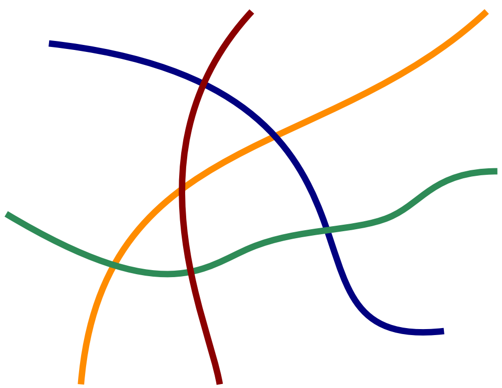
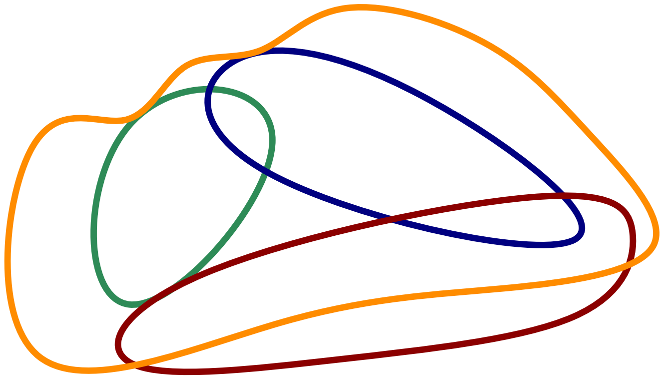
Joint Workshop Berlin-Graz-Zürich

Arrangements of Pseudolines and Pseudocircles





Sandro Roch

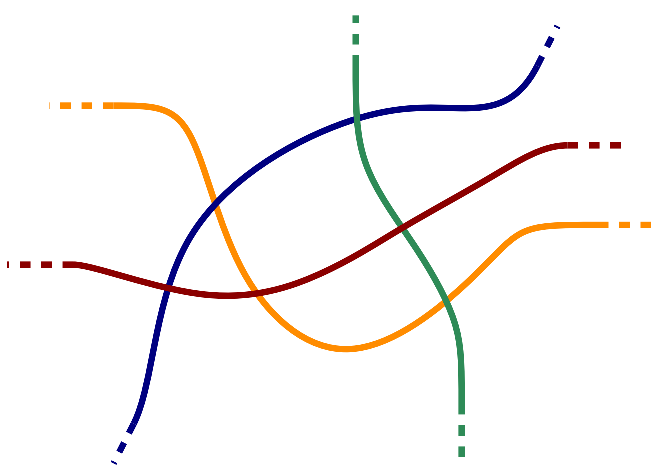
Arrangements of pseudolines

Def: *Pseudoline arrangement*:

- Continuous curves $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}^2$ with

$$\lim_{t \to \infty} \|f_i(t)\| = \lim_{t \to -\infty} \|f_i(t)\| = \infty$$

- Each two have exactly one intersection where they cross



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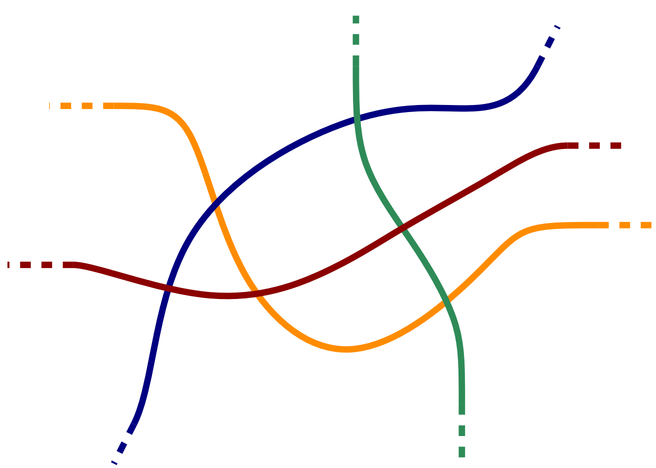
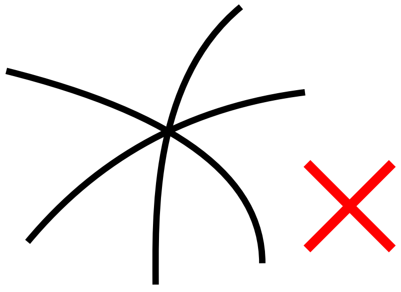
- No point in which ≥ 3 pseudolines cross





Wiring diagrams

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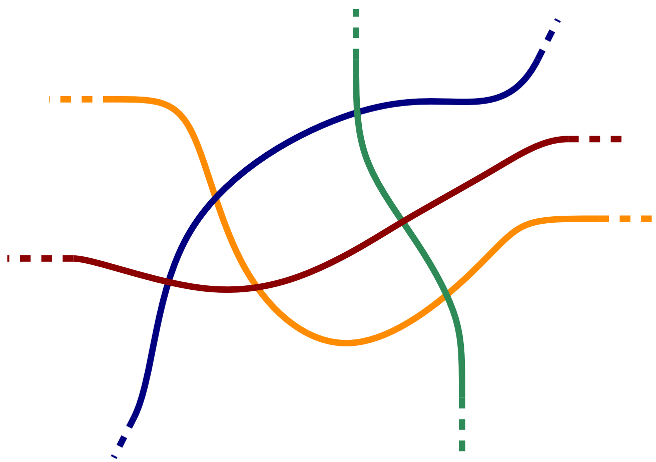

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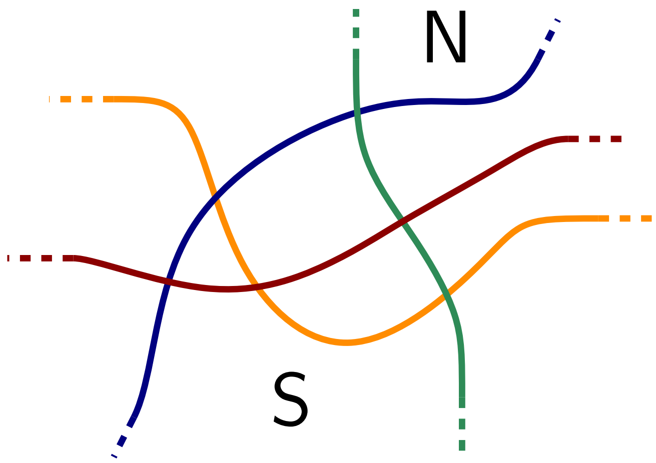

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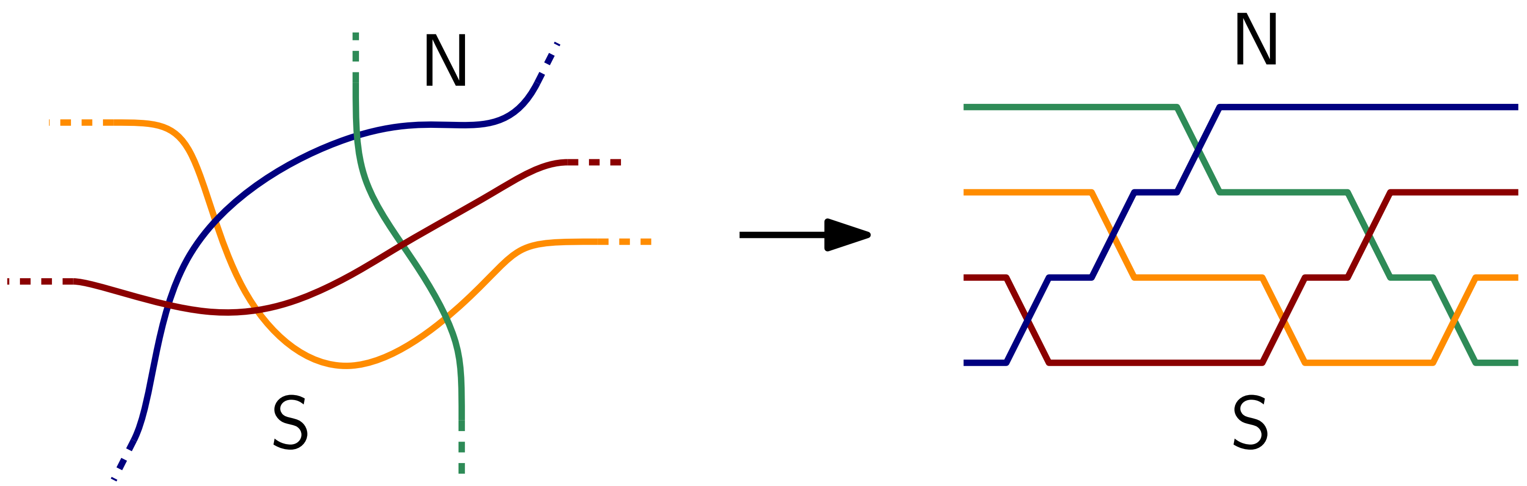

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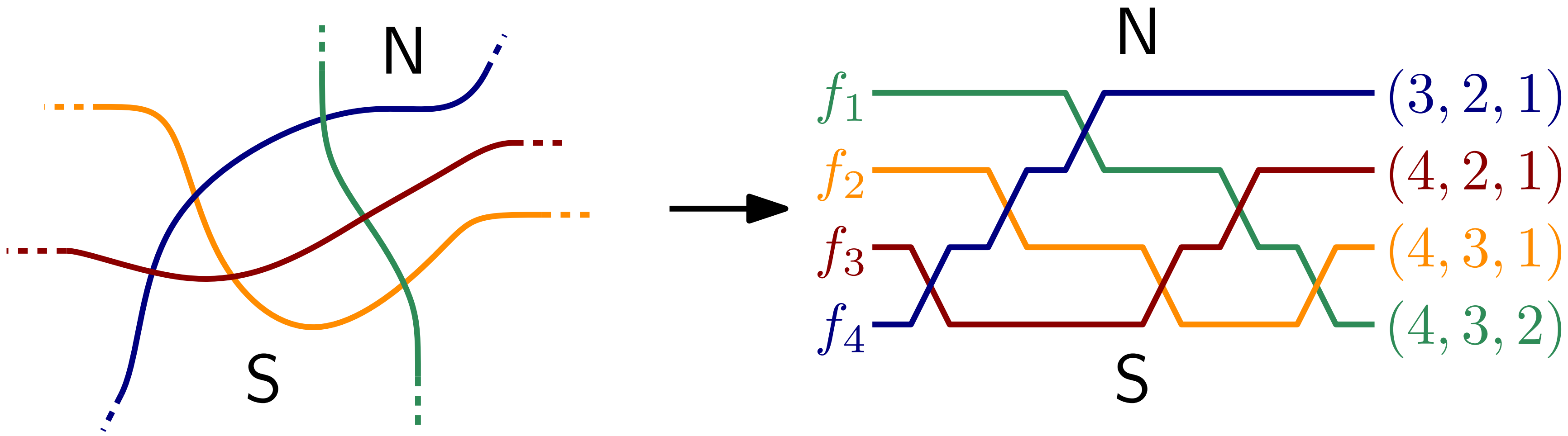

Encoding by intersection orders:
Permutation $\pi_i \in S_{n-1}$ encodes intersection order of f_i.

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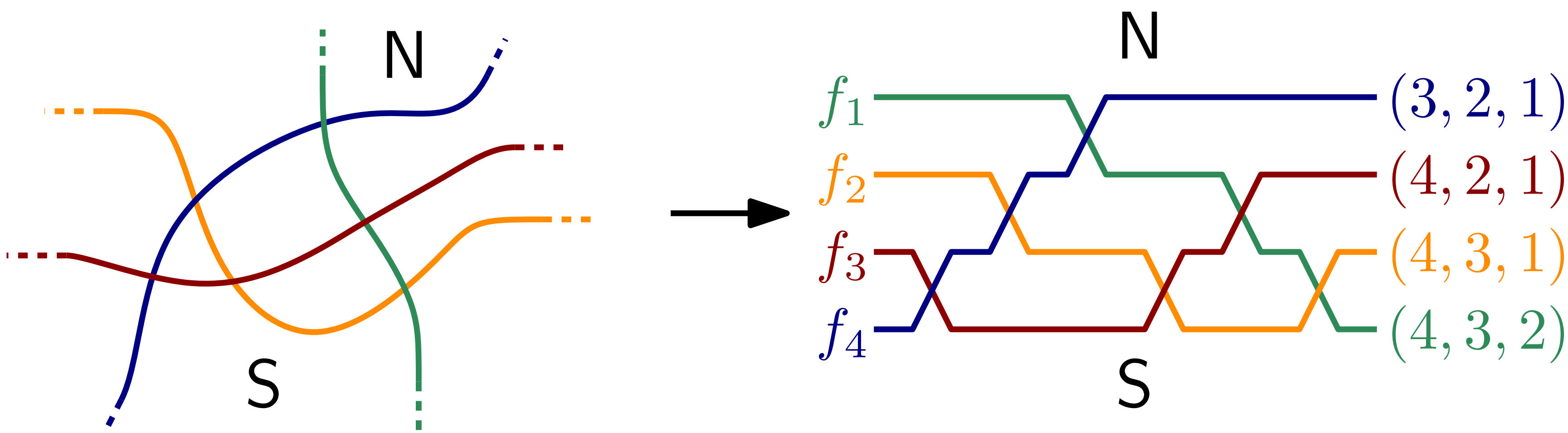

