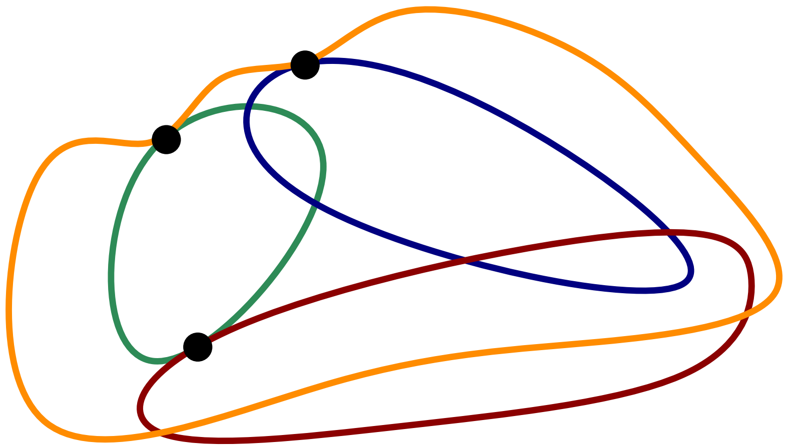
FoC Monday Colloquium 16.05.2022

Arrangements of Pseudocircles: On Digons and Triangles

Stefan Felsner, Sandro Roch, Manfred Scheucher

Pseudocircle arrangements

Example:



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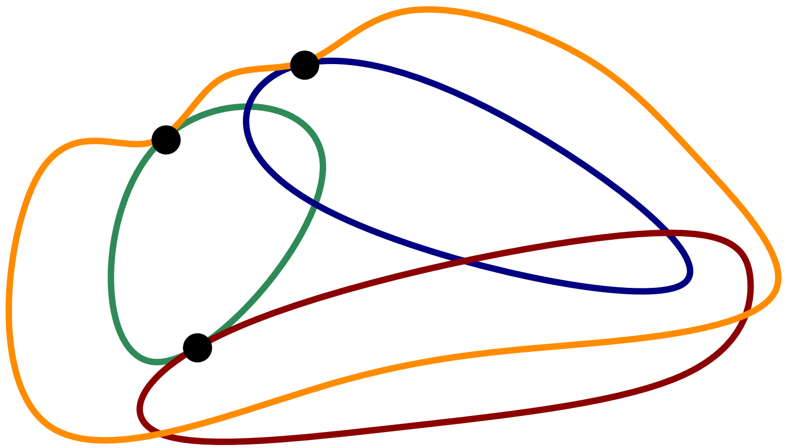

Each two pseudocircles...

...either cross exactly twice, ... **or** ...have a single touching

Pseudocircle arrangements

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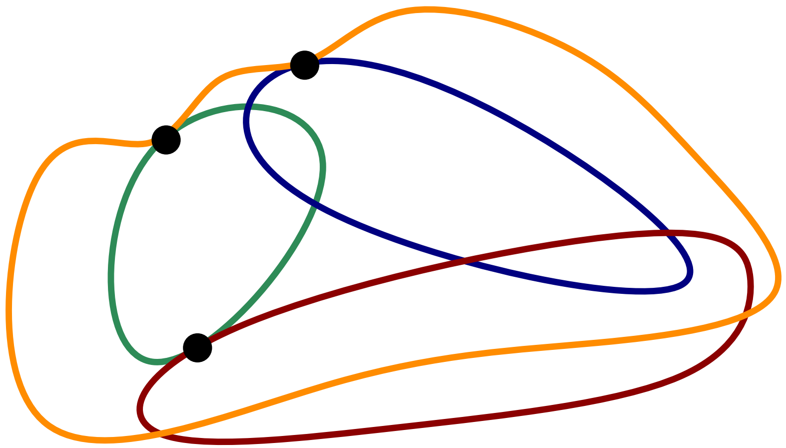

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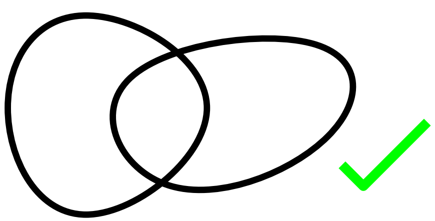
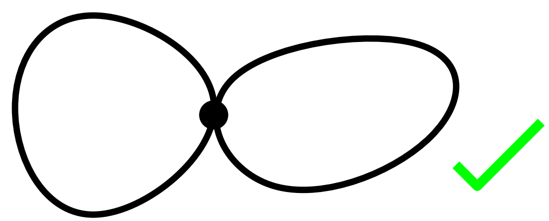
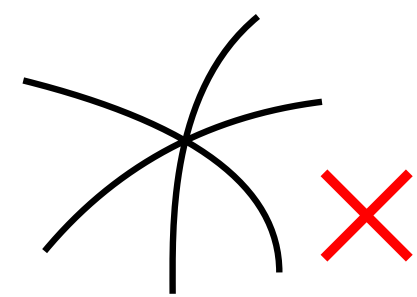


or





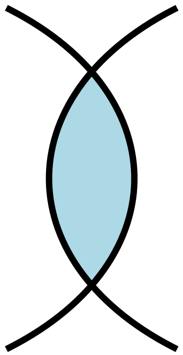
...either cross exactly twice, ...

...have a single touching

No intersection of ≥ 3 pseudocircles in single point

Grünbaum's conjecture on digons

Conjecture: In every pseudocircle arrangement, there are at most $2n - 2$ digons. (Grünbaum, 1972)



digon

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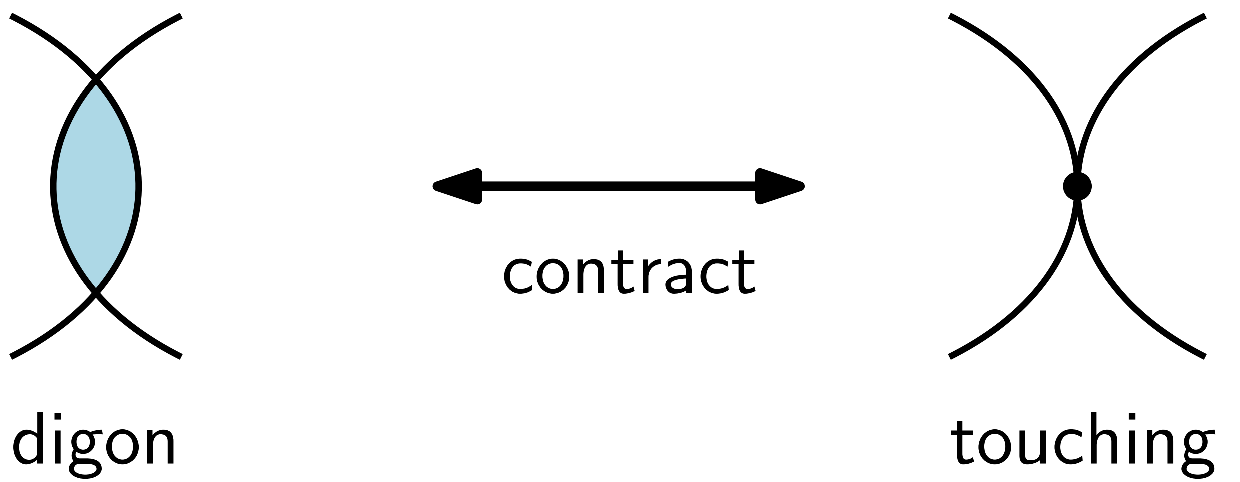

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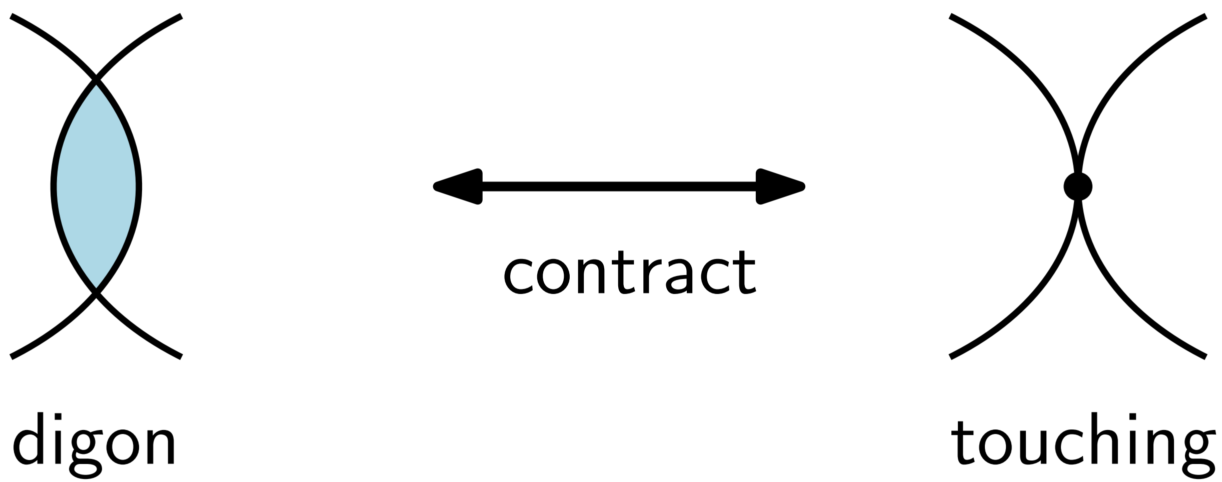
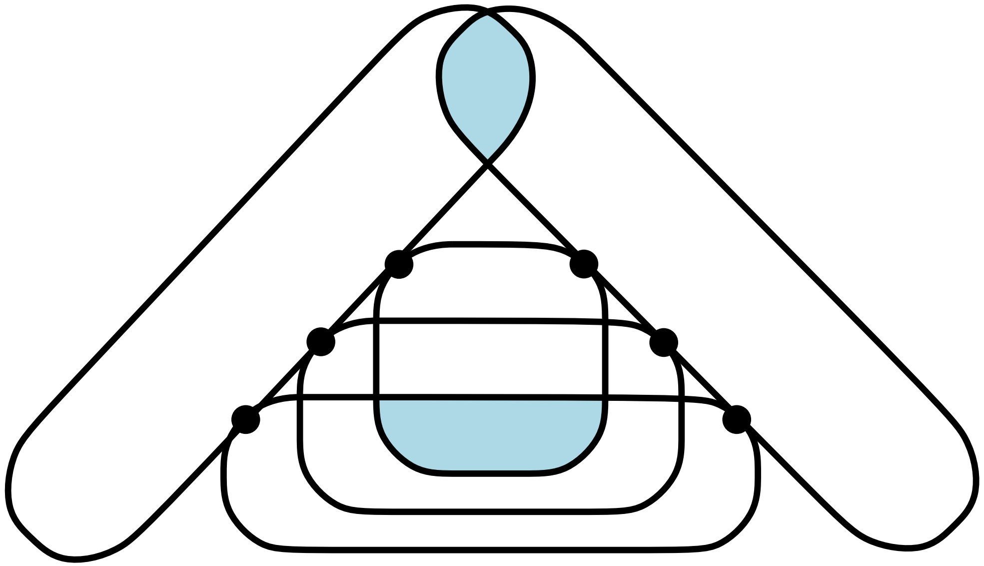


Equivalent: At most $2n-2$ touchings.

