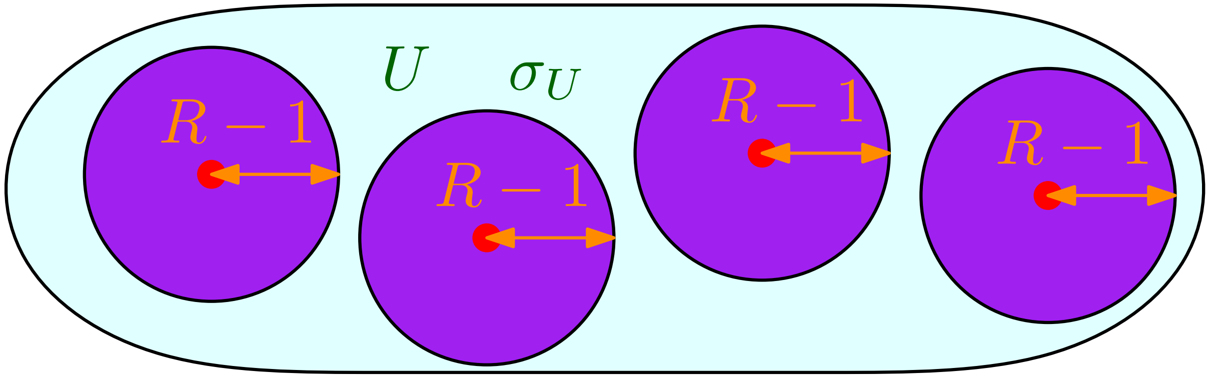
$$\left|\mathbb{P}[f(X_T) \geq \hat{\mu}] - \mathbb{P}_\pi[f(X) \geq \hat{\mu}]\right| > \frac{1}{2e} \quad \Longrightarrow \quad \tau \in \Omega(n \log n)$$



Goal: Specify distribution on X_0 and threshold $\hat{\mu} > 0$
s.t. with $T := \frac{\log n}{8e\Delta \log \Delta}$ we have

$$\Big|\mathbb{P}[f(X_T) \geq \hat{\mu}] - \mathbb{P}_\pi[f(X) \geq \hat{\mu}]\Big| > \frac{1}{2e} \implies \tau \in \Omega(n \log n)$$

Let $U := V \setminus \big(\cup_{v \in C} B_{R-1}(v) \big)$, choose $\sigma_u : U \to Q$ arbitrary.



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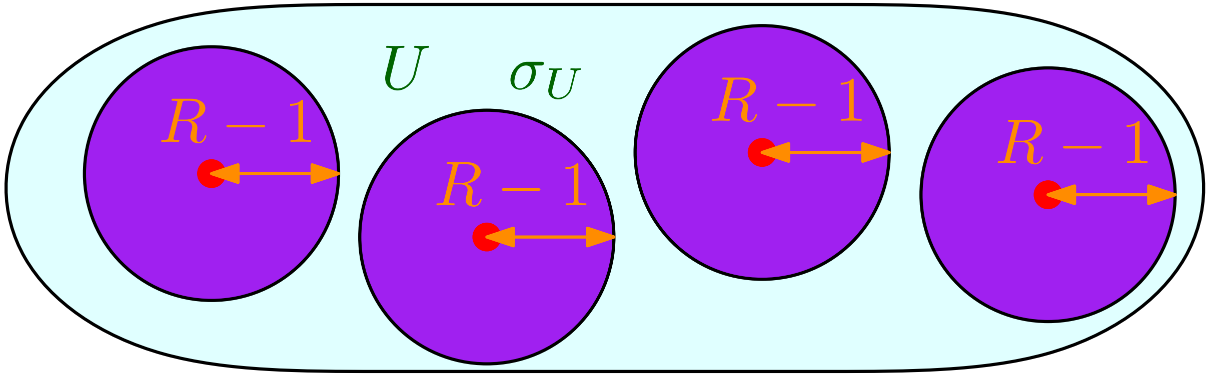
$$\mu := \mathbb{E}_{(\pi|\sigma_u)}[f(X)] \qquad \varepsilon := \frac{1}{4e^{2T}} \qquad \hat{\mu} := \begin{cases} \mu - \varepsilon & : \mu > \frac{1}{2} \\ \frac{1}{2} & : \mu = \frac{1}{2} \\ \mu + \varepsilon & : \mu < \frac{1}{2} \end{cases}$$



