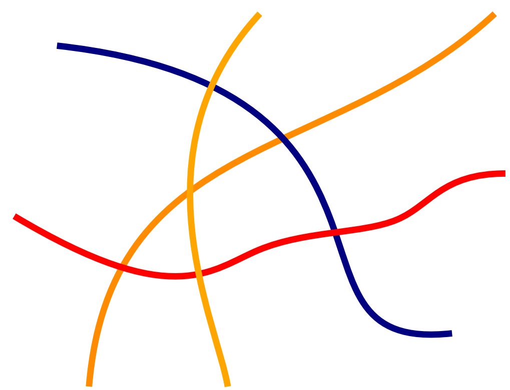
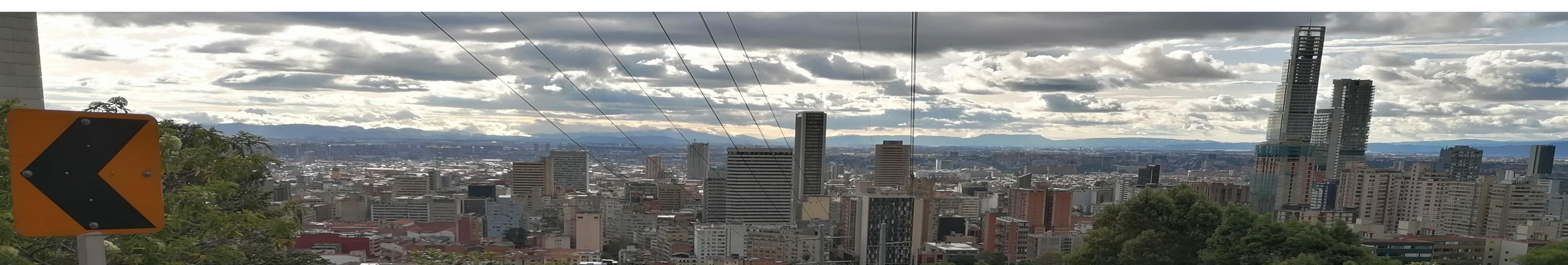
Mini course at Universidad Distrital Francisco José de Caldas

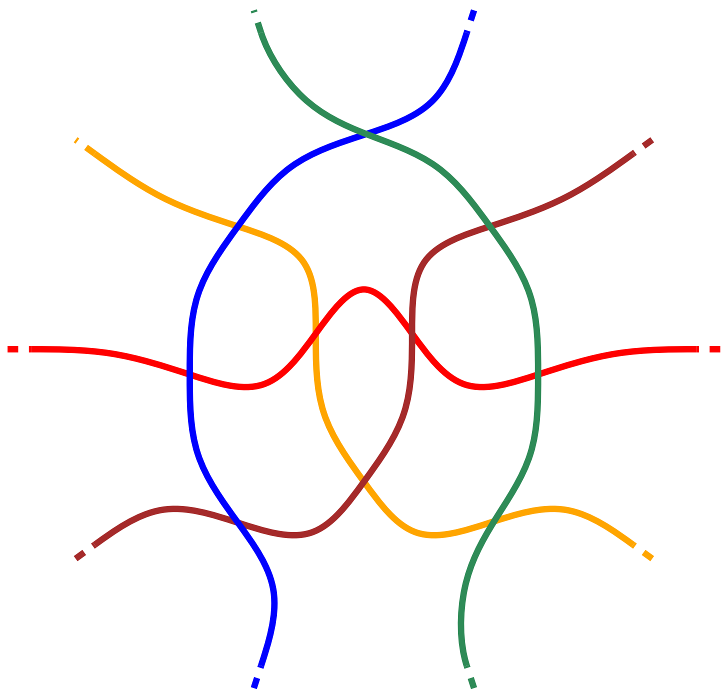
Introduction to Arrangements of Pseudolines



Sandro M. Roch



pseudoline arrangements



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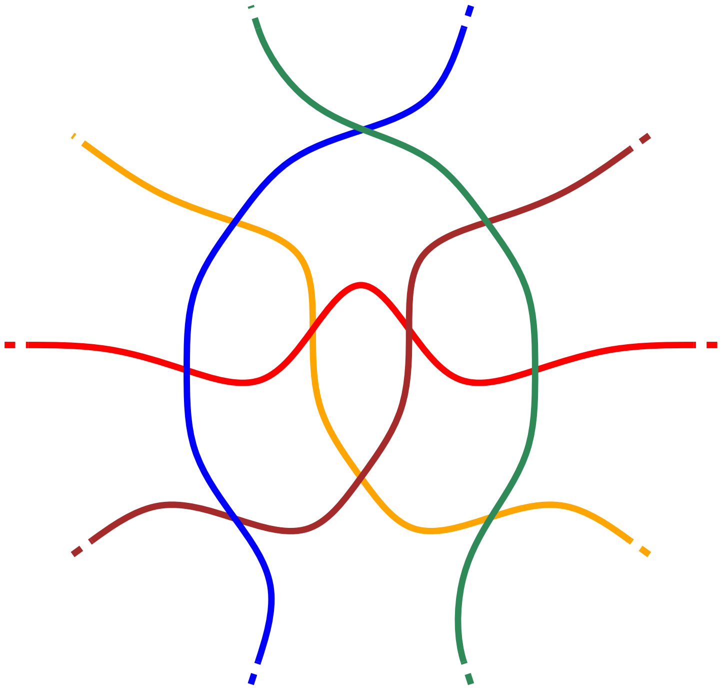

Def: Family of continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}^2$ *(pseudolines)*

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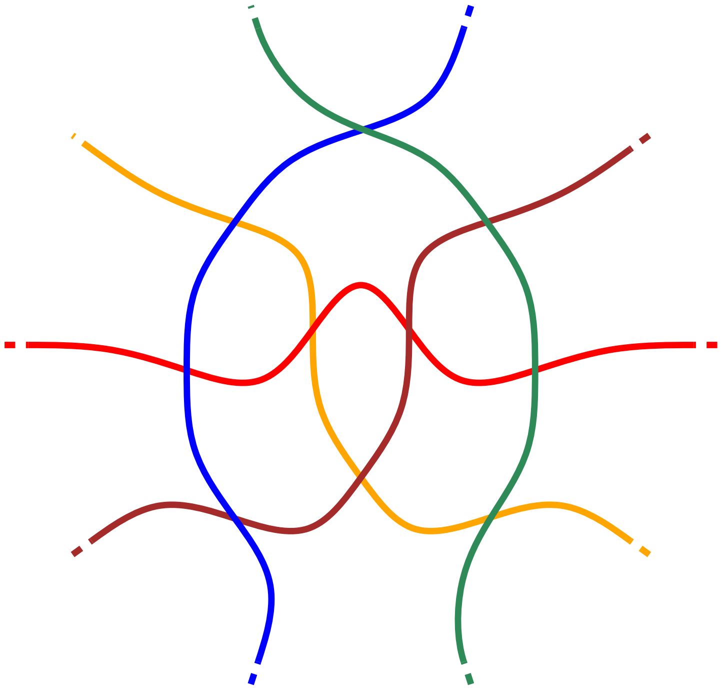

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- $\lim_{x\to\infty} \|f_i(x)\| = \lim_{x\to-\infty} \|f_i(x)\| = \infty$

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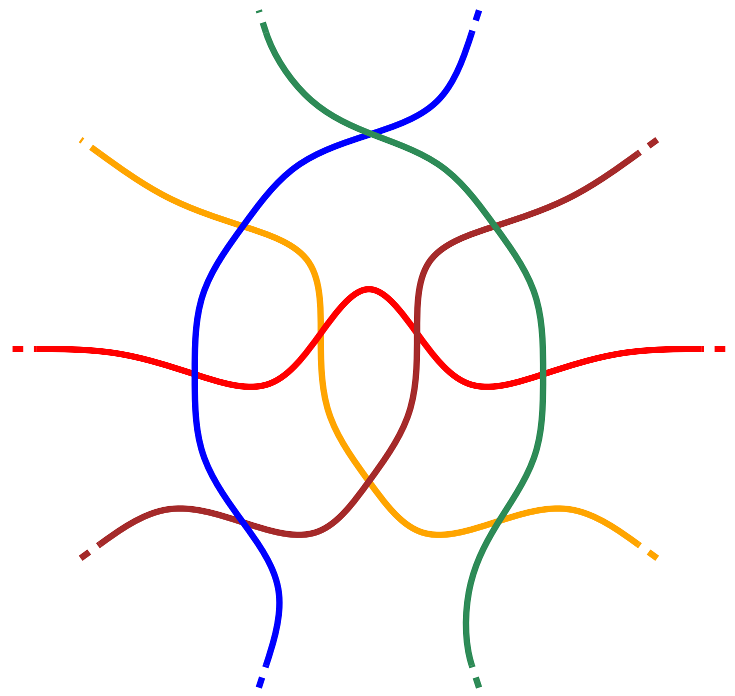

Def: Family of continuous curves $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R}^2$ *(pseudolines)*

- $\lim_{x\to\infty} \|f_i(x)\| = \lim_{x\to-\infty} \|f_i(x)\| = \infty$
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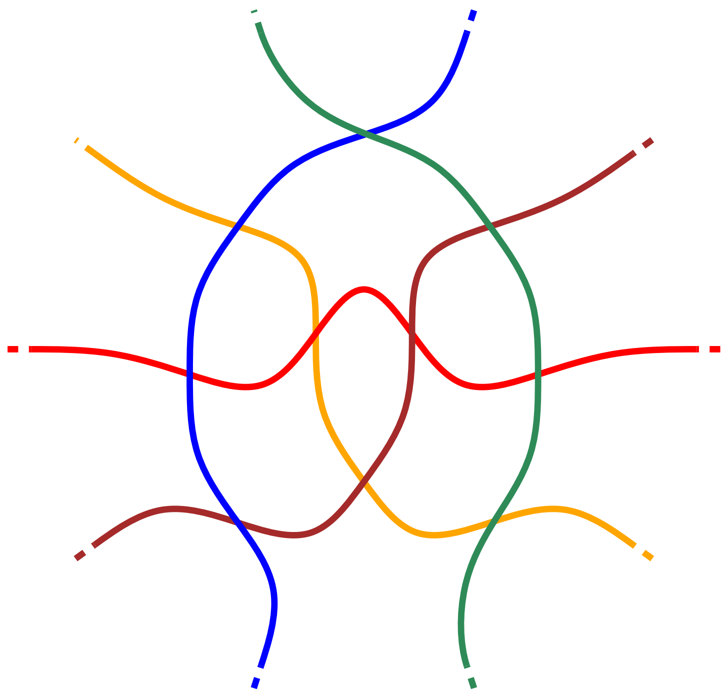

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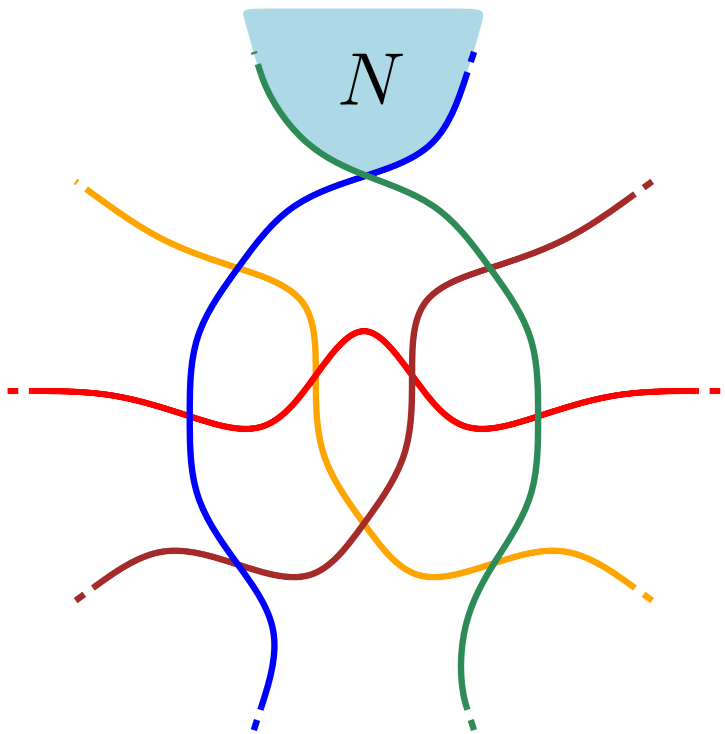
- At each point, no more than two pseudolines cross.
  (simple arrangement)

marked pseudoline arrangements

marked pseudoline arrangement: pseudoline arrangment $\mathscr{A}$ with distinguished unbounded *north cell* N.



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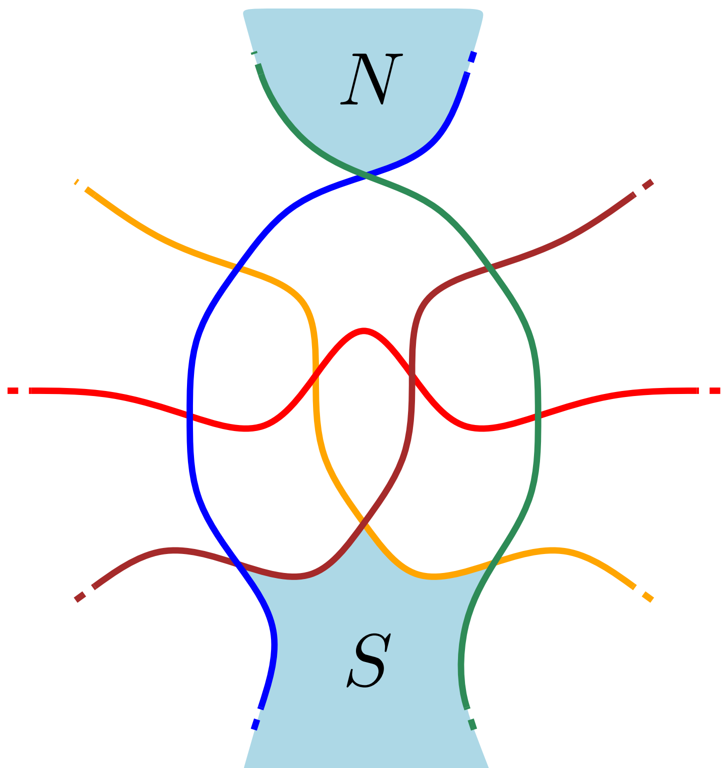

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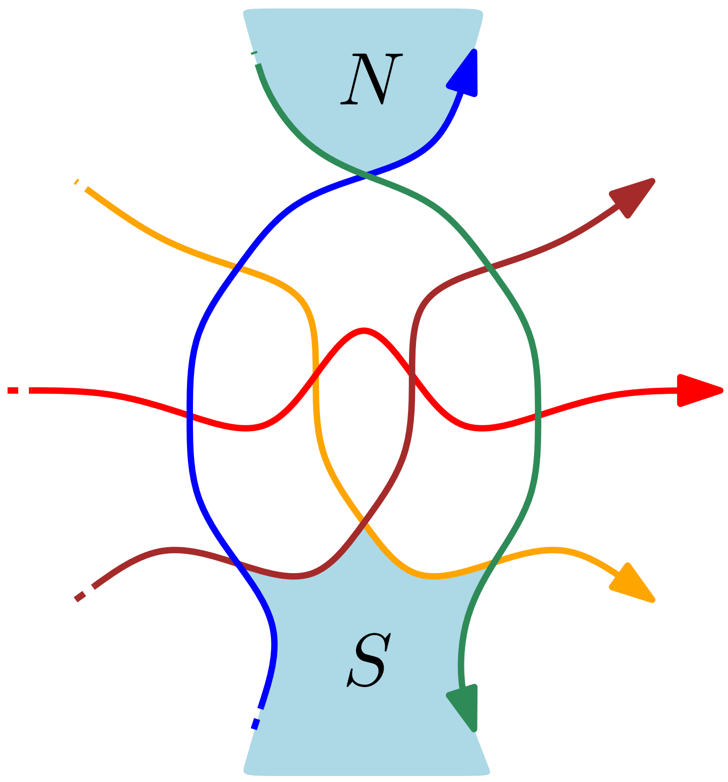

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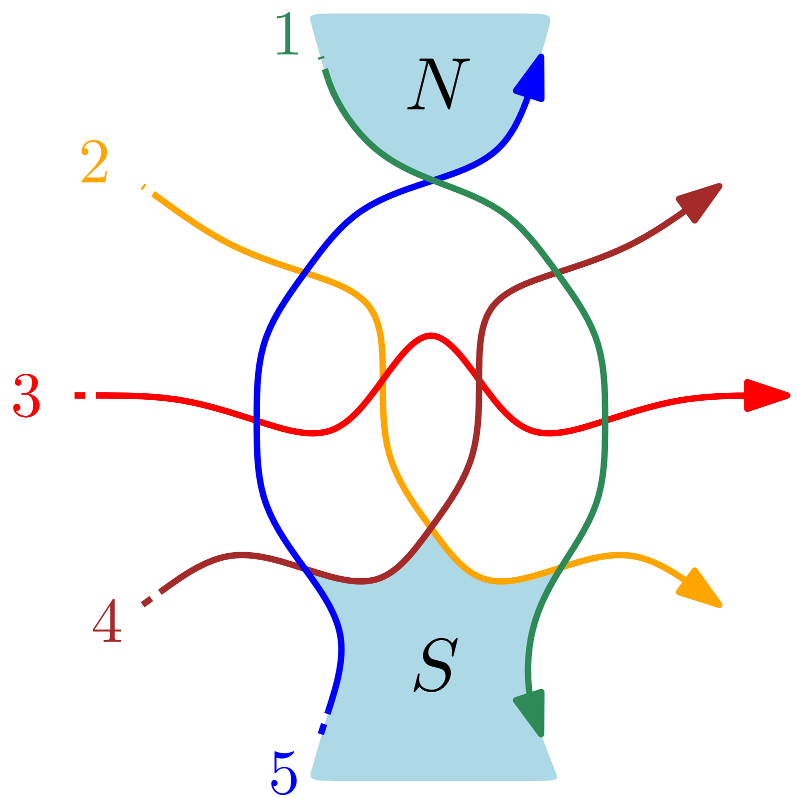
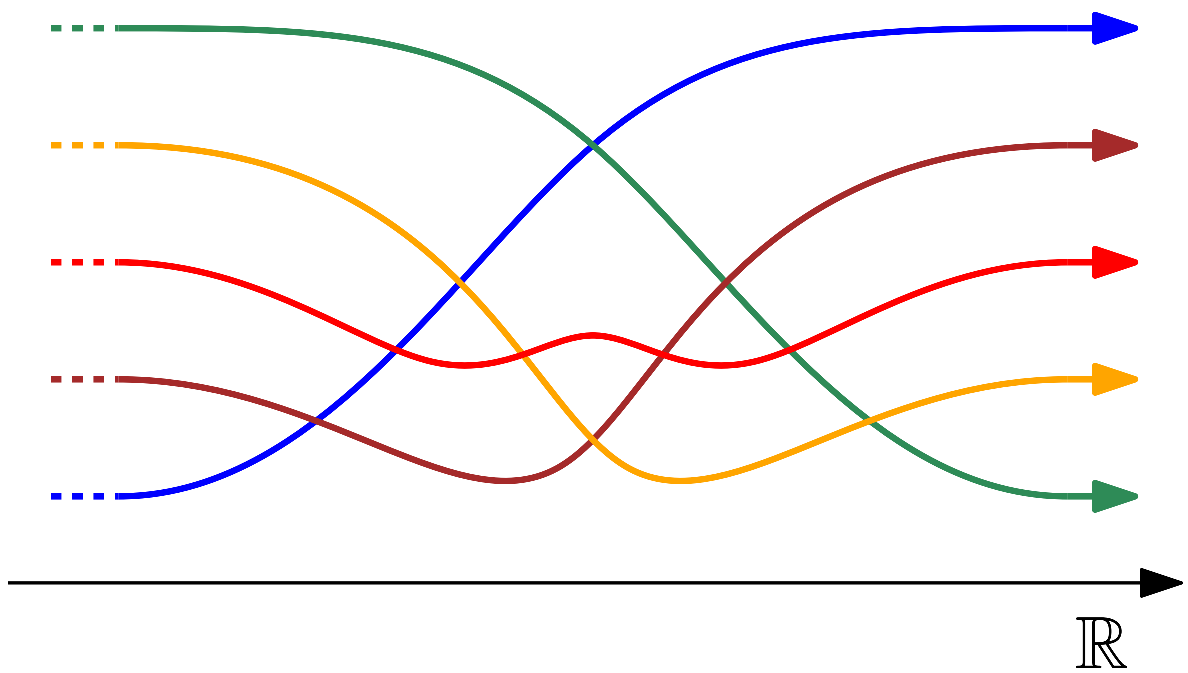


alternative definition via x-monotonic curves



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- Each two $f_i \neq f_j$ intersect in exactly one point where they cross.
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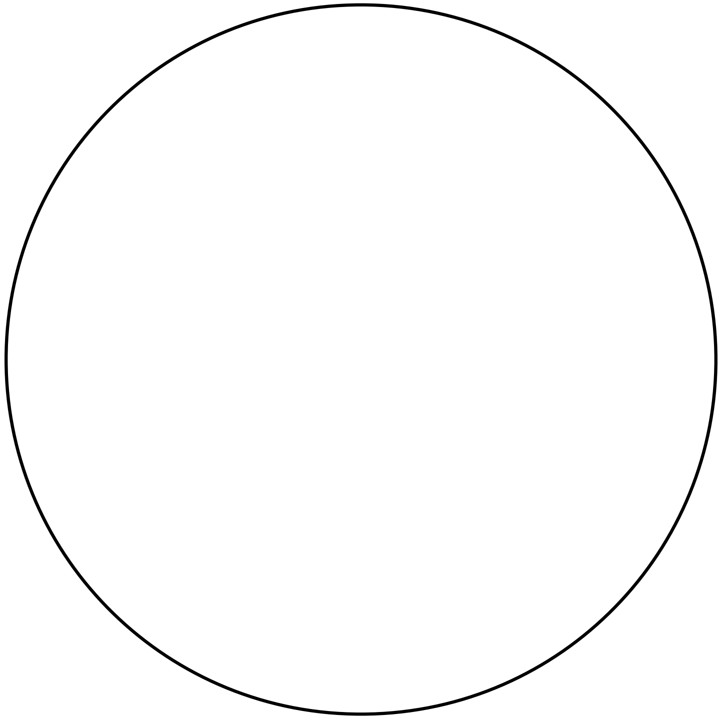
alternative definition via projective plane

alternative definition via projective plane

Real projective plane

$\mathbb{P}^2$



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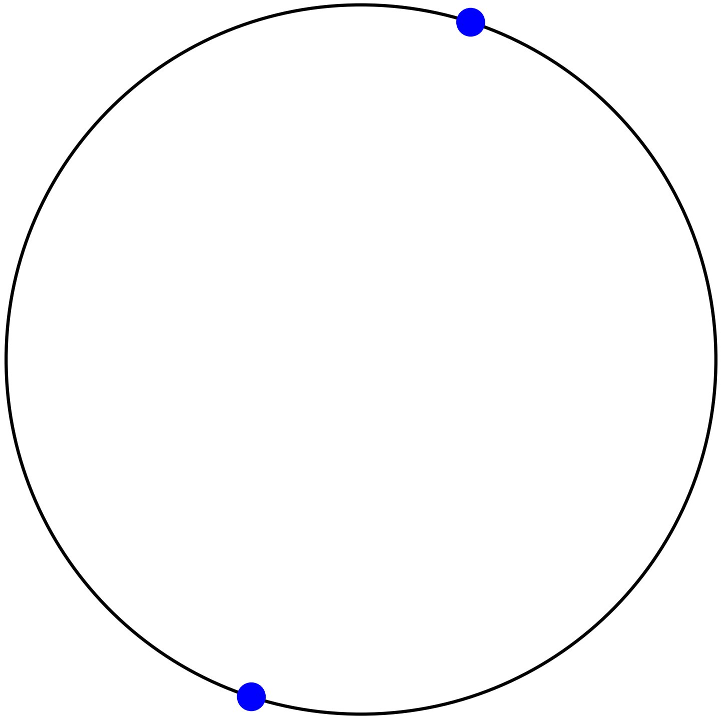

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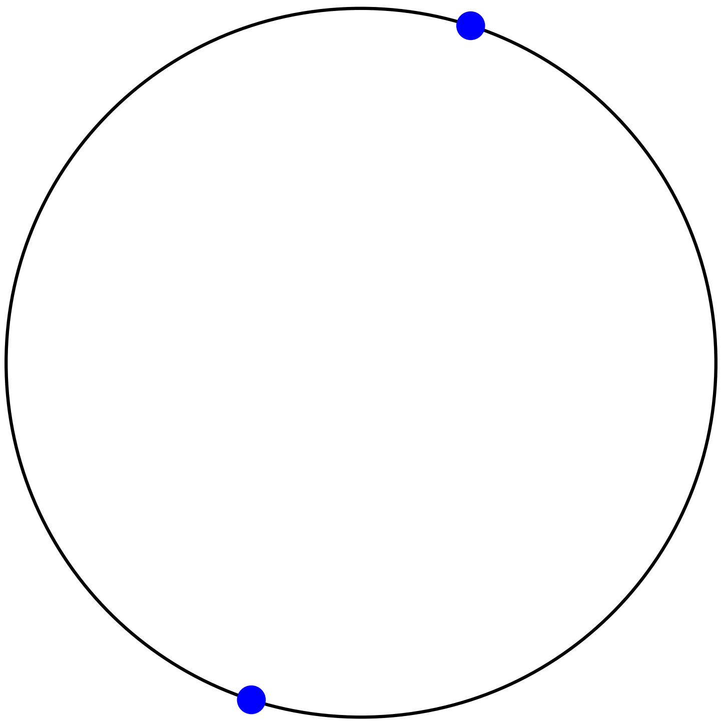

