Mittagsseminar

Algorithms for sampling random pseudoline arrangements

Sandro M. Roch

16.12.2021

Rhombic tilings of zonotopes

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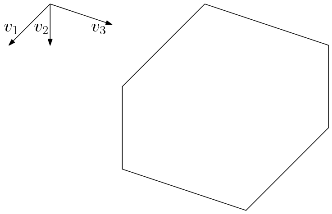

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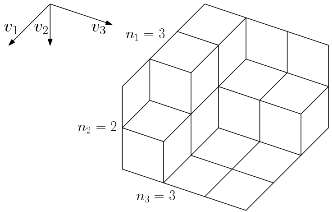

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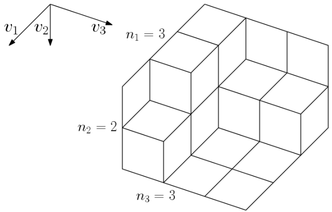

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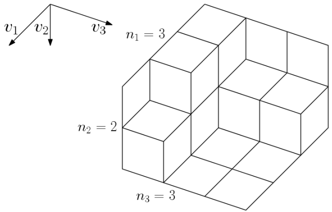

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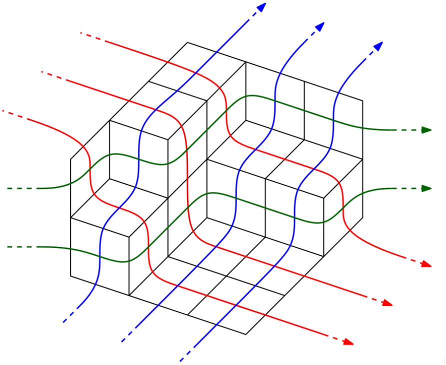
- *Rhombic tiling*: For fixed *shape* $n_1, \ldots, n_r \in \mathbb{N}$, tiling of $Z(v_1, \ldots, v_r)$ by rhombi of type $Z\left(\frac{v_i}{n_i}, \frac{v_j}{n_j}\right)$.

Rhombic tilings and pseudoline arrangements

- Draw curves through ribbons of parallel edges:



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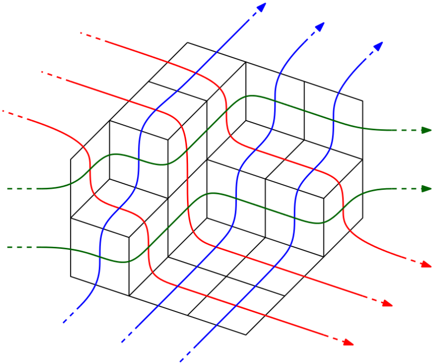

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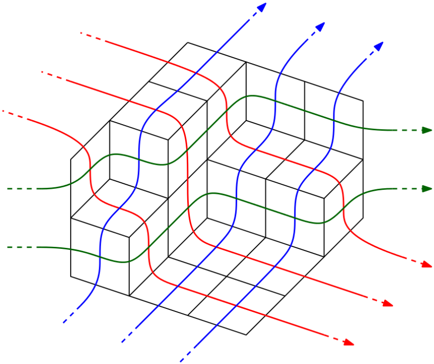

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- Generalizable to non-simple arrangements

Uniform sampling of pseudoline arrangements

- **Big open problem:** Polynomial algorithm for uniform sampling PLA.
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  - Motivation: Determine average characteristics
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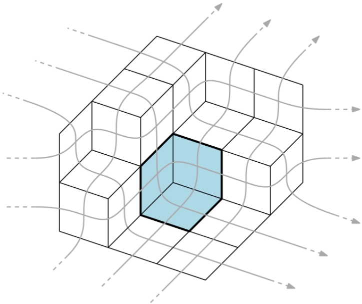

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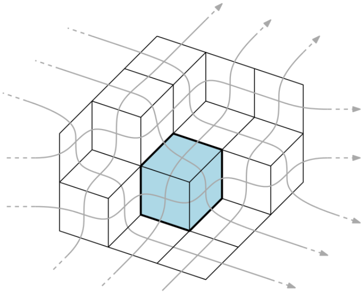