Grünbaum's conjecture on digons

Cylindrical pseudocircle arrangement:

Exist two cells separated by each pseudocircle



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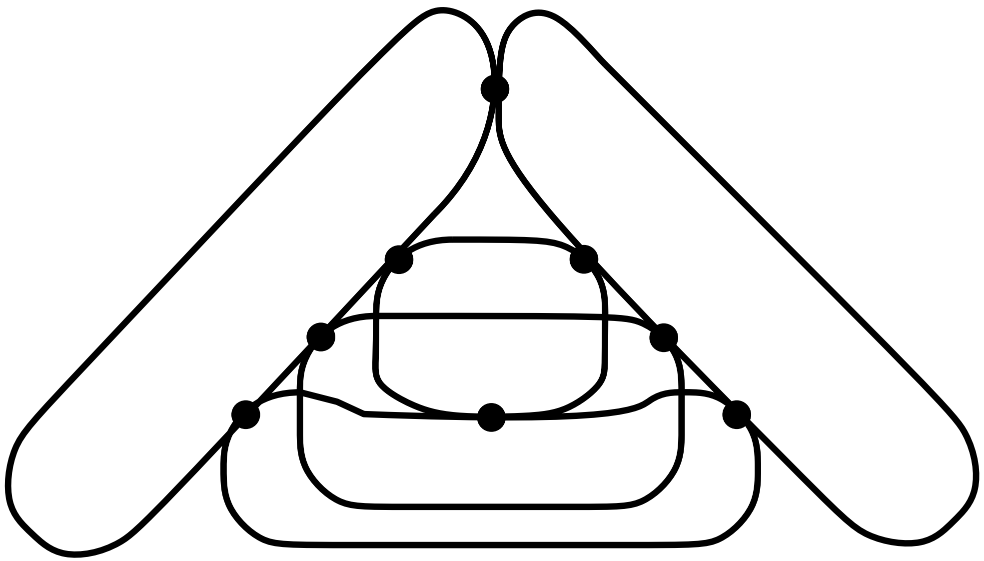

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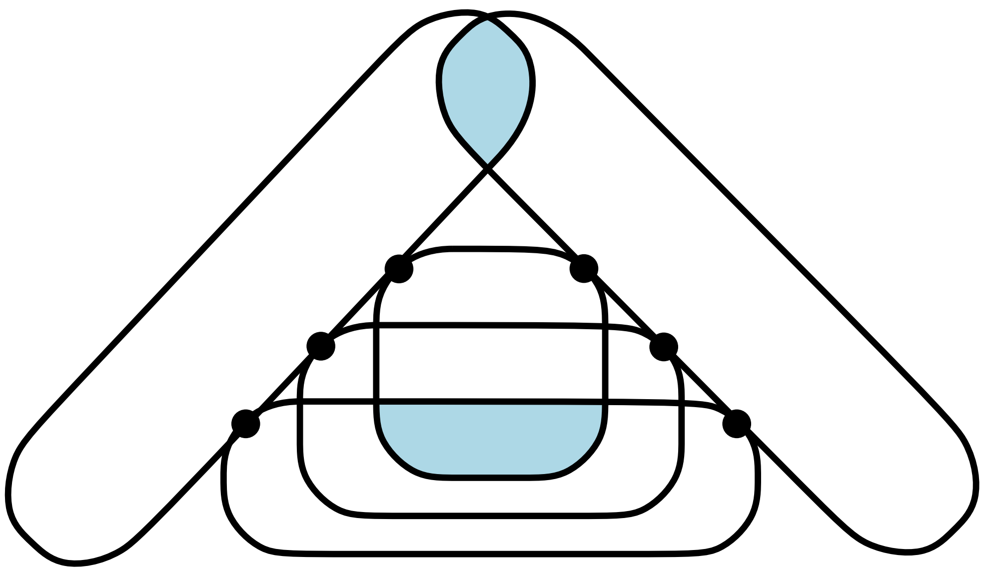
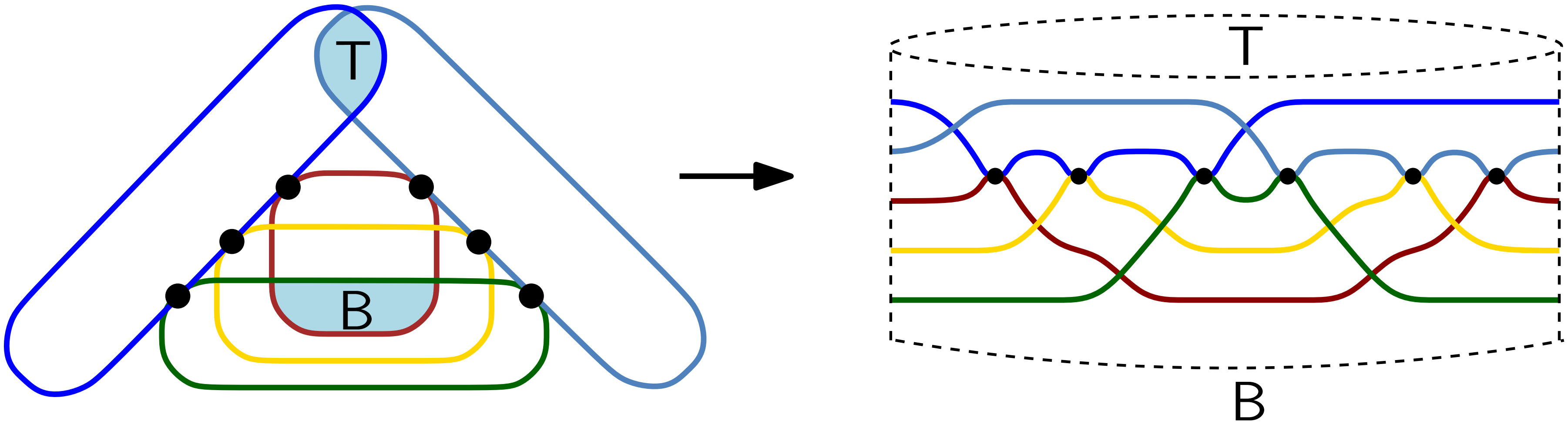


Agarwal et al. (2004):

- Cylindrical case: At most $2n-2$ touchings
- General case: At most $O(n)$ touchings

Cylindrical case

- Cylindrical arrangements can be drawn on a cylinder (Bultena, Grünbaum, Ruskey, 1998)



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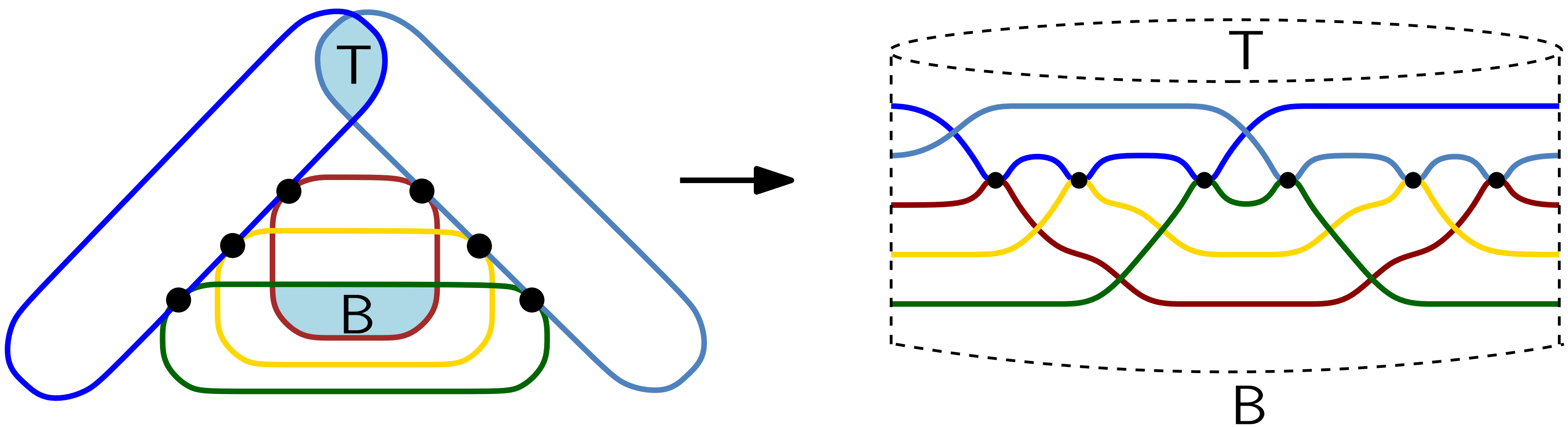

Def: *pseudoparabola arrangement*:

- Continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}$
- Each two $f_i \neq f_j$ either cross twice or touch once

Cylindrical case

Theorem (Agarwal et al., 2004):
Pseudoparabola arrangements have at most $2n - 4$ touchings.

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Proof sketch:

- Define *touching graph* $G = (V, E)$:
  - Vertices: $V = \{f_1, \cdots, f_n\}$
  - Edges: $E = \{(f_i, f_j) : f_i$ and f_j form a touching $\}$

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  - Edges: $E = \{(f_i, f_j) : f_i$ and f_j form a touching $\}$
- Show that G is planar and bipartite
- $\implies$ At most $|E| \leq 2n - 4$ touchings

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G bipartite:

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G bipartite: Each parabola f_i has...

- ... **either** all touchings from below
- ... **or** all touchings from above.

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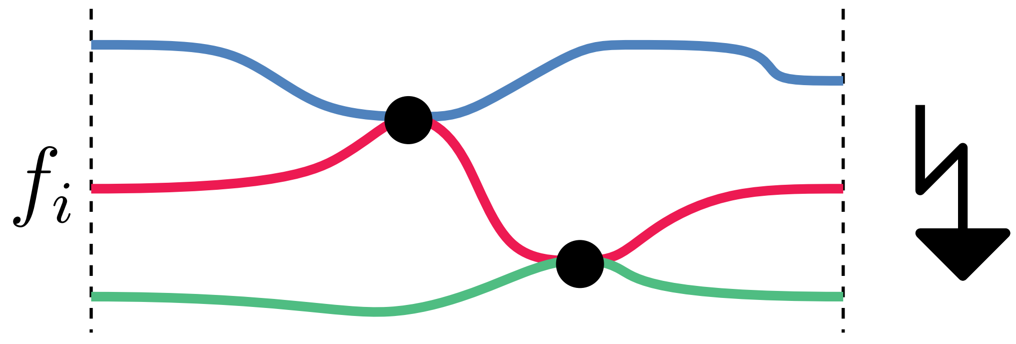
Suppose not...