Def: Family of simple closed curves $f_1, \cdots, f_n : [0,1] \to \mathbb{P}^2$

- Each f_i does not separate $\mathbb{P}^2$
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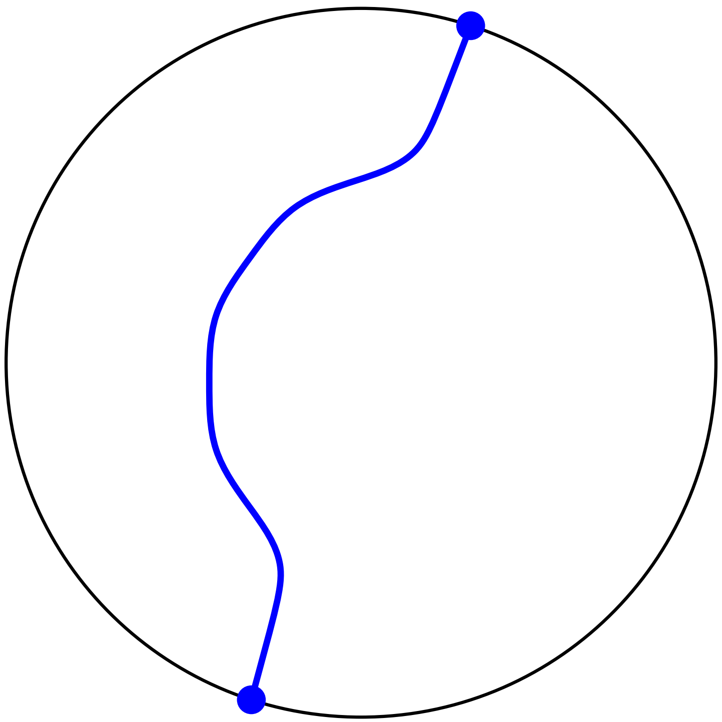

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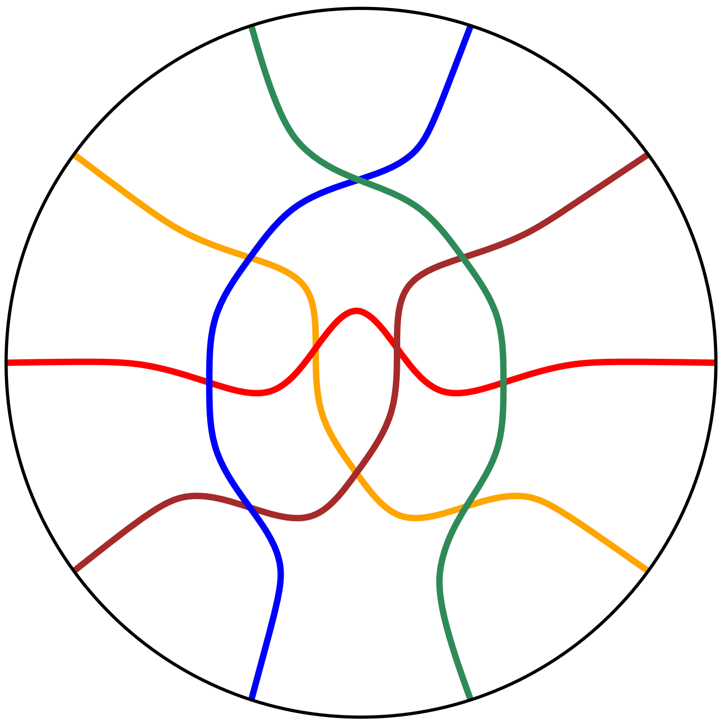

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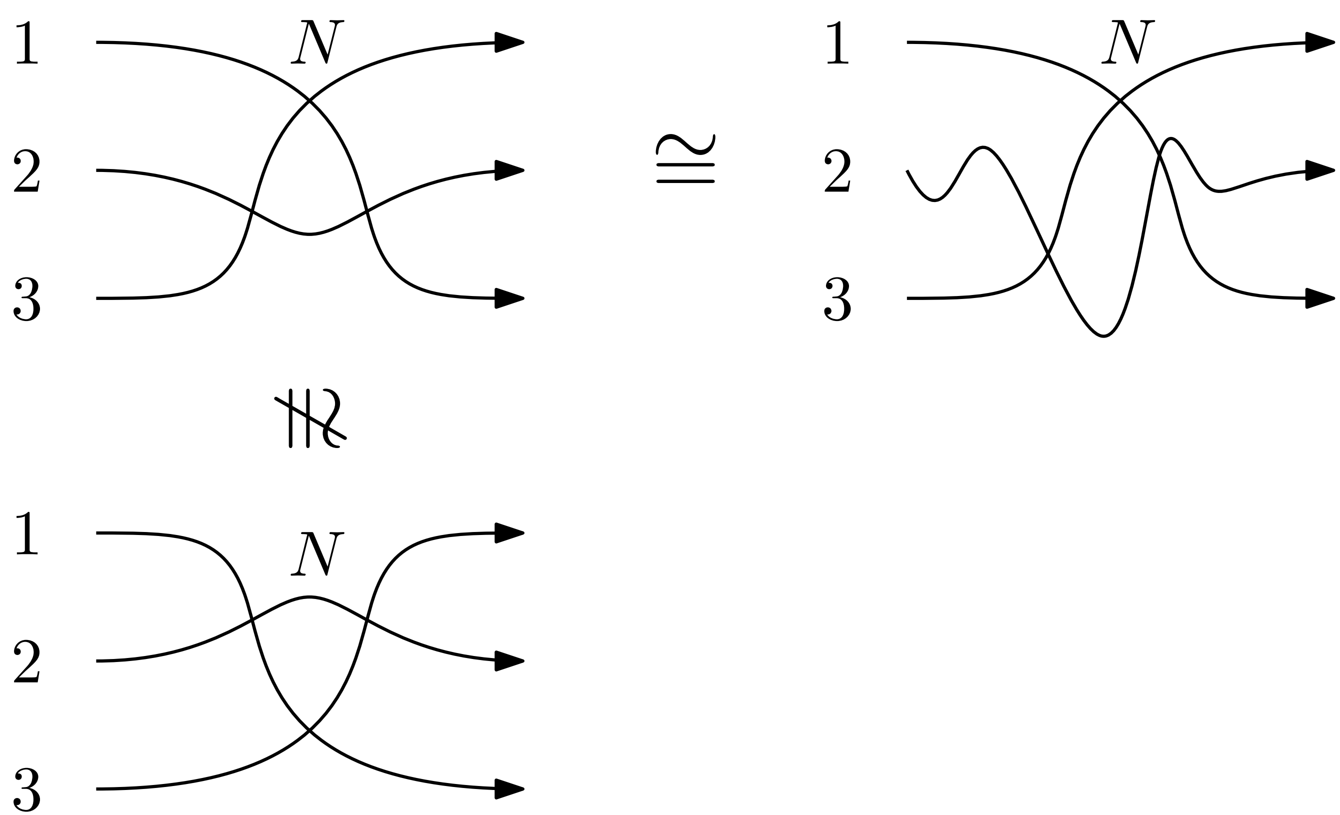
Equality relation: Consider two (marked) arrangements as equal, if they differ only by homeomorphic transformation.



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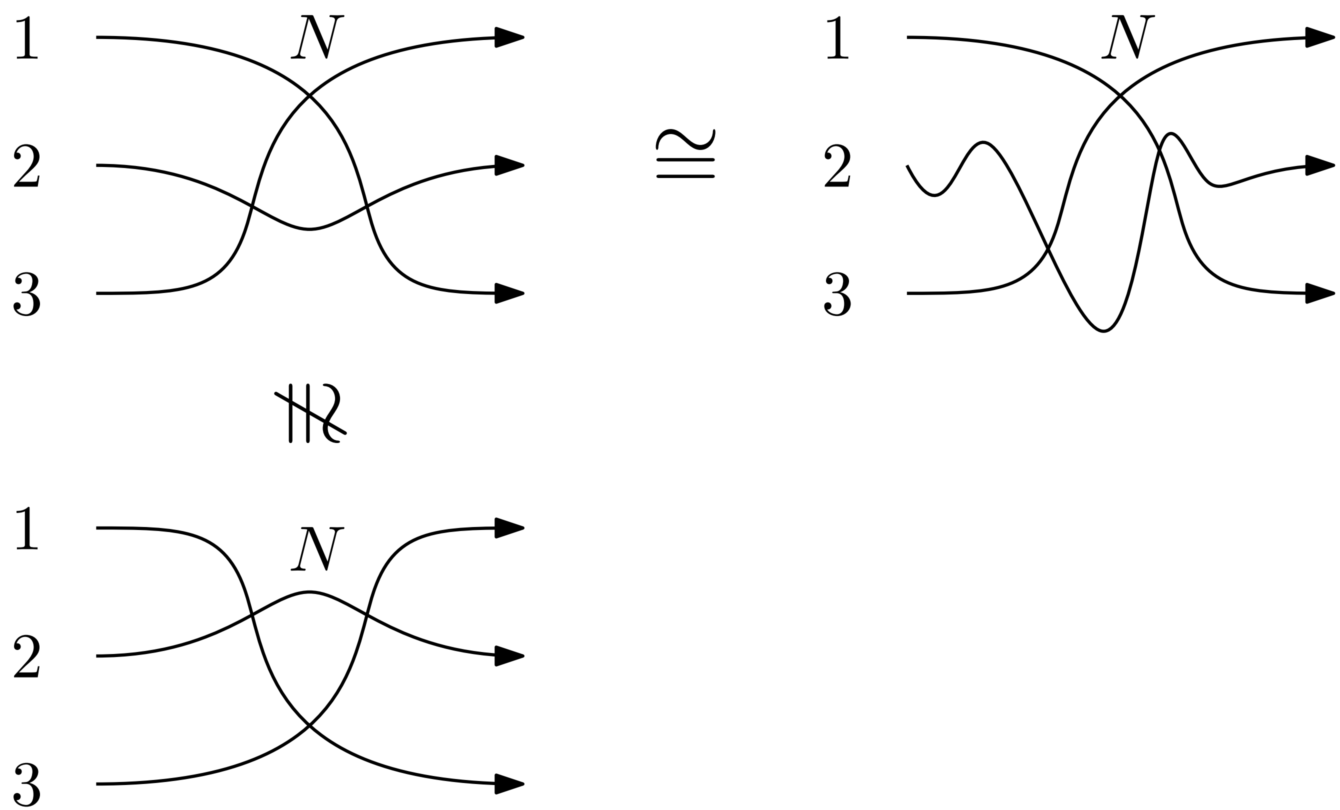

Two marked arrangements are equal iff all intersection orders coincide!

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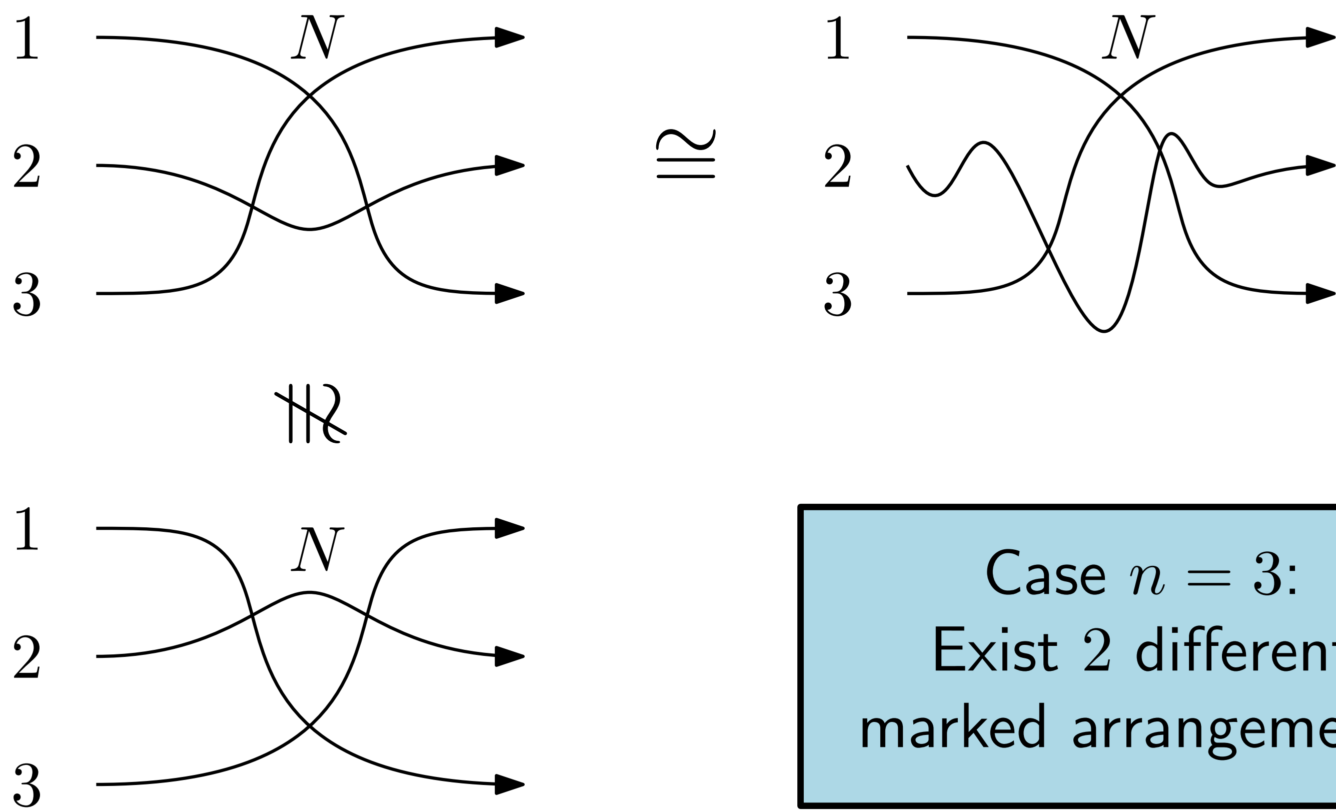

Case $n = 3$:
Exist 2 different
marked arrangements

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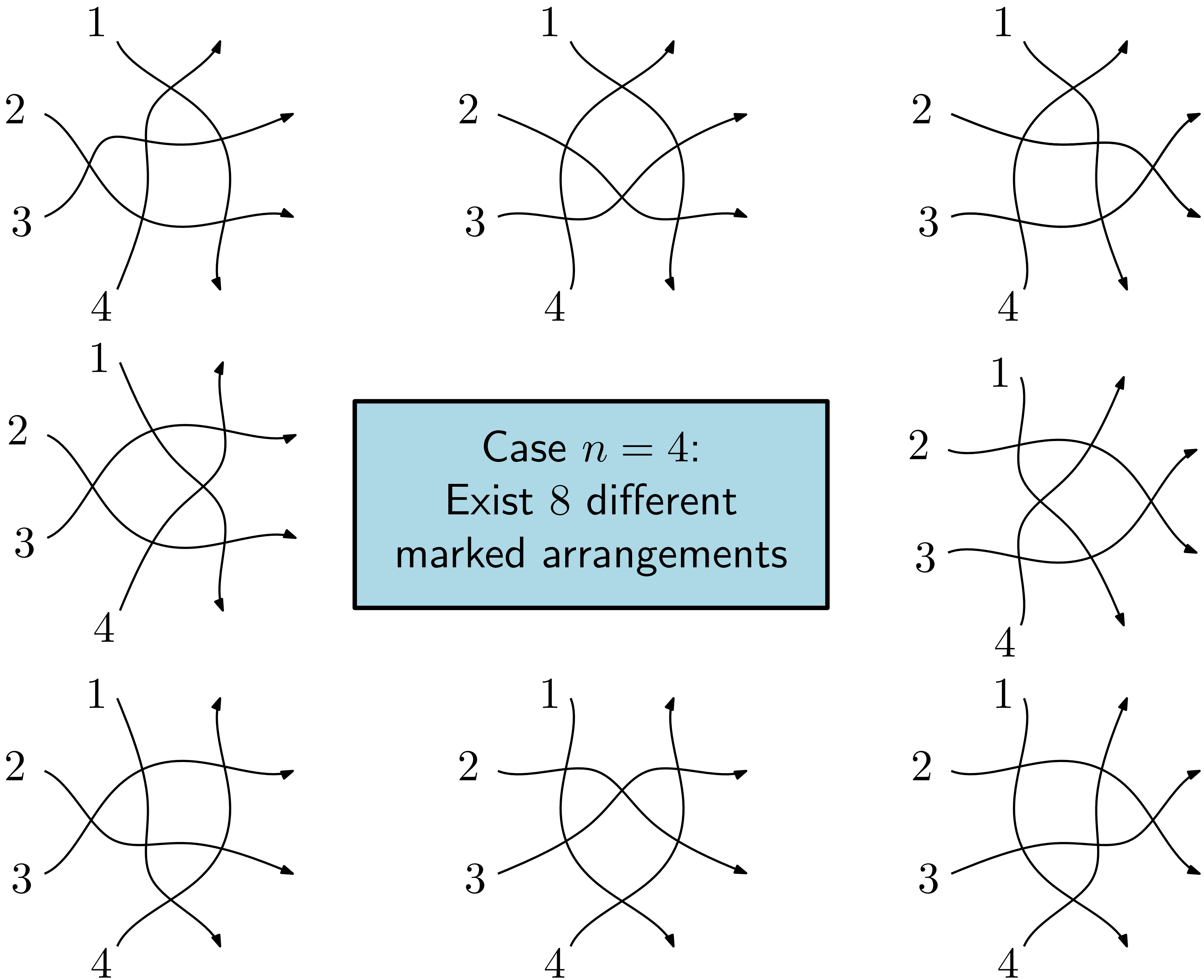
Case $n = 4$:
Exist 8 different
marked arrangements

stretchability

stretchability

Stretchable pseudoline arrangements can be drawn using straight lines.

stretchability

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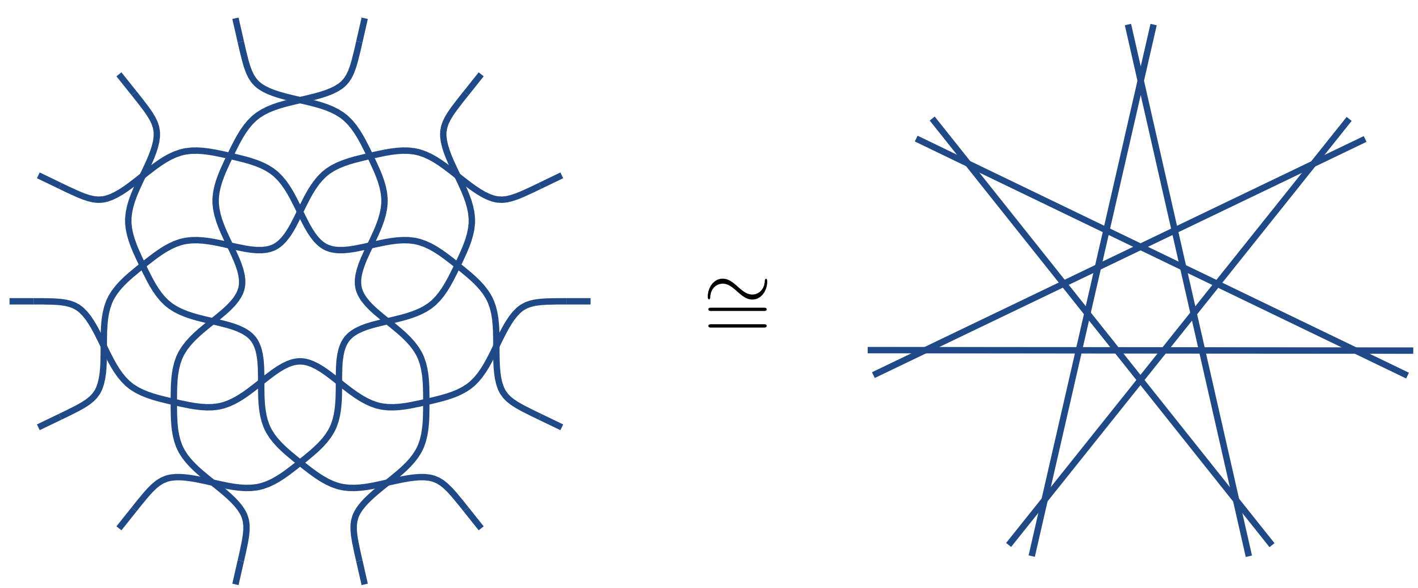
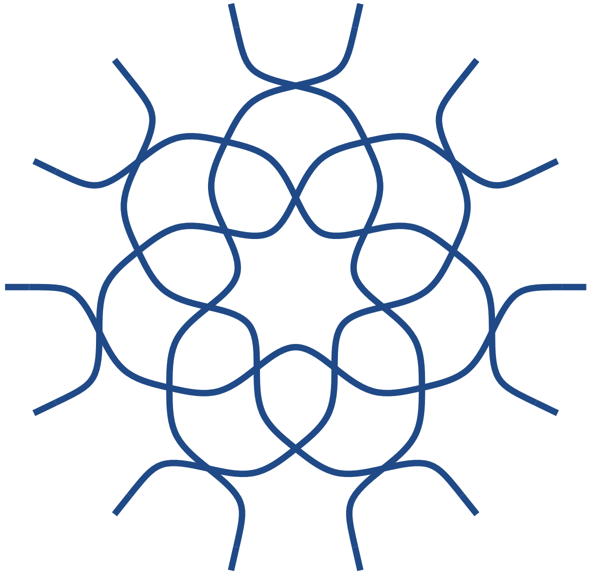


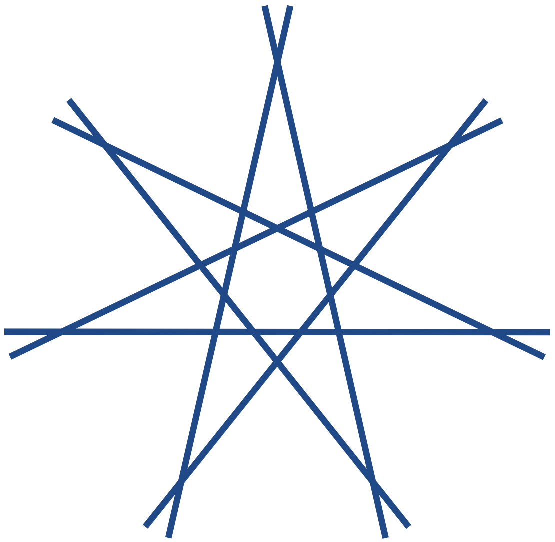
stretchability

Stretchable pseudoline arrangements can be drawn using straight lines.

Theorem (Goodman & Pollack, 1980)
All arrangements of $n \leq 7$ pseudolines are stretchable.



$\cong$



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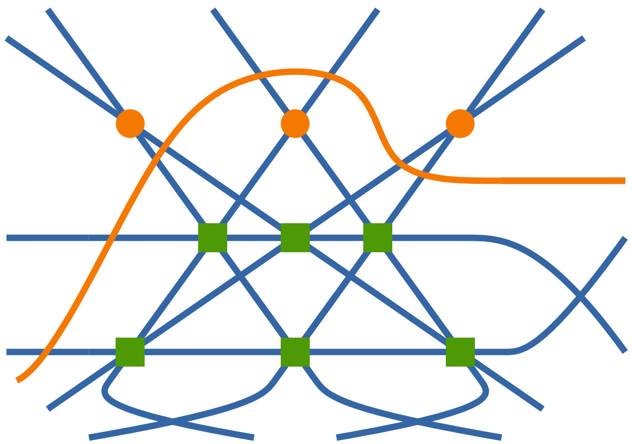
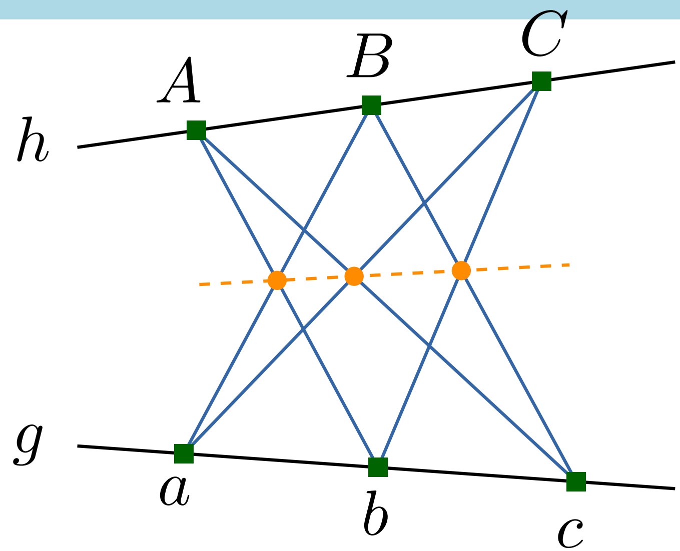


non-stretchable arrangement
with multicrossings (non-simple)
(Ringel, 1957)



Pappus Theorem:
Orange intersections
are colinear.

stretchability

Stretchable pseudoline arrangements can be drawn using straight lines.

Theorem (Goodman & Pollack, 1980)

All arrangements of $n \leq 7$ pseudolines are stretchable.