Rapidly mixing Markov chains

Markov chain (X_t), state space $\mathcal{X}$, transition prob. $P : \mathcal{X} \times \mathcal{X} \to [0, 1]$

- If *irreducible* and *aperiodic*: Converges to *stationary distribution* π:

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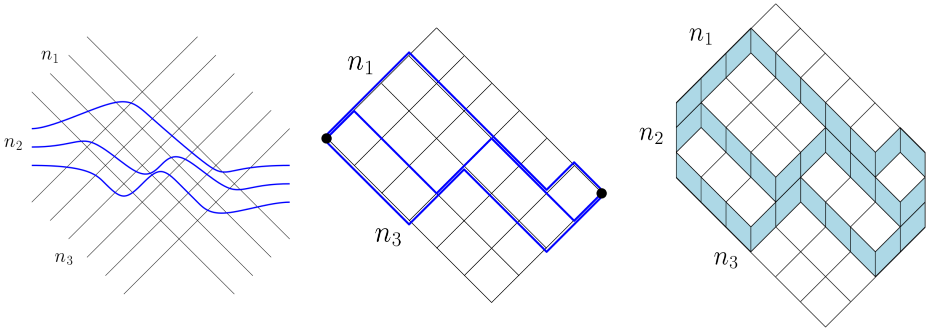
Definition

A class of Markov chains is *rapidly mixing* if for each of them $\tau(\varepsilon) \in \mathcal{O}\left(p\left(\log \frac{|\mathcal{X}|}{\varepsilon}\right)\right)$ for some $p \in \mathbb{R}[X]$.

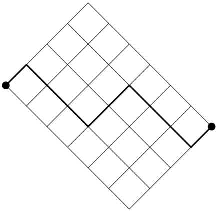
Pseudoline arrangements with 3 parallel classes

- **Ruby, Randall, Sinclair, 2001:**
  Rapidly mixing Marcov chain for sampling PLA with 3 parallel classes
- Idea: Extension of grid along non-crossing monotonic paths



- On non-crossing path systems perform flips over grid cells.

- Simple case: Only one insertion path



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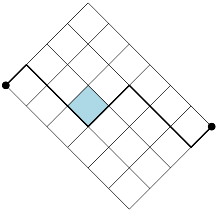

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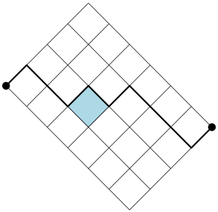

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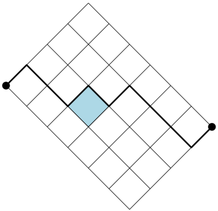

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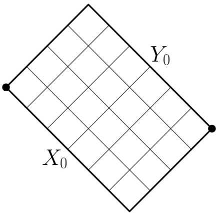

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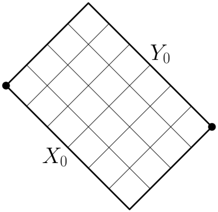

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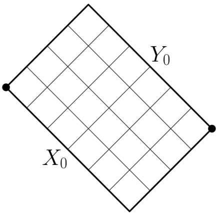

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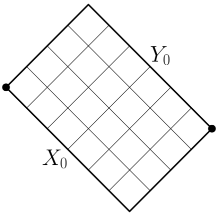

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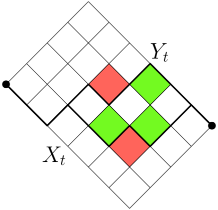

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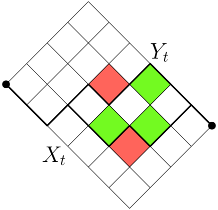

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- From theory: $\tau(\varepsilon) \leq 6 \cdot \mathbb{E}[\tau_C] \left(1 + \log\left(\frac{1}{\varepsilon}\right)\right)$

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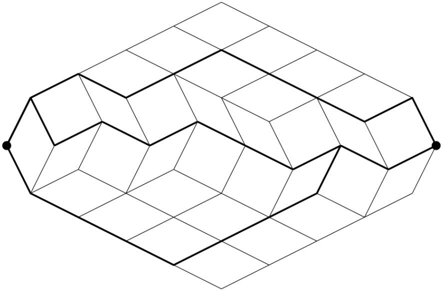