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G planar:

Cylindrical case

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G planar:

- Find drawing s.t. indep. edges cross even times
- Hanani-Tutte-Theorem: $\implies$ G planar

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Drawing of G:

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Drawing of G:

- Draw vertices on vertical line

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- Draw edges as y-monotone curves

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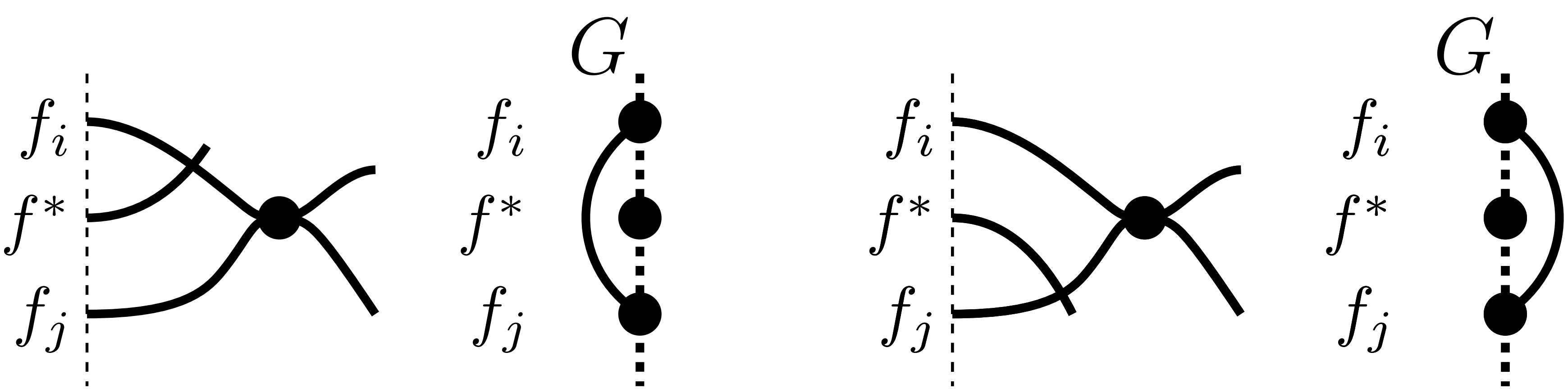
Drawing of G:

- Draw vertices on vertical line
- Draw edges as y-monotone curves
- Two cases for edge $\{f_i, f_j\}$:



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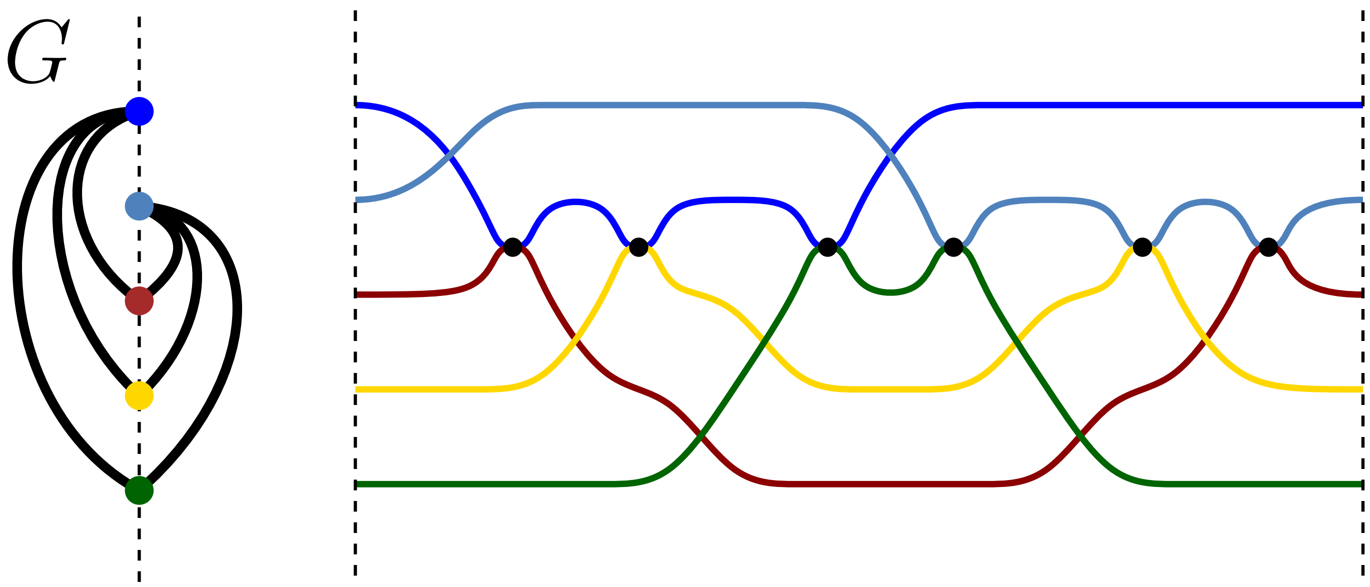
Drawing of G: Example



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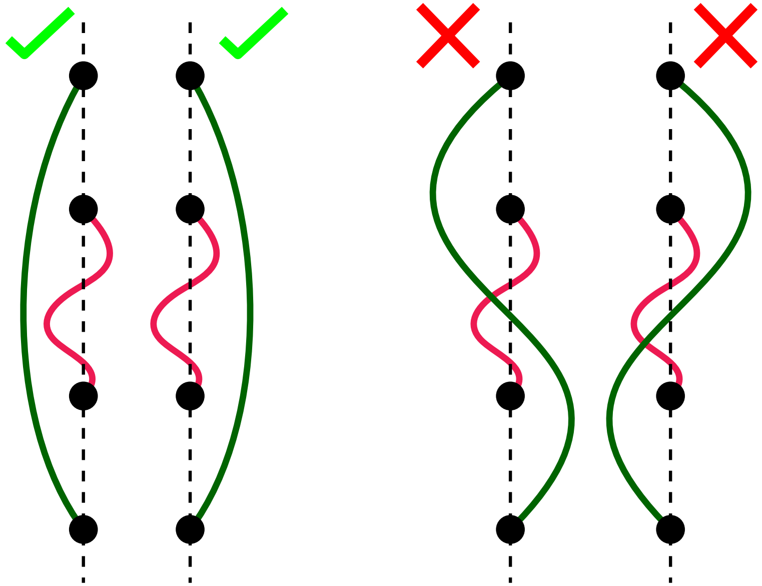

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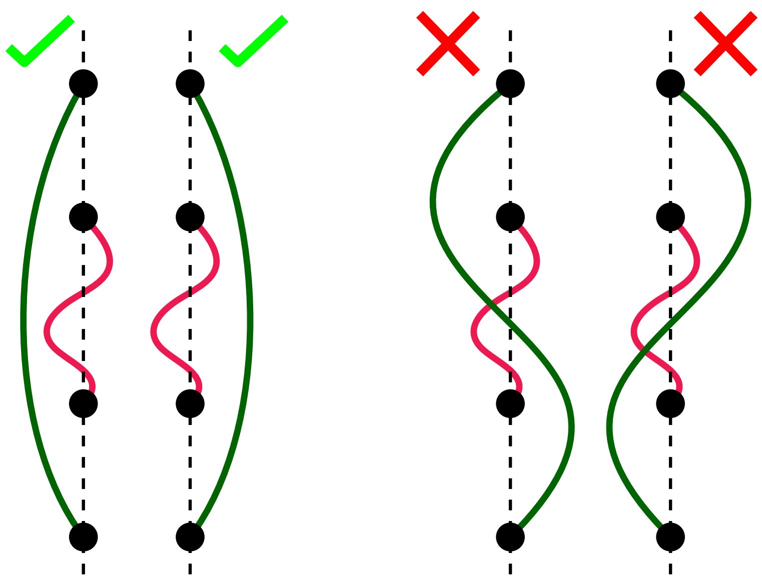
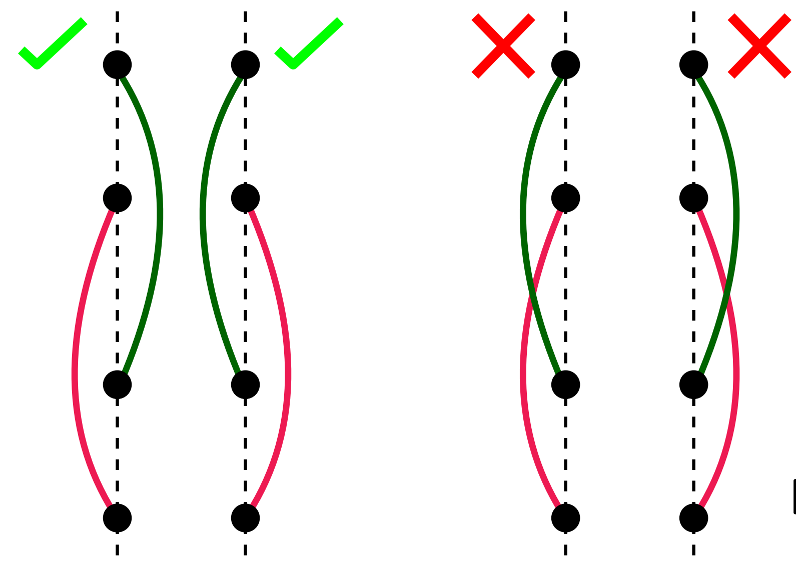


Overlapping case:



$\square$

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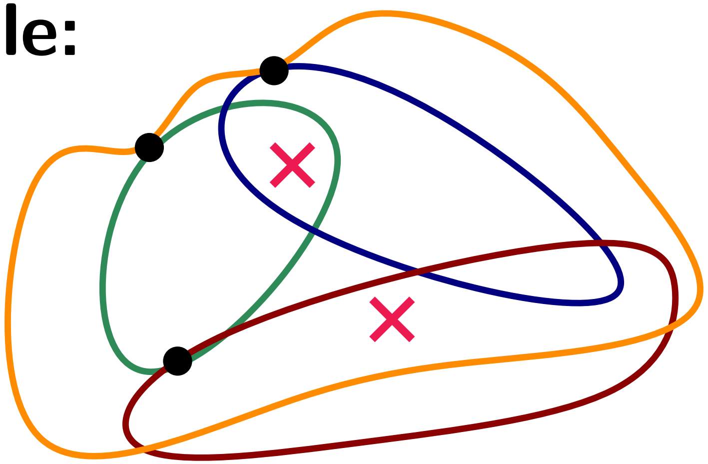

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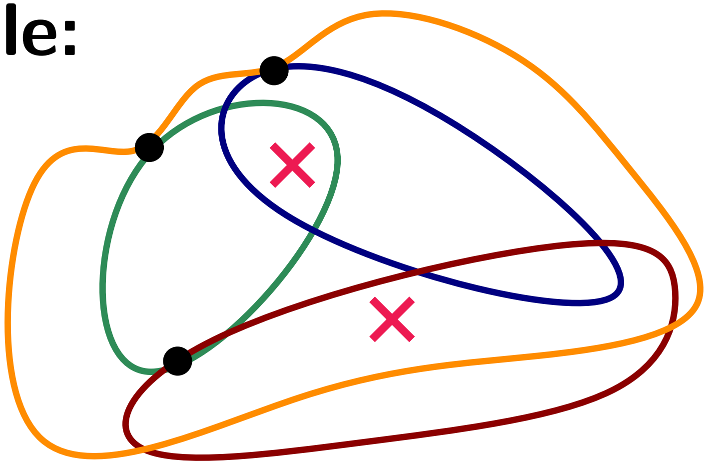

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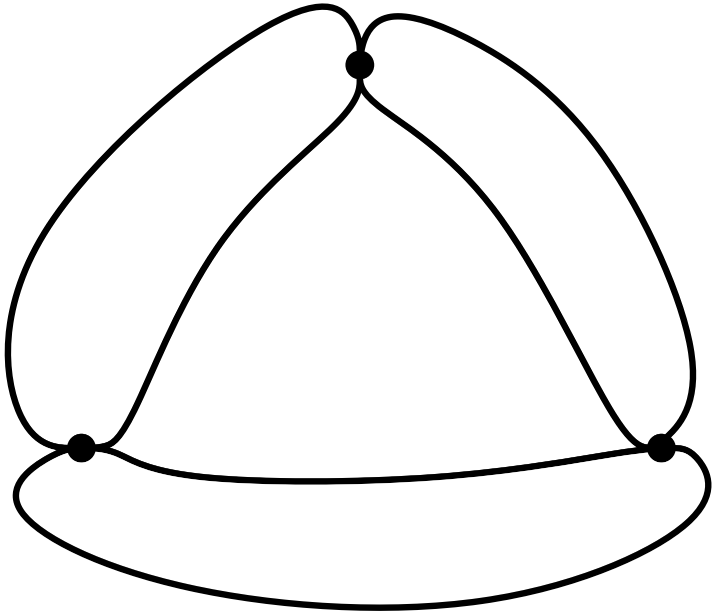
Theorem (Felsner, R., Scheucher)

If three pseudocircles pairwise touch, then the arrangement has at most $2n-2$ touchings.