Theorem (Shor, 1991)

It is NP-hard to decide whether a given arrangement is stretchable.

wiring diagrams

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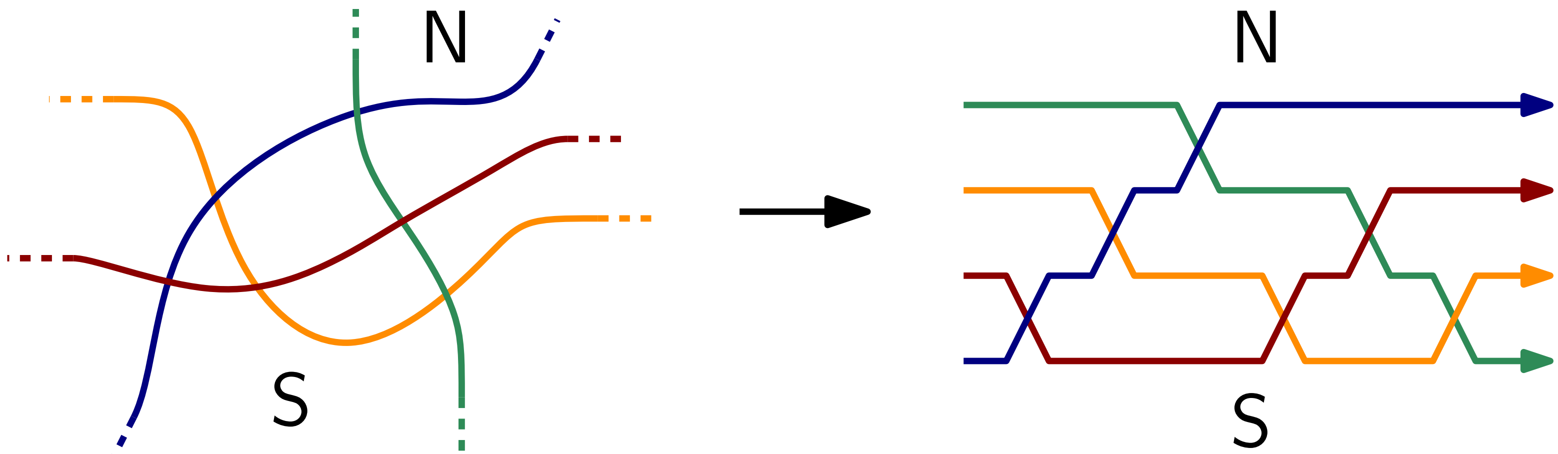
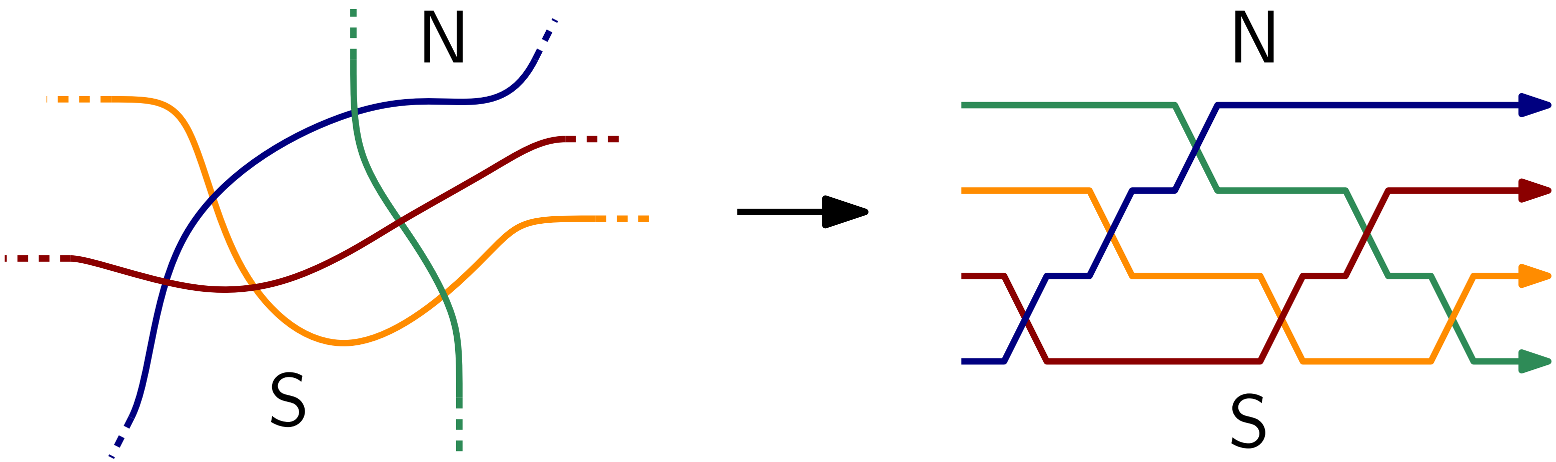


Wiring diagram: canonical drawing of a pseudoline arrangement

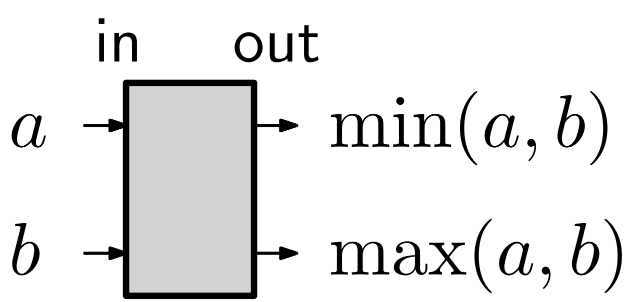
- pseudolines $\sim$ horizontal wires
- crossings $\sim$ sequence of switches between wires
- North cell N lies above all wires.

Will see: Every arrangement can be drawn as a wiring diagram!

wiring diagrams



Wiring diagrams as sorting networks:



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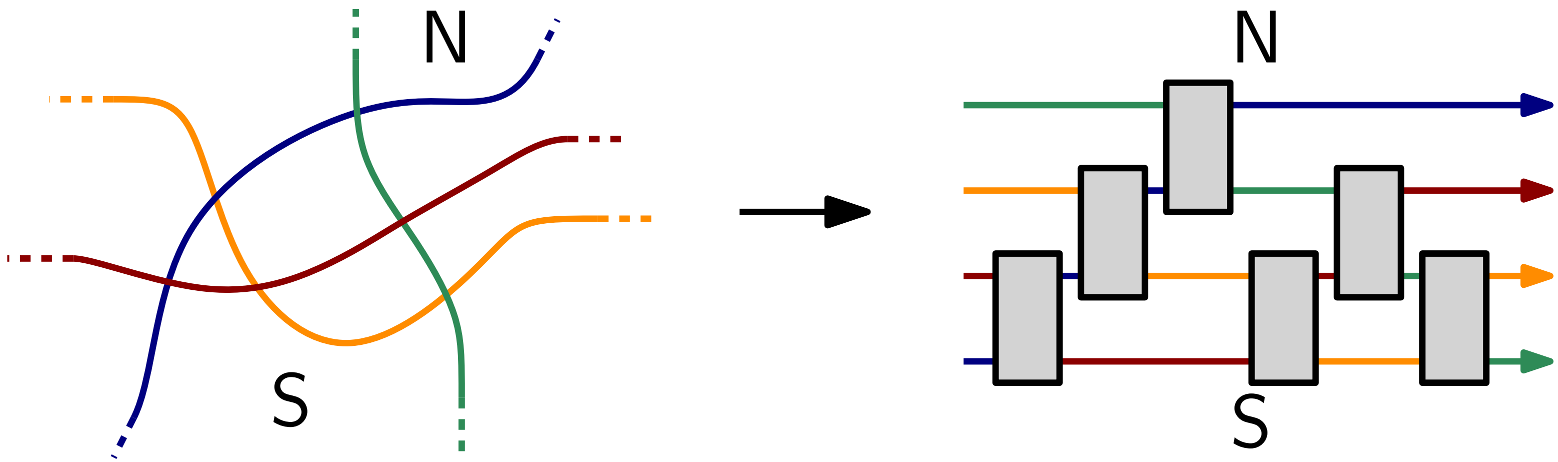

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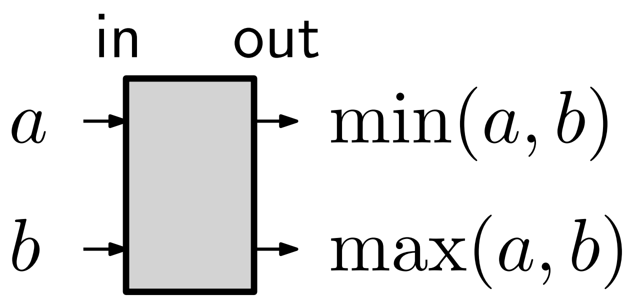

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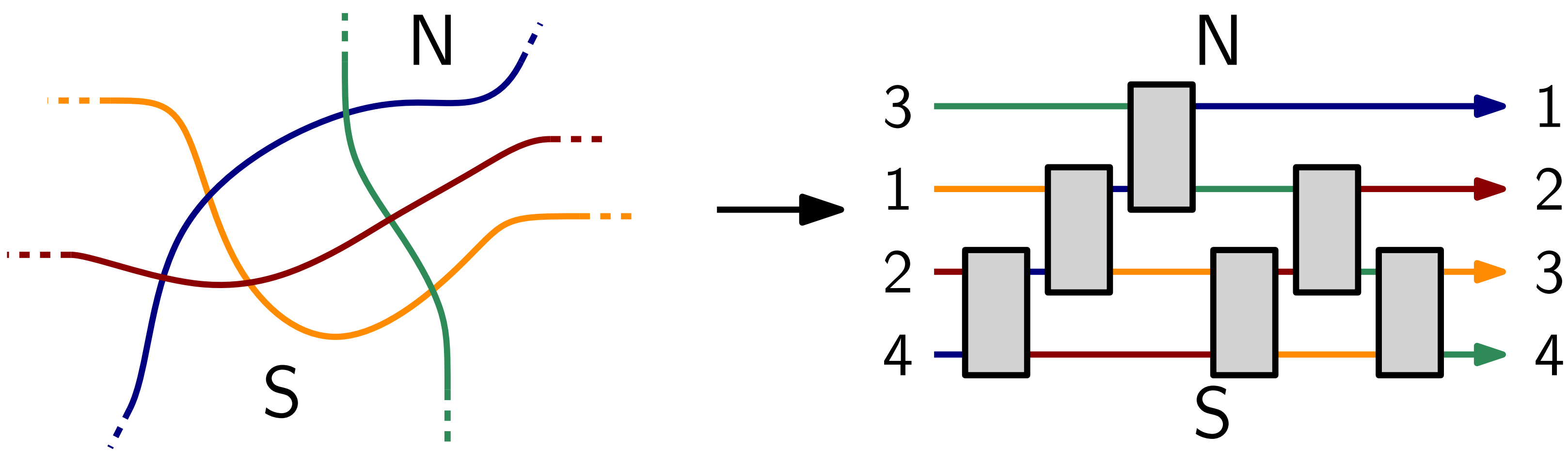

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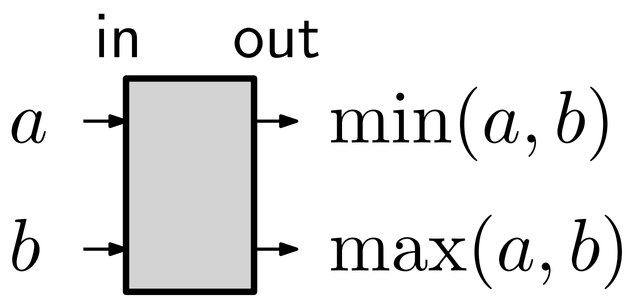

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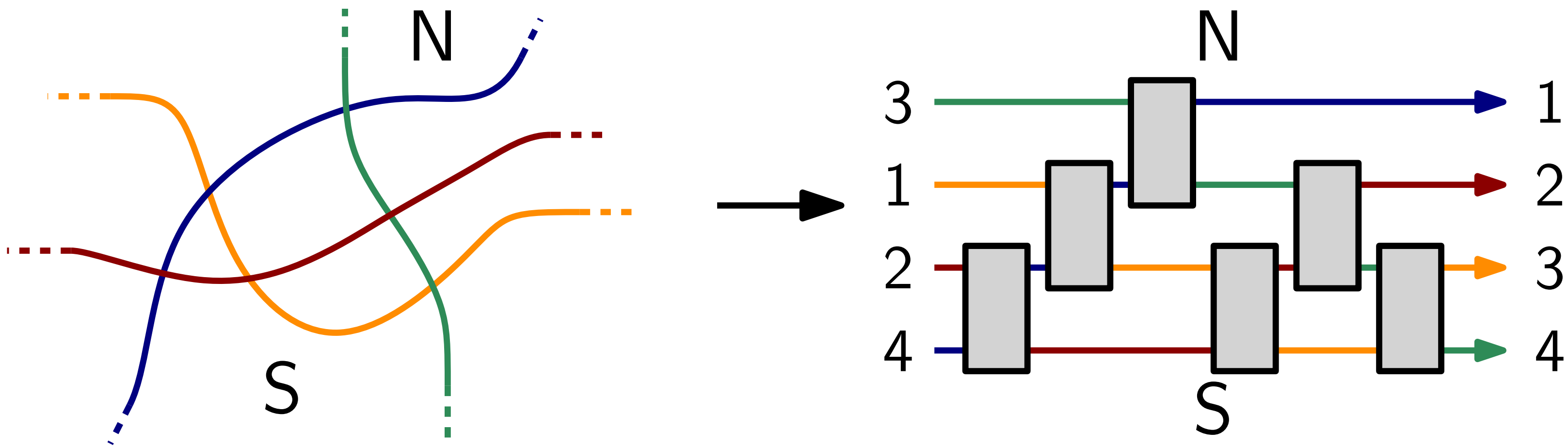

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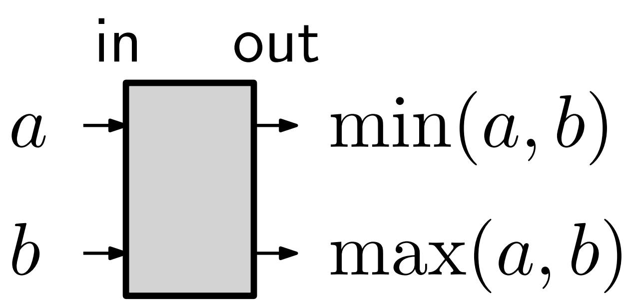
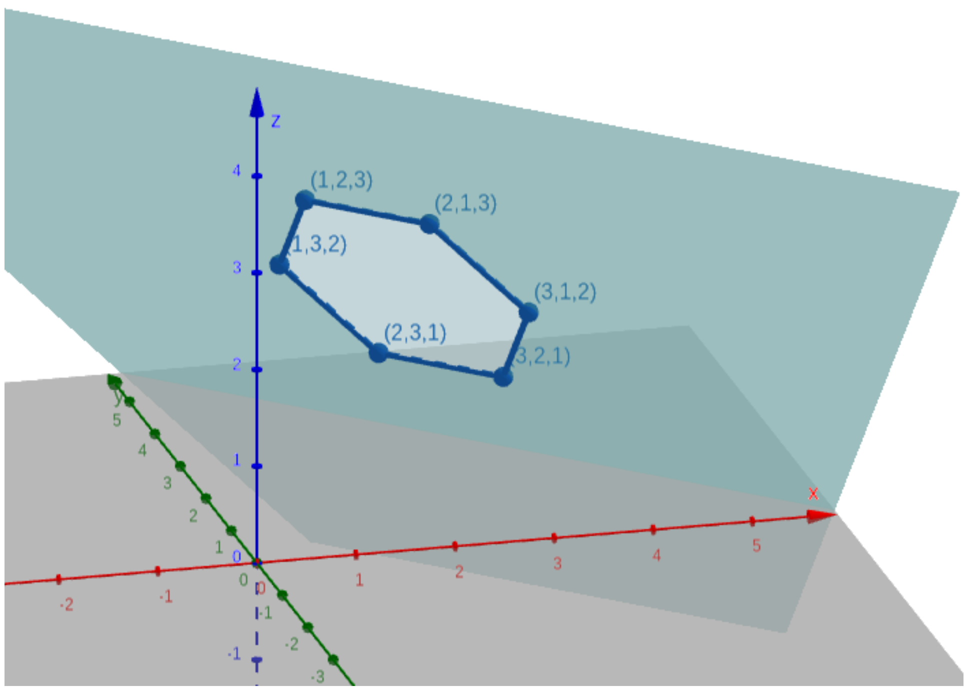


Sorting networks encode sorting algorithms that are based on *comparison & exchange* of neighbor elements.

monotonic paths on permutahedron

Permutahedron of order n:

$$P_n := \mathrm{conv}(\{(\pi(1), \cdots, \pi(n)) \in \mathbb{R}^n \mid \pi \in S_n\})$$



Example: $n = 3$

$$x_1 + \cdots + x_n = n(n+1)/2$$

$$\implies \dim(P_n) = n - 1$$

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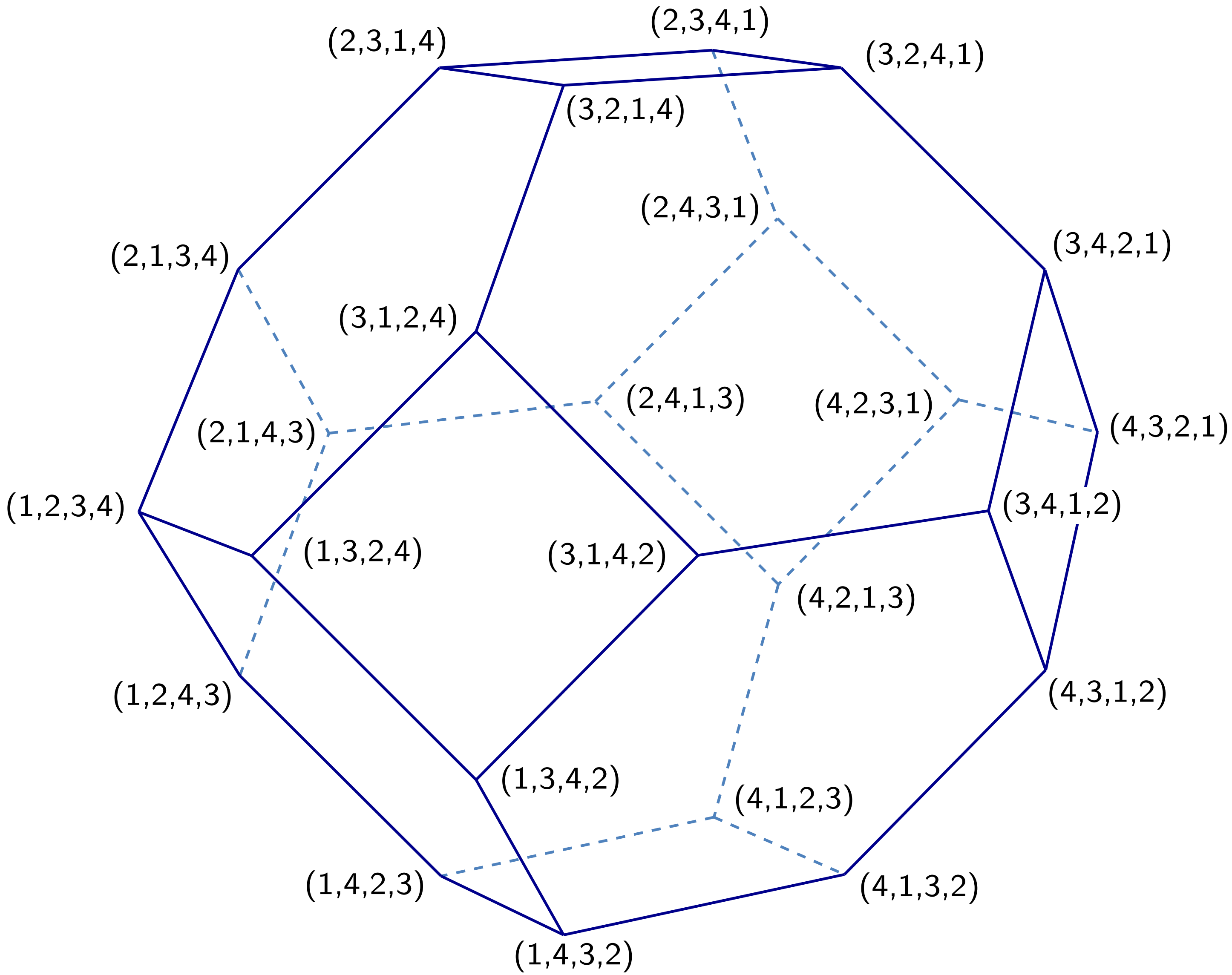

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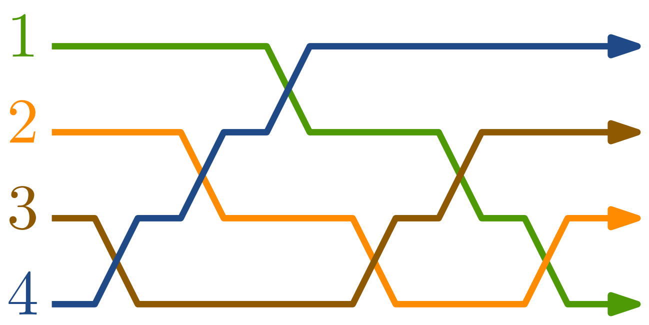

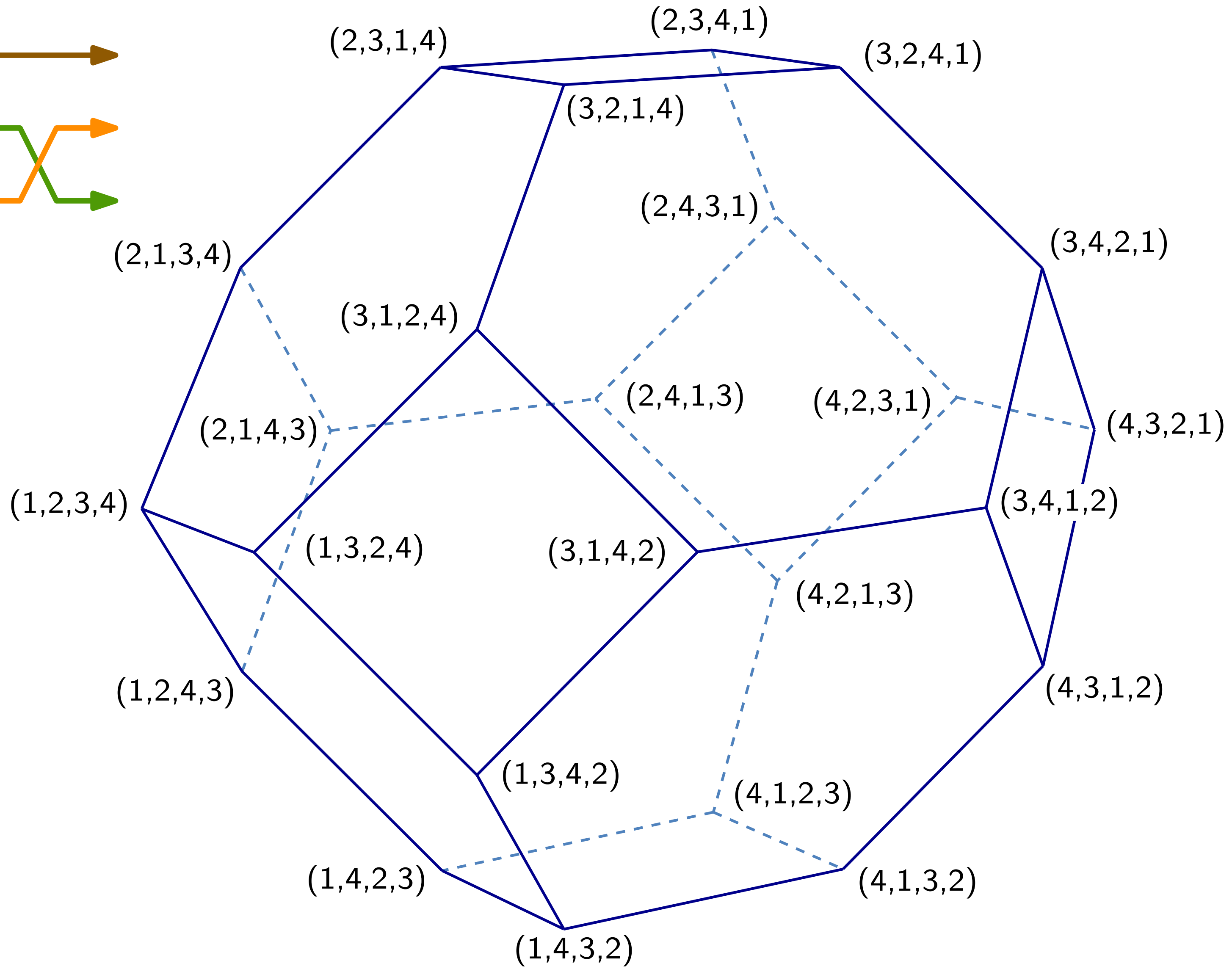

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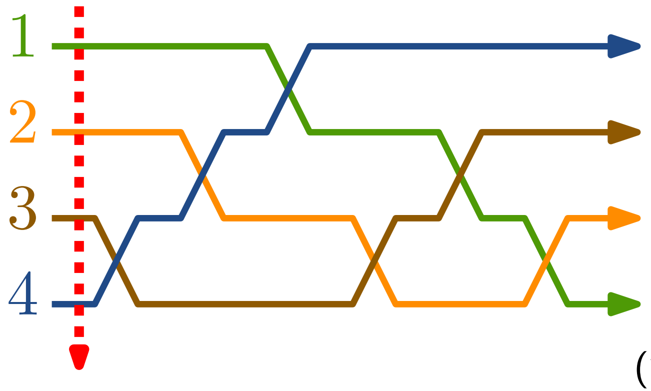