Encoding by intersection orders:
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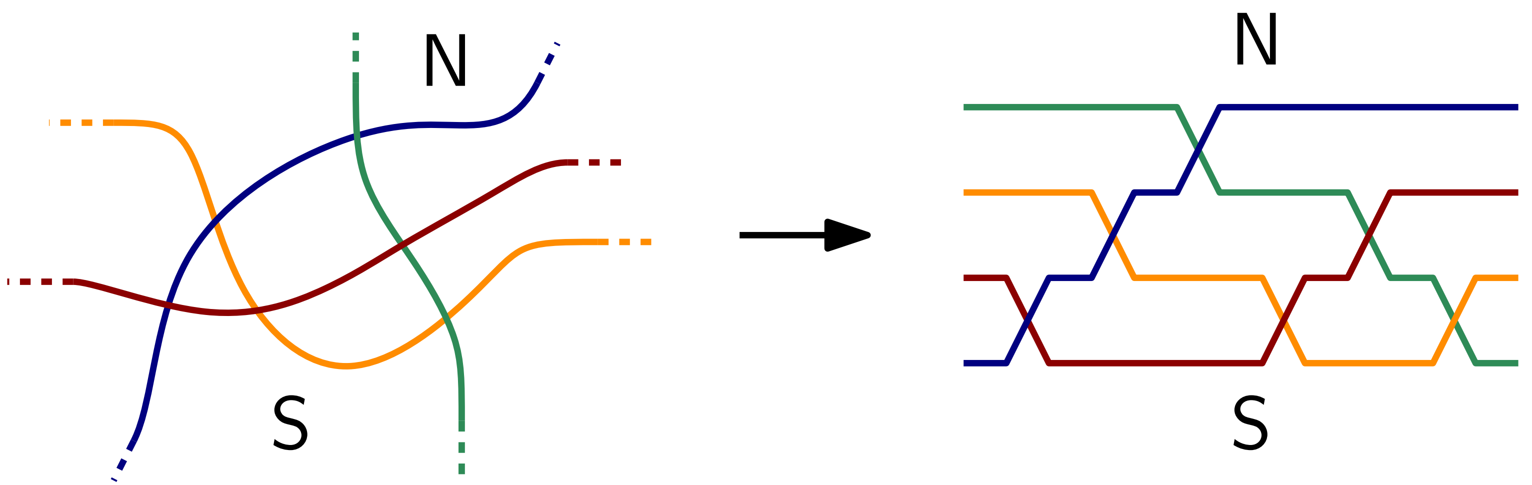
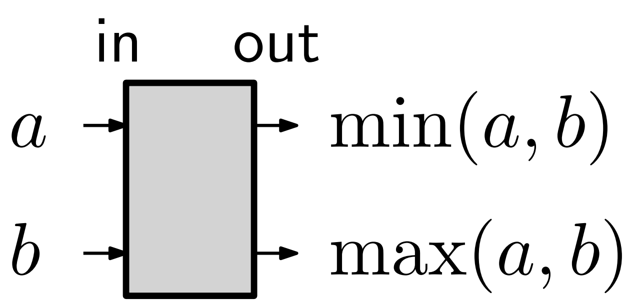
Theorem (Felsner & Weil, 2001):
$(\pi_1, \ldots, \pi_n) \in (S_{n-1})^n$ describe arrangement iff for all $i < j < k$ the pairs $(i,j), (j,k), (i,k)$ appear in π_k, π_i, π_j all well ordered or all reversed.

Wiring diagrams



Wiring diagrams as sorting networks:



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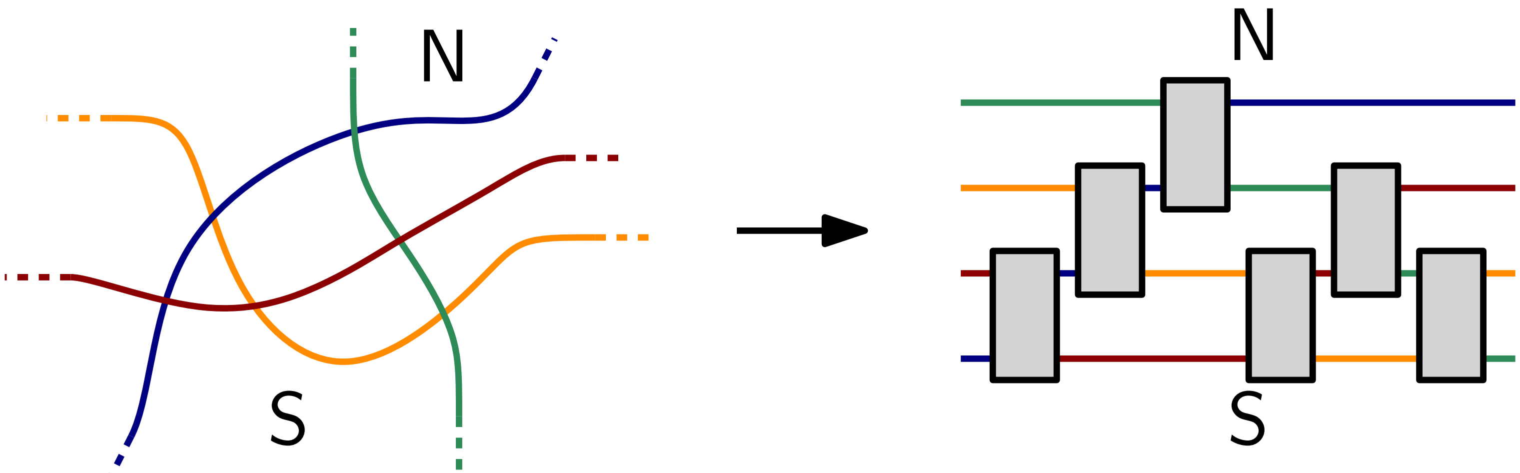

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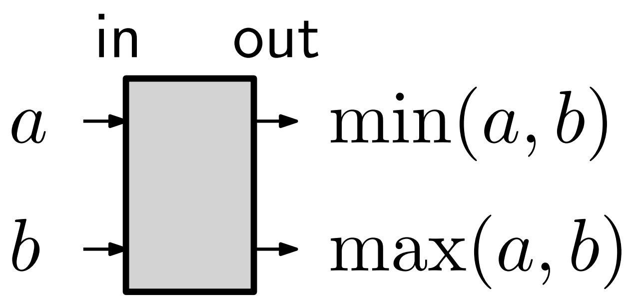

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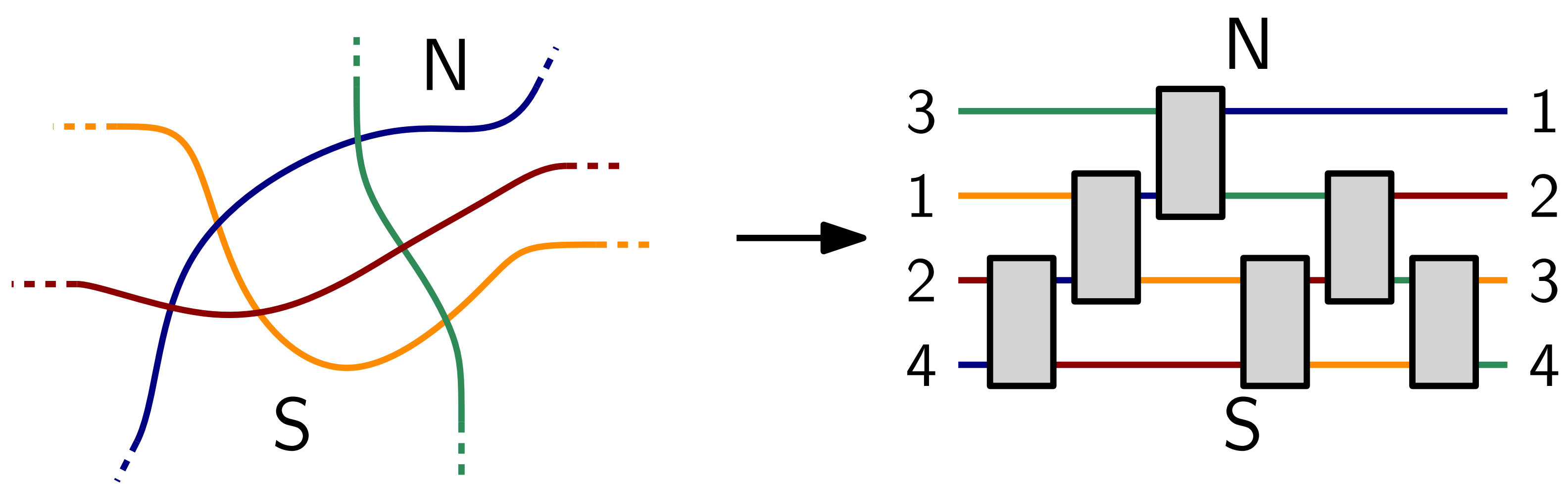

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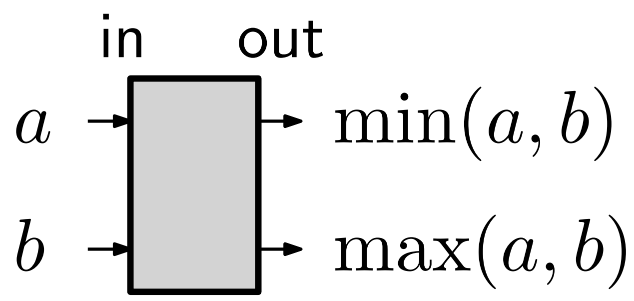

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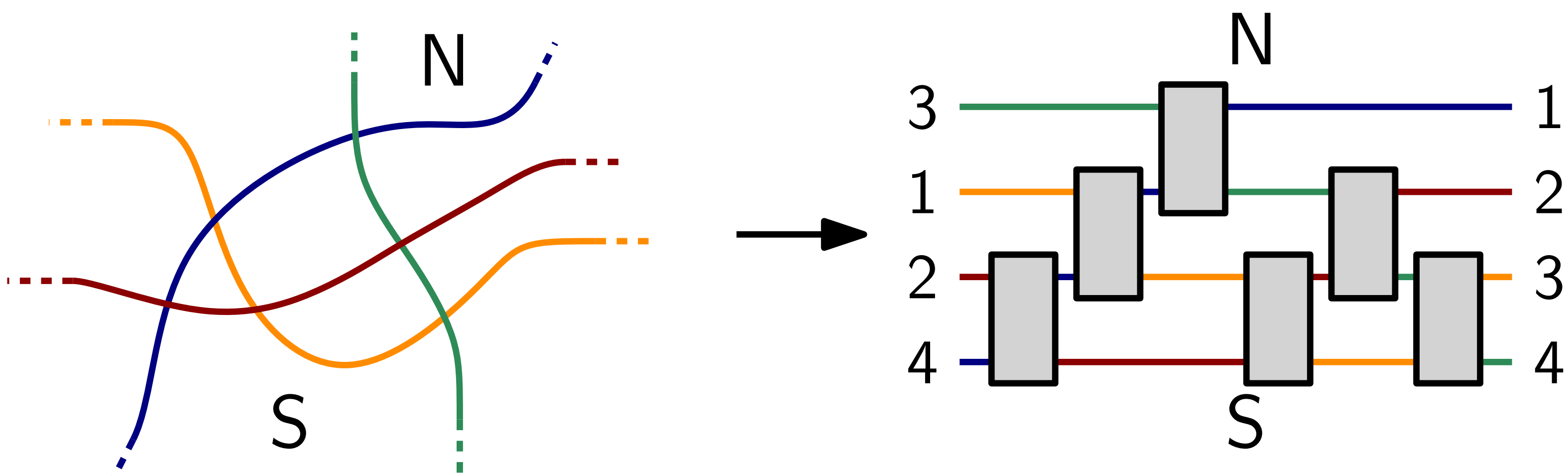

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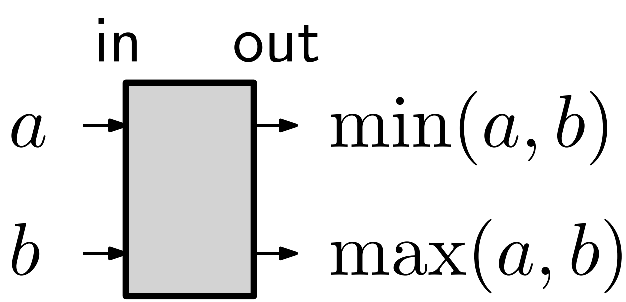
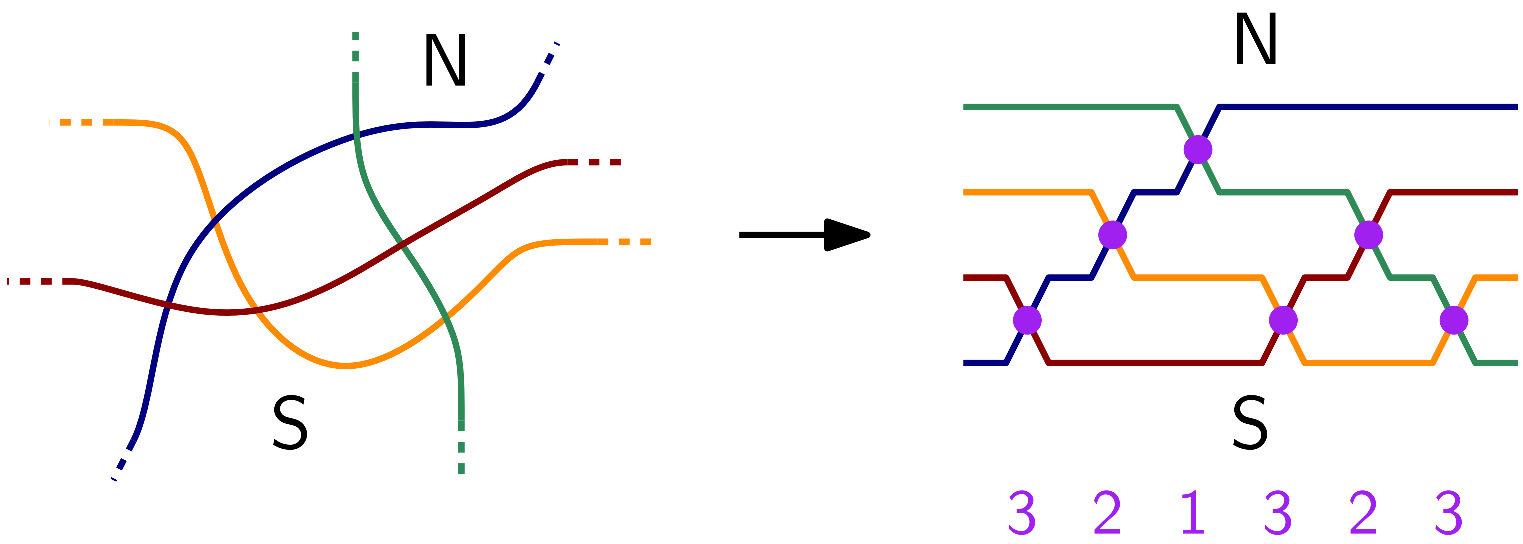


Sorting networks encode sorting algorithms based on *adjacent comparison & transposition*.