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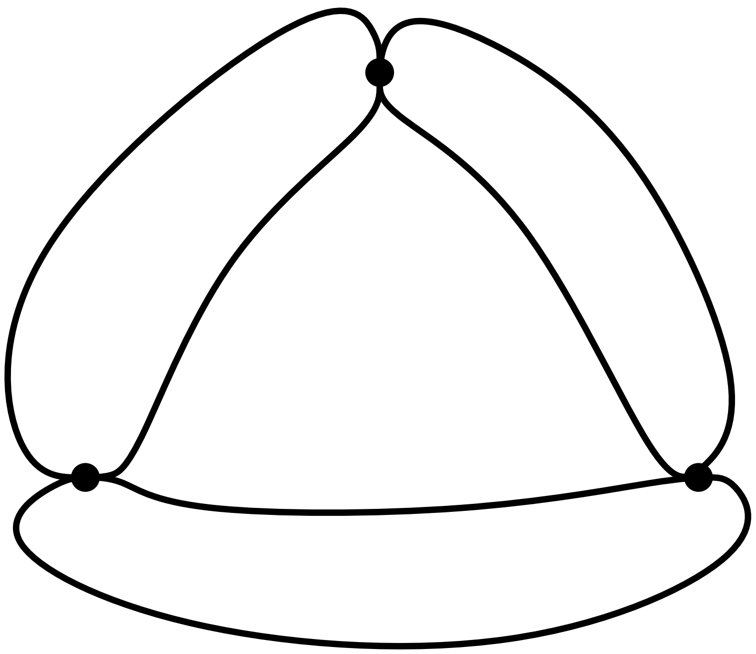
Sketch of proof:



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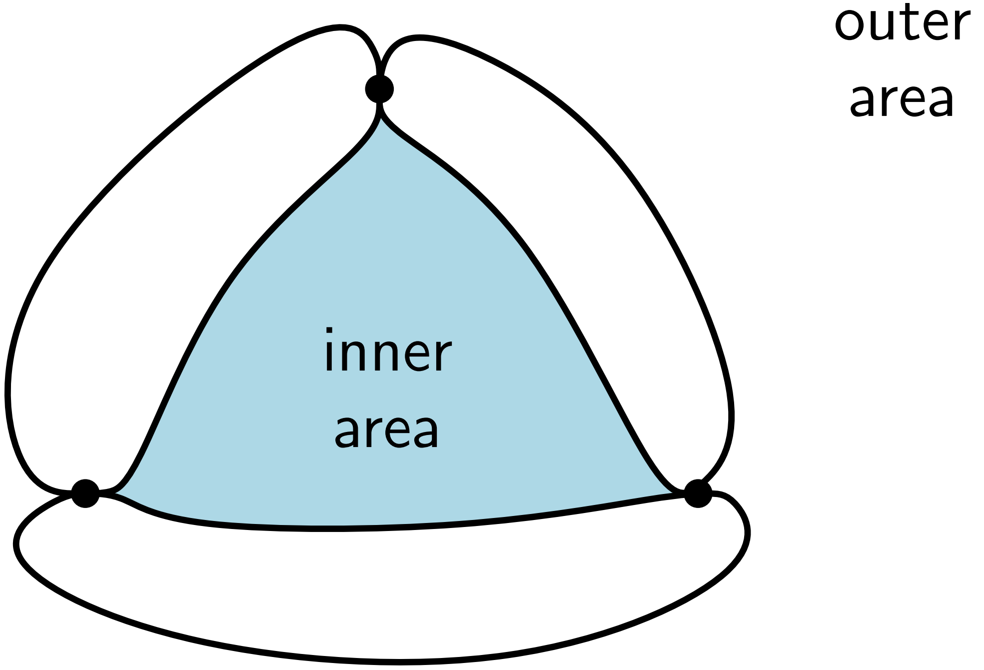

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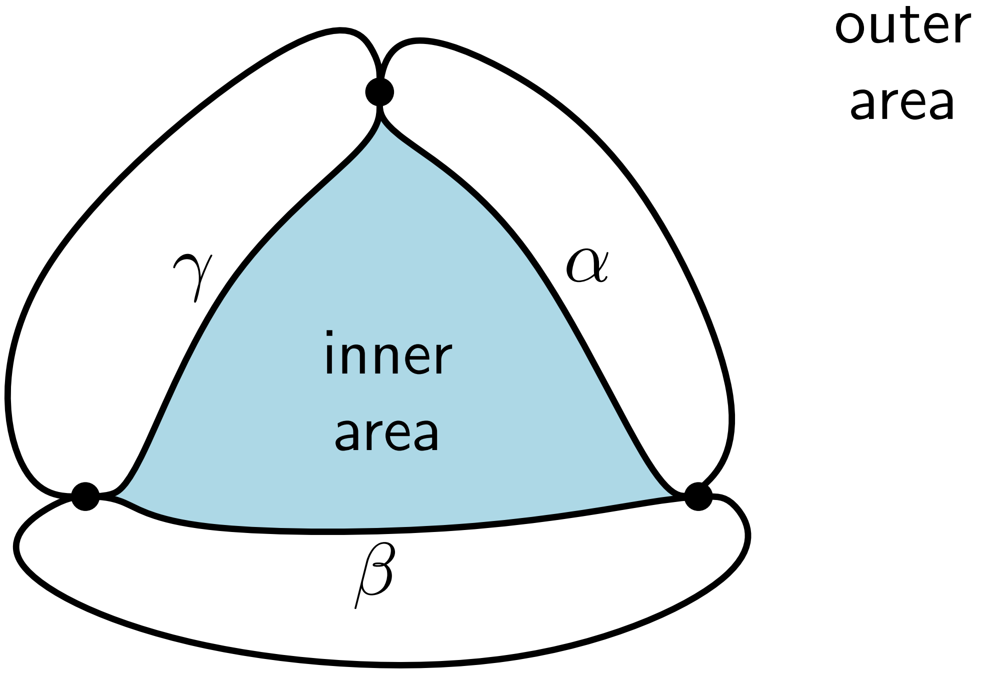

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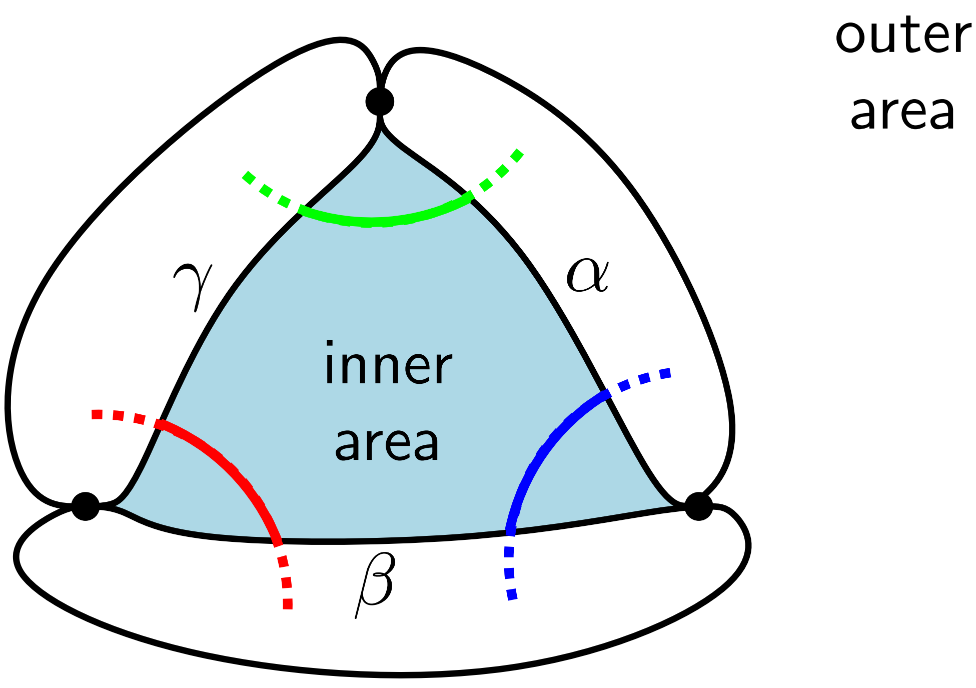
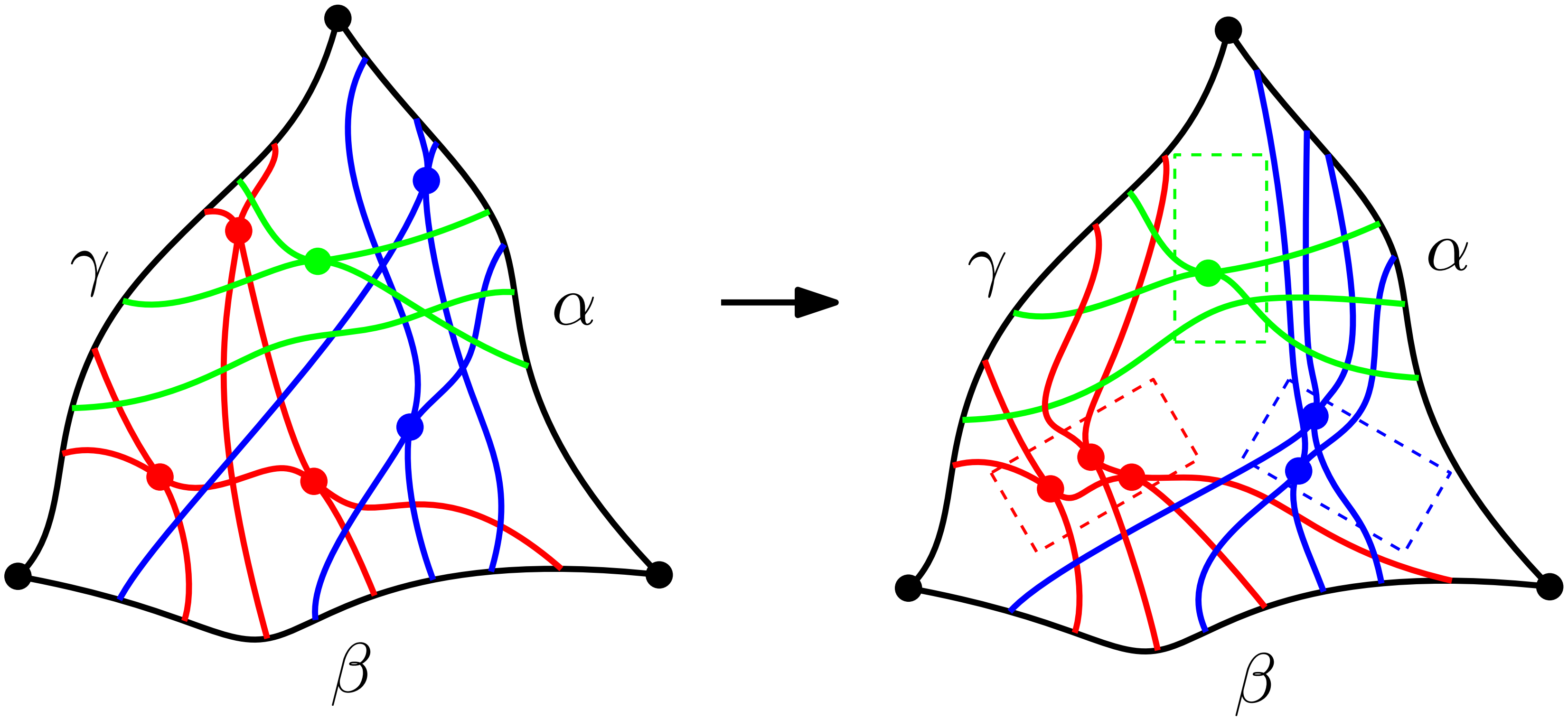