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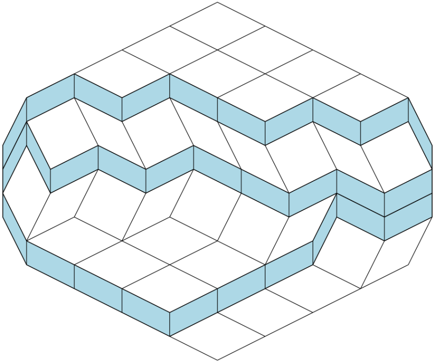
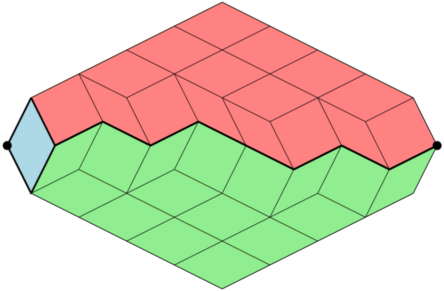
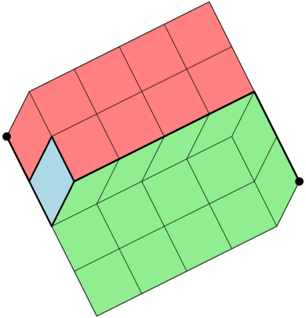


Bottlenecks in path insertion chain



Extension of $r = 4$ classes



Extension of $r = 3$ classes

- Partition of paths into two classes:
  - A: Paths **above blue cell**, never going through green cells
  - $A^{\complement}$: Paths **below blue cell**, never going through red cells
- Flip on blue cell is only transition between A and $A^{\complement}$.
- **Result:** Path insertion Markov chain not rapidly mixing for $r \geq 3$.

Repeated reinsertion of pseudolines

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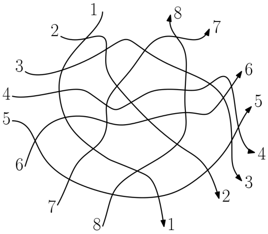

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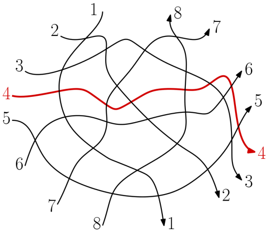

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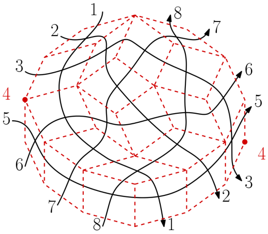

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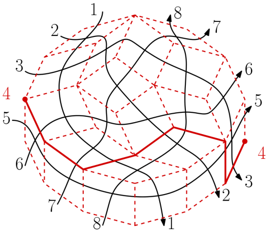

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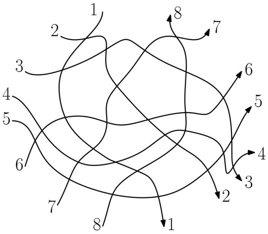

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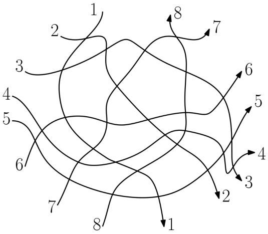

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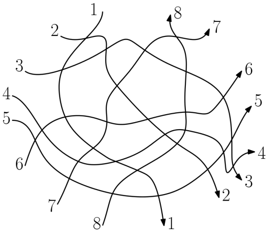

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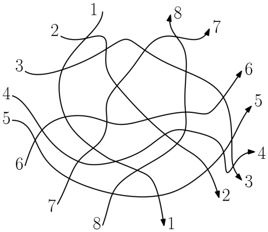

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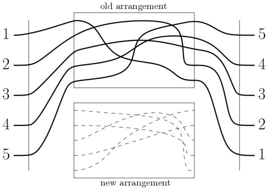

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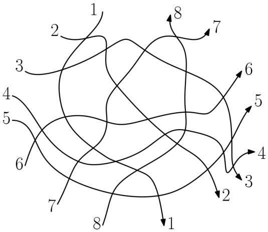

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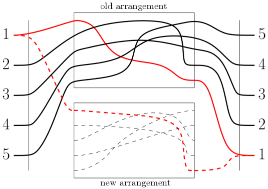