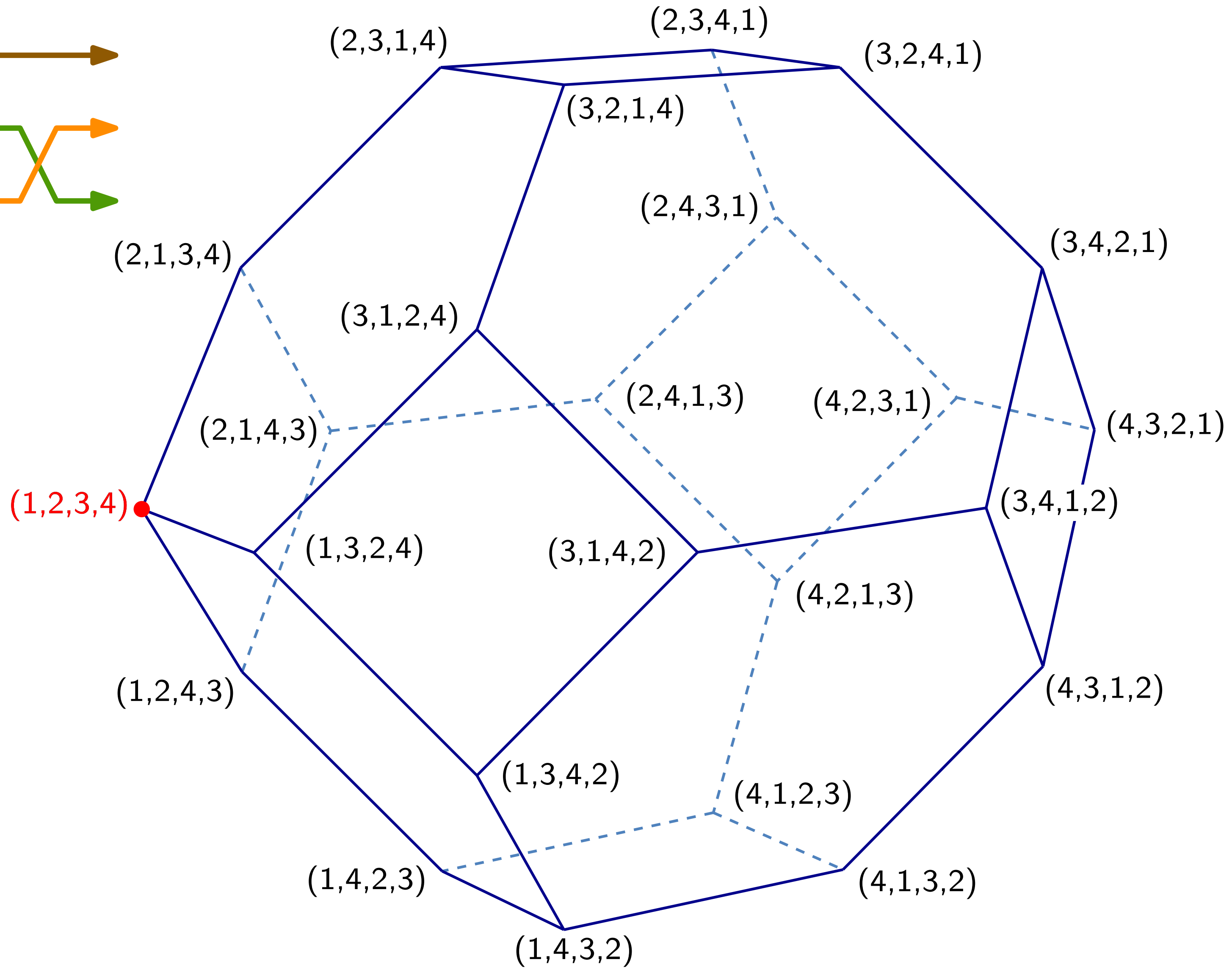

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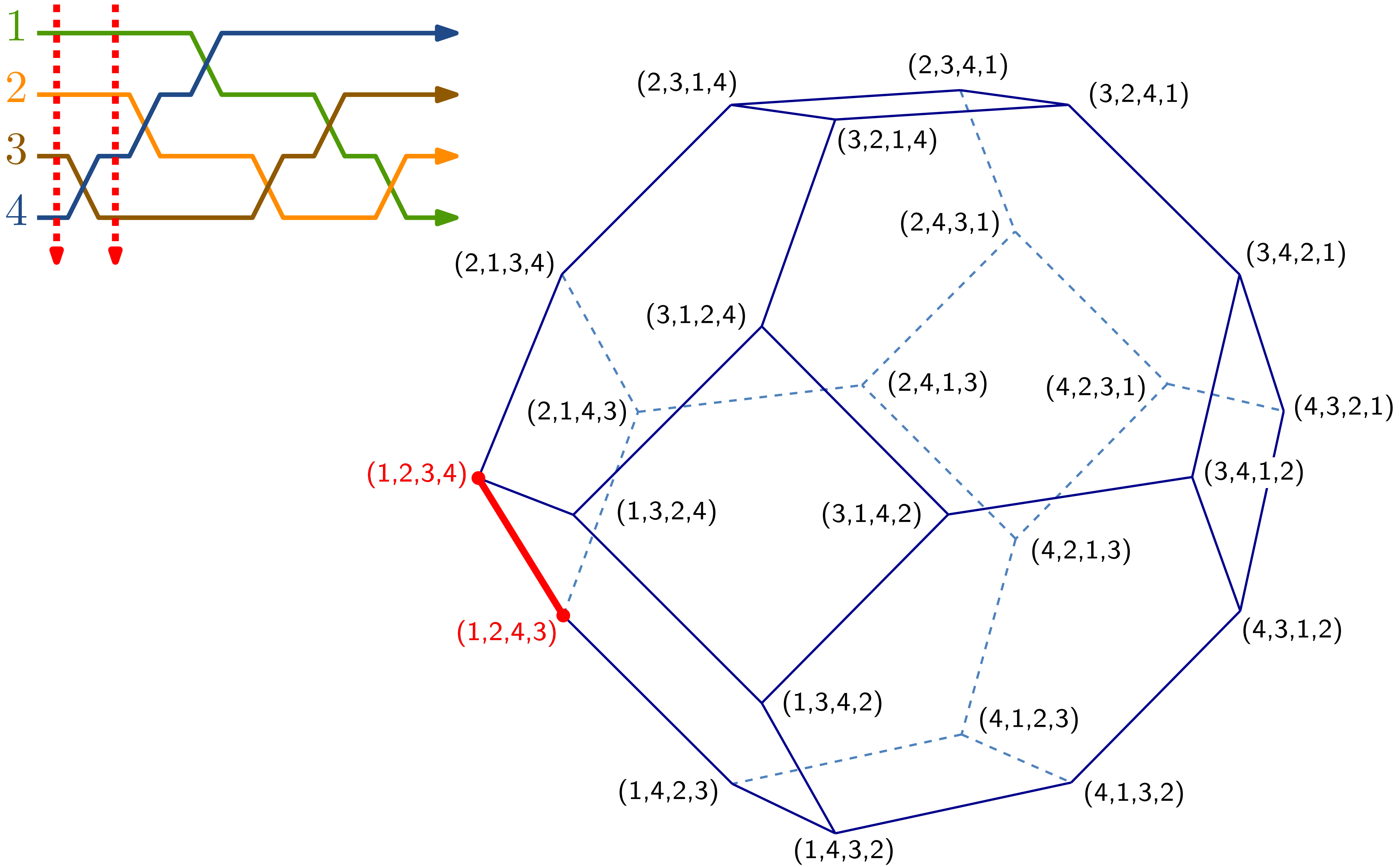

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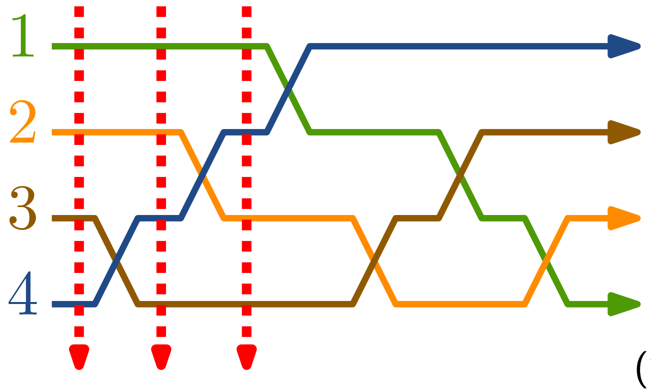

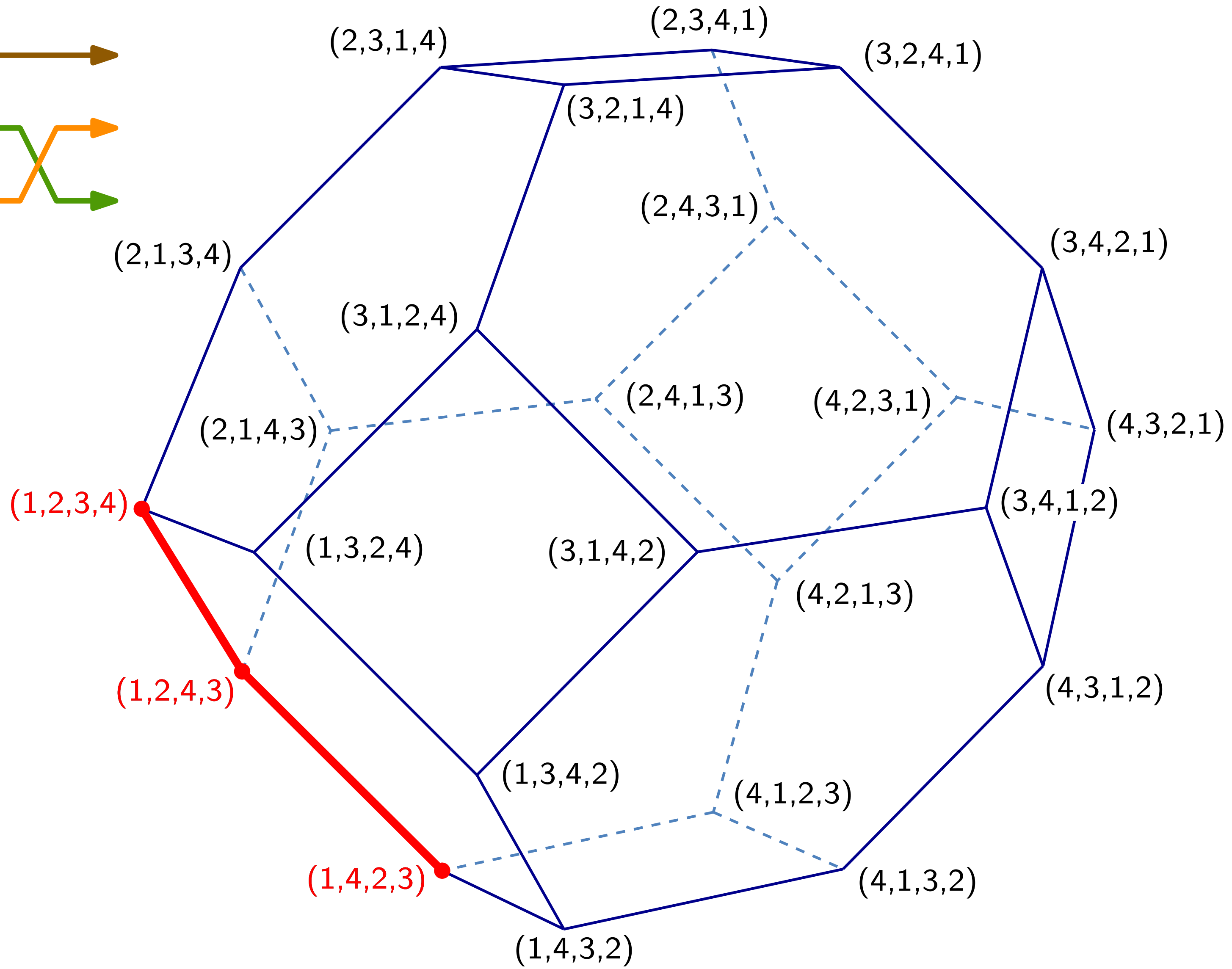

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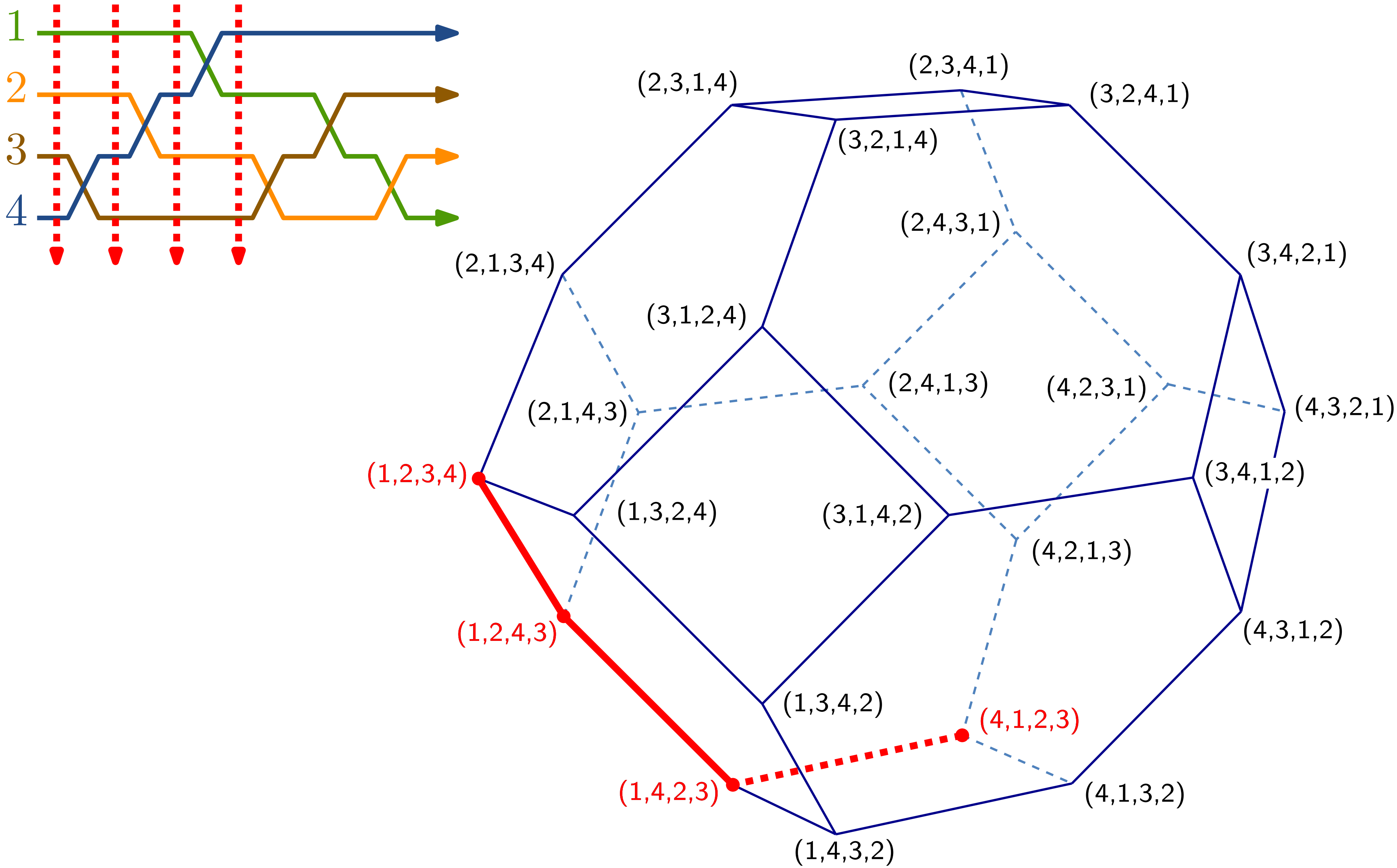

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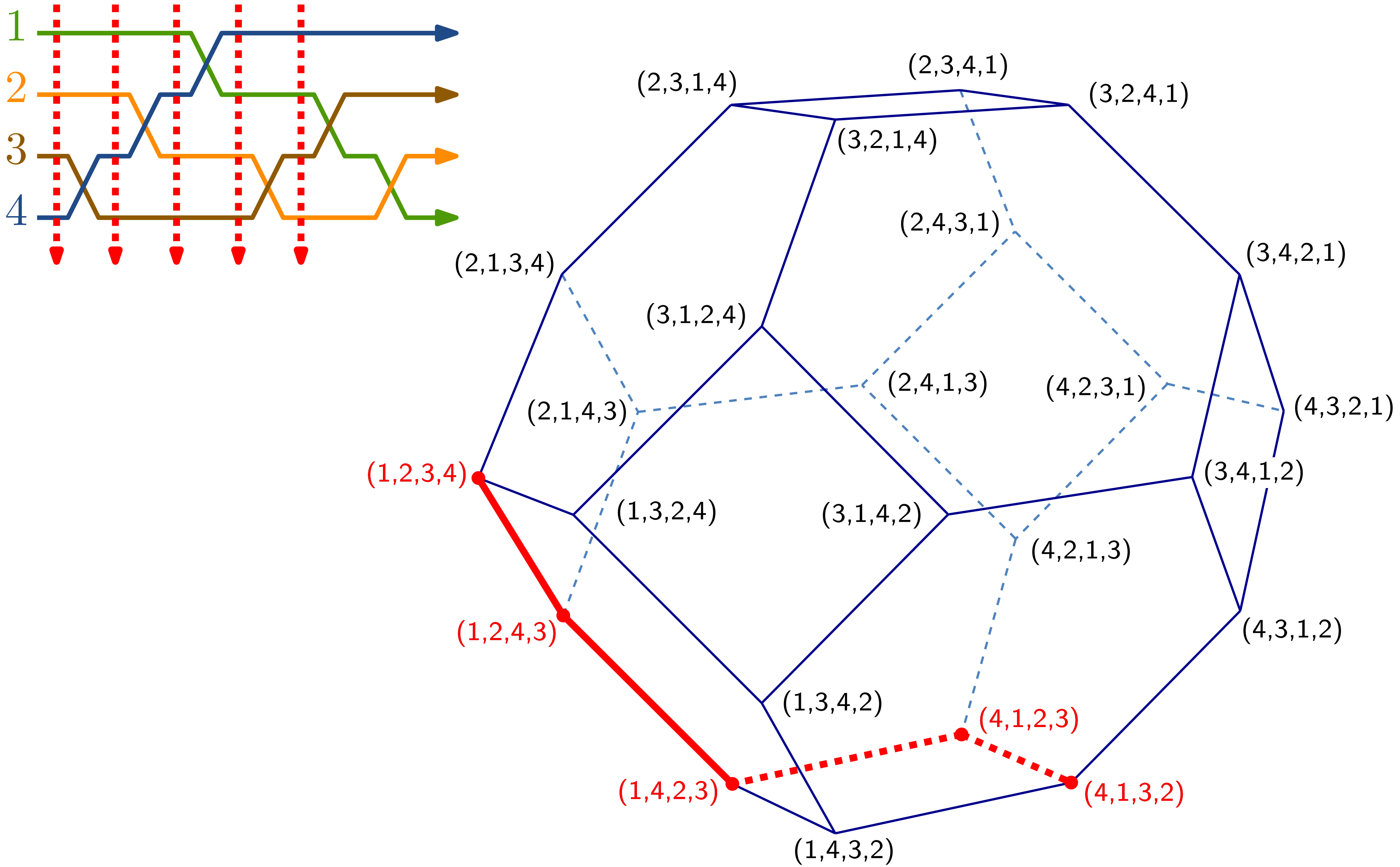

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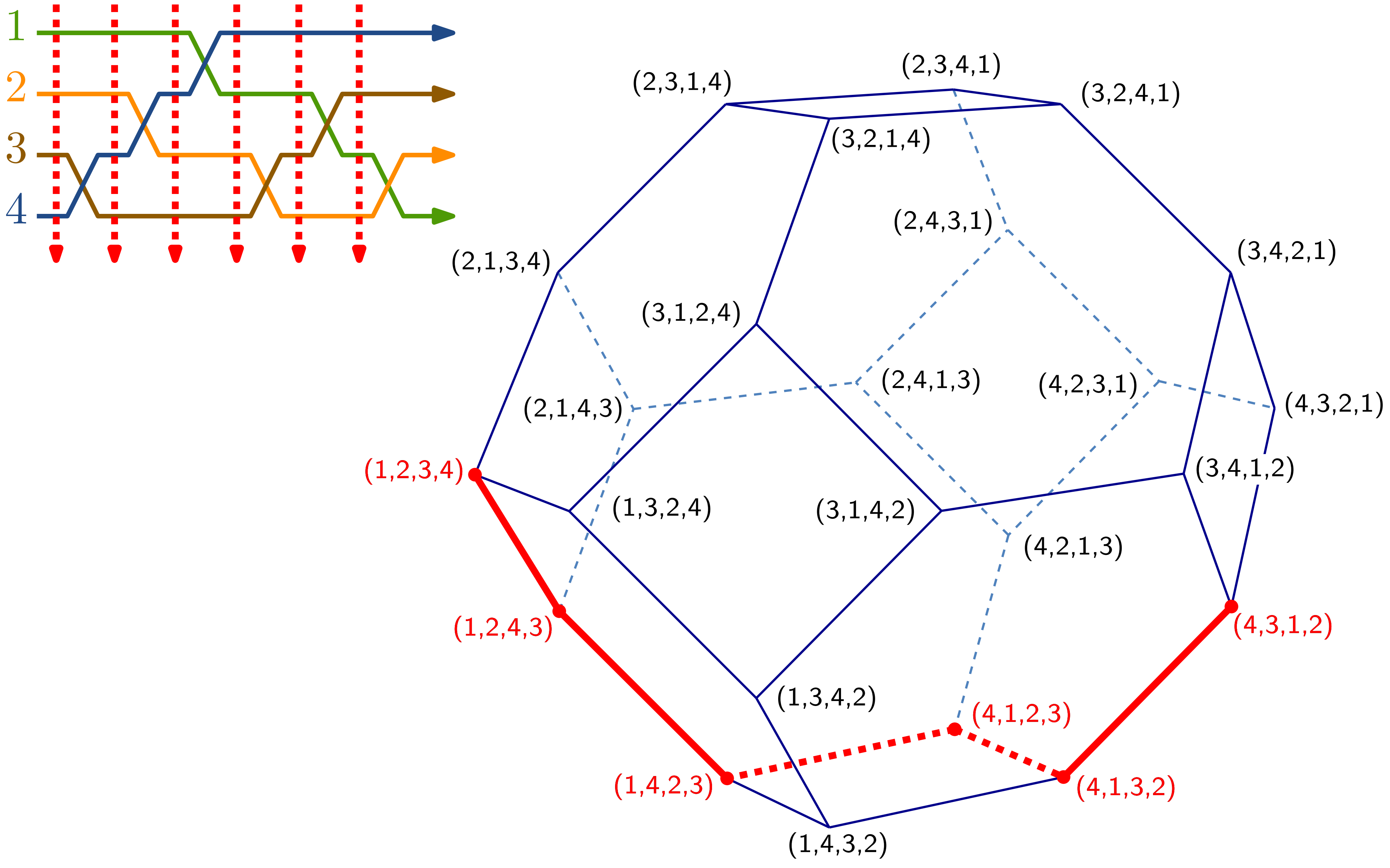

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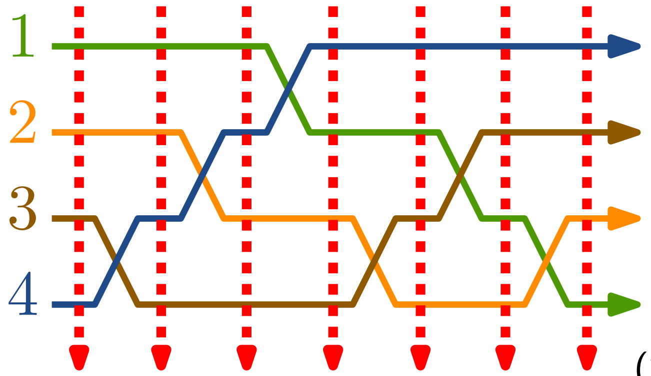

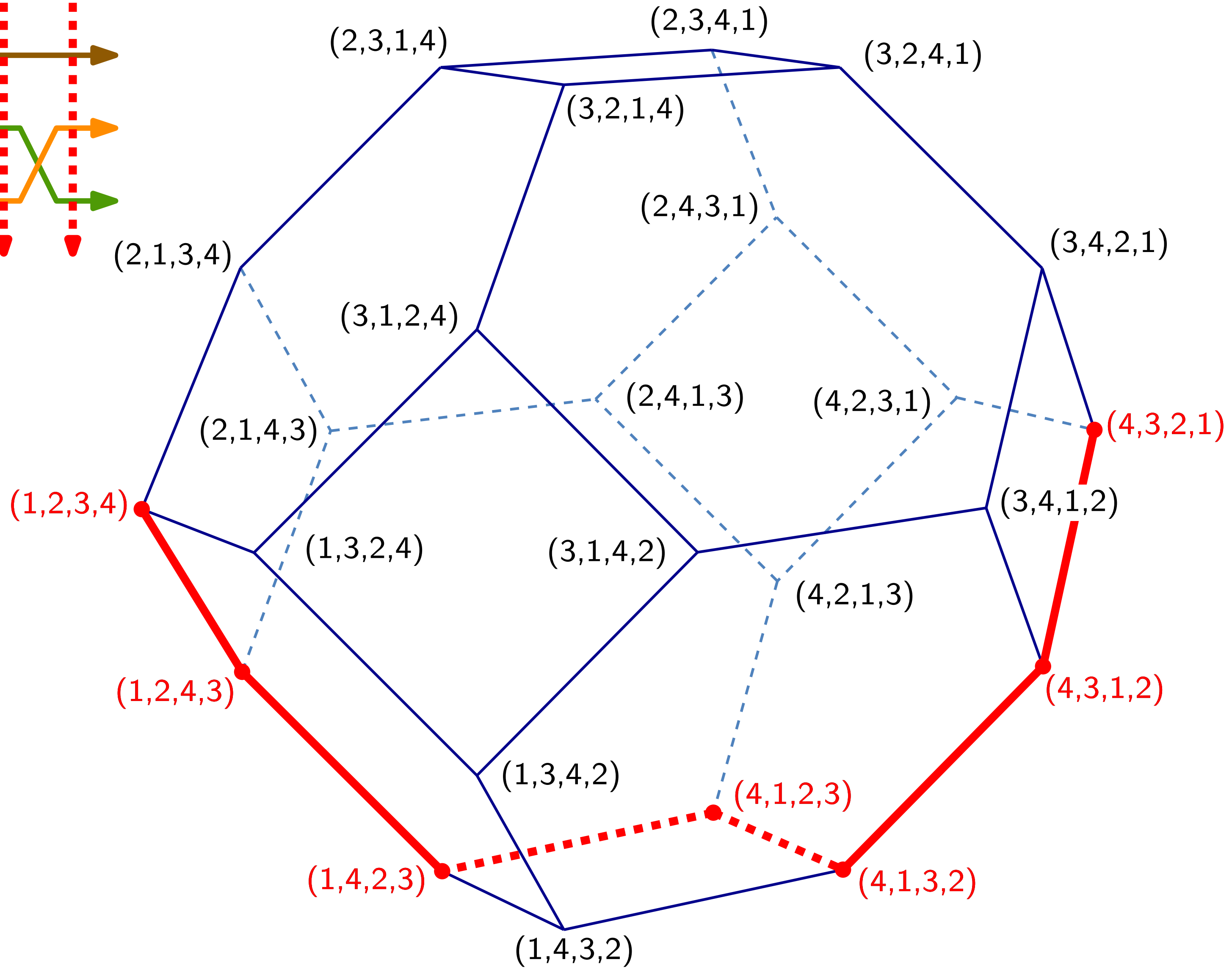

allowable sequences

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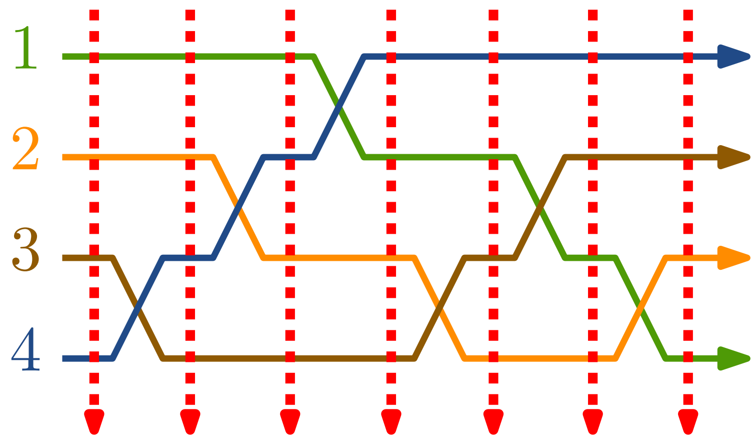

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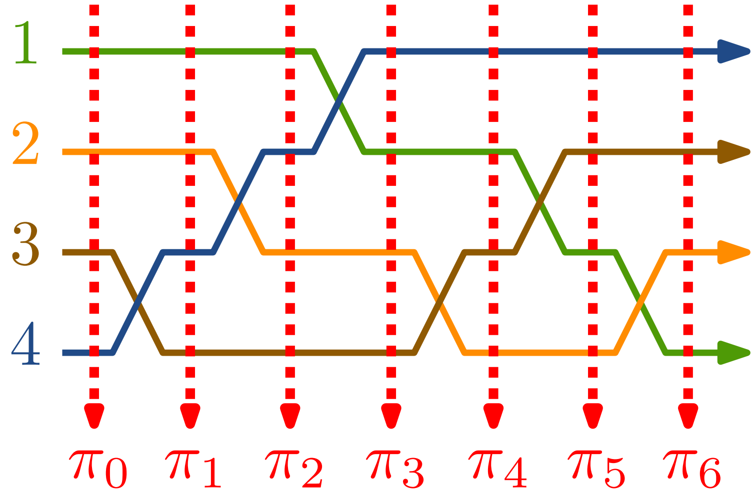

Def: *allowable sequence:* sequence of permutations $\pi_0, \cdots, \pi_{\binom{n}{2}}$

- Starts with $\pi_0 = [1, \cdots, n]$.
- Ends with $\pi_{\binom{n}{2}} = [n, \cdots, 1]$.
- $\pi_i = \tau_i \circ \pi_{i-1}$ for some neighbor transposition $\tau_i = (s_i, s_i + 1)$.