Grünbaum's conjecture on digons

Main idea: Reduction to cylindrical case

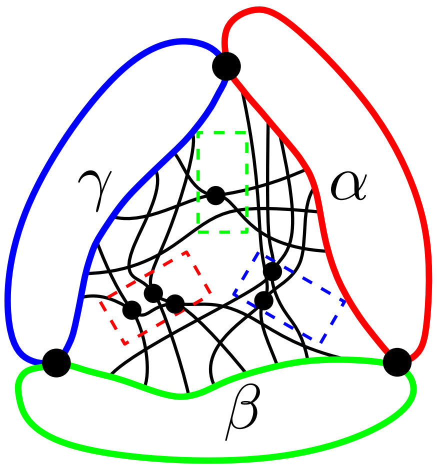
Step 1: Transformation of the inner and outer area



Concentrate intersections between same type in small areas

Grünbaum's conjecture on digons

Step 2: Make arrangement cylindrical



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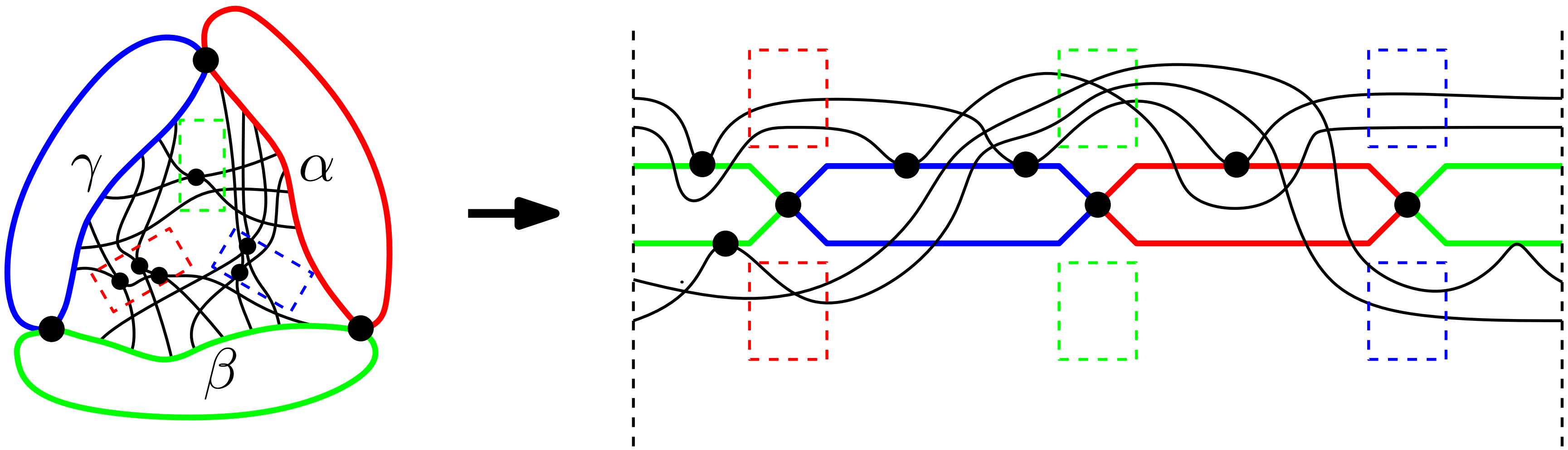

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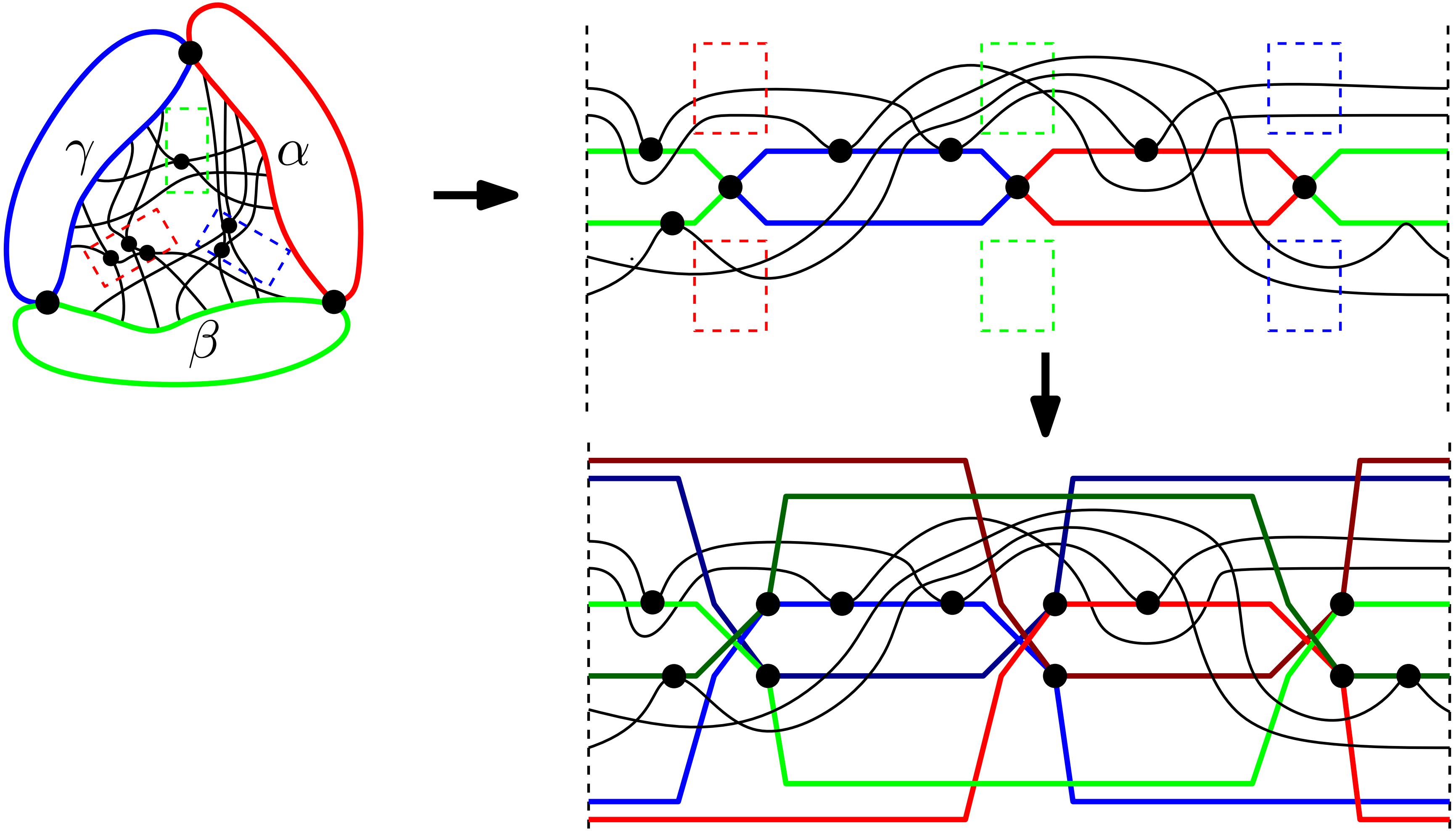

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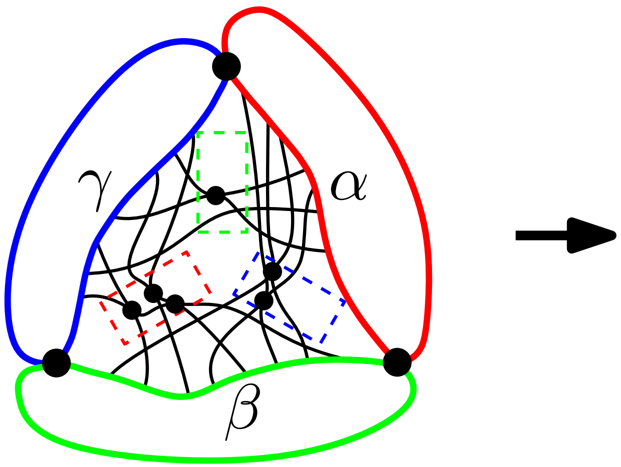

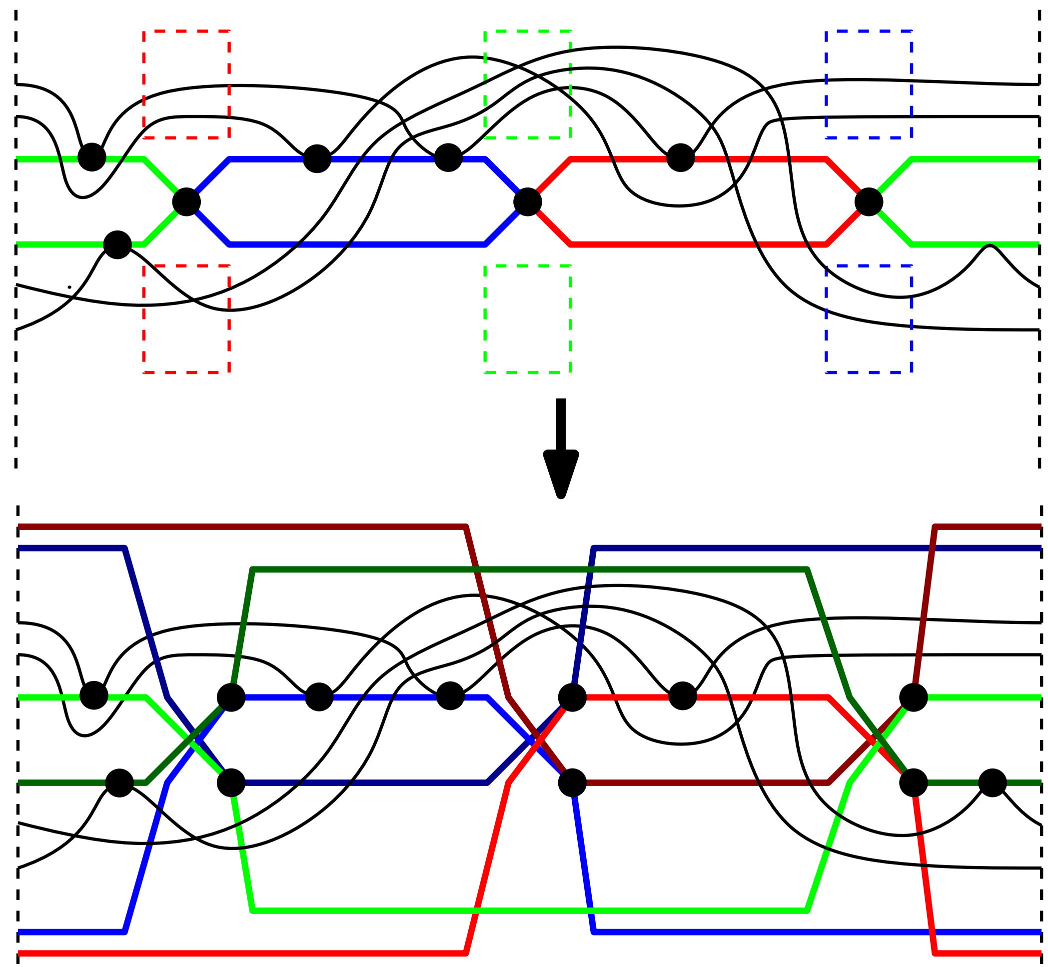

Agarwal et al. (2004):
At most $2n-2$ touchings.