Wiring diagrams



Bijection to standard Young tableaux:

- Levels of crossings give *allowable sequence*.

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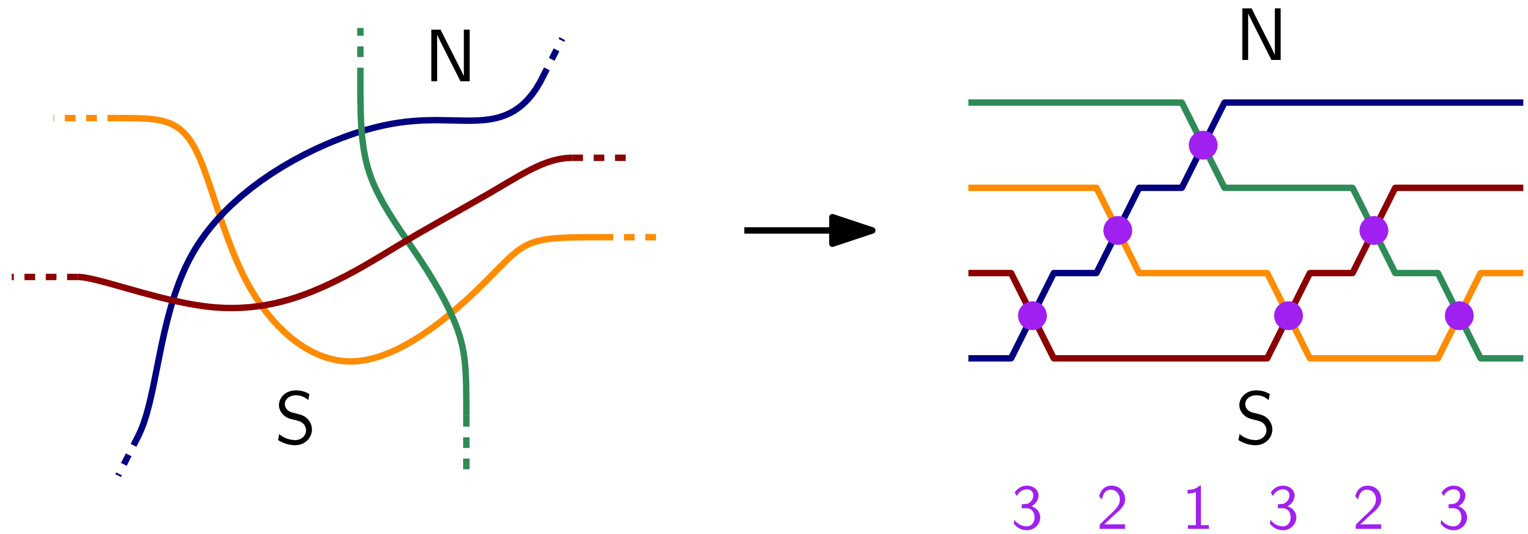

Bijection to standard Young tableaux:

- Levels of crossings give *allowable sequence*.
- Felsner 2001: Allowable sequences in bijection to *standard young tableaux* of staircase shape

(3, 2, 1, 3, 2, 1) $\mapsto$

1	4	6
2	5	
3		

Schensted insertion!

Rhombic tilings of zonotopes

Rhombic tilings of zonotopes

Def. $v_1, \ldots, v_r \in \mathbb{R}^2$ pw. indep. define zonotope:

$$Z(v_1, ..., v_r) := \left\{ \sum \lambda_i v_i : \lambda_i \in [-1, 1] \right\}$$

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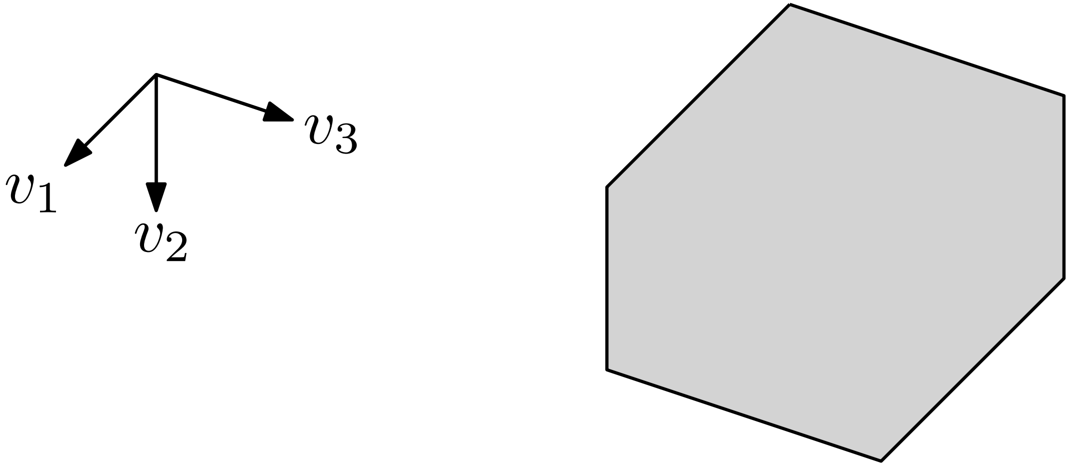

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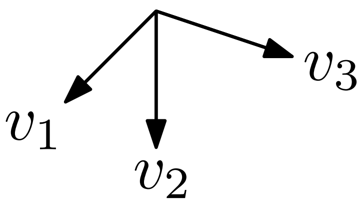

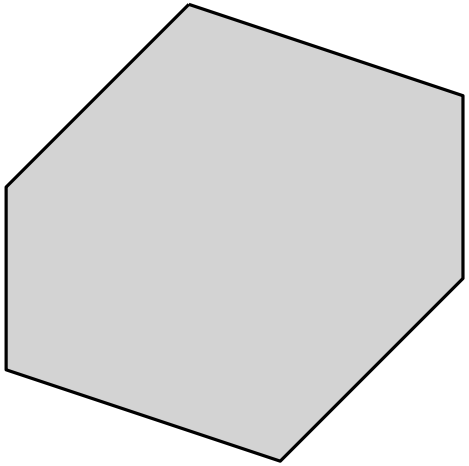

- Zonotopes $Z(v_i, v_j)$, $v_i \neq v_j$ are *rhombi* (*lozenges*).

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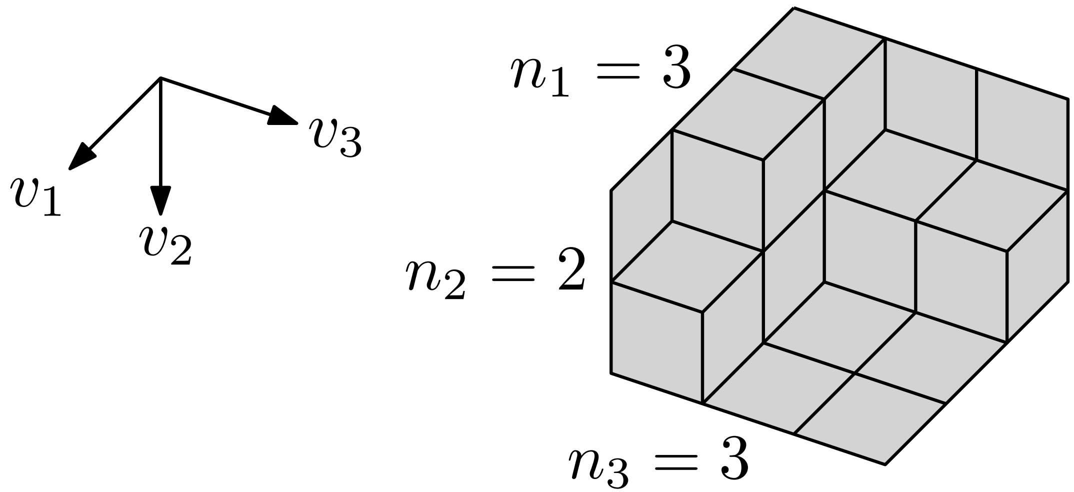
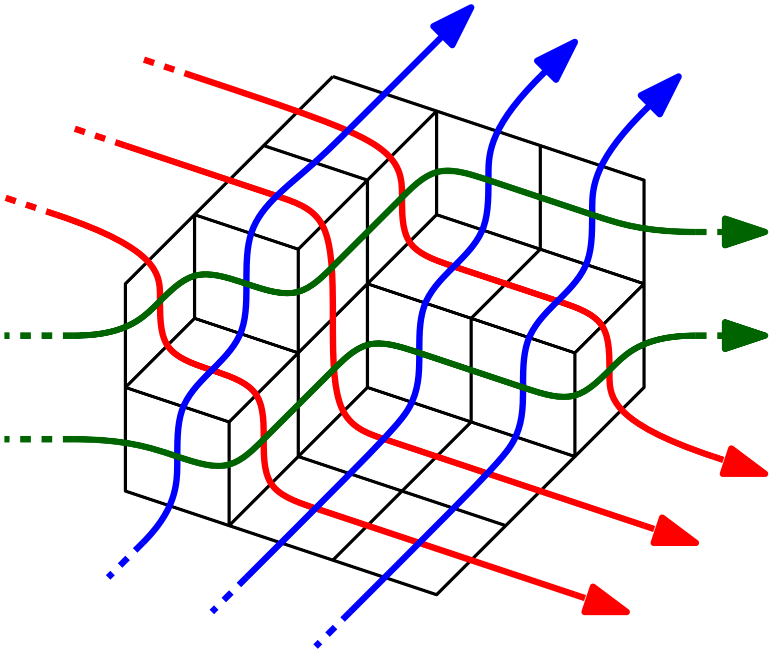


- Zonotopes $Z(v_i, v_j)$, $v_i \neq v_j$ are *rhombi* (*lozenges*).
- *Rhombic tiling*: For fixed $n_1, \ldots, n_r \in \mathbb{N}$, tiling of $Z(v_1, \ldots, v_r)$ by rhombi of shape $Z\left(\frac{v_i}{n_i}, \frac{v_j}{n_j}\right)$

Rhombic tilings of zonotopes

Draw curves through ribbons of parallel edges:



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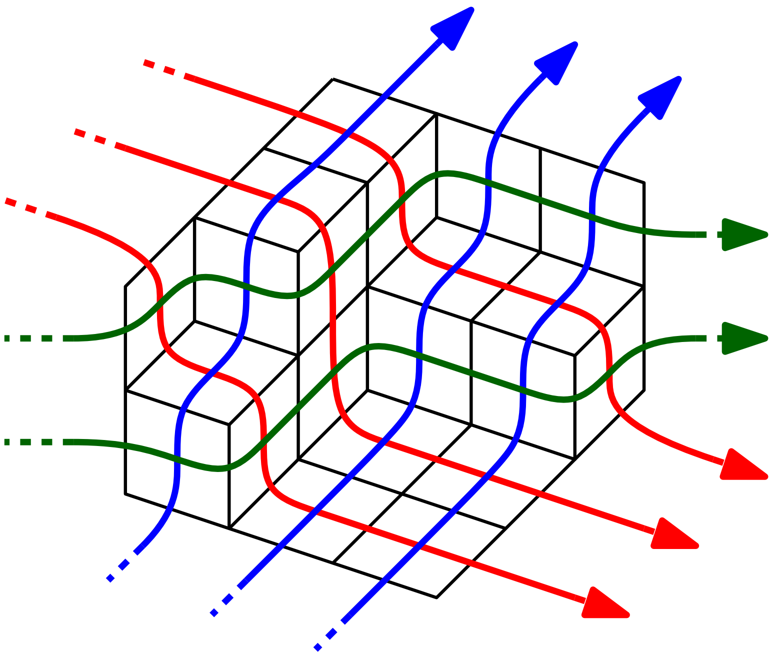

Yields *generalized* pseudoline arrangement:

- r classes of $n_1, ..., n_r$ non-intersecting pseudolines.
- Pseudolines of different classes cross each other exactly once.

Plane partitions and grid paths

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Def: Matrix $[h_{i,j}] \in {\mathbb{N}_0}^{r \times s}$ is called *plane partition*, if rows and columns are monotonically increasing.