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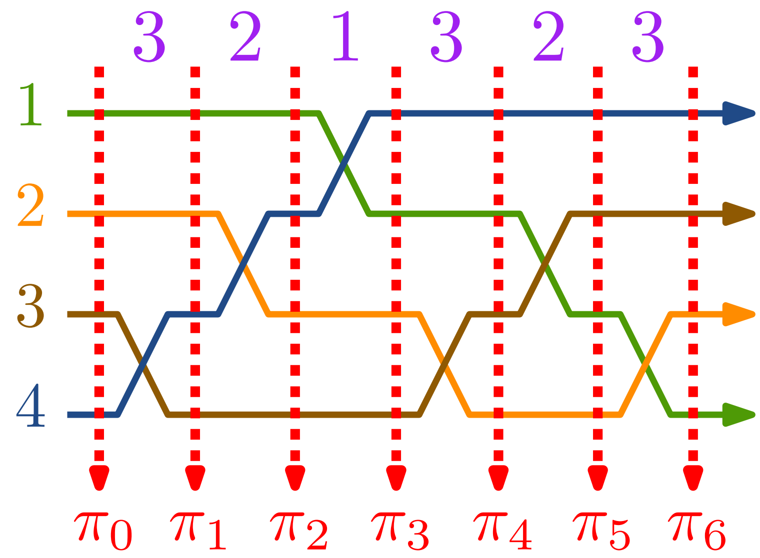

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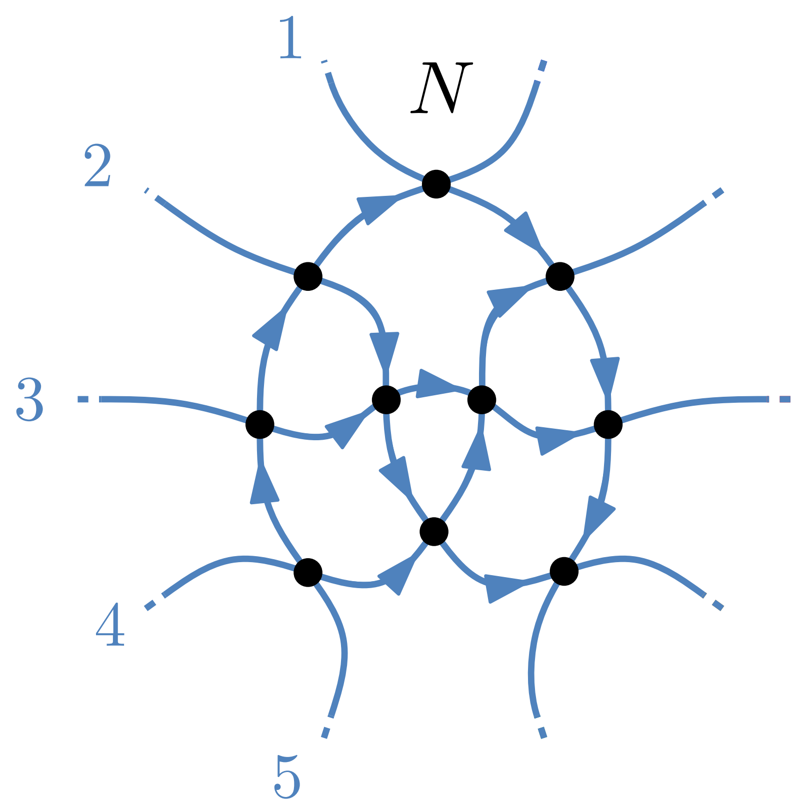
Suffices to write only transposition indices $(s_1, \cdots, s_{\binom{n}{2}})$!

Example: $(s_1, \cdots, s_6) = (3, 2, 1, 3, 2, 3)$

arrangement graph

Marked arrangement $\mathscr{A}$ defines digraph $G_{\mathscr{A}}$ on set of crossings.



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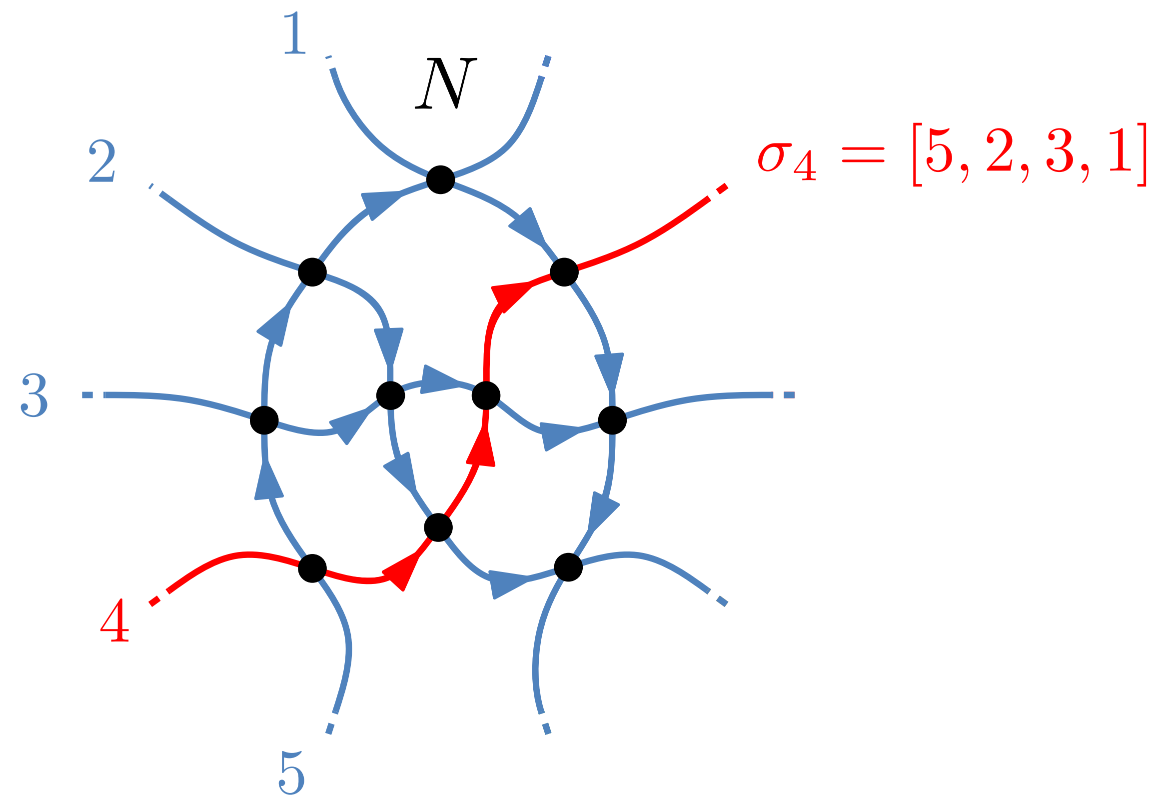

Local intersection orders $\sigma_1, \cdots, \sigma_n$ and $G_{\mathscr{A}}$ determine each other.

arrangement graph

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Lemma

The arrangement graph $G_{\mathscr{A}}$ is acyclic.

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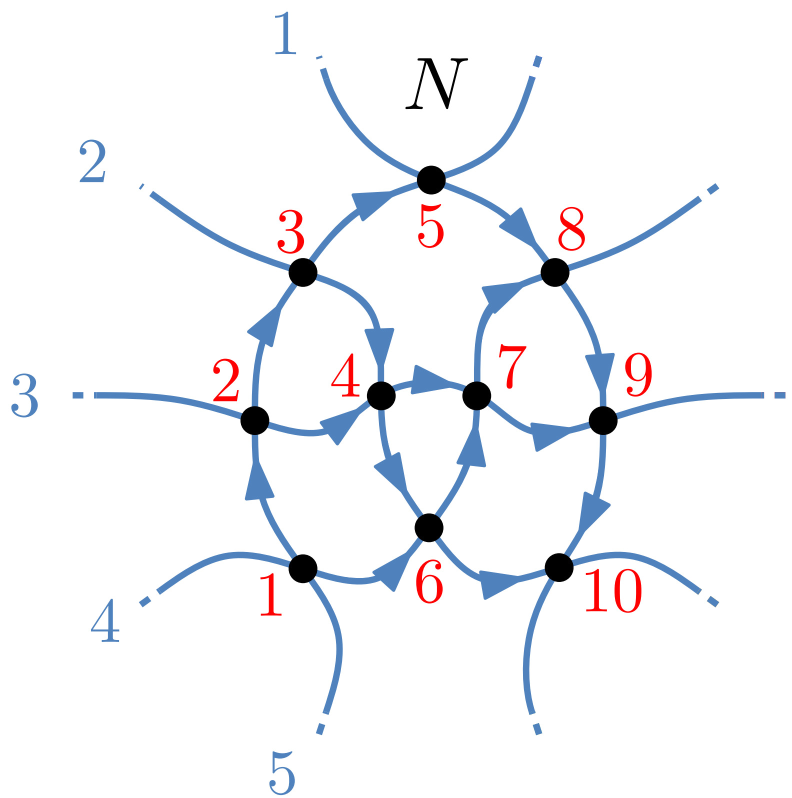
Consequence: There exists a topological sorting of the crossings.



sweeping arrangements

Lemma

Every pseudoline arrangement can be drawn as a wiring diagram.

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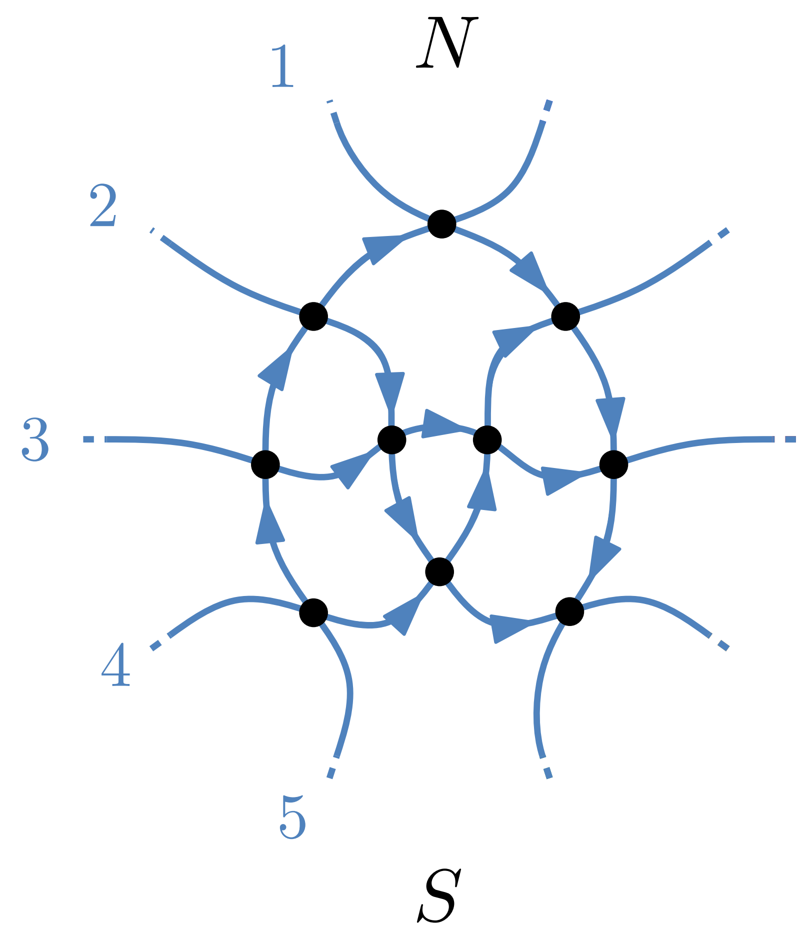
Proof sketch:



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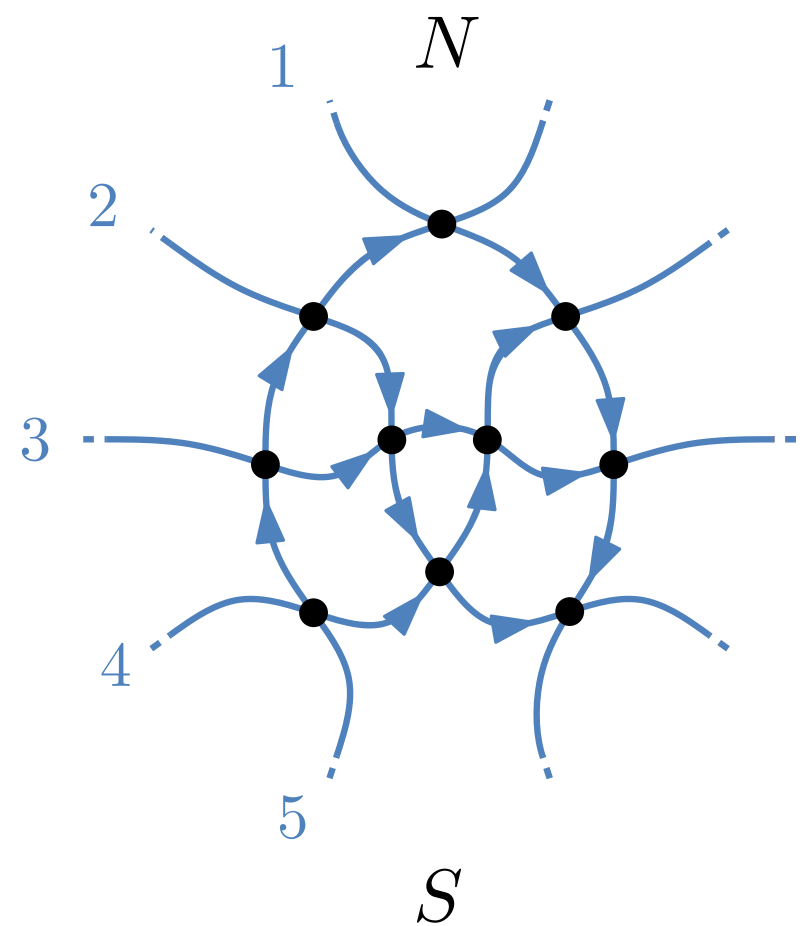

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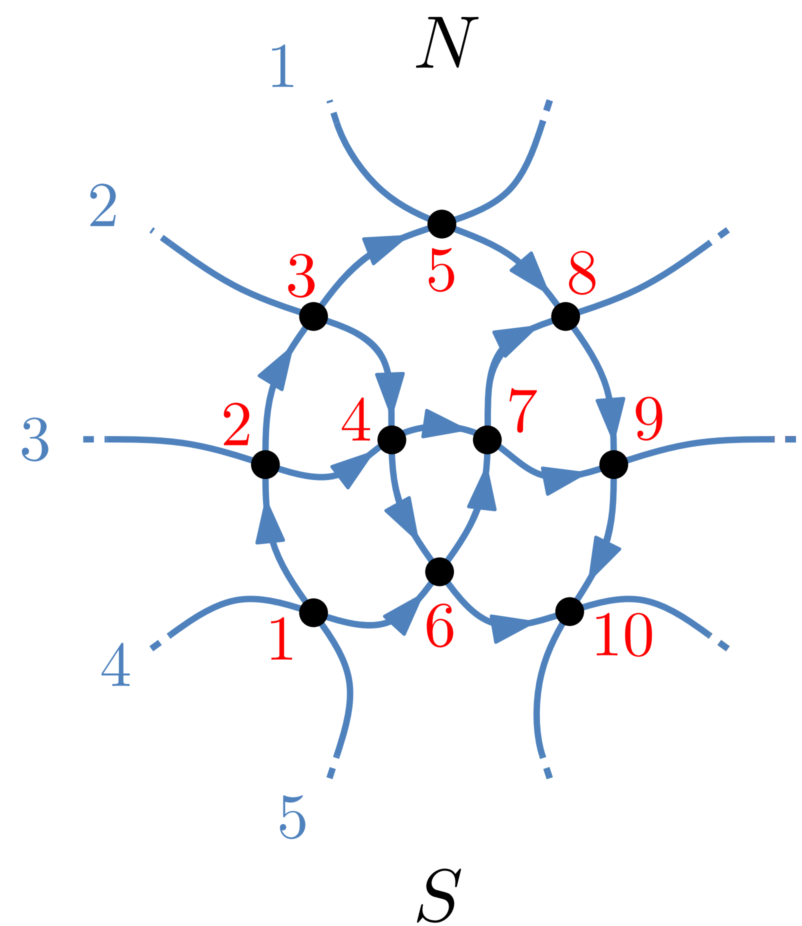

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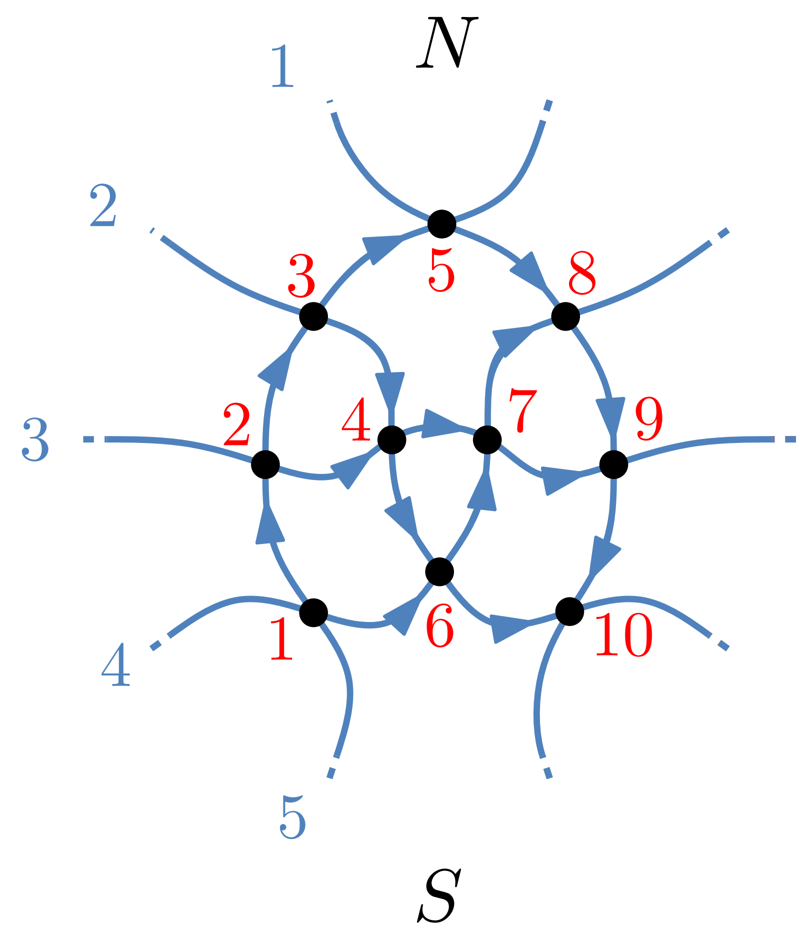

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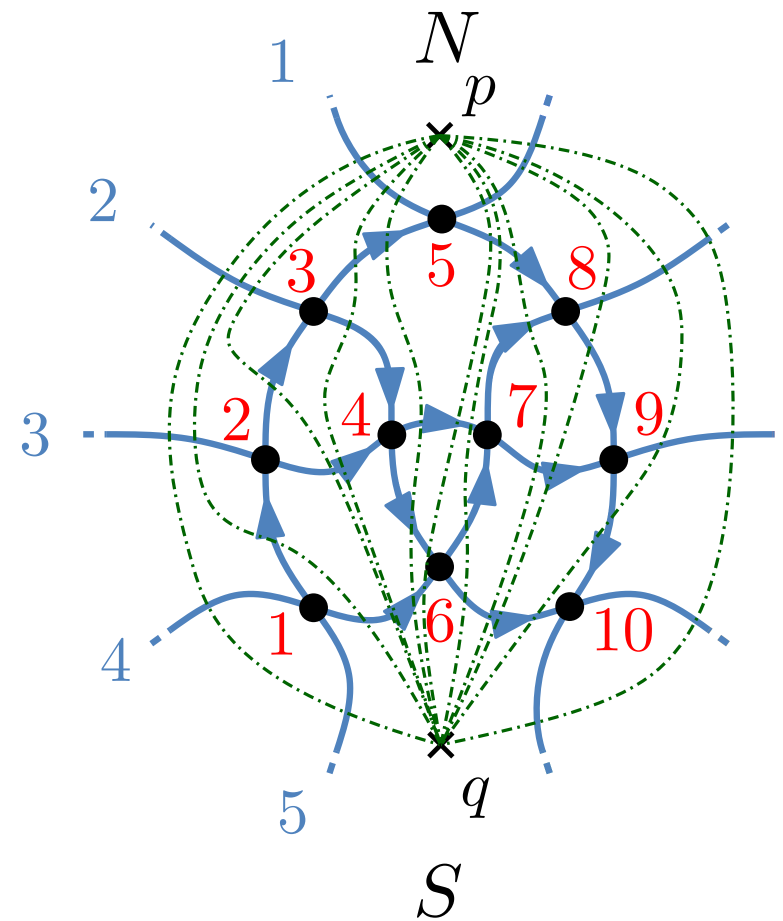

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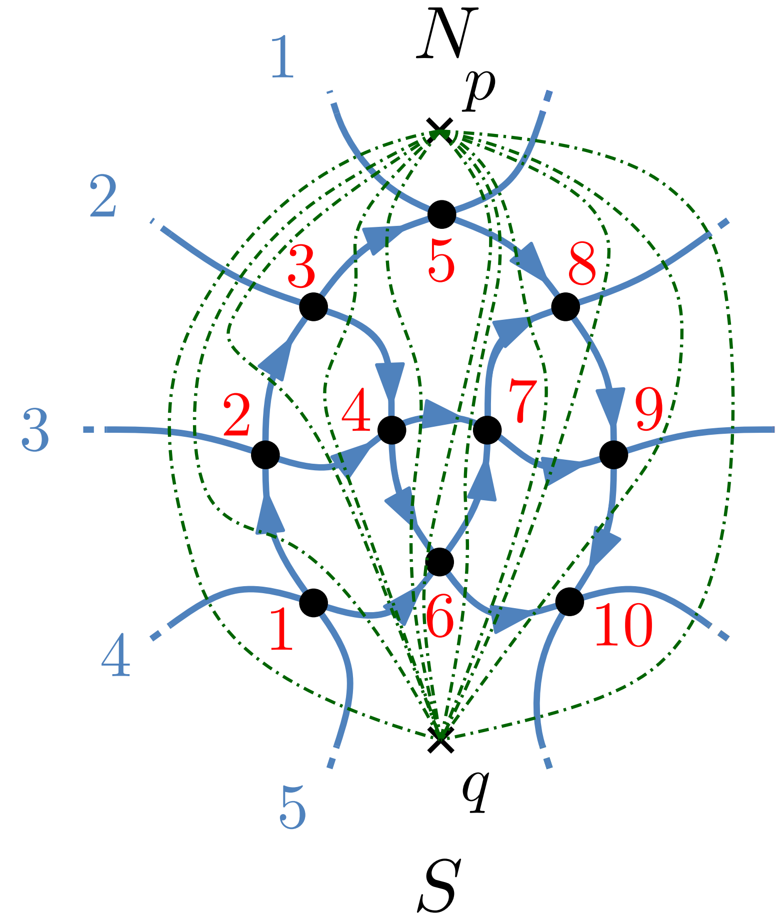