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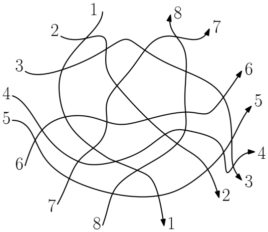

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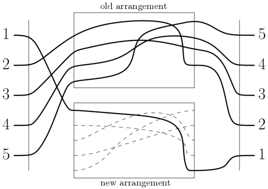

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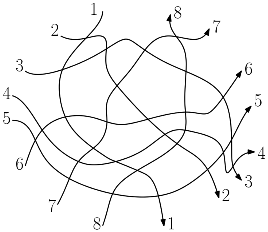

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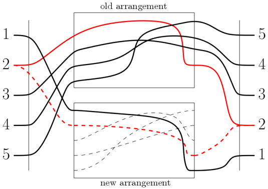

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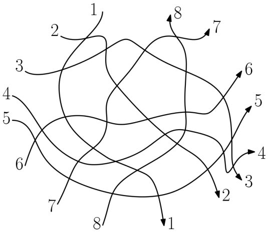

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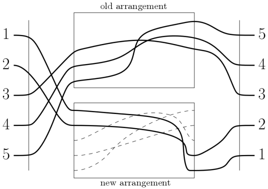

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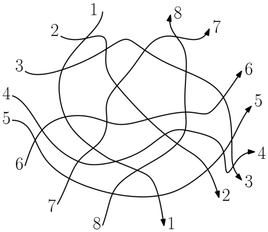

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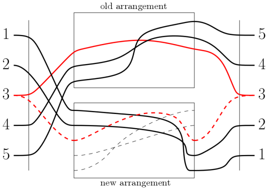

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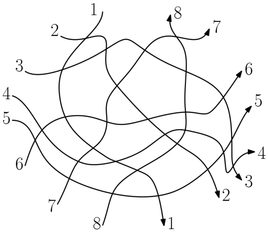

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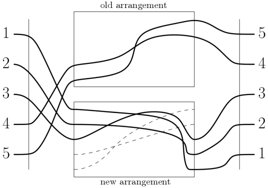

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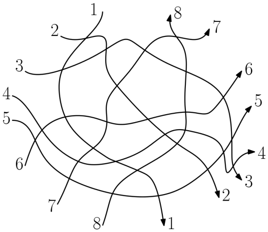

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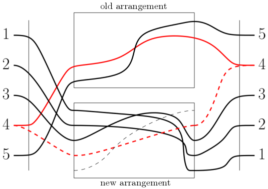

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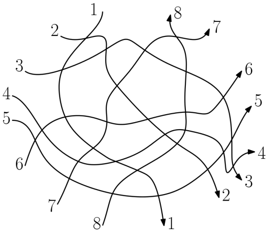

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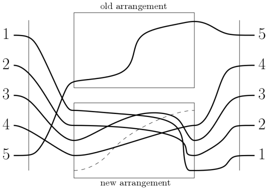

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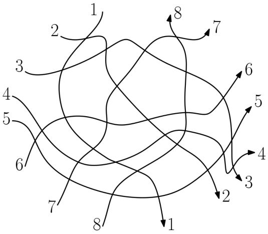

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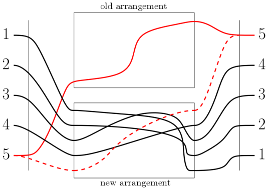

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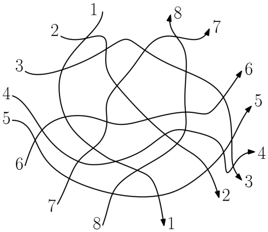

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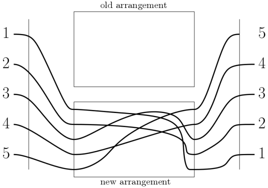