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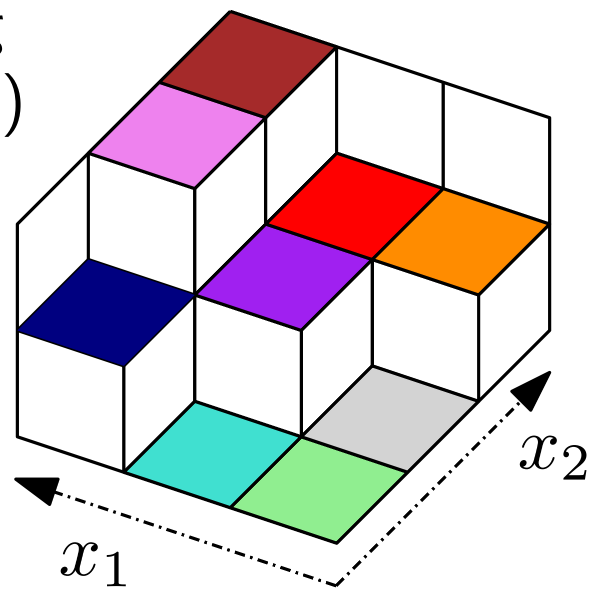
shape (3, 2, 3)



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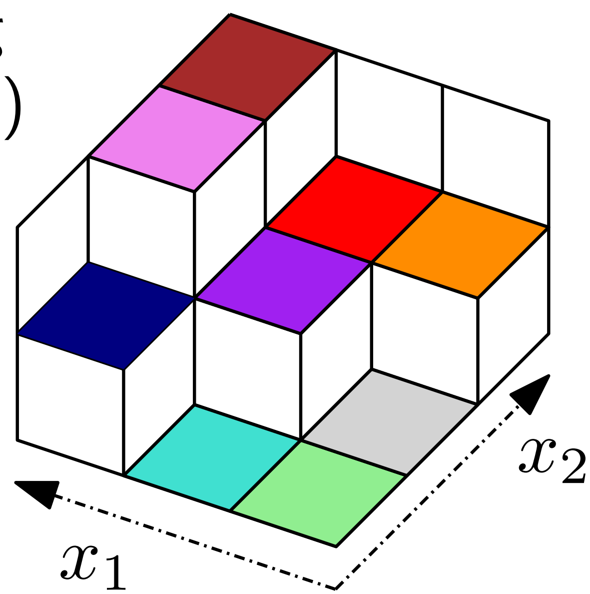

x_2 →, x_1 ↓

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

plane partition
with entries
$h_{i,j} \leq 2$

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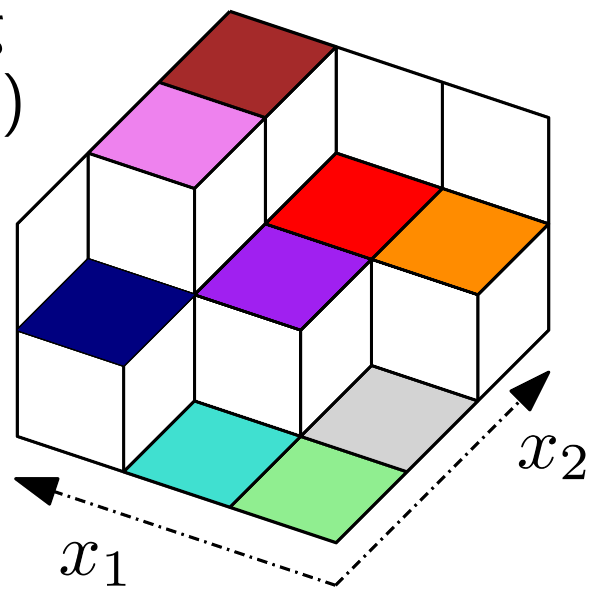

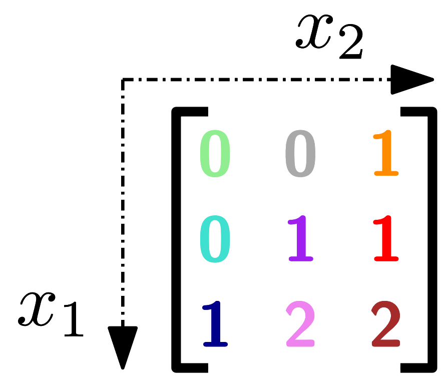

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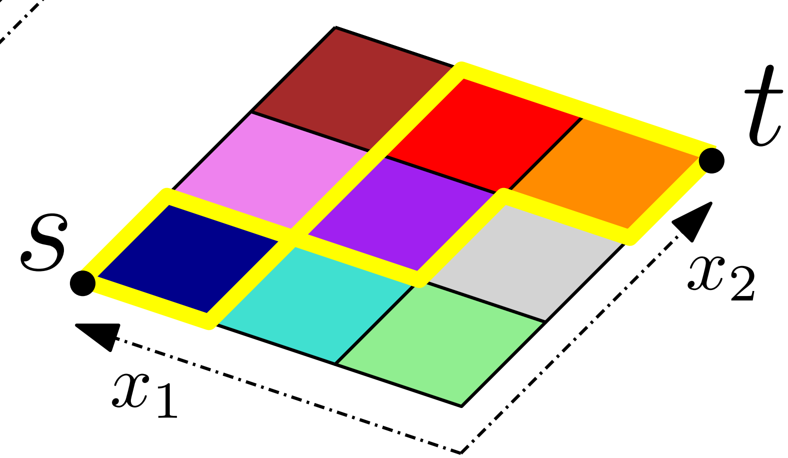

monotonic
non-crossing
paths in grid

Pseudoline arrangements

wiring diagram

signotope

system of inter-
section orders

plane partition

rhombic tiling
of zonotope

sorting network

hiher Bruhat order

standard Young tableau

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oriented matroid of rank 3

Pseudoline arrangements

wiring diagram

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Question I:
How can pseudoline arrangements efficiently be sampled with uniform distribution?

rhombic tiling of zonotope

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Random sampling using Markov chains

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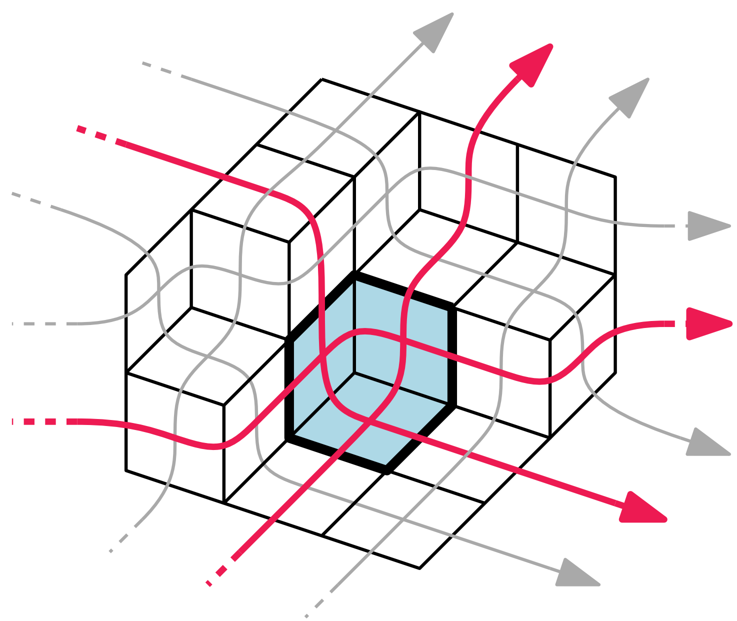

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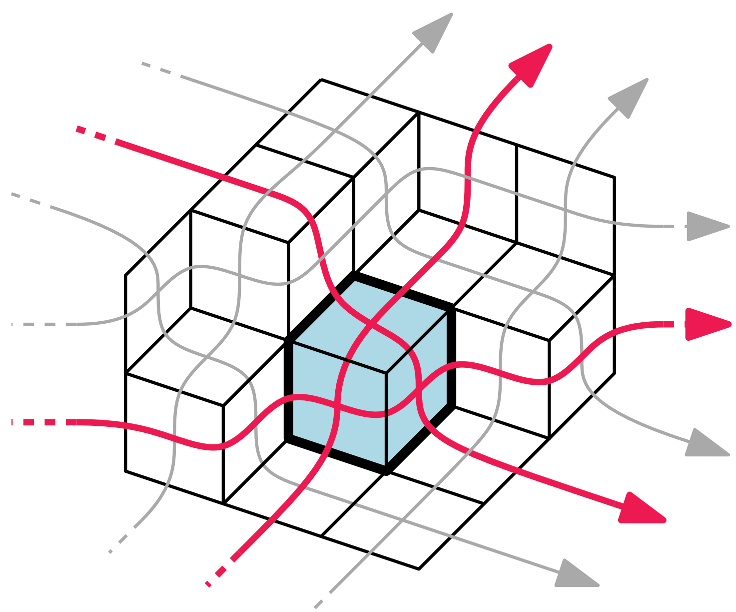

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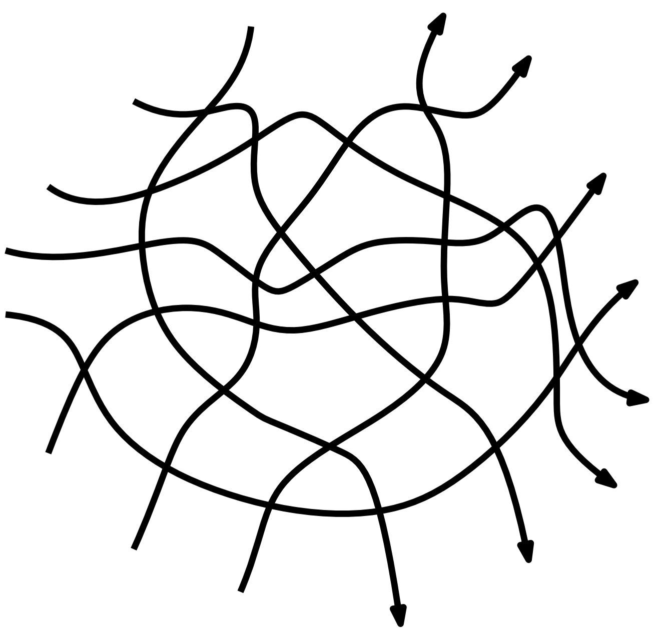

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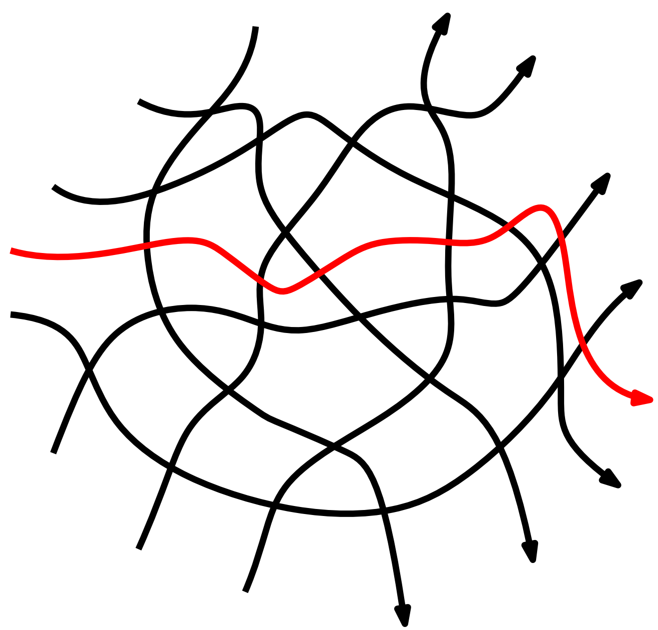

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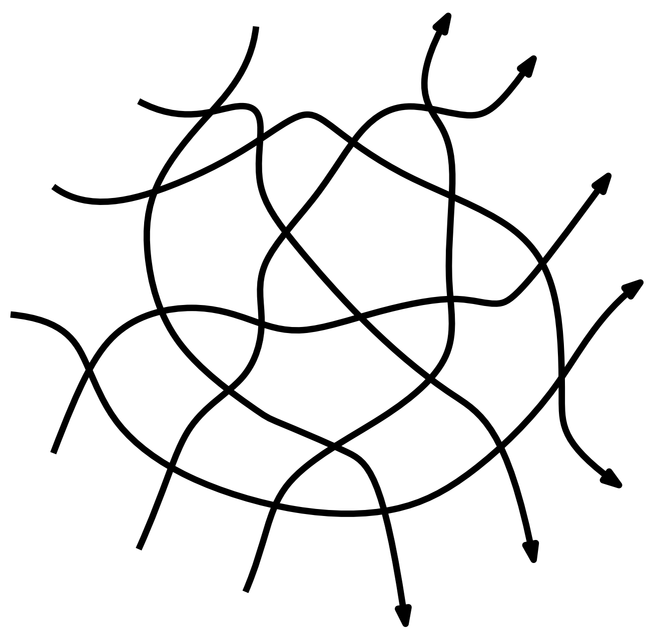

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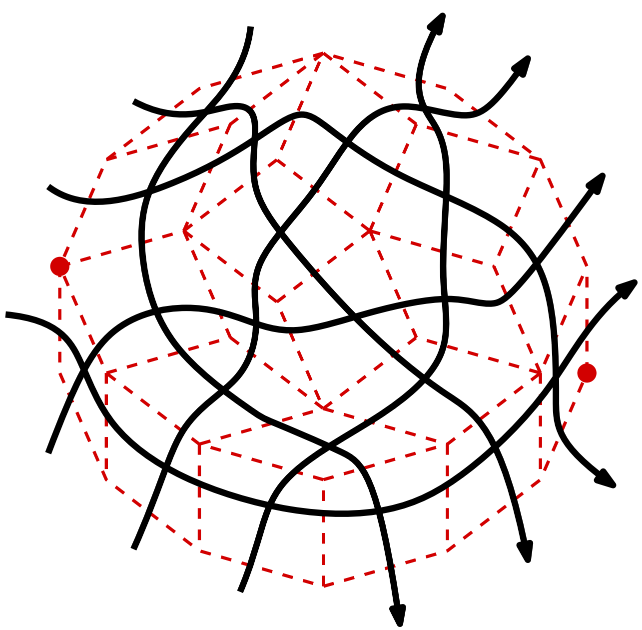

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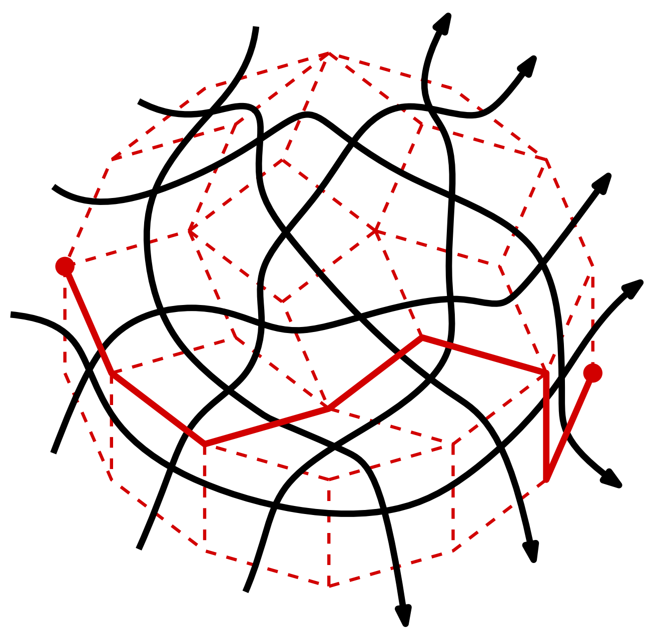

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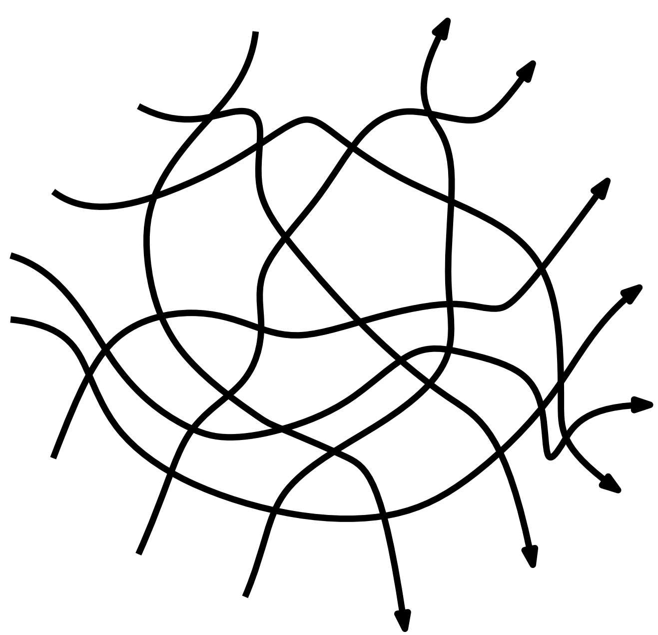

