Exercises

Exercice 1 :

In the class we have seen that a sorting network (of order n) is a sequence of sorting boxes on vertical positions $s_1, \cdots, s_r \in \{1, \cdots, n-1\}$, each connecting the wires on levels s_i and $s_i + 1$, which is able to sort every permutation $\pi \in S_n$. Show that the wiring diagrams of pseudoline arrangements are exactly the minimal sorting networks.

This includes the following statements :

1) A wiring diagram indeed generates a sorting network of length $\binom{n}{2}$.
2) Every sorting network has length at least $\binom{n}{2}$.
3) From any sorting network longer than $\binom{n}{2}$, a sorting box can be removed so that the remaining sequence of sorting boxes is still a sorting network.

Exercice 2 :

Show that for every n there exists a marked arrangement of n pseudolines $\mathcal{A}_n$ that has a unique allowable sequence. Equivalently, the arrangement graph $G_{\mathcal{A}_n}$ has a unique topological sorting.

Exercice 3 :

We have seen that, unfortunately, marked arrangements of pseudolines are not directly in bijection to allowable sequences but to equivalence classes of them. We call an allowable sequence $(s_1, \cdots, s_{\binom{n}{2}})$ *left-extreme*, if there is no i with $s_i \geq s_{i+1} + 2$, i.e. there is no „steep descent“ in the sequence of switches.

1) Show that for every marked arrangement $\mathcal{A}$ there is exactly one corresponding allowable sequence that is left-extreme. In other words, there is a one-to-one correspondence between marked arrangements and left-extreme allowable sequences.
2) Find the left-extreme allowable sequence that corresponds to the marked pseudoline arrangement that we saw in the beginning (Figure 1) and compute the corresponding standard-Young-tableau.

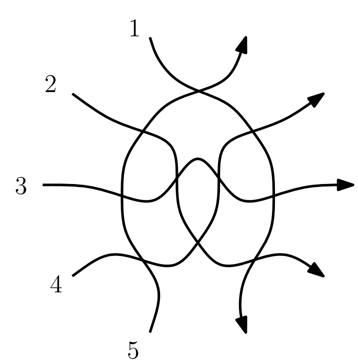

FIGURE 1 – Marked pseudoline arrangement, $n = 5$.

Exercice 4 :

How could the bijection between marked pseudoline arrangements and rhombic tilings be generalized so that it includes non-simple arrangements, i.e. arrangements that contain crossings of three or more pseudolines ? The idea may be sufficient.

Exercice 5 :

Repeated application of the Schützenberger operator gives a one-to-one correspondence between standard-Young-tableaux and allowable sequences. What is the relation between two allowable sequences that correspond to transposed tableaux ?

Exercice 6 :

In the class we stated that signotopes of rank 3 are exactly the sign functions of marked arrangements. Prove one direction : The sign function of a marked arrangement is a signotope.

Hint : One possible way to show that the sequence $S := (\chi_{\mathcal{A}}(jkl), \chi_{\mathcal{A}}(ikl), \chi_{\mathcal{A}}(ijl), \chi_{\mathcal{A}}(jkl))$ has at most one sign change is to show :

— If $S = (-, +, *, *)$, then $S = (-, +, +, +)$.

— If $S = (-, -, +, *)$, then $S = (-, -, +, +)$.

The implications $(+, -, *, *) \implies (+, -, -, -)$ and $(+, +, -, *) \implies (+, +, -, -)$ follow analogously.

Exercice 7 :

Let $\mathcal{A}$ be a pseudoline arrangement and l one of its pseudolines. The pseudoline l decomposes the Euclidean plane into a *left side* L and a *right side* R, w.r.t. an arbitrary orientation of l. Prove : Pseudoline l is incident to a triangle in L, unless all crossings of A lie in R or on l itself.

Exercice 8 :

Prove the equivalence that we saw in the class :

1) The bichromatic triangle conjecture holds.
2) For every pseudoline arrangement $\mathcal{A}$ with $n \geq 3$ pseudolines, the following graph $G_l(\mathcal{A})$ is connected : consider the n pseudolines as vertices with an edge between two if they are incident to the same triangle cell.
3) For every simple pseudoline arrangement $\mathcal{A}$ with $n \geq 3$ pseudolines, the following graph $G_\Delta(\mathcal{A})$ is connected : Consider the triangle cells as vertices with an edge between two if they share a pseudoline.

Exercice 9 :

(Probably harder)

Let $\mathcal{A}$ be an arrangement of n pseudolines. Show that the crossings of $\mathcal{A}$ can be colored using n colors so that no color appears twice on the boundary of any cell.