Canonical paths

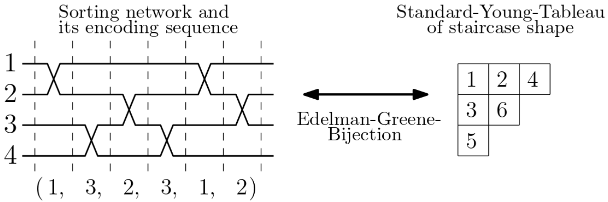
- For all $s, t \in \mathcal{X}$ find *canonical path* γ_{st} of transitions from s to t.
- Define *path congestion*:

$$\rho := \max_{\substack{x,y \in \mathcal{X} \\ P(x,y) \neq 0}} \frac{1}{\pi(x) P(x,y)} \sum_{\gamma_{st} \ni (x,y)} \pi(s)\pi(t)$$

- Low path congestion $\rho \Rightarrow$ No bottlenecks $\Rightarrow$ Low mixing time $\tau(\varepsilon)$
- Could be useful tool for „Reinsertion markov chain“.

Rejection sampling via Standard-Young-Tableaux

- PLAs of shape $(n_1, ..., n_r) = (1, ..., 1)$ as *Sorting networks*:



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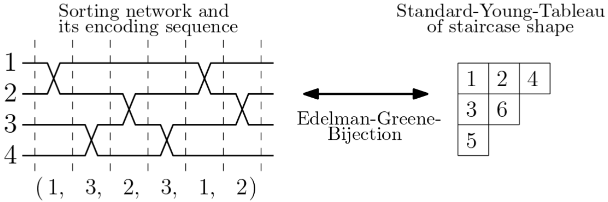

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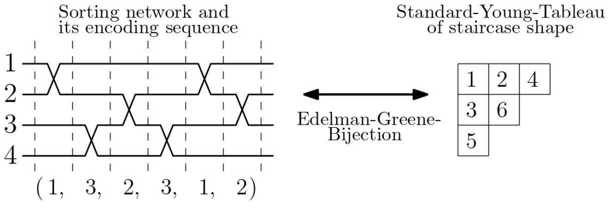

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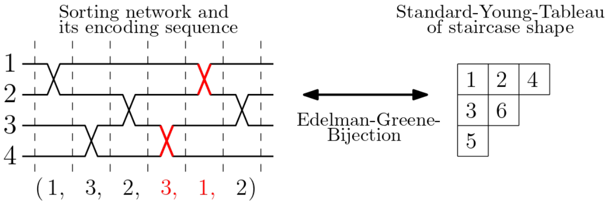

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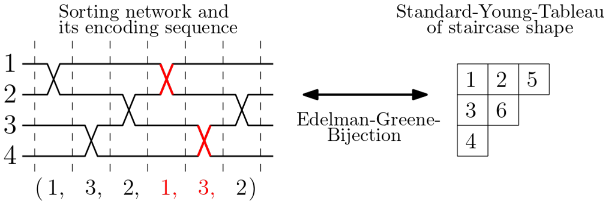

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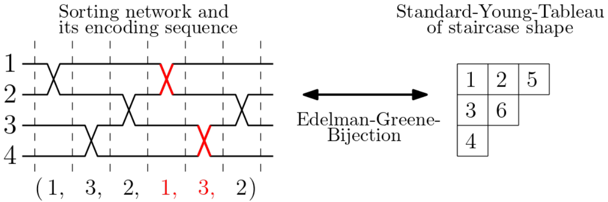

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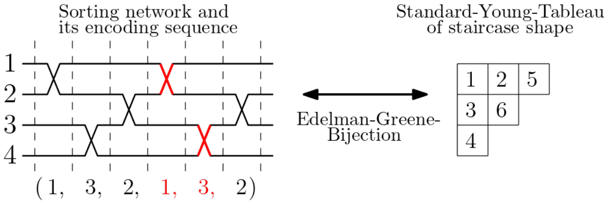

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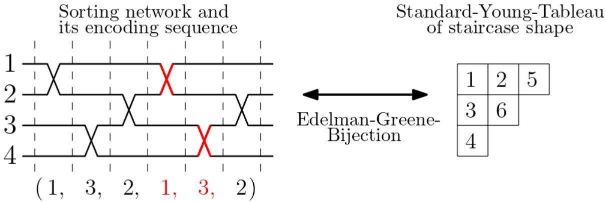

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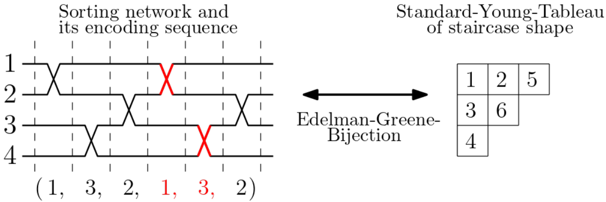

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- **Experimental result:** Sampling PLAs with $r \leq 8$ in reasonable time

Optimal plane partitions

- **Def:** Matrix $[h_{i,j}] \in {\mathbb{N}_0}^{r \times s}$ is called *plane partition*, if rows and columns are monotonically increasing.