Pseudocircle arrangements

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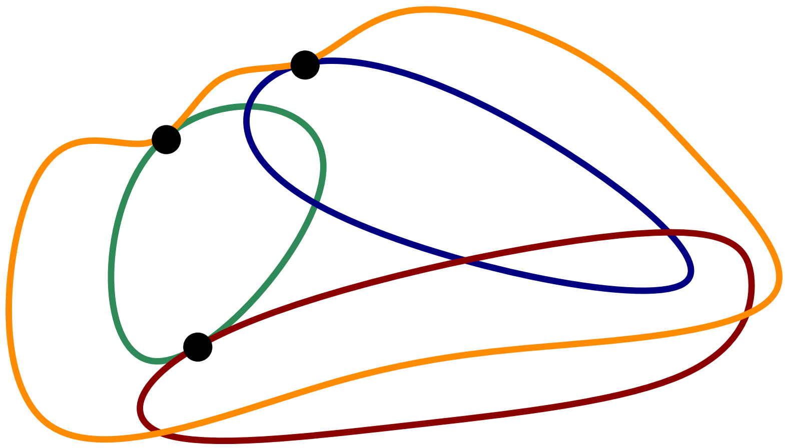
Example:



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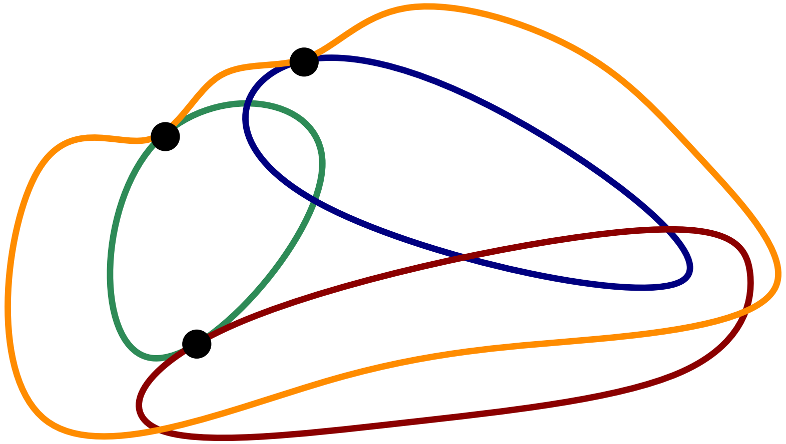
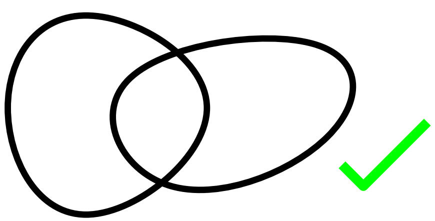


Each two pseudocircles...



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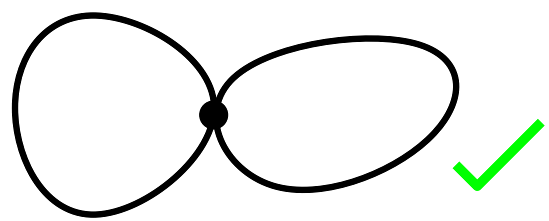

...either cross exactly twice, ...

...have a single touching

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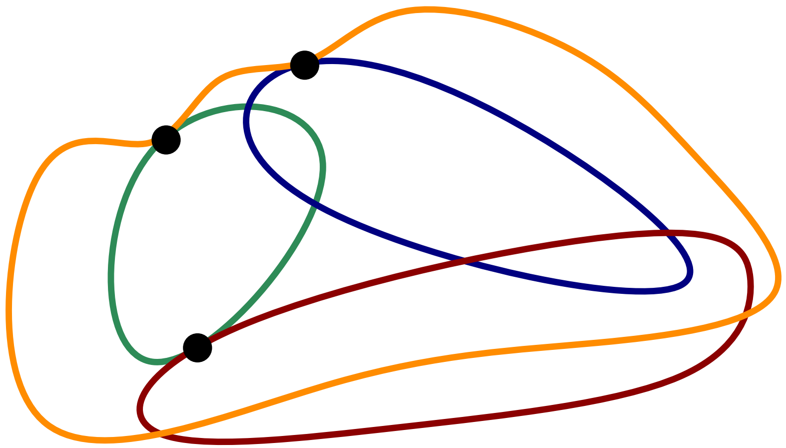

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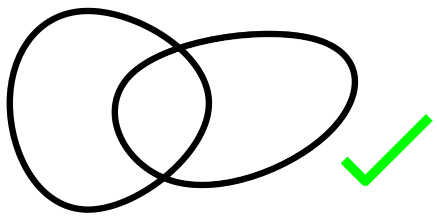

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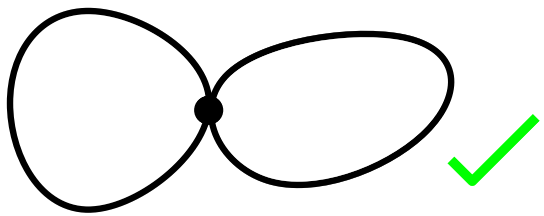

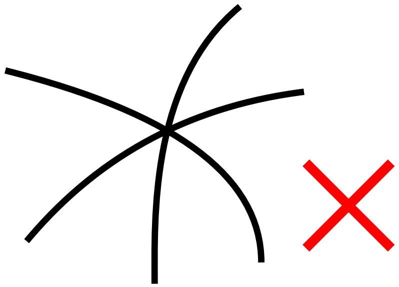

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No intersection of ≥ 3 pseudocircles in single point

Arrangements of great-pseudocircles

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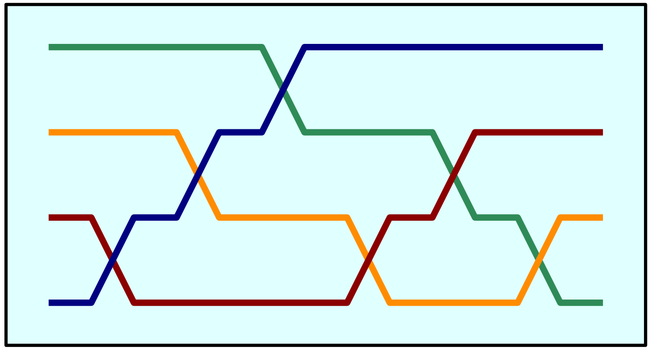
Glue together two copies of pseudoline arrangement:



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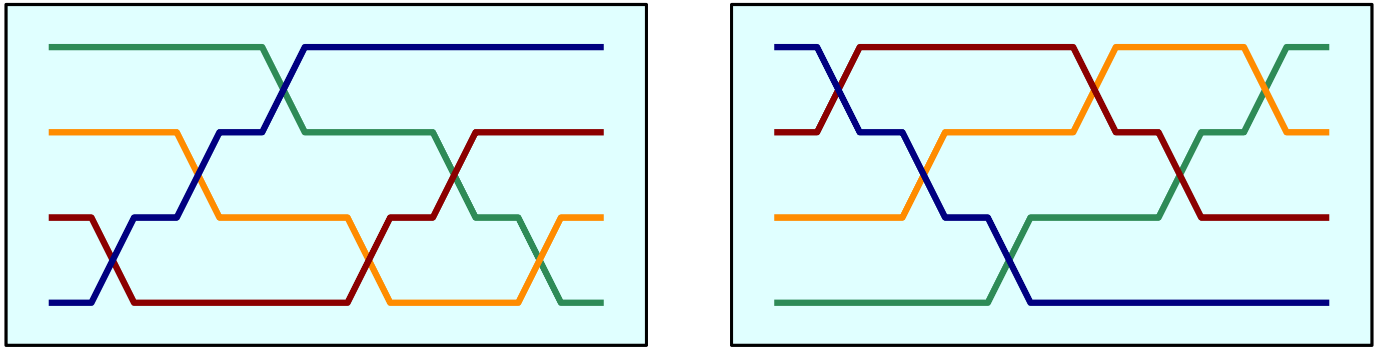

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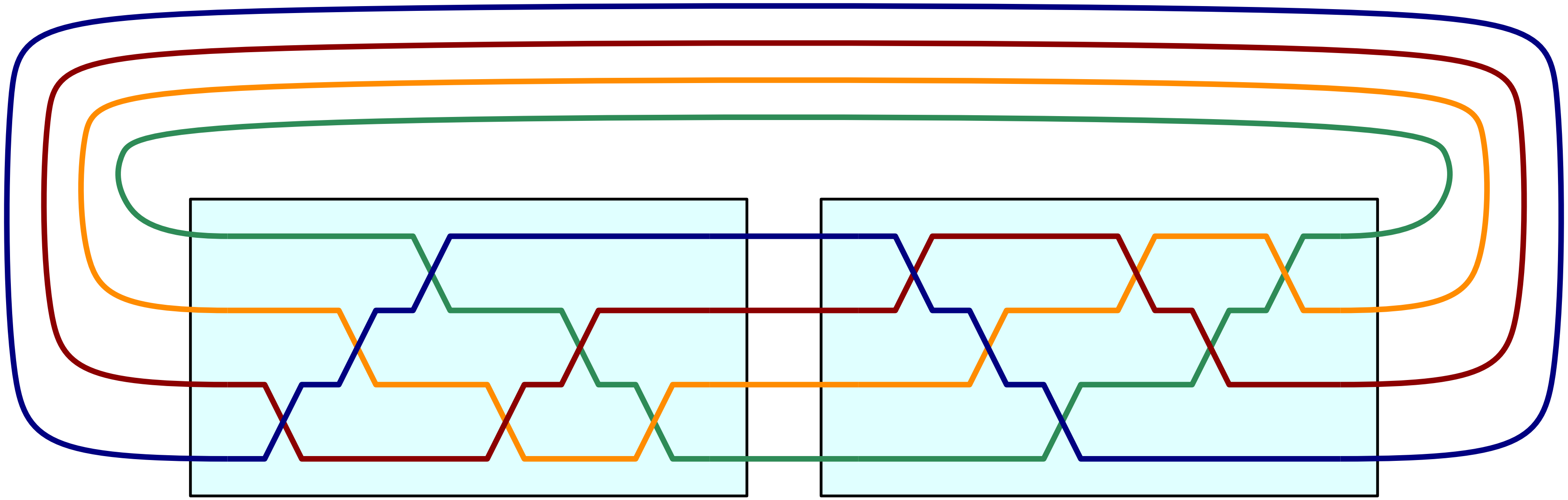

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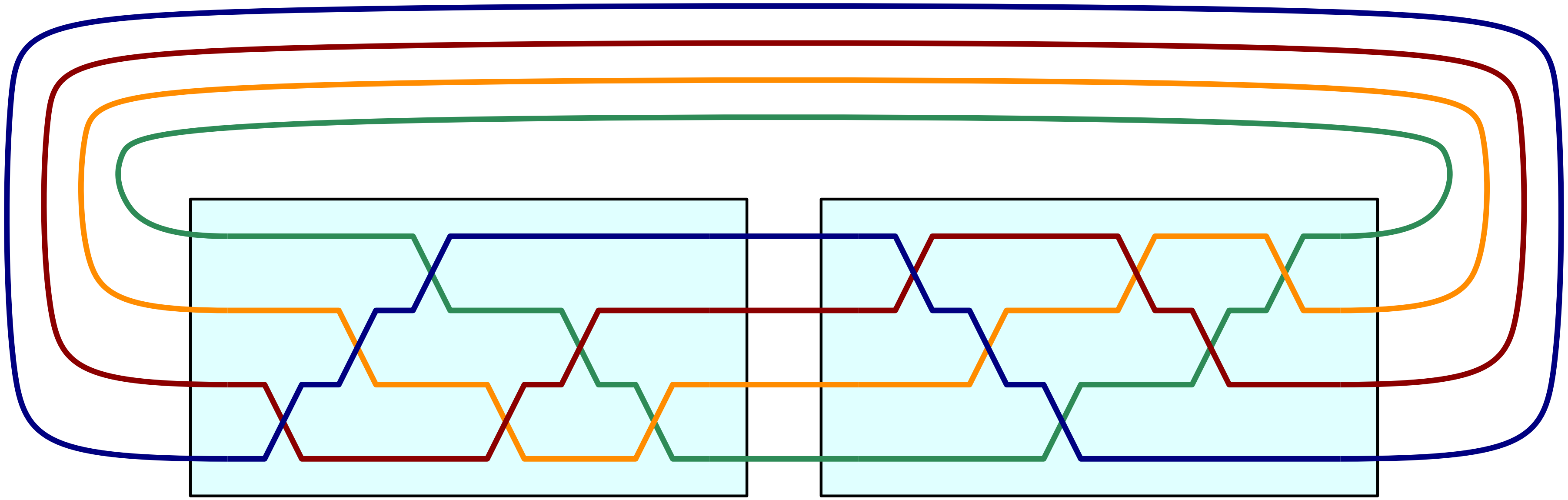