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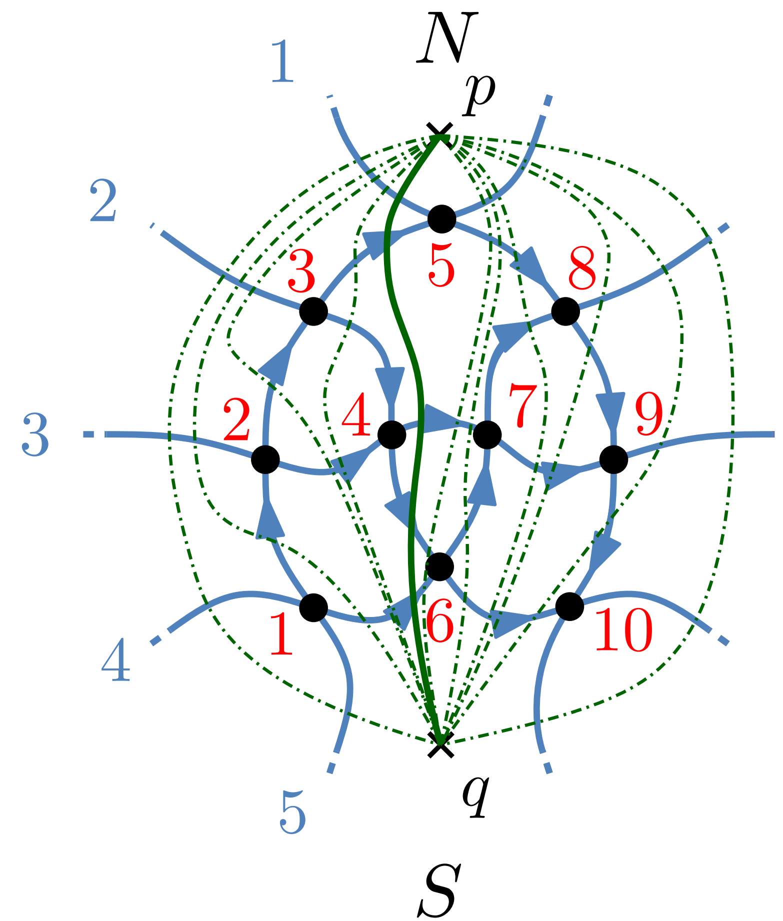

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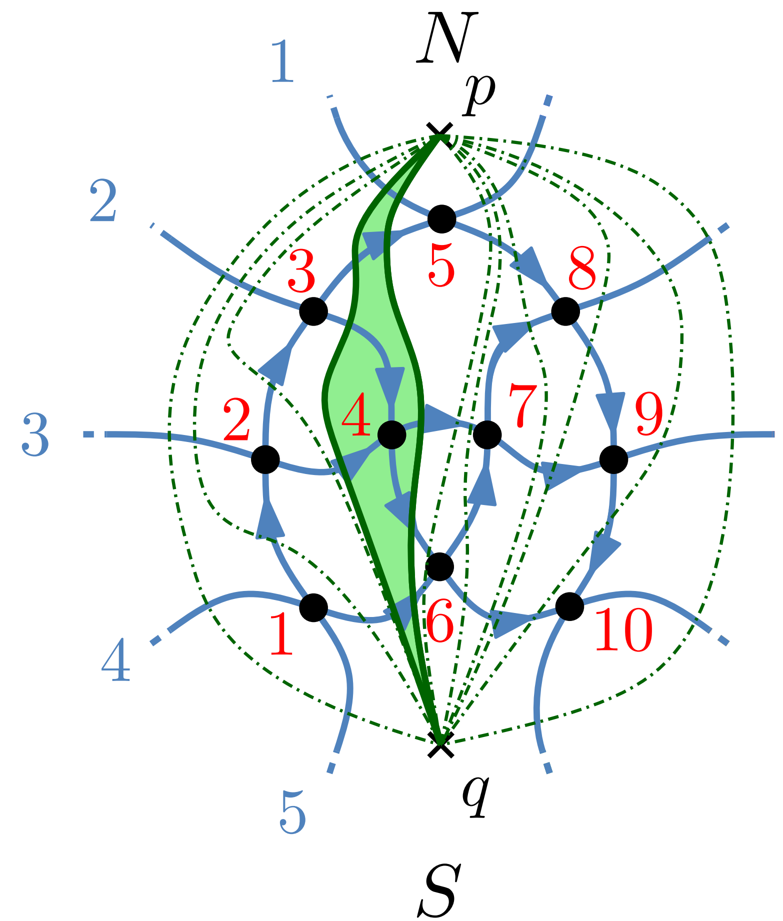
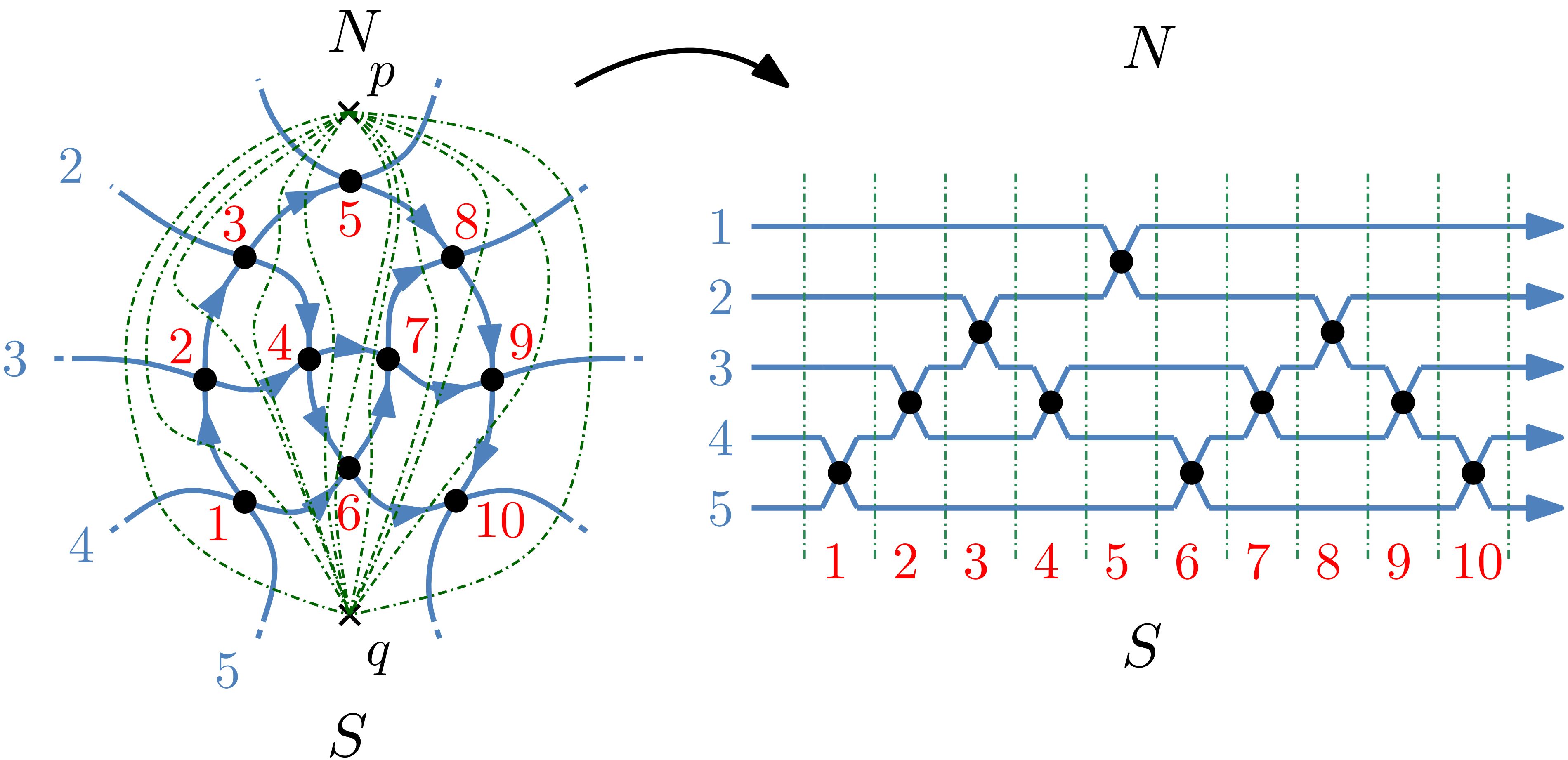
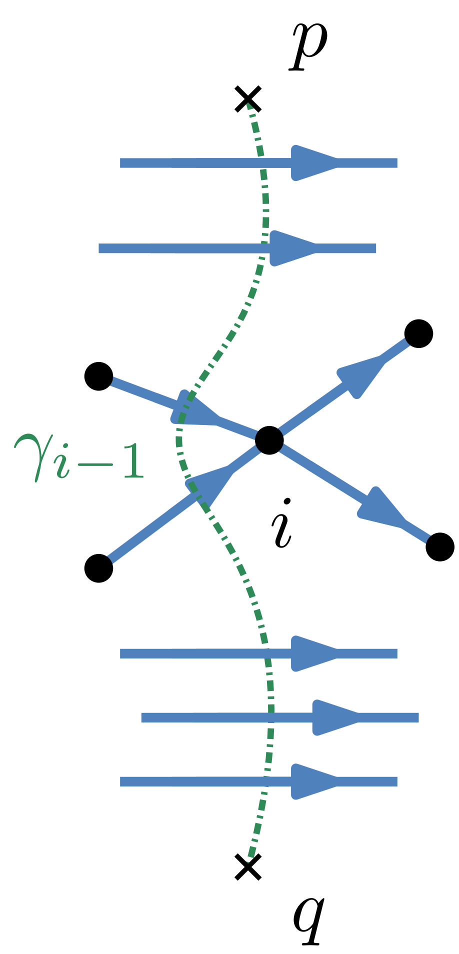
- Exactly one crossing lies in area between γ_i and γ_{i+1}.



If we can sweep $\mathscr{A}$, then we can draw $\mathscr{A}$ as a wiring diagram!



Iteratively obtain γ_i from γ_{i-1}:



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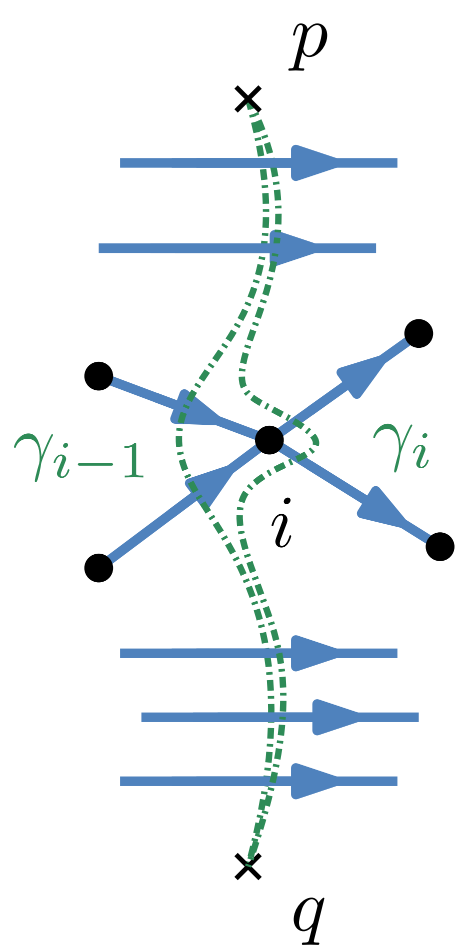
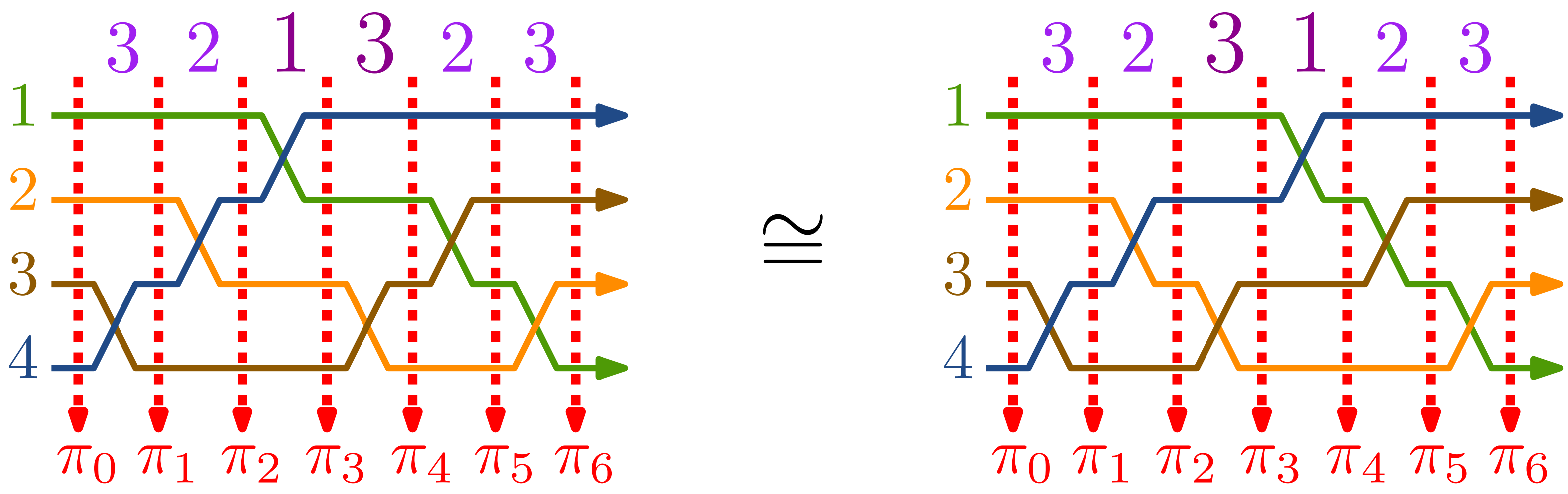


□

equivalence of allowable sequences

Different allowable sequences correspond to the same arrangement!



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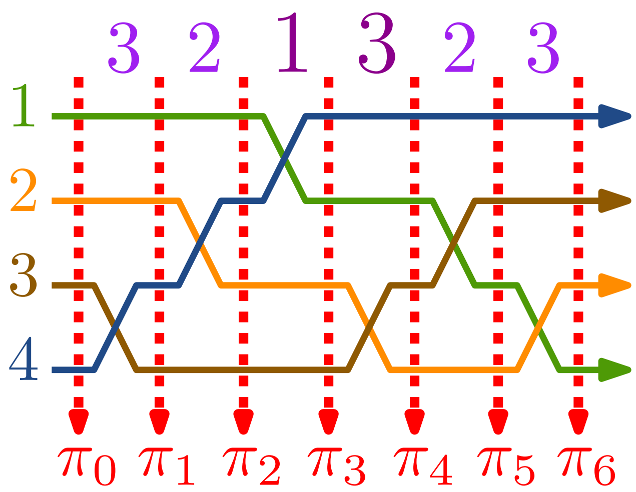

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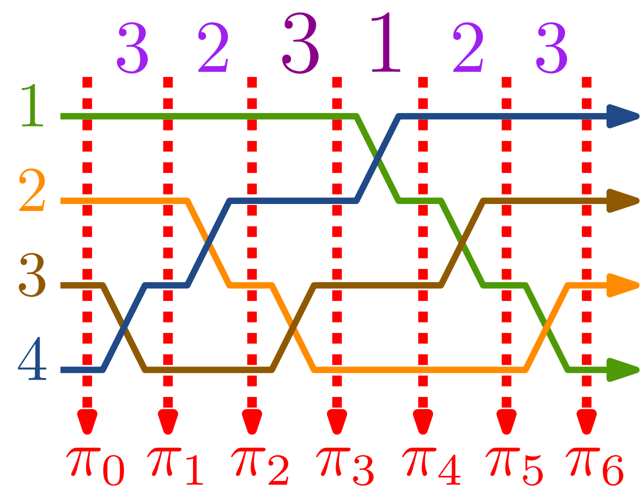

Def:
- Allowable sequences

$$S = (s_1, \cdots, s_i, s_{i+1}, \cdots, s_{\binom{n}{2}})$$

and $$S' = (s_1, \cdots, s_{i+1}, s_i, \cdots, s_{\binom{n}{2}})$$

are called *directly equivalent*, if $|s_i - s_{i+1}| \geq 2$.

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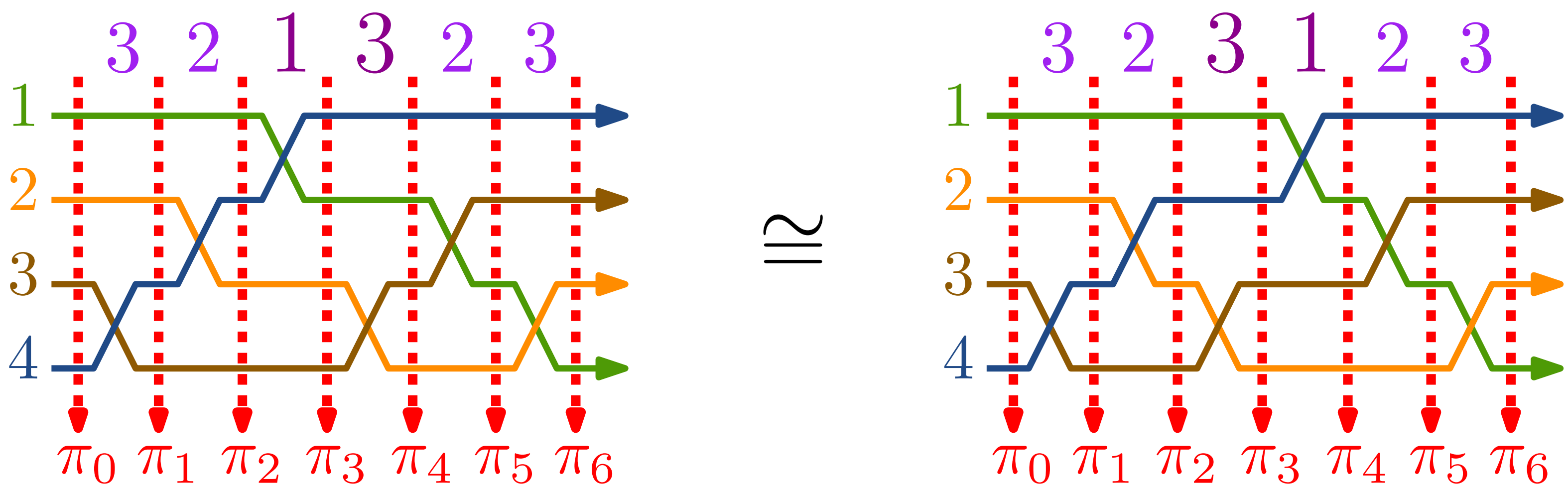

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- S and S' are called *equivalent* ($S \sim S'$), if there are $S = S_1, \cdots, S_r = S'$ with S_i and S_{i+1} directly equivalent.

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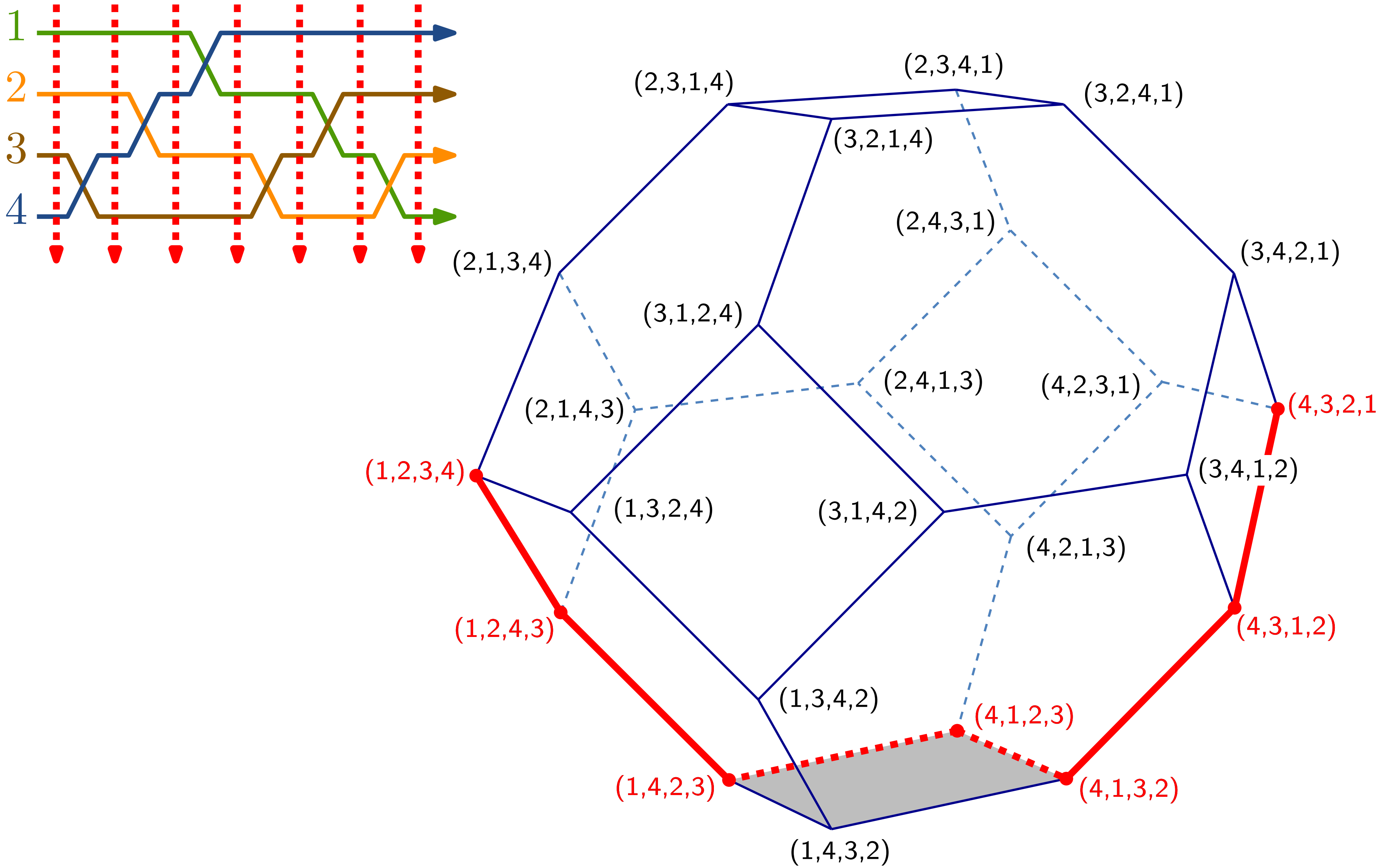

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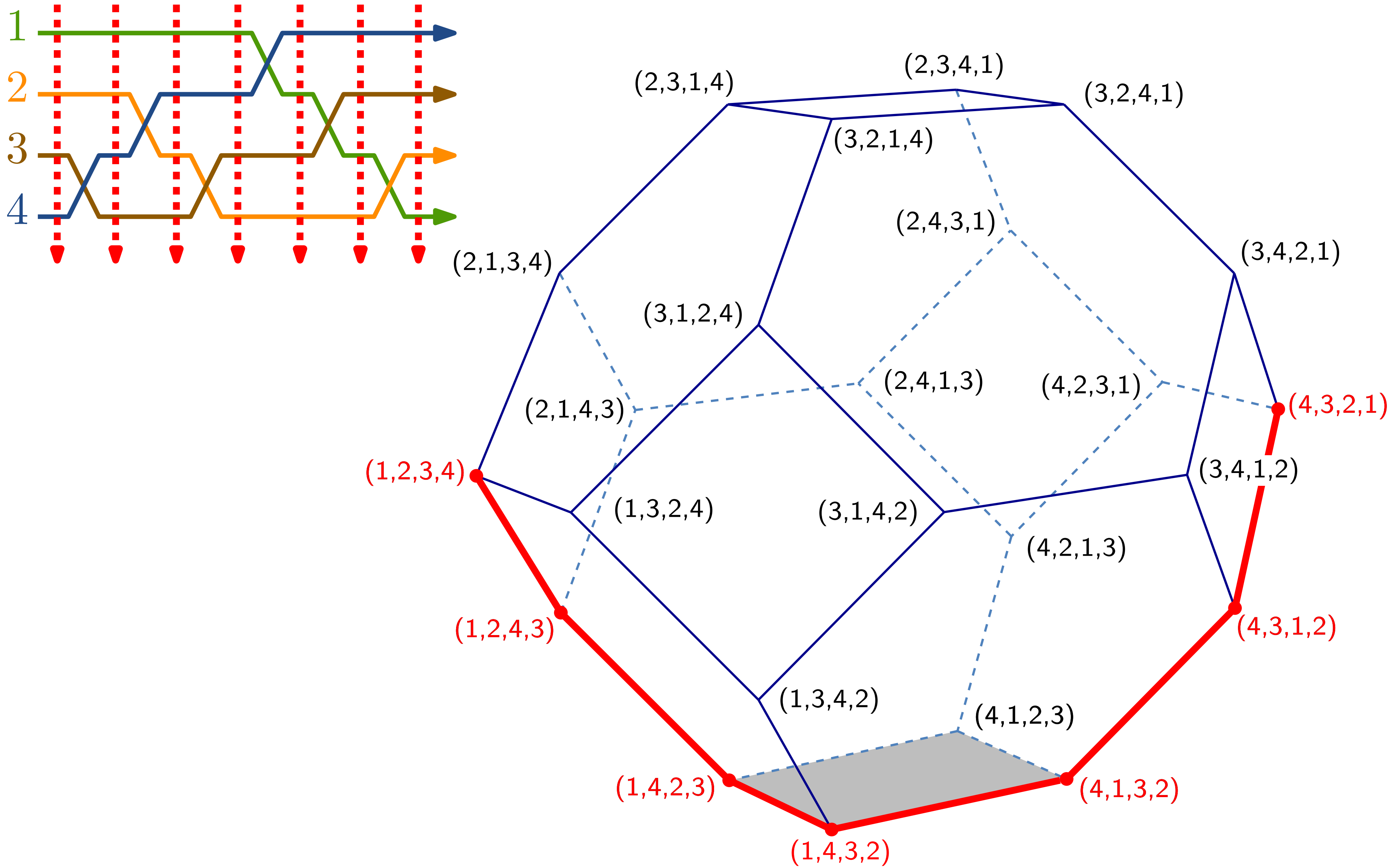

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There is a one-to-one correspondence between marked arrangements of pseudolines and equivalence classes of allowable sequences.