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- For objective functions $f_{i,j} : \{0, ..., h\} \to \mathbb{Q}$ consider the problem

$$
\begin{aligned}
\textbf{min} \quad & \sum_{i=1}^{r} \sum_{j=1}^{s} f_{i,j}(h_{i,j}) & \\
\text{s.t.} \quad & h_{i,j} \leq h_{i,j+1} & \text{für alle } i \in [r], j \in [s-1] \\
& h_{i,j} \leq h_{i+1,j} & \text{für alle } i \in [r-1], j \in [s] \\
& 0 \leq h_{i,j} \leq h & \\
& h_{i,j} \in \mathbb{Z} &
\end{aligned}
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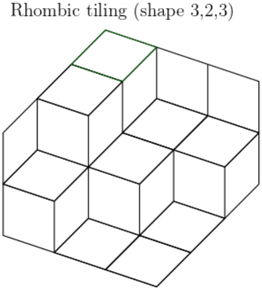
- *Plane partitions*, *rhombic tilings* and *systems of non-crossing paths* in bijection:



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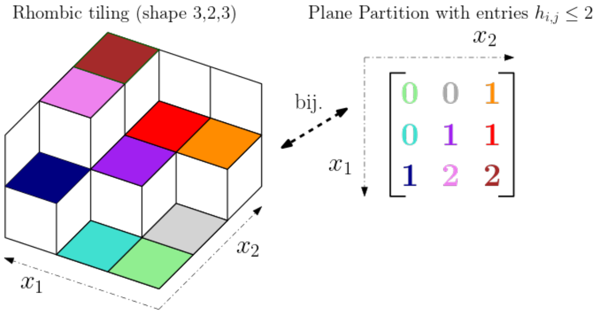

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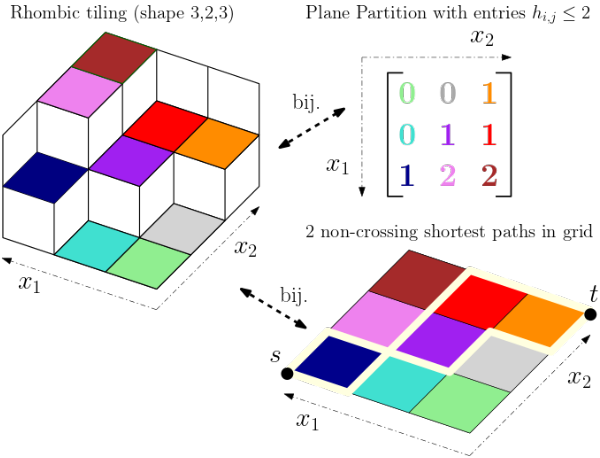

- **Equivalent problem:** On set $\mathcal{P}$ of systems of h non-crossing monotonic paths in $(r \times s)$-grid graph:

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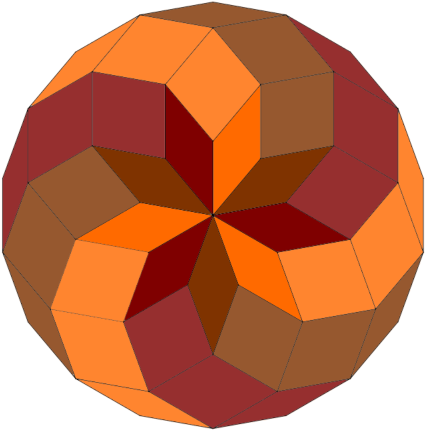
- Can be solved by dynamic programming approach in time polynomial in r, s but exponential in h.

Questions



Exact references in submitted thesis or on demand.