Observation: Obtain pseudocircle arrangement

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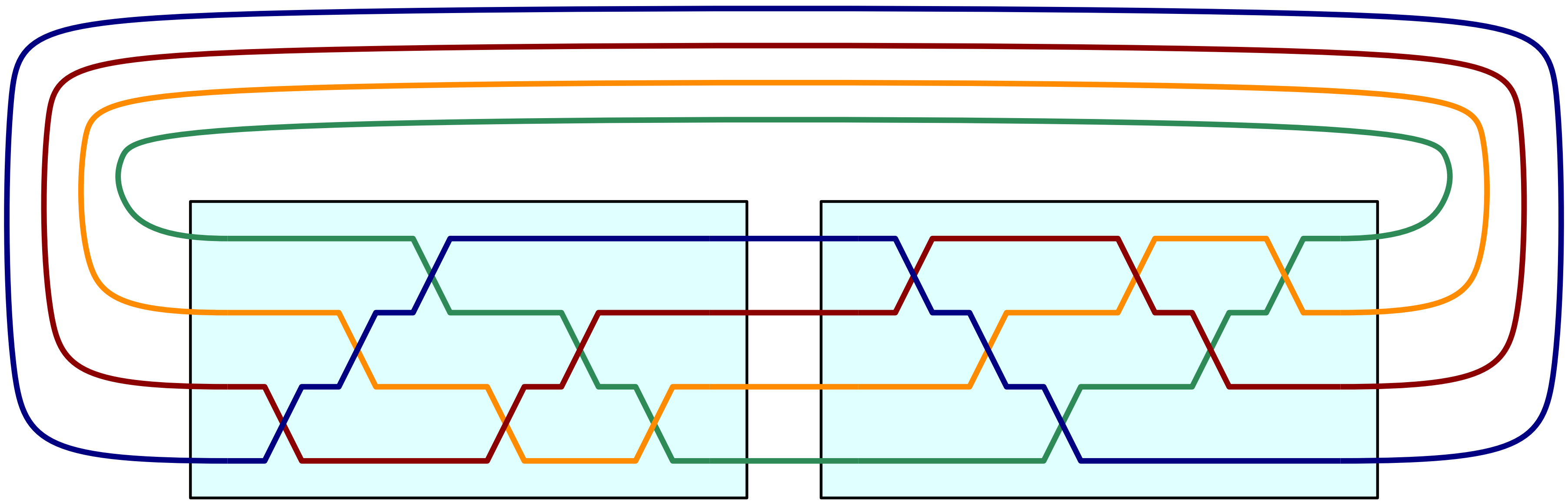
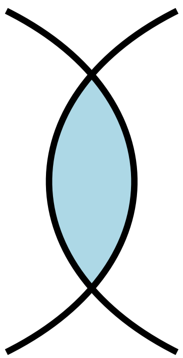


Observation: Obtain pseudocircle arrangement
$\implies$ Special class *great-pseudocircle arrangement*

Grünbaum's conjecture on digons

Conjecture: In every pseudocircle arrangement, there are at most $2n-2$ digons. (Grünbaum, 1972)


digon

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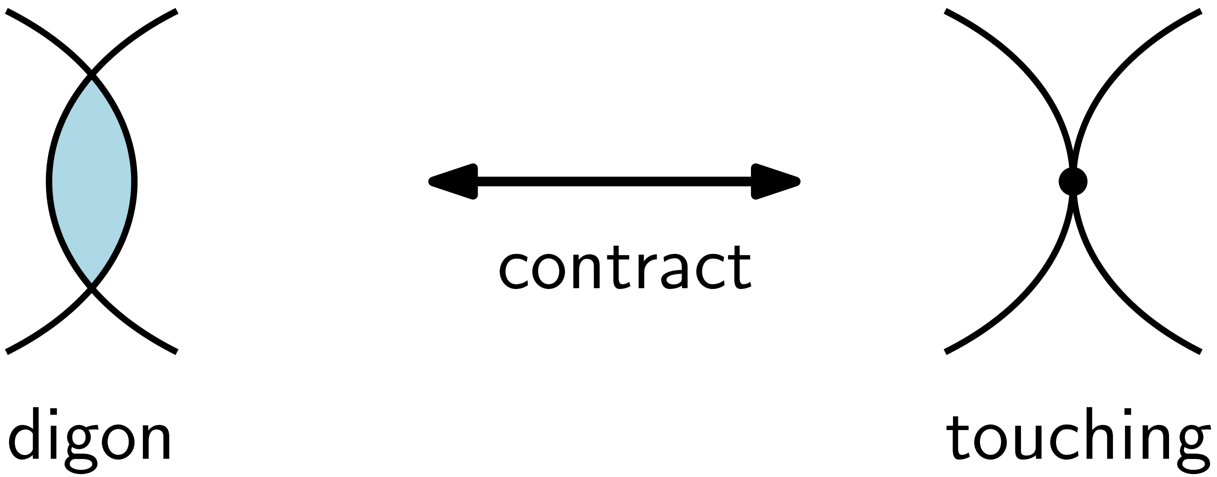

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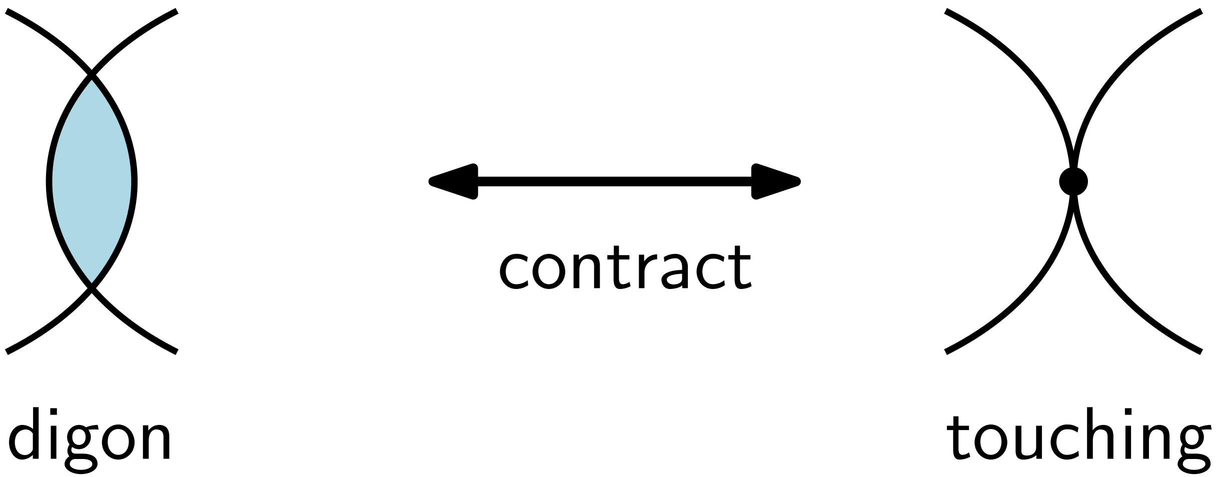
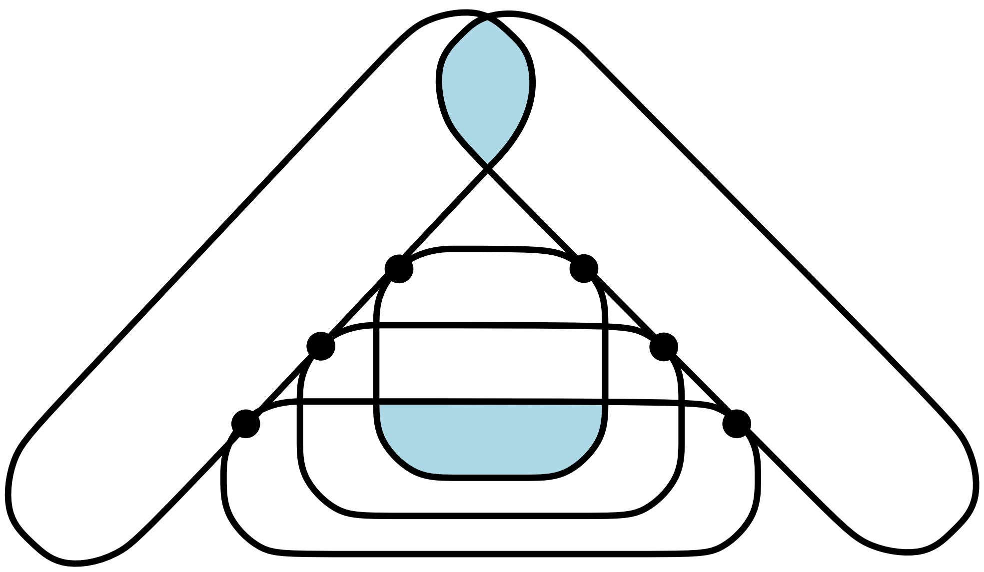


Equivalent: At most $2n-2$ touchings.

Grünbaum's conjecture on digons

Cylindrical pseudocircle arrangement:

Exist two cells separated by each pseudocircle



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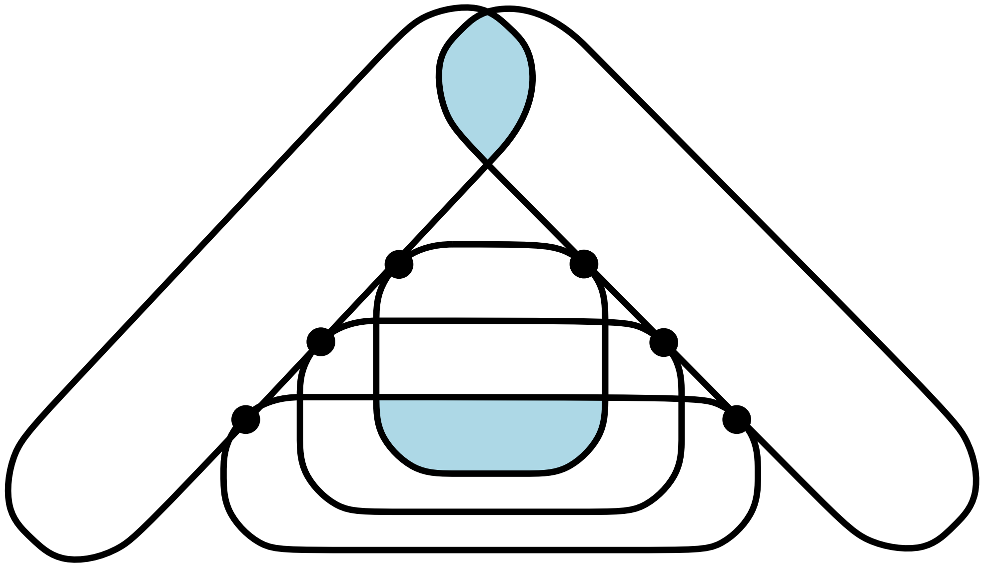


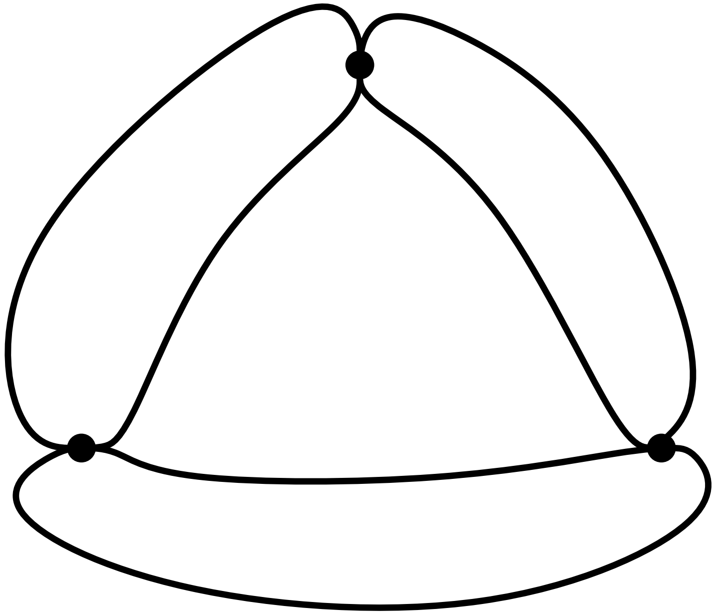
Agarwal et al. (2004):

- Cylindrical case: At most $2n - 2$ touchings
- General case: At most $O(n)$ touchings

Grünbaum's conjecture on digons

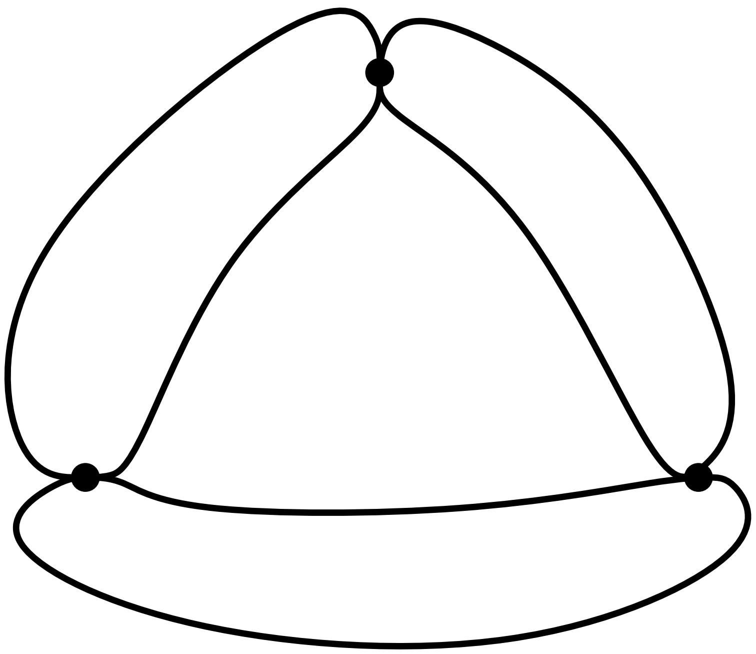
Theorem (Felsner, R., Scheucher)

If three pseudocircles pairwise touch, then the arrangement has at most $2n - 2$ touchings.