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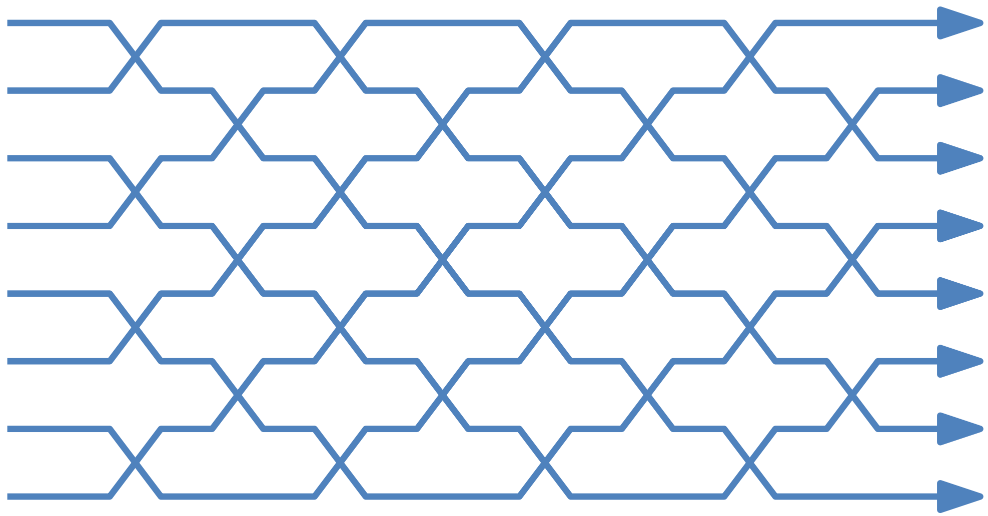
Ingredients for a formal proof:

- Arrangement $\mathscr{A}$ together with a top. sorting π of $G_{\mathscr{A}}$ yields an allowable sequence $S_{\mathscr{A},\pi}$.
- Every allowable sequence can be obtained this way.
- Different top. sortings of $G_{\mathscr{A}}$ correspond exactly to allowable sequences equivalent to $S_{\mathscr{A},\pi}$.

„brick wall conjecture“

Conjecture (Gutierres, Mamede, Santos, 2020)

The *wall arrangements* are the marked arrangements that maximize the number of corresponding allowable sequences.



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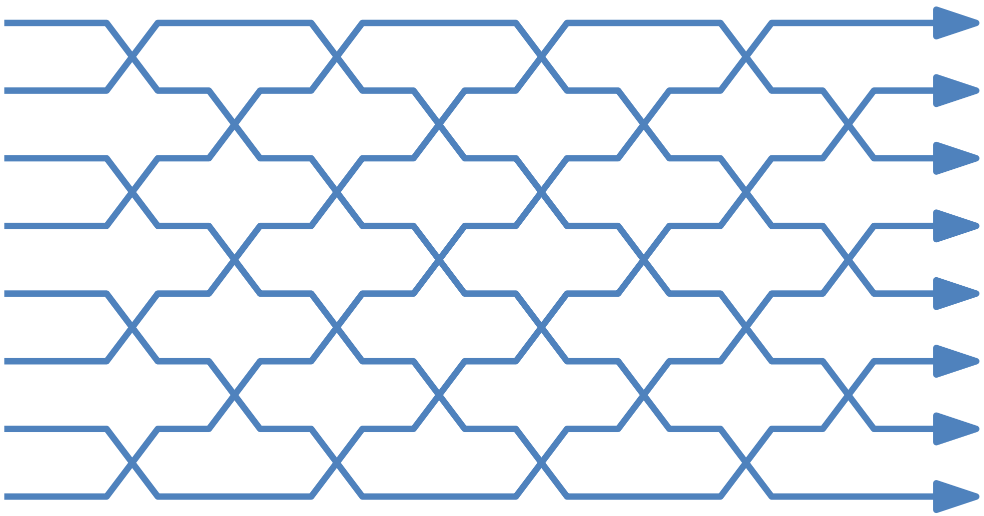

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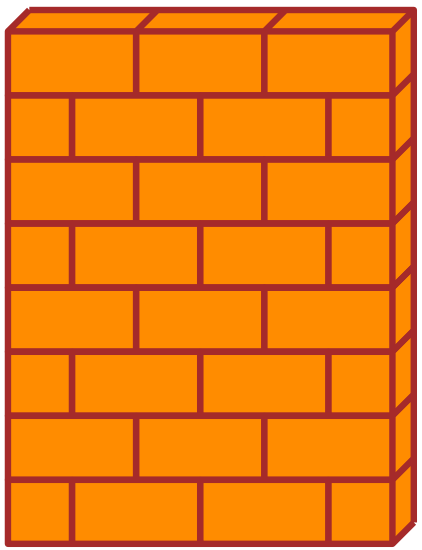

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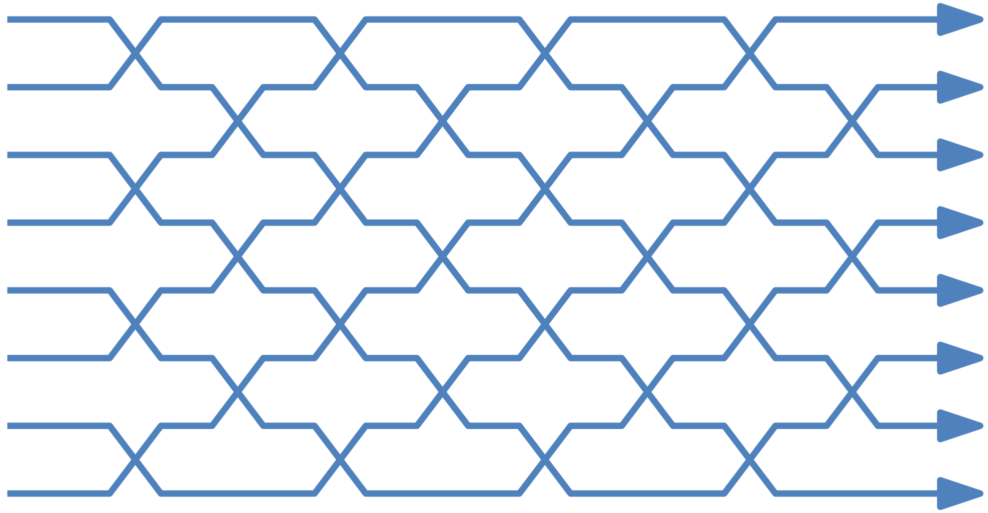

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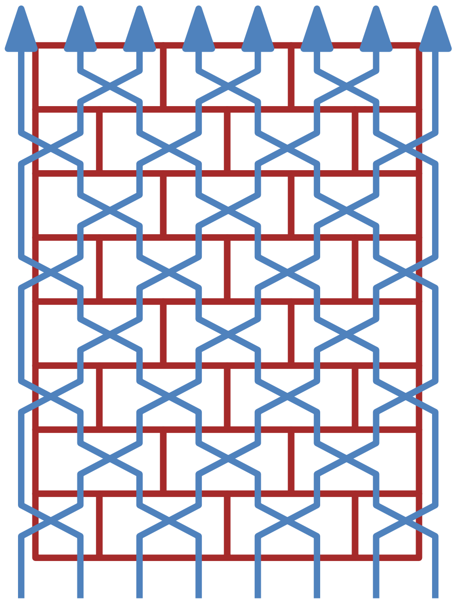

rhombic tilings

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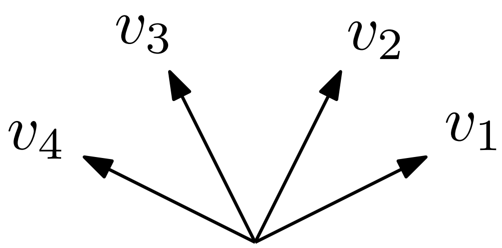
Def: $v_1, \cdots, v_n \in \mathbb{R}^2$ pw. lin. indep.



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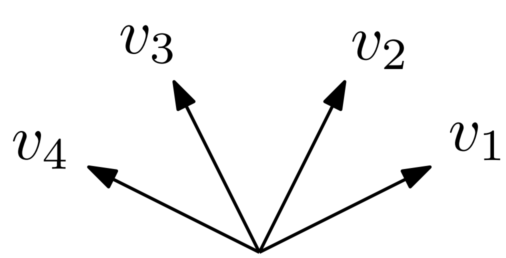
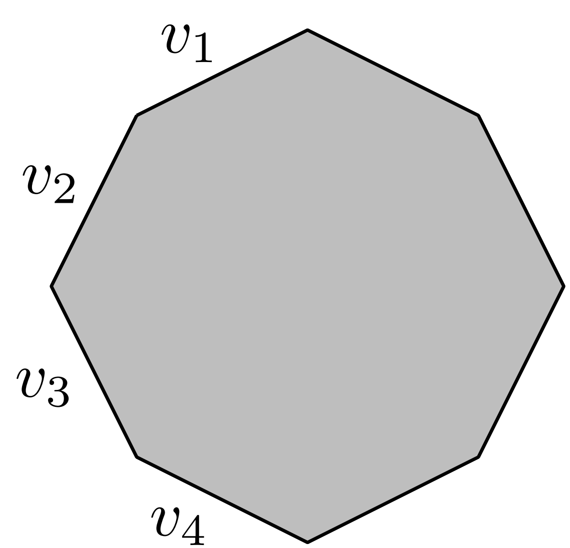
$$Z(v_1, \cdots, v_n) := \left\{ \sum_{i=1}^{n} \lambda_i v_i \ : \ \lambda_1, \cdots, \lambda_n \in [-1, 1] \right\}$$





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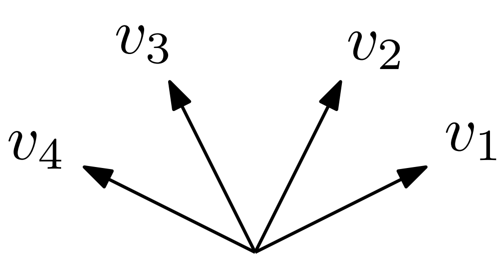

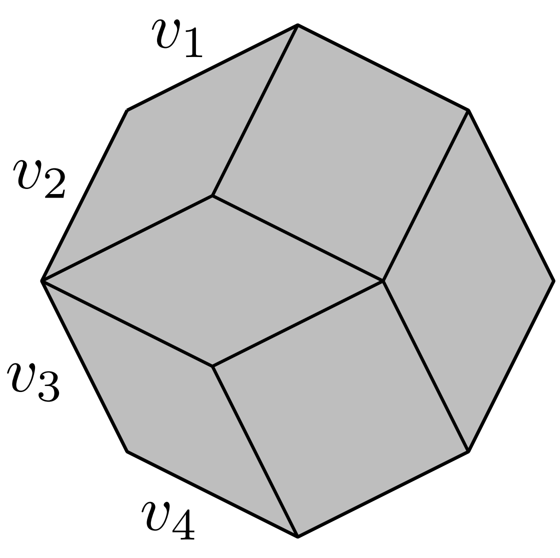

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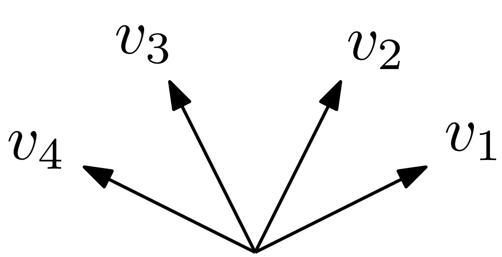

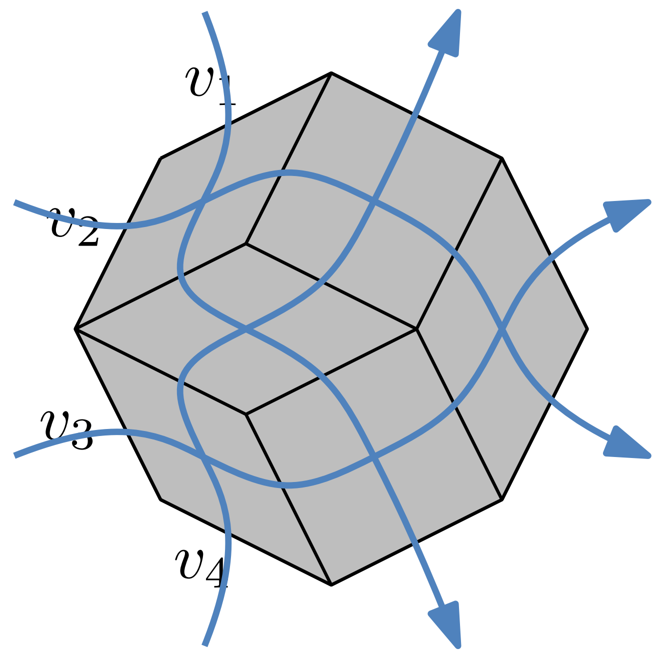

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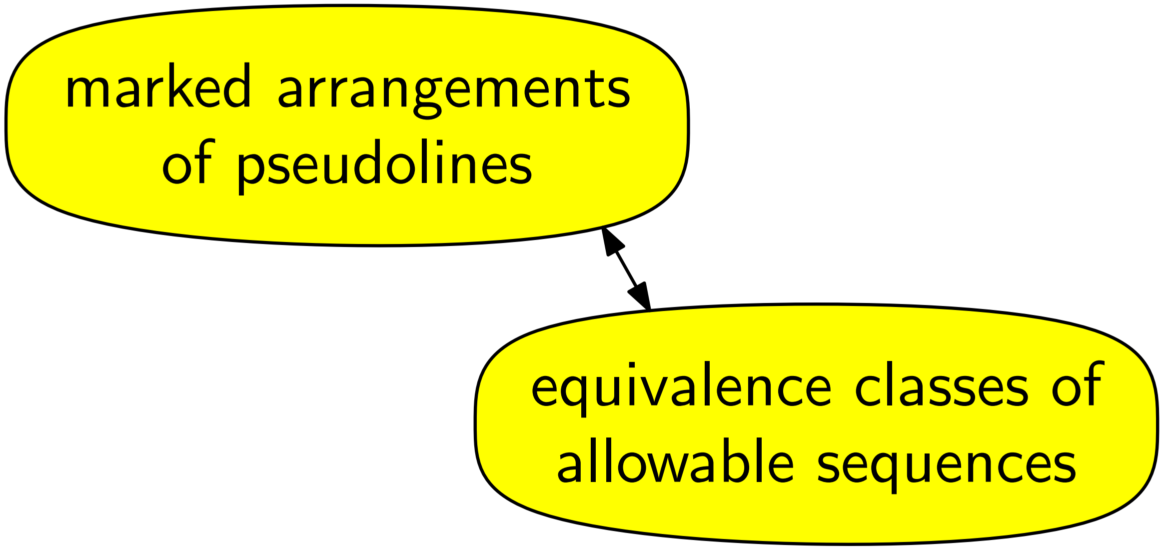

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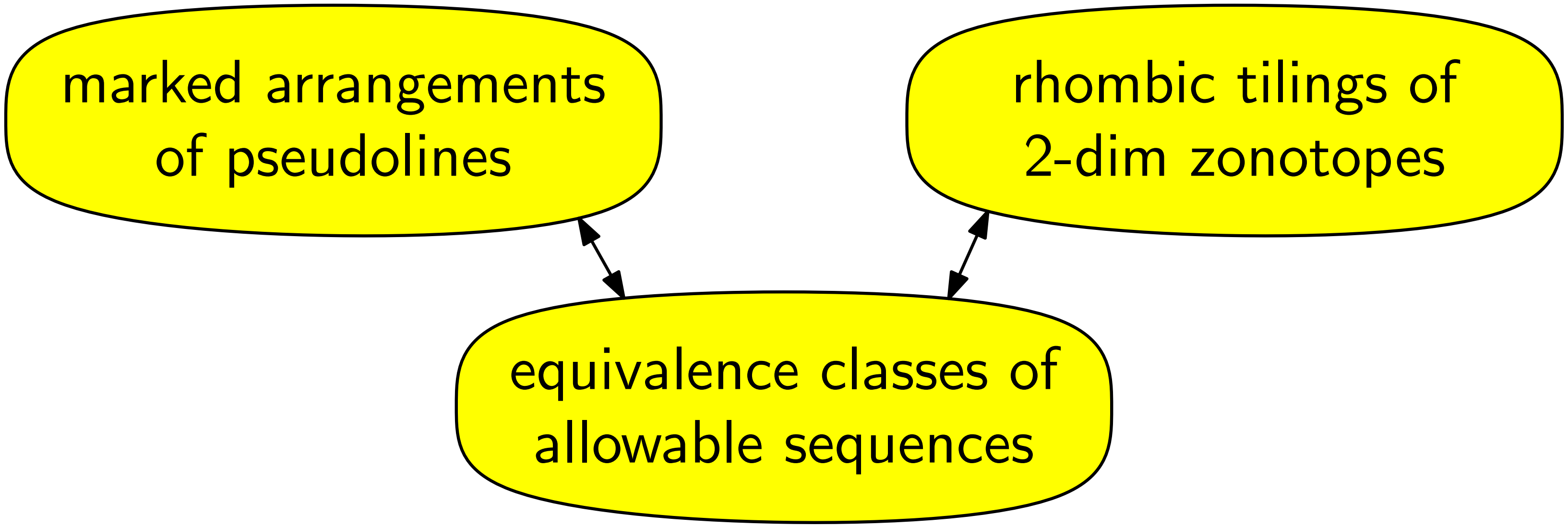

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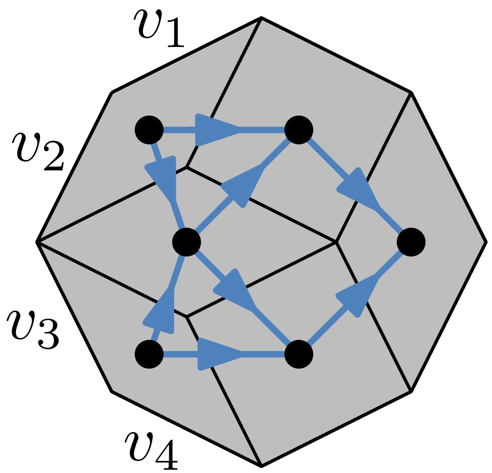

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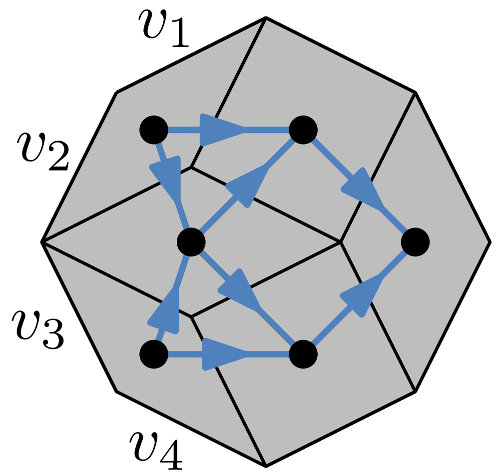

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Lemma (Guibas & Yao, 1980)

The tiling graph $G_{\mathscr{T}}$ is acyclic.

Proof: There is a topological sorting of $G_{\mathcal{T}}$.

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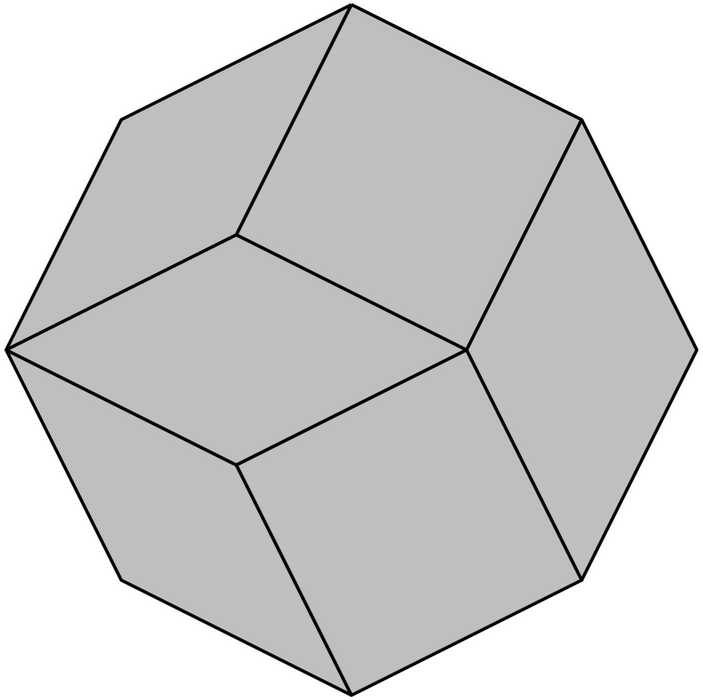

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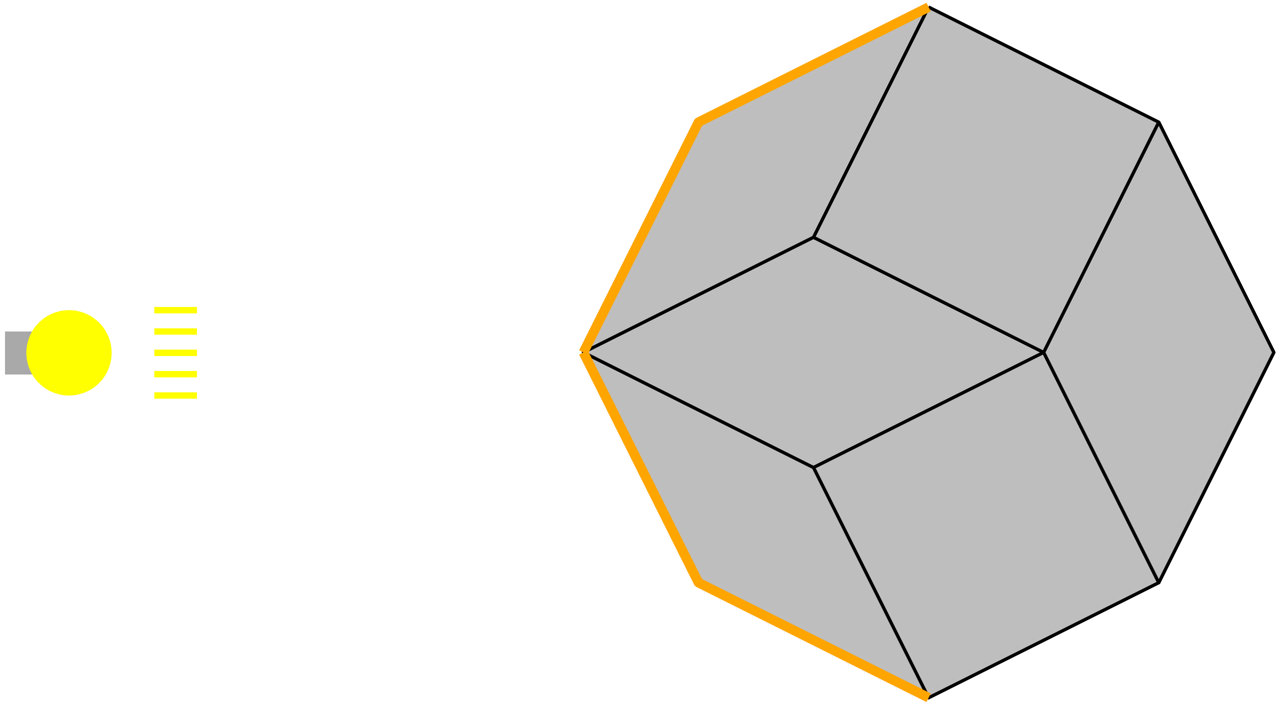

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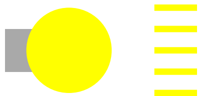

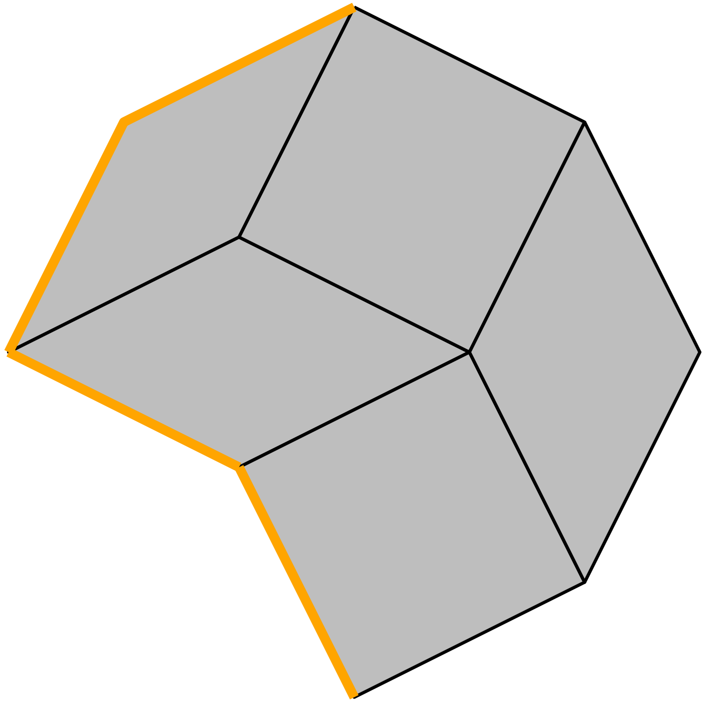

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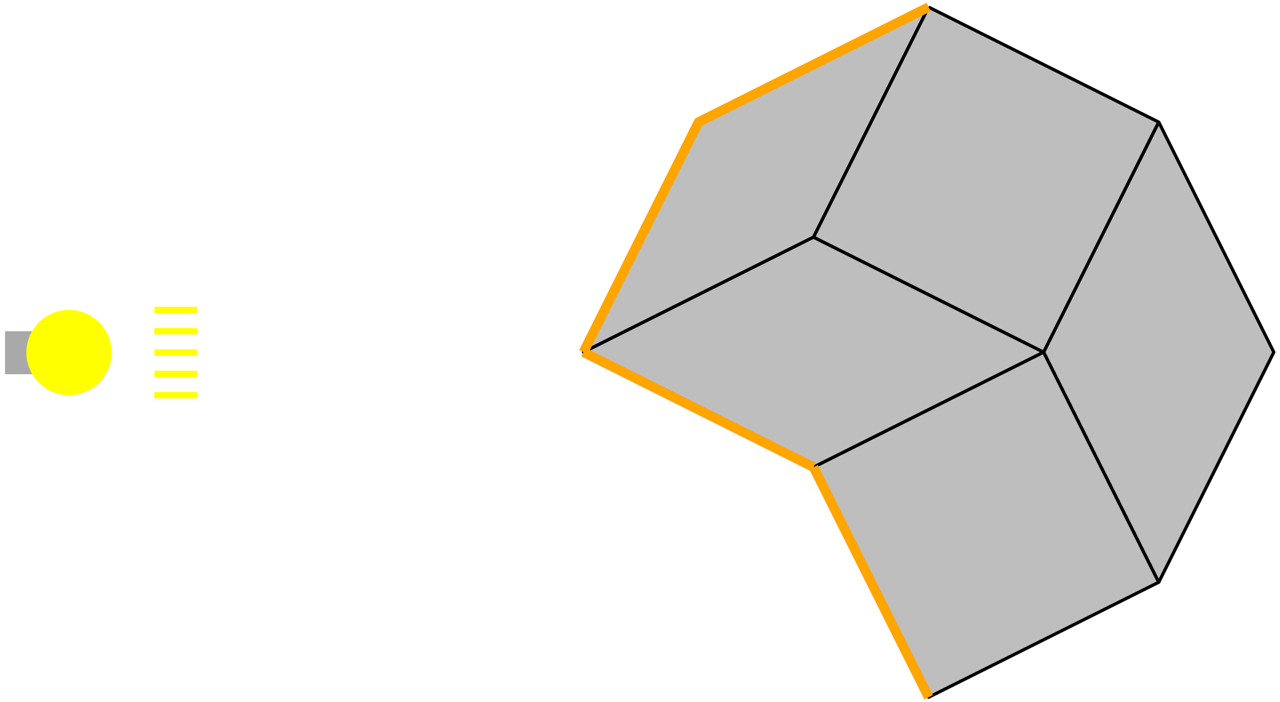

Claim: There exists a tile whose left side is completely lit.