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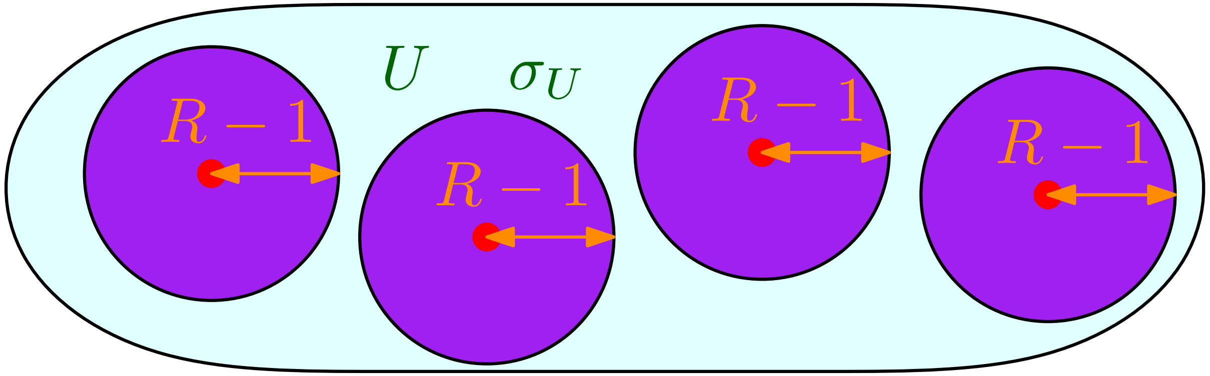
W.l.o.g. $\mathbb{P}_\pi[f(X) \geq \hat{\mu}] \leq \frac{1}{2}$ (otw. replace all Q_v by $V \setminus Q_v$)



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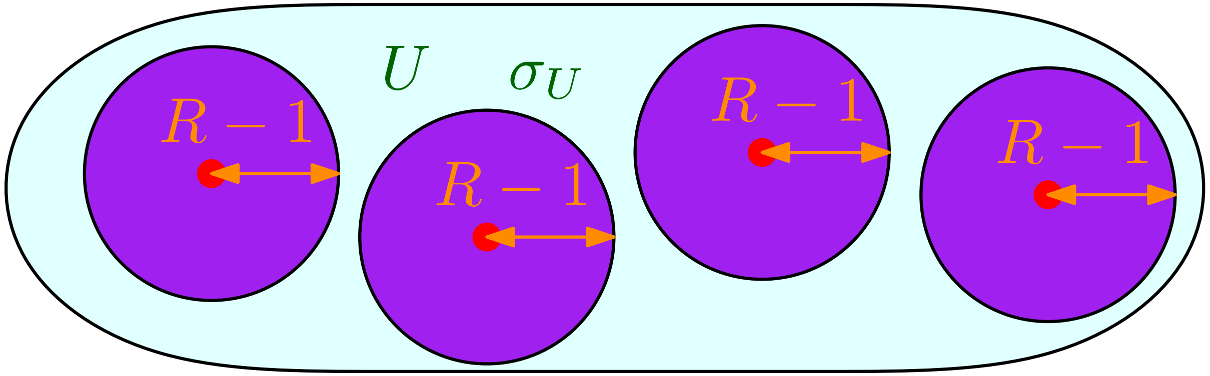
Initial distribution $X_0 \sim (\pi \mid \sigma_U, X_0(v) \in Q_v$ f.a. $v \in C)$



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Claim: $\mathbb{P}[f(X_T) \geq \hat{\mu}] > \frac{1}{2} + \frac{1}{2e}$ (blackboard)