Grünbaum's conjecture on digons

Sketch of proof:



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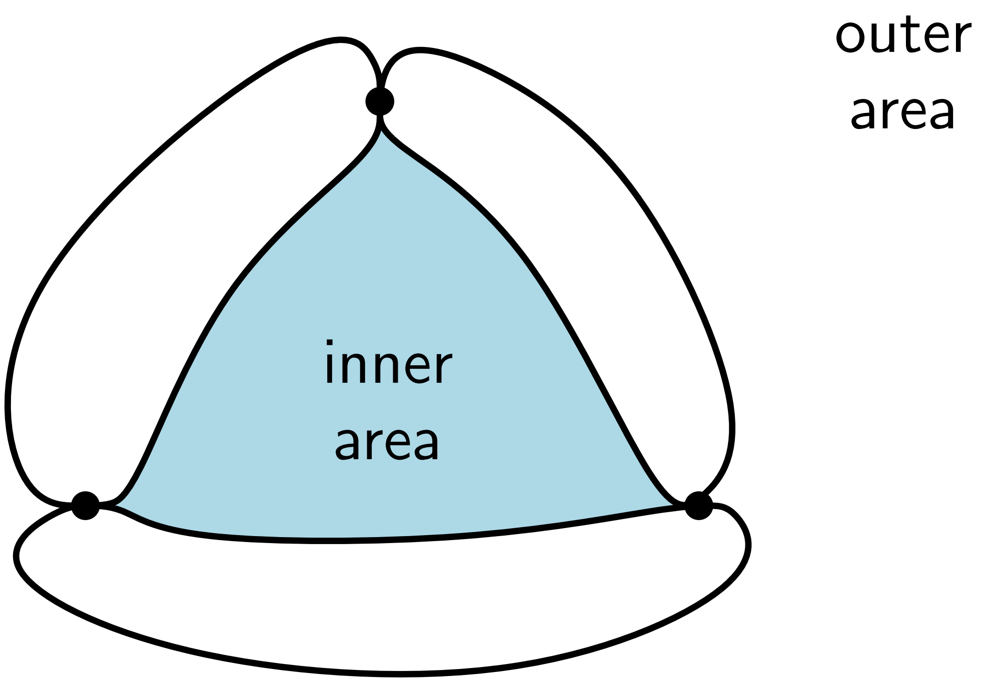

Grünbaum's conjecture on digons

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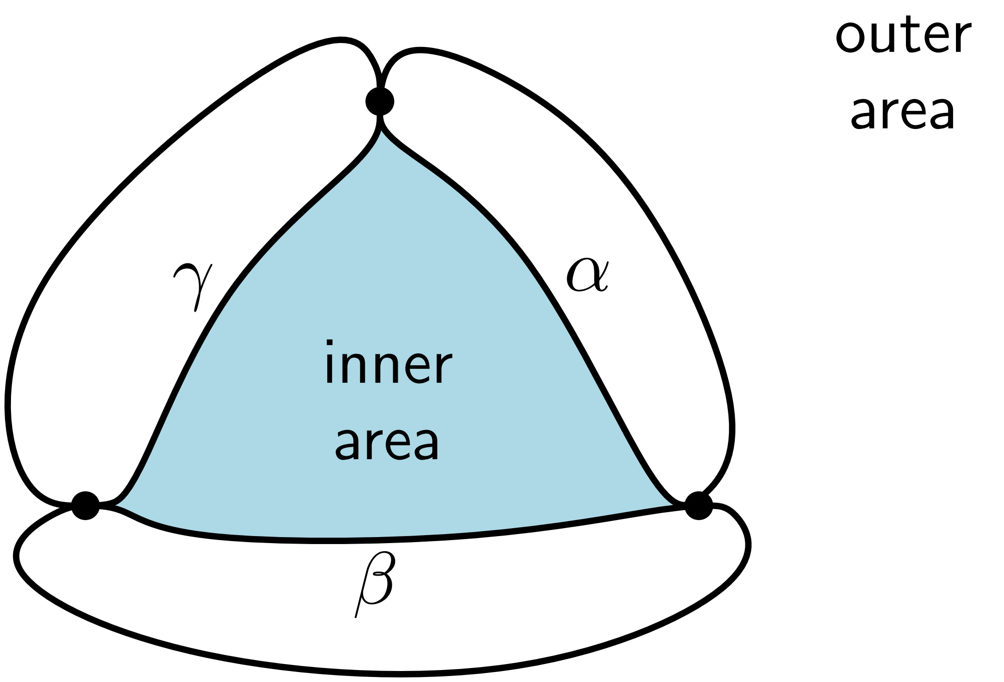

Grünbaum's conjecture on digons

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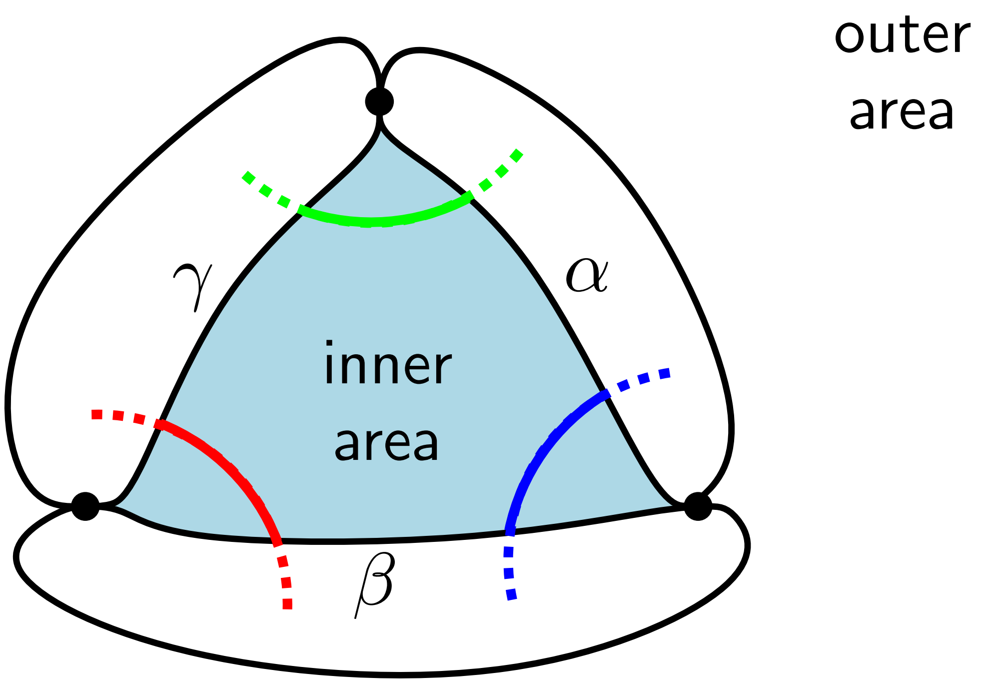
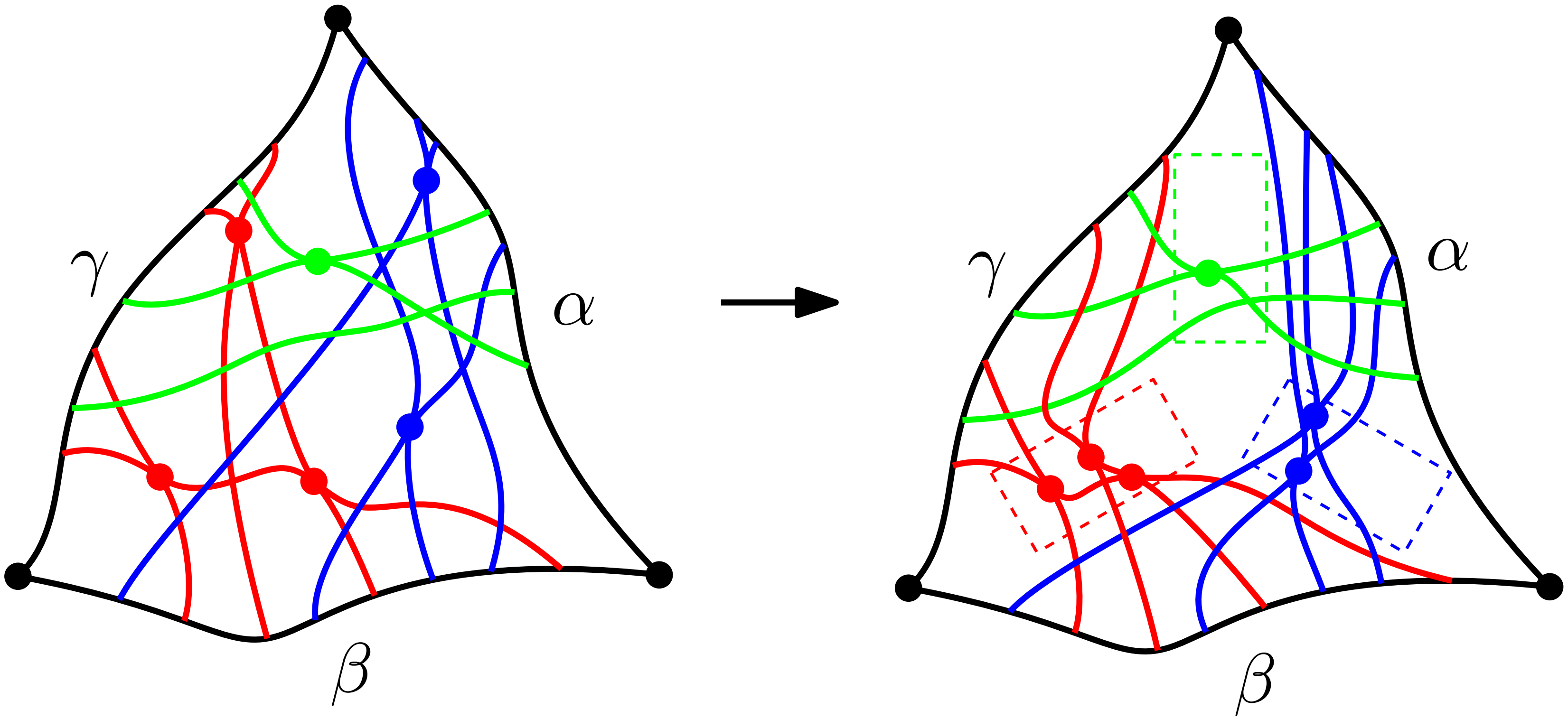


Grünbaum's conjecture on digons

Main idea: Reduction to cylindrical case

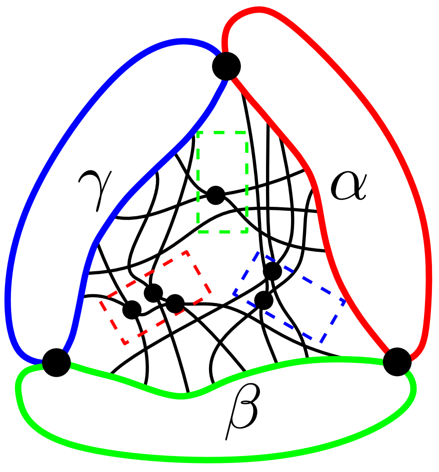
Step 1: Transformation of the inner and outer area



Concentrate intersections between same type in small areas

Grünbaum's conjecture on digons

Step 2: Make arrangement cylindrical



Grünbaum's conjecture on digons

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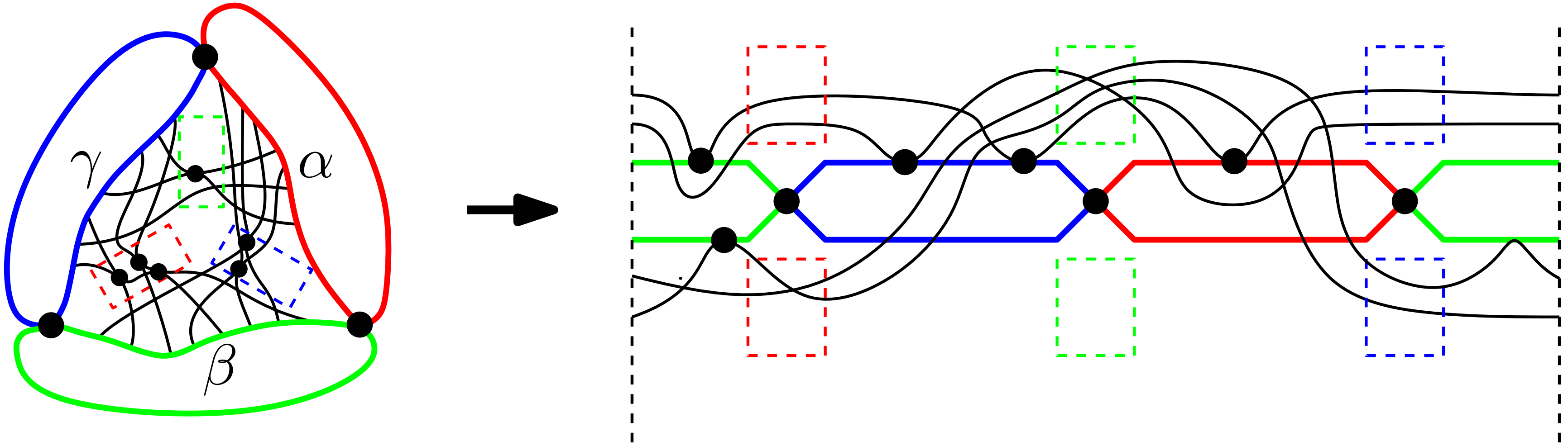