□

Many touchings in absence of touching triple

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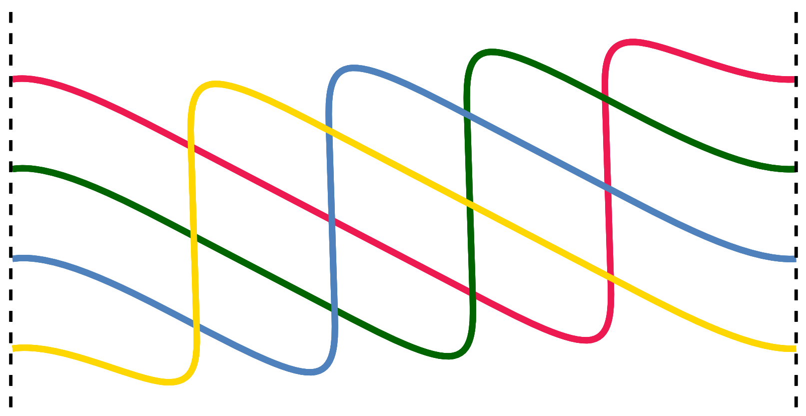
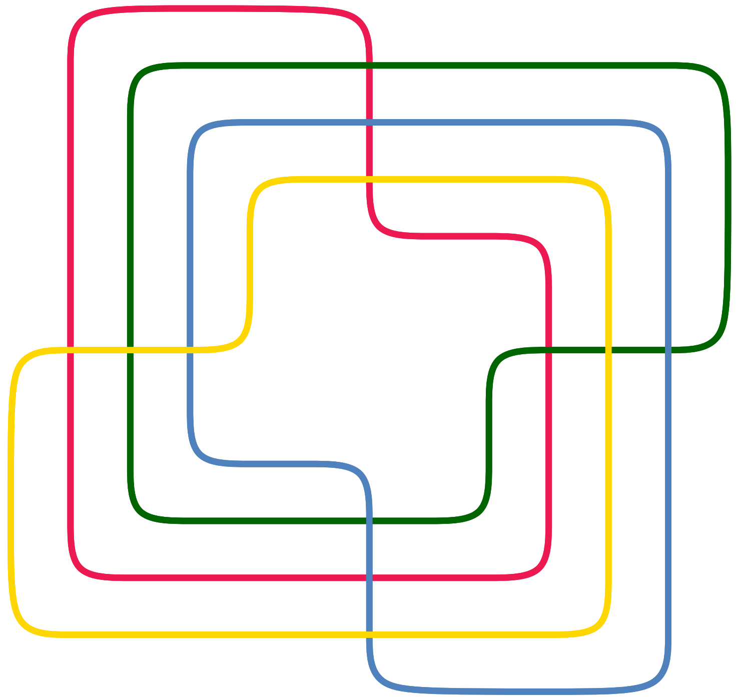
Standard cylindrical arrangement:

Example on 4 pseudocircles:

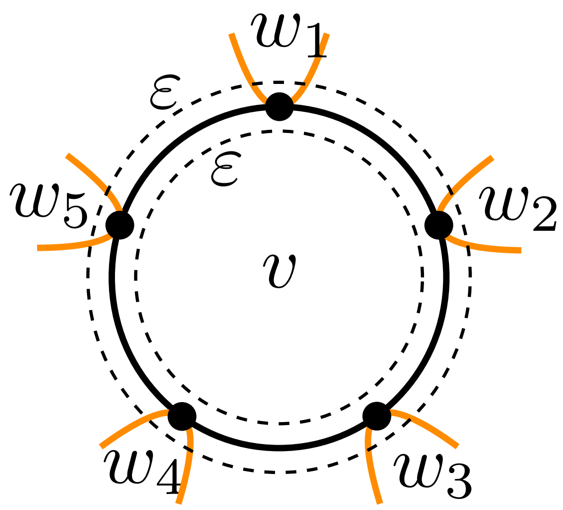




Many touchings in absence of touching triple

Blossom operation:

- Take pseudocircle v with k touchings $w_1, \cdots, w_k$
- Insert ε-closely standard arrangement of k pseudocircles $v'_1, \cdots, v'_k$ inside v



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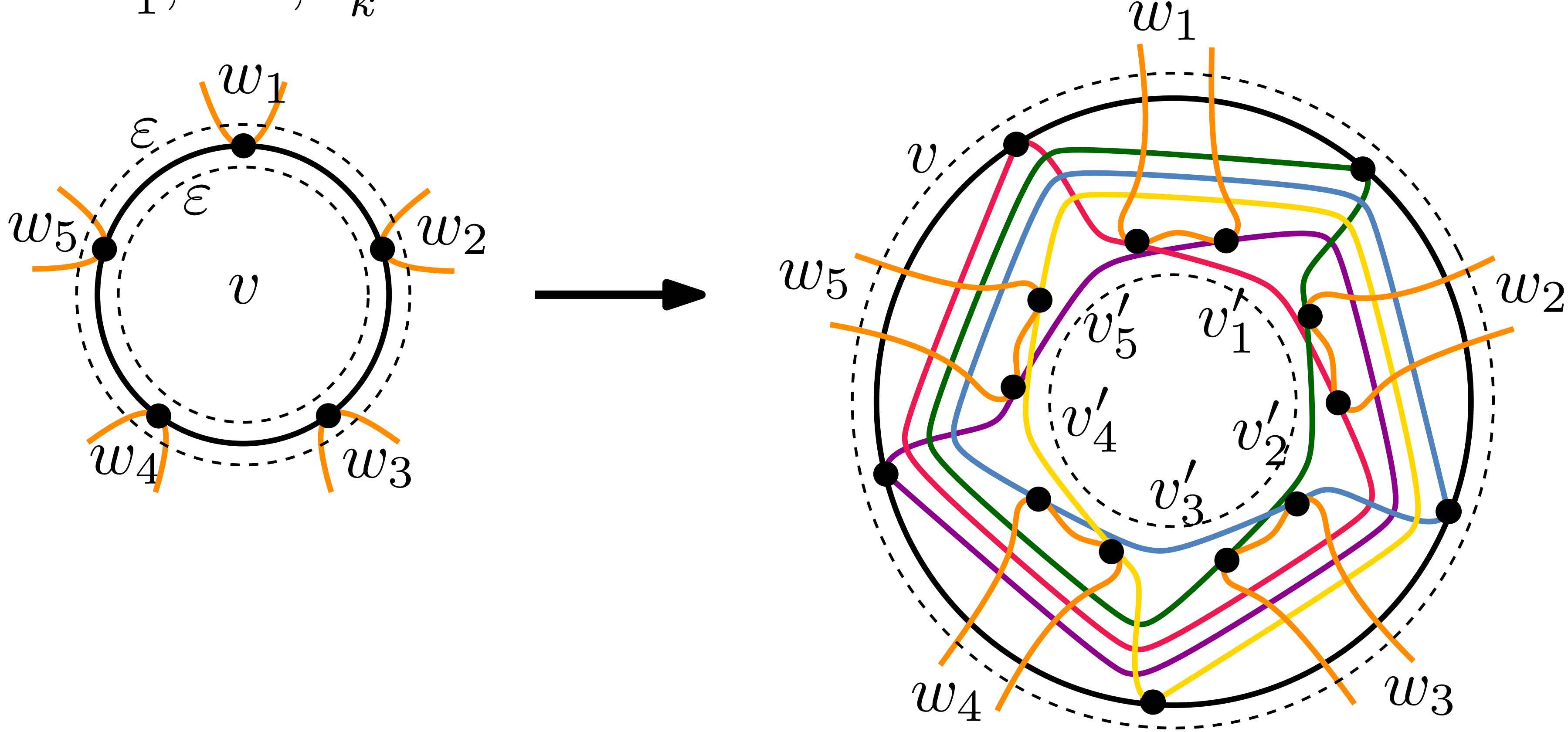