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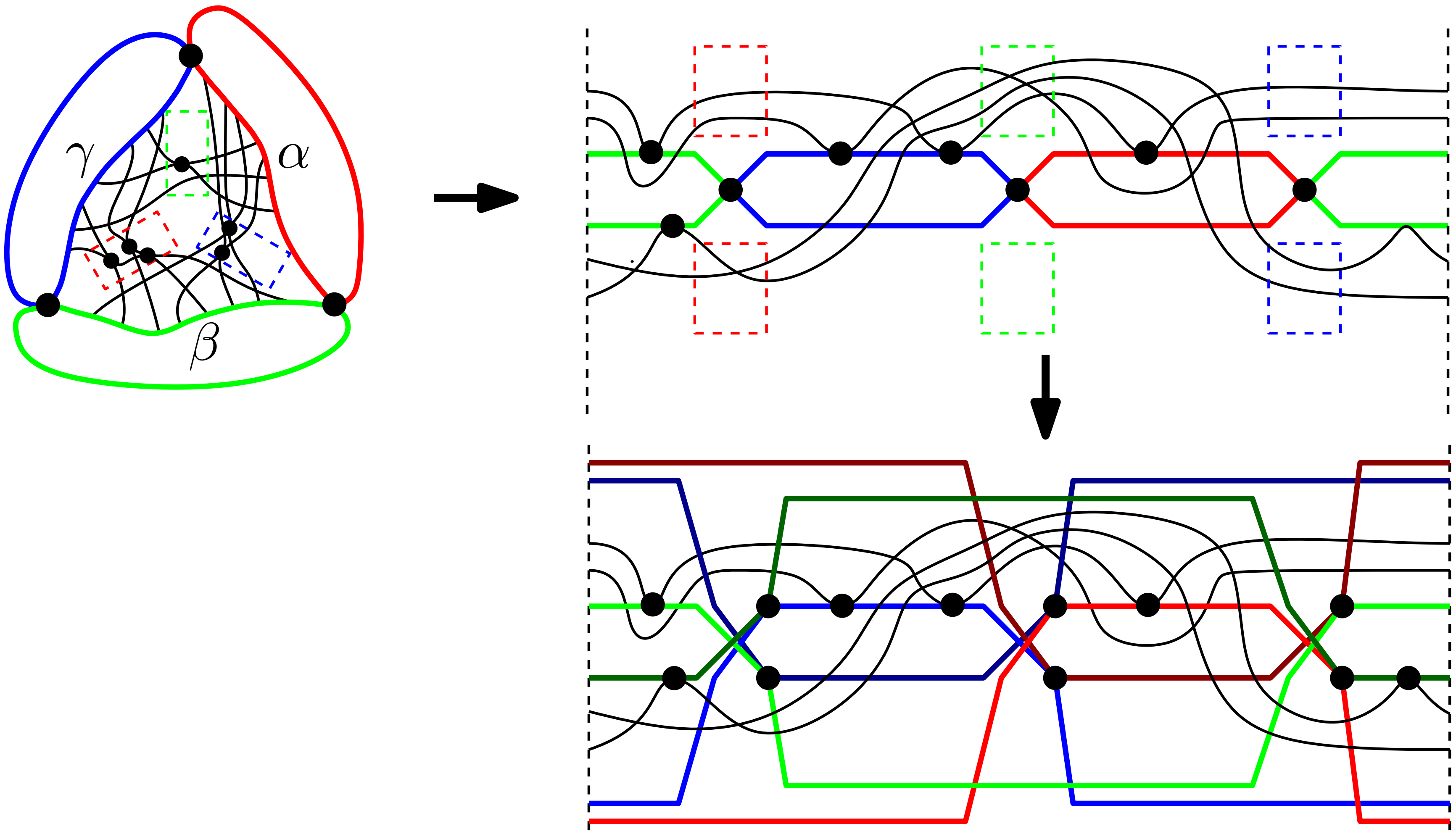

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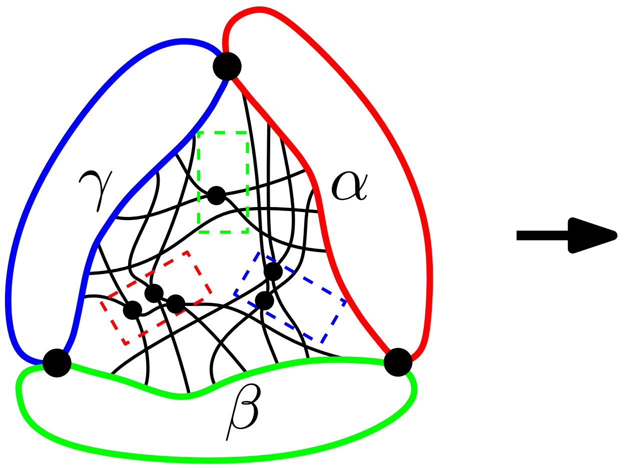

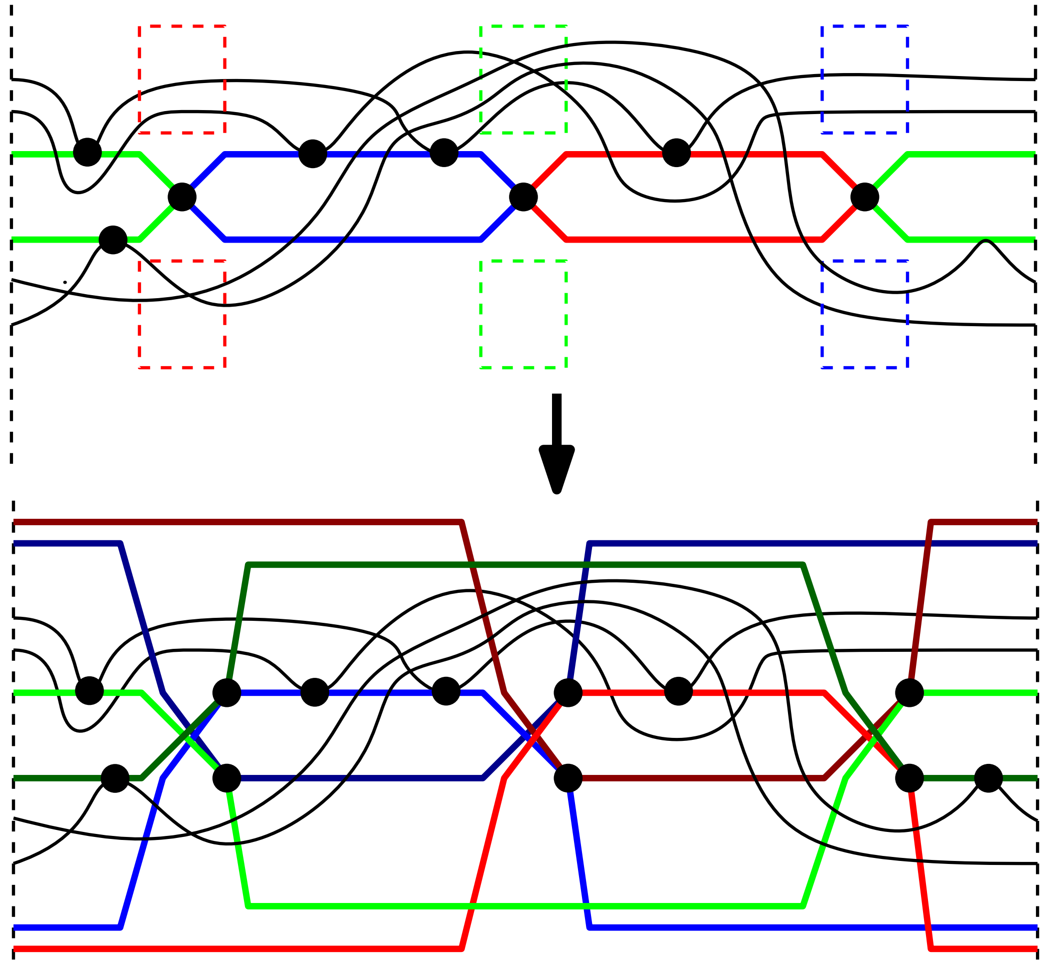

Agarwal et al. (2004):
At most $2n - 2$ touchings.

Triangles in digon and touching free arrangements

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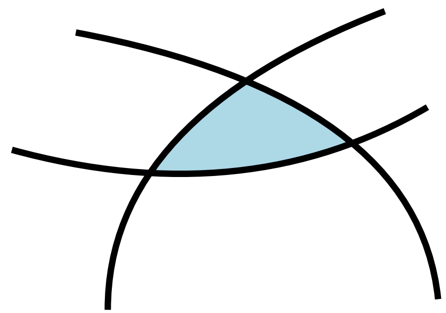
Conjecture (Grünbaum 1972): Arrangements without digons and touchings have $p_3 \geq 2n - 4$ triangles.



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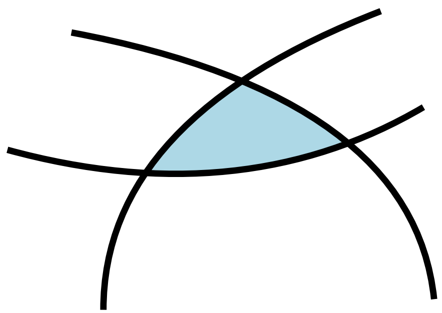
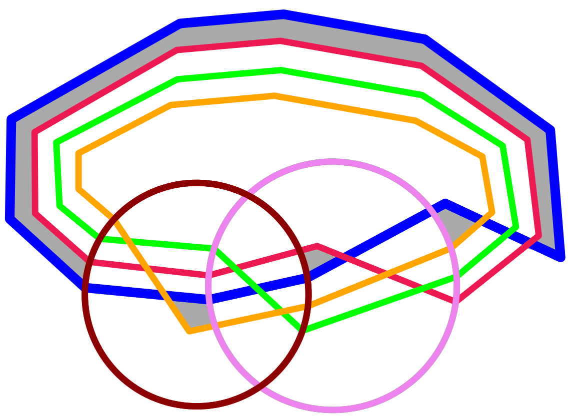
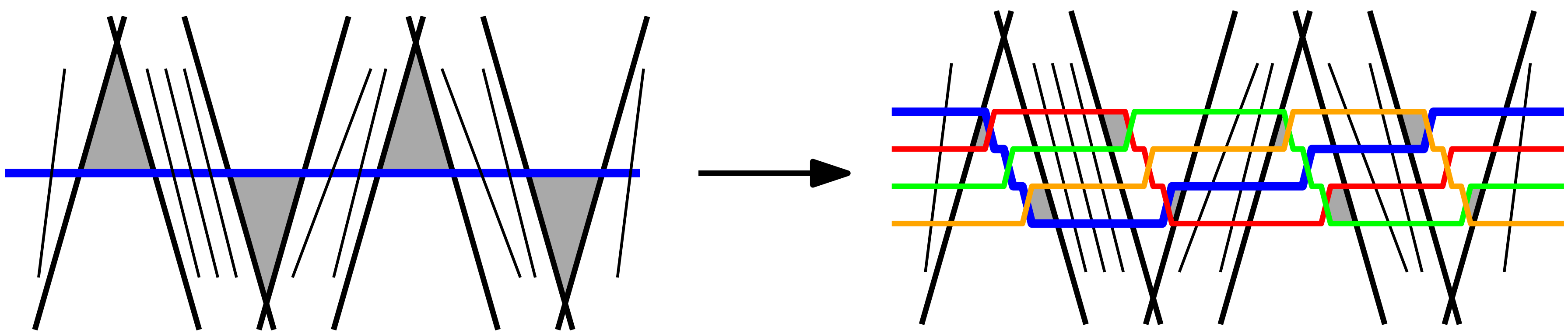


- Snoeyink and Hershberger (1991): $p_3 \geq \frac{4}{3}n$
- Felsner and Scheucher (EuroCG 2017):
  Examples with $p_3 < \frac{16}{11}n$, Grünbaum's conjecture disproved
- Felsner, R., Scheucher (2022):
  Theorem: For $n \geq 6$ there exist examples with $p_3 = \lceil \frac{4}{3}n \rceil$.

Triangles in digon and touching free arrangements

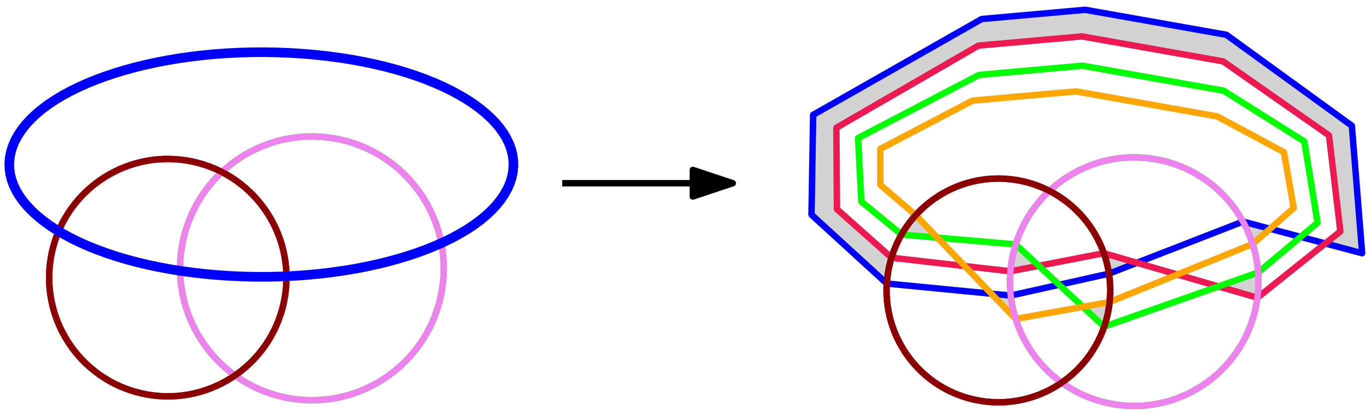
Replace iteratively blue pseudocircle by 4 twisted pseudocircles:



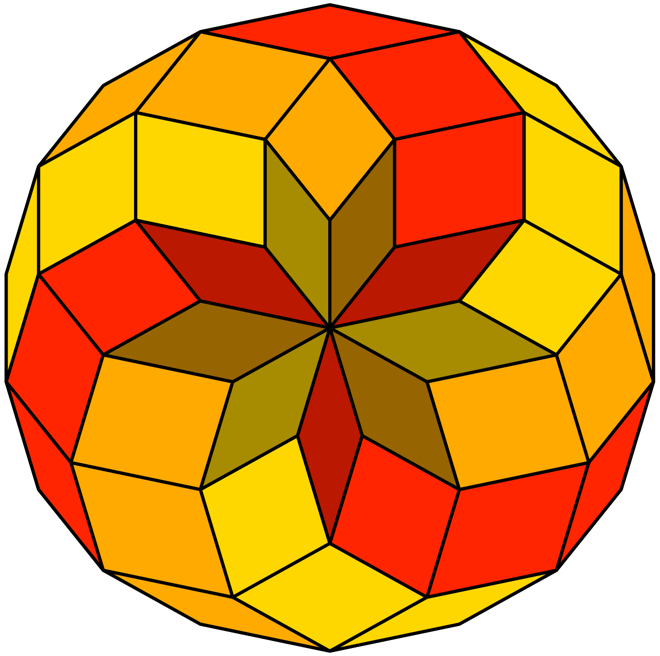


Each iteration increases n by 3 and p_3 by 4.

Triangles in digon and touching free arrangements



Questions?



Question I: How can pseudoline arrangements efficiently be sampled with uniform distribution? In particular, is there a rapidly mixing Markov chain?

Question II: Does Grünbaum's conjecture ($\leq 2n-2$ touchings) hold for all (simple) pseudocircle arrangements?