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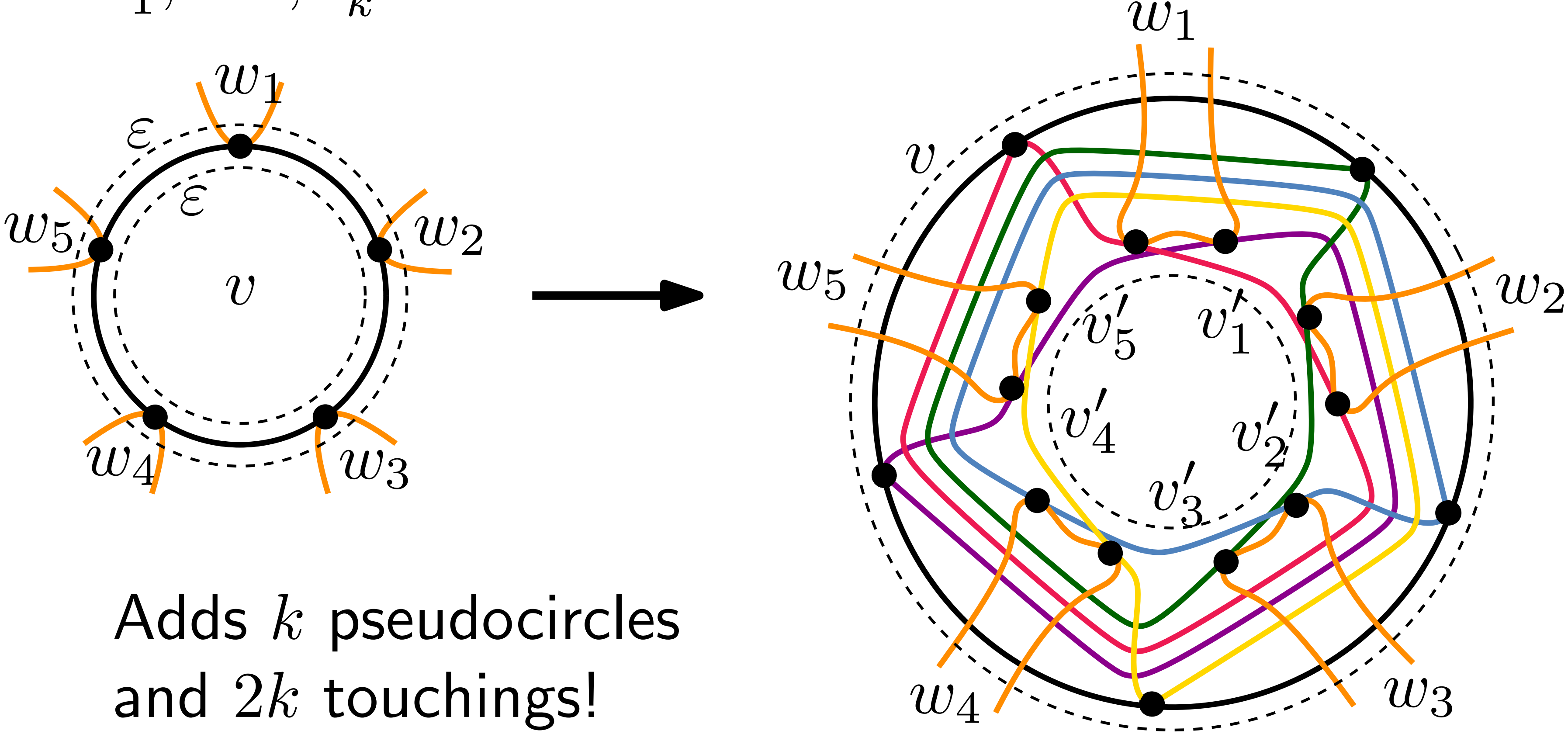
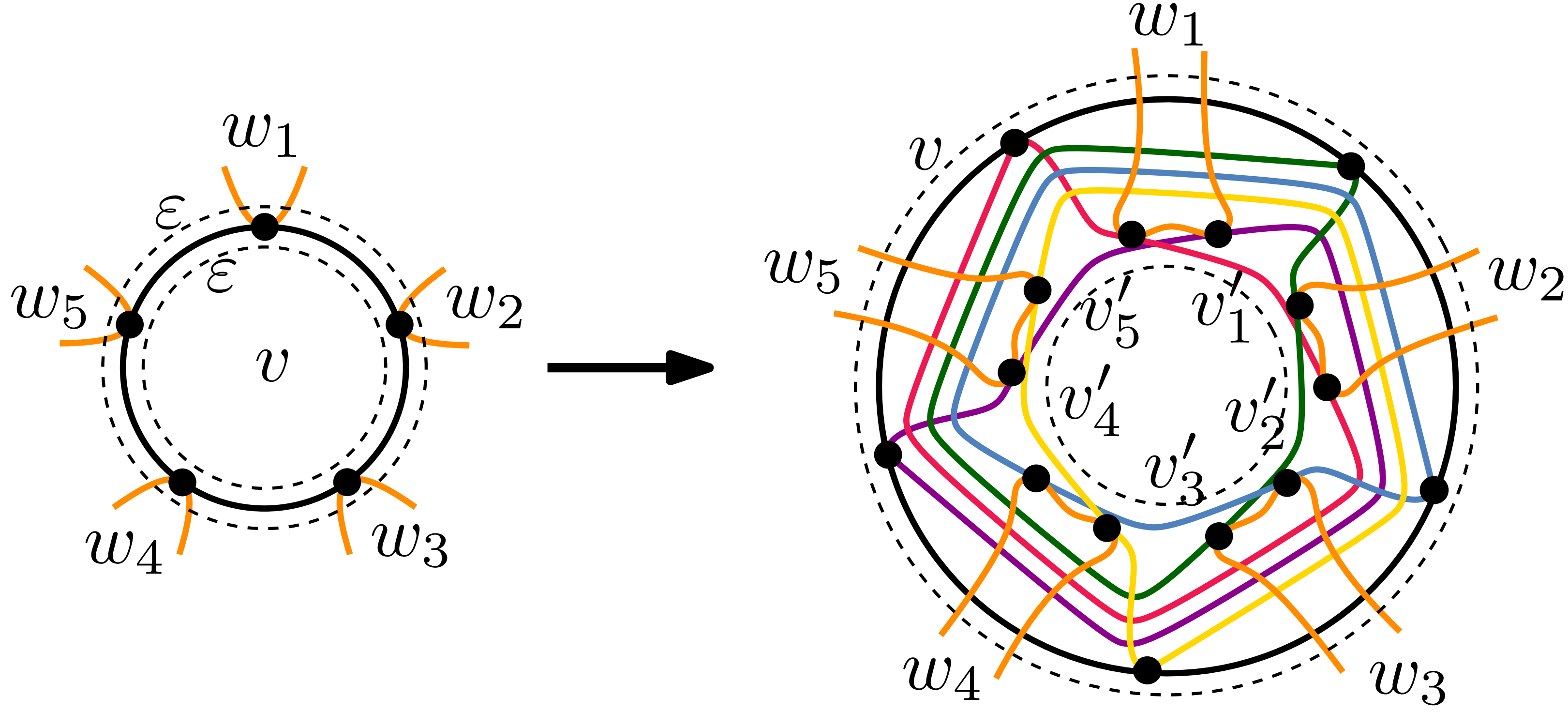


Adds k pseudocircles
and $2k$ touchings!

Many touchings in absence of touching triple

Blossom operation: Change in the touching graph G



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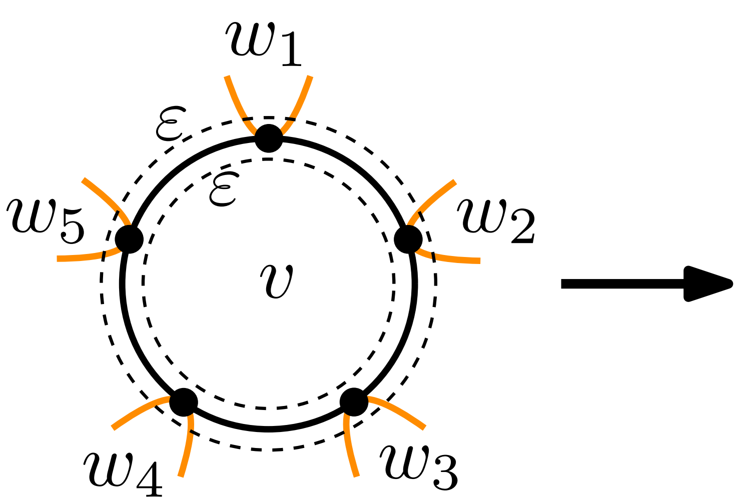
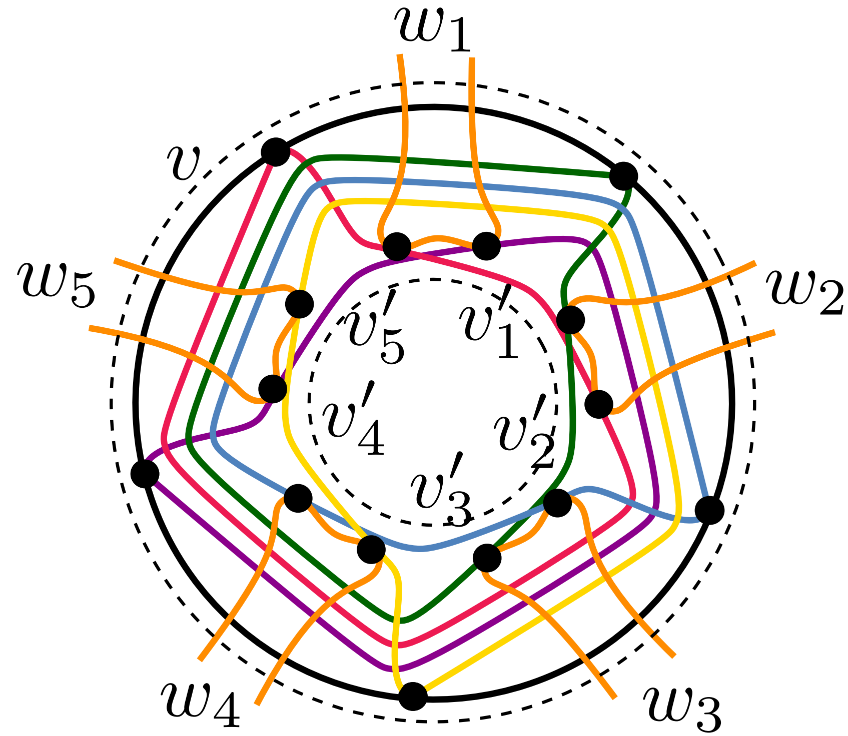
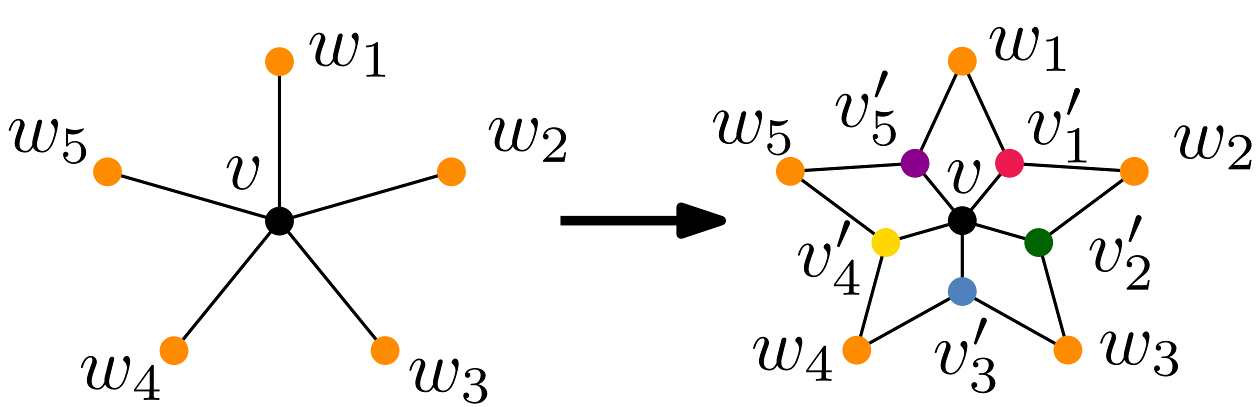




Touching graph G:



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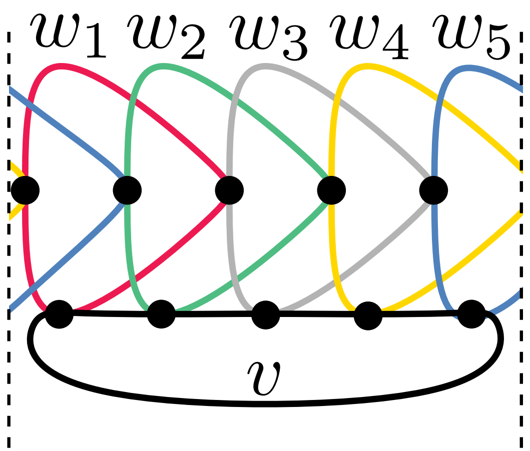
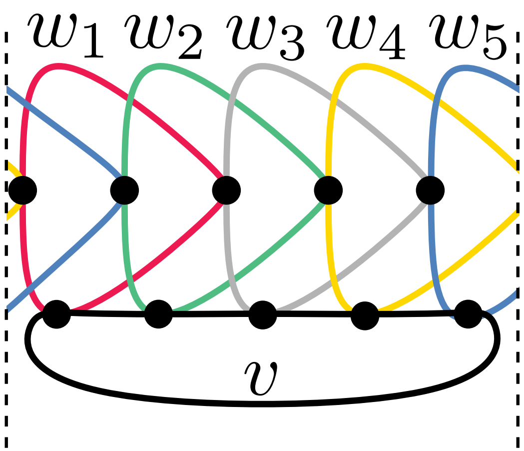


$|V| = 6$ pseudocircles

$|E| = 10$ touchings

Many touchings in absence of touching triple



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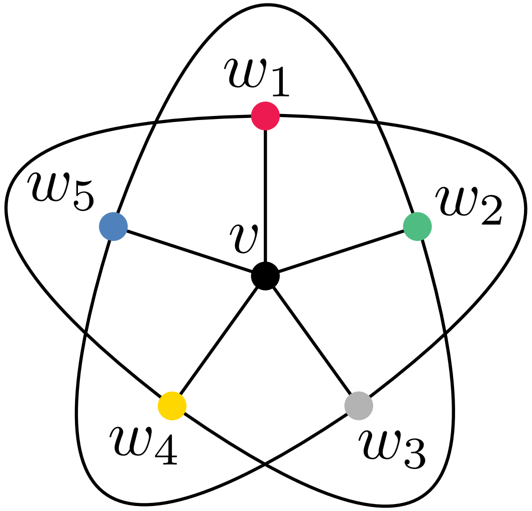

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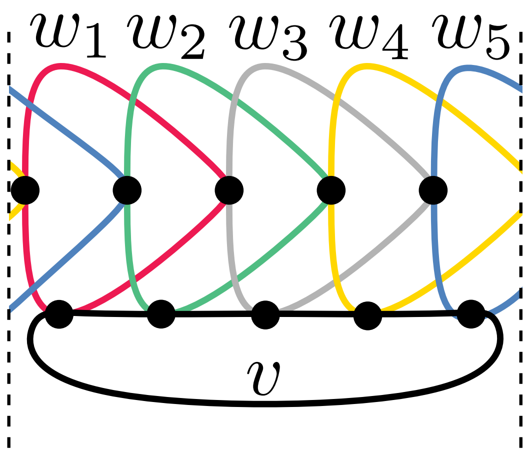

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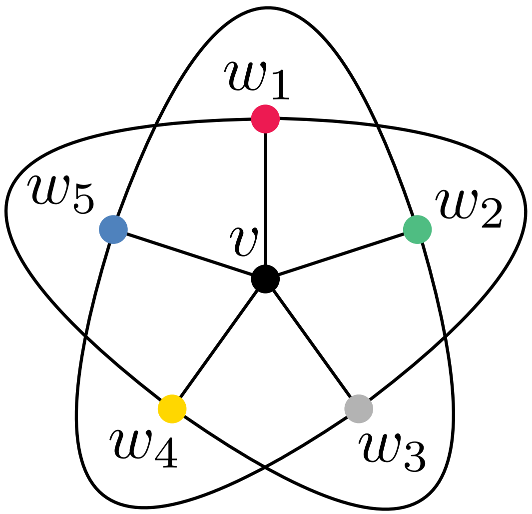
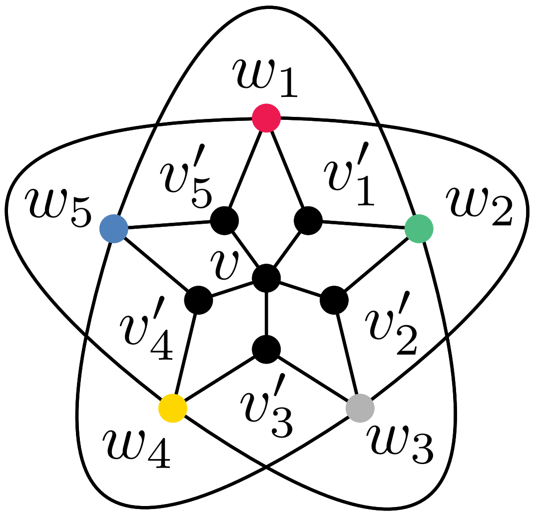


Blossom operation



Triangles in digon and touching free arrangements

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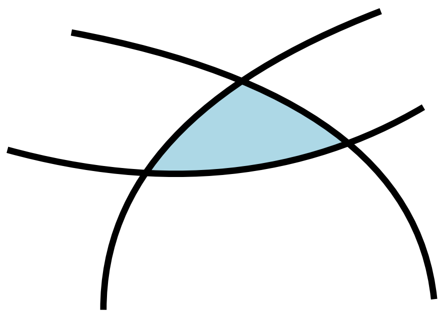
Conjecture (Grünbaum 1972): Arrangements without digons and touchings have $p_3 \geq 2n - 4$ triangles.



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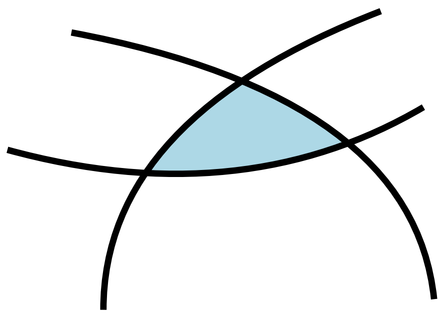
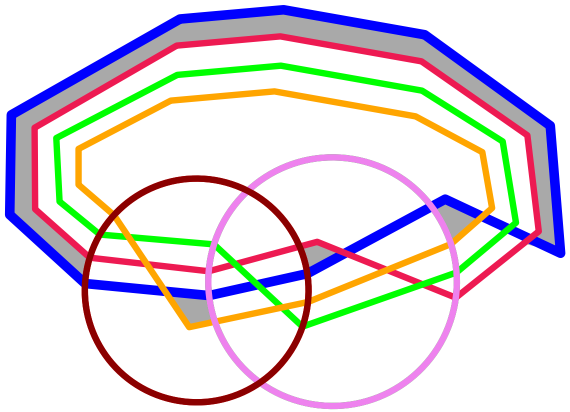
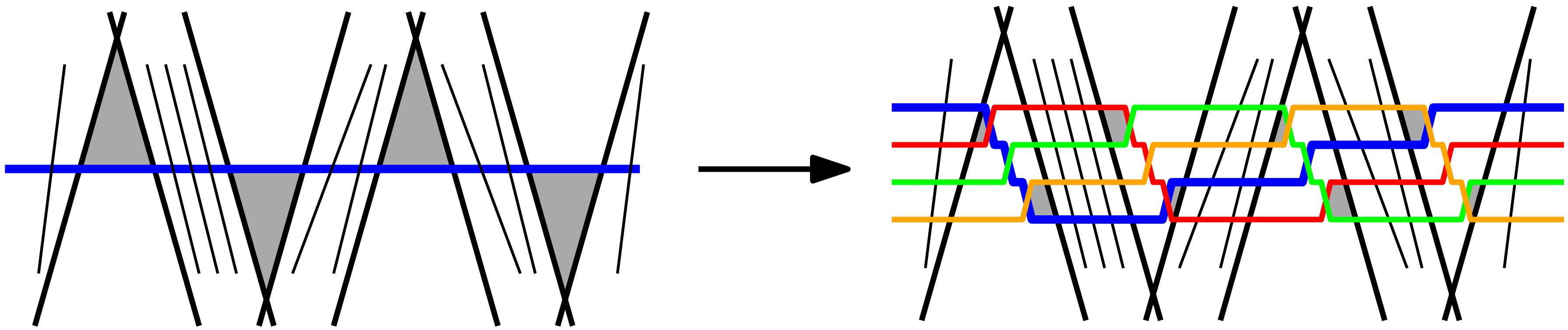


- Snoeyink and Hershberger (1991): $p_3 \geq \frac{4}{3}n$
- Felsner and Scheucher (EuroCG 2017):
  Examples with $p_3 < \frac{16}{11}n$, Grünbaum's conjecture disproved
- Felsner, R., Scheucher (2022):
  Theorem: For $n \geq 6$ there exist examples with $p_3 = \lceil \frac{4}{3}n \rceil$.

Triangles in digon and touching free arrangements

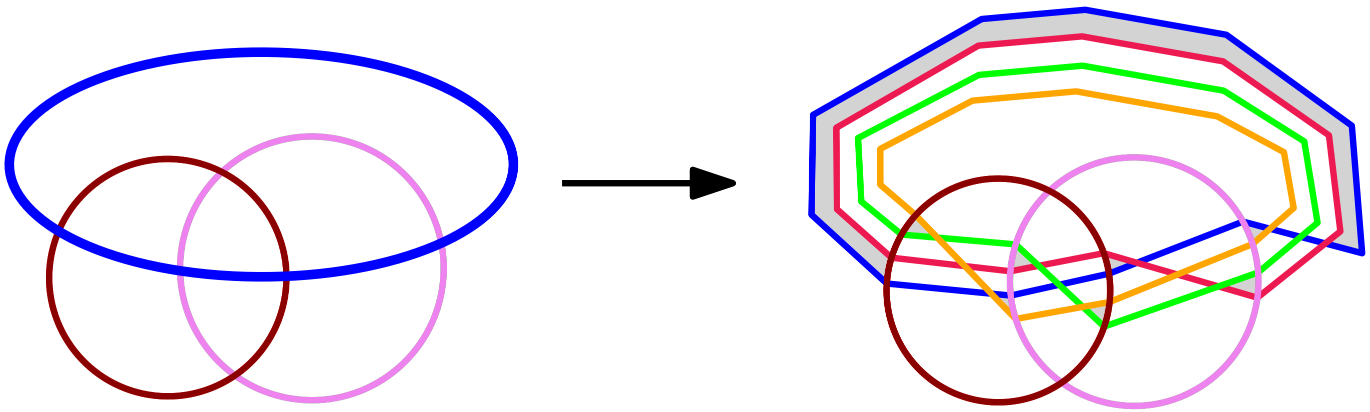
Replace iteratively blue pseudocircle by 4 twisted pseudocircles:



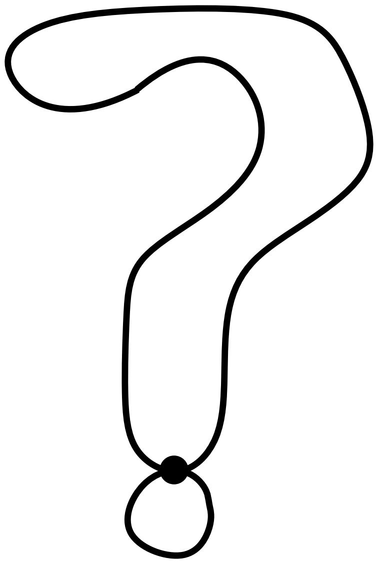


Each iteration increases n by 3 and p_3 by 4.

Triangles in digon and touching free arrangements



Questions?



Theorem 1: If three pseudocircles pairwise touch, then the pseudocircle arrangement has at most $2n - 2$ touchings.

Theorem 2: There exist digon and touching free pseudocircle arrangements with $p_3 = \lceil \frac{4}{3}n \rceil$ triangles.