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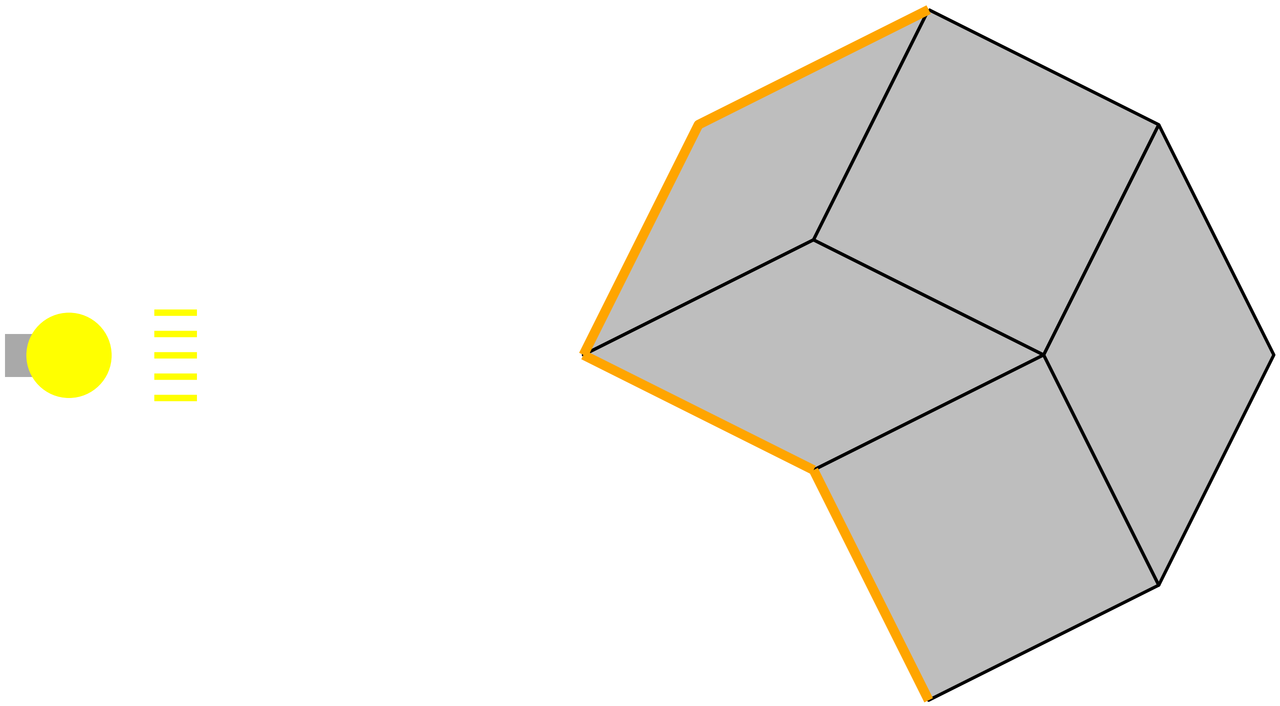

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$\Rightarrow$ Among the tiles with lit top point the one with lowest top point.

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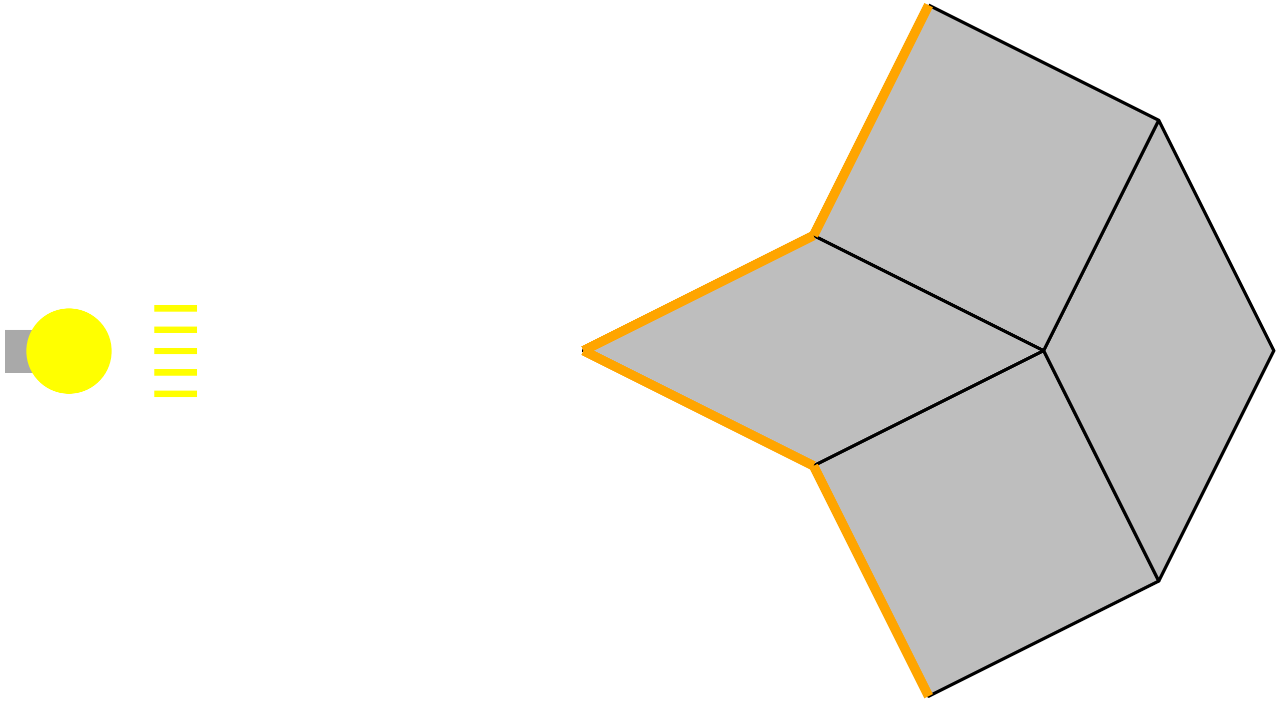

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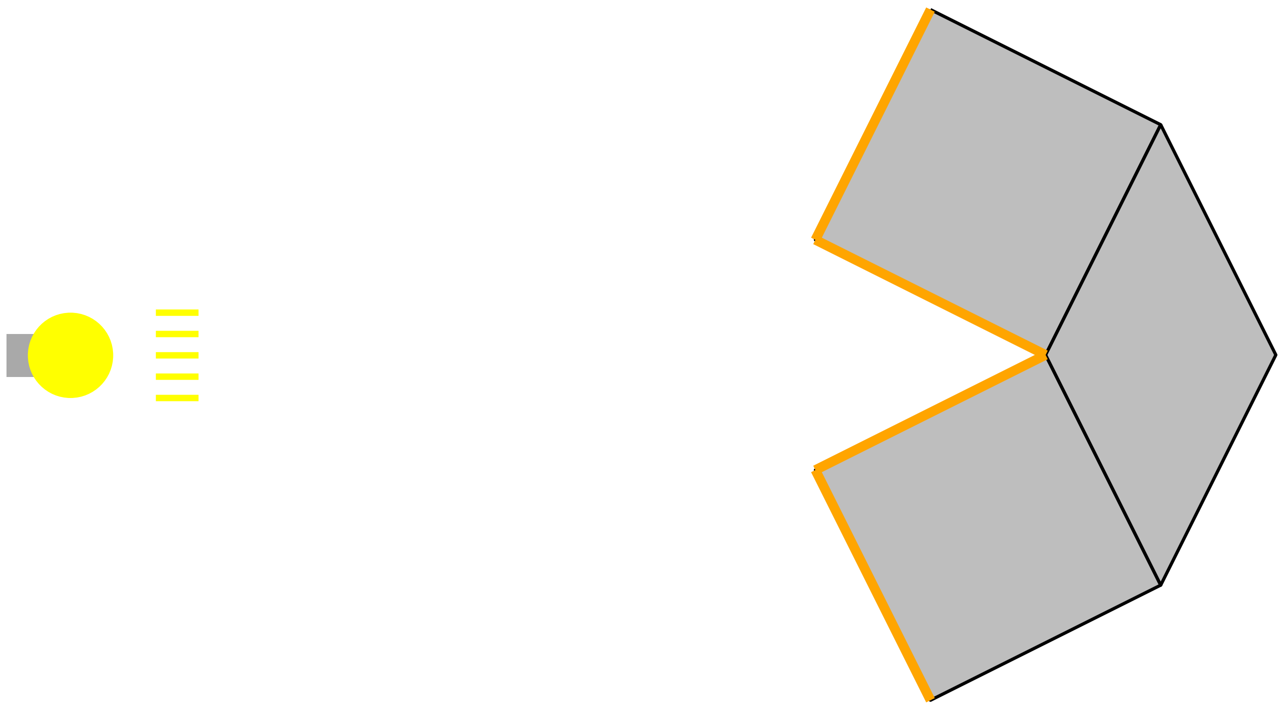

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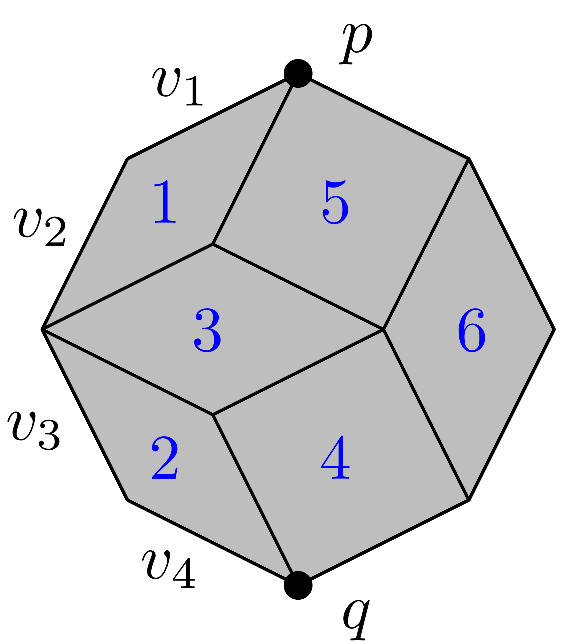

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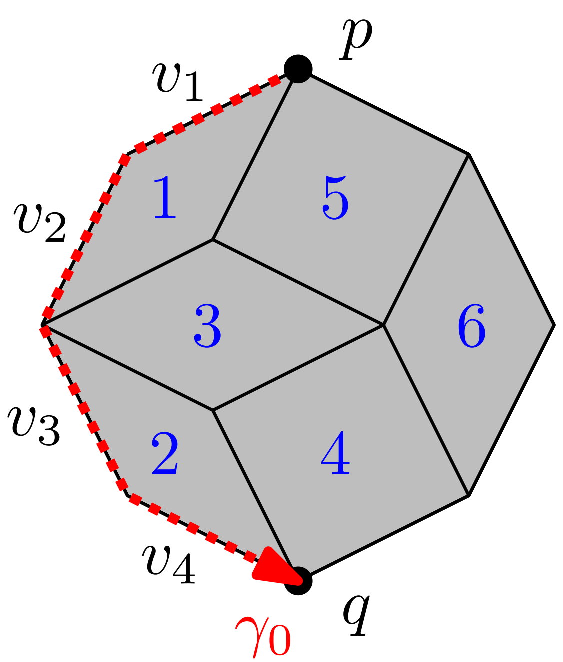

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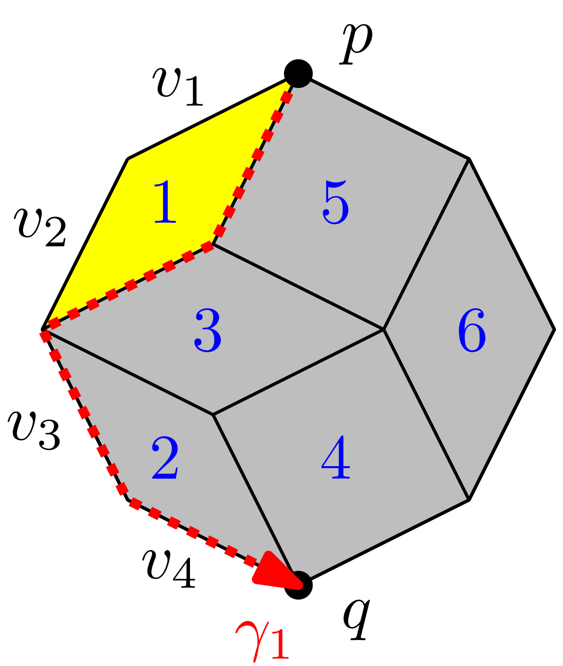

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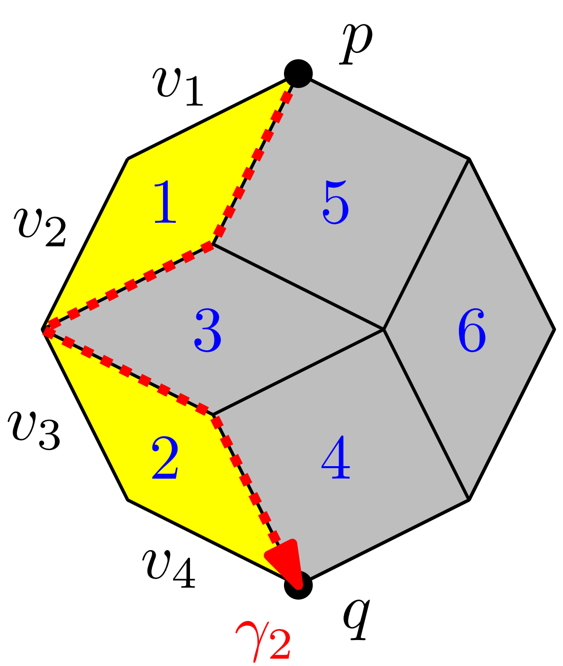

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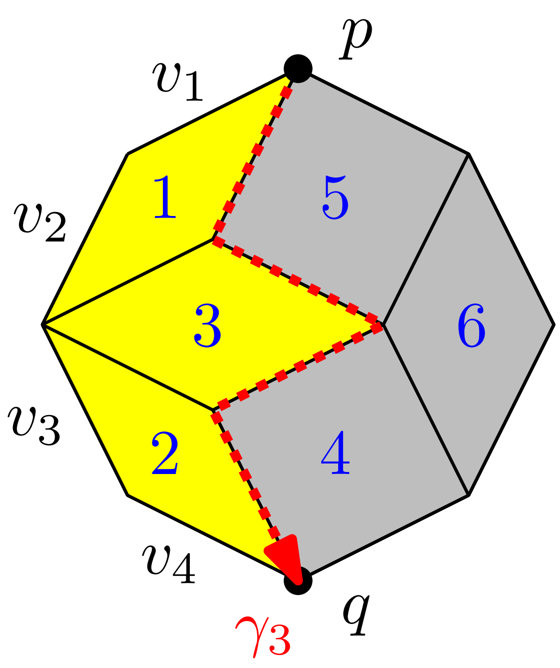

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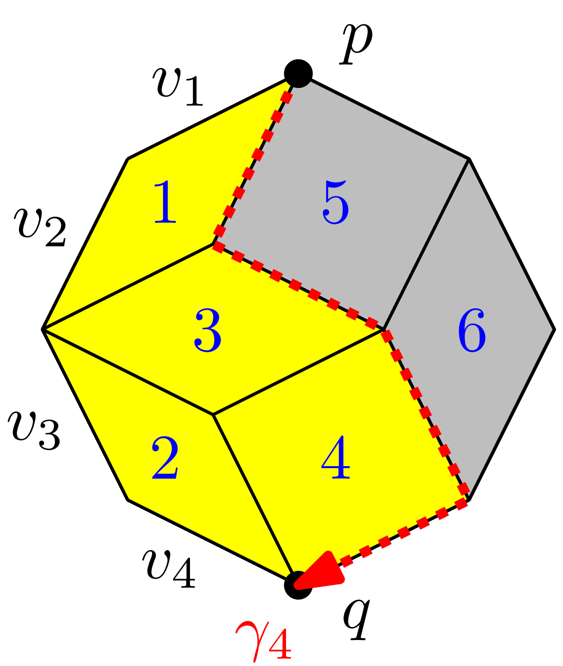

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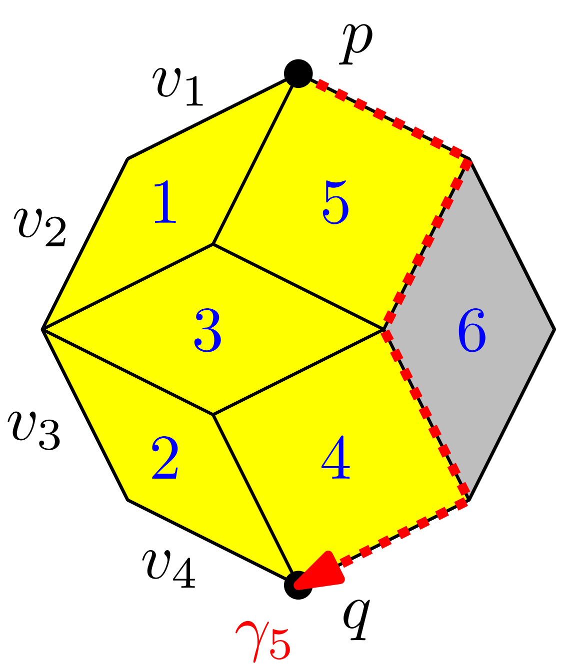
rhombic tilings

Theorem

There is a one-to-one correspondence between rhombic tilings and equivalence classes of allowable sequences.

Proof: Assoc. (tiling $\mathscr{T}$, top. sorting of $G_{\mathscr{T}}$) $\mapsto$ allowable sequence by doing a sweep in the order of a topological sorting of $G_{\mathscr{T}}$.



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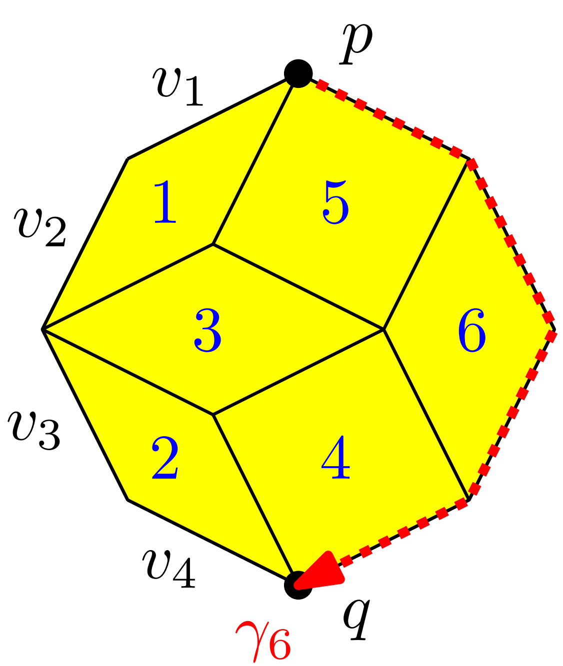

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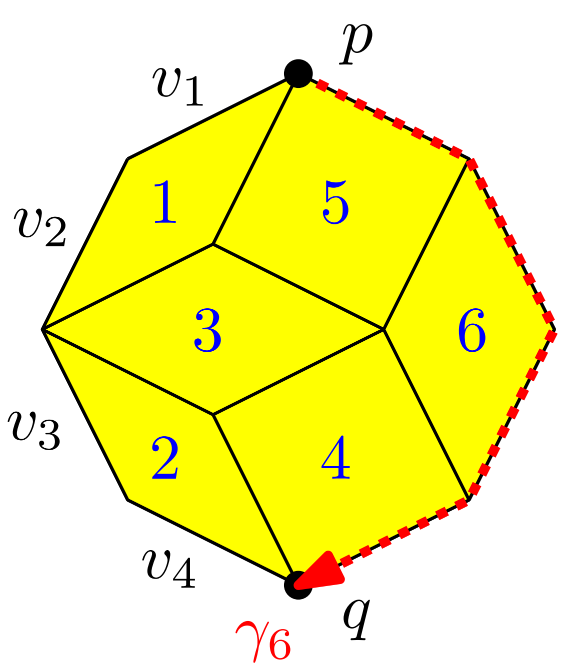

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valid allowable sequence???

Claim: In obtained permutation sequence, every pair $i \neq j$ is swapped exactly once ($\Rightarrow$ obtain valid allowable sequence).

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- At least once: Clear! Because get from $[1, \cdots, n]$ to $[n, \cdots, 1]$.
- Swap of pair $i \neq j$ happens on flip over rhombus $Z(v_i, v_j)$ with

$$\text{Vol}\,(Z(v_i, v_j)) = 4 \cdot |\det([v_i, v_j])|.$$

- These swaps exhaust entire volume, because

$$\sum_{i<j} 4 \cdot |\det([v_i, v_j])| = \text{Vol}\,(Z(v_1, \cdots, v_n))\,.$$

- Hence, there cannot have been further swaps.

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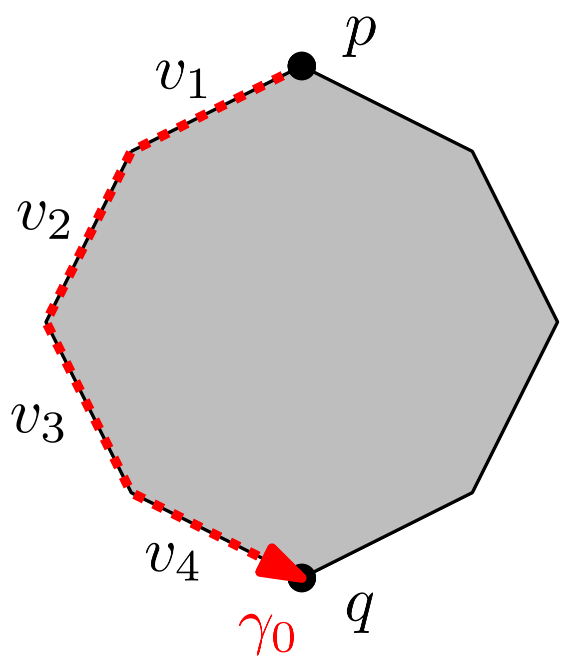

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Invariant:

- Current path γ_i contains vectors $v_1, \cdots, v_n$ in order π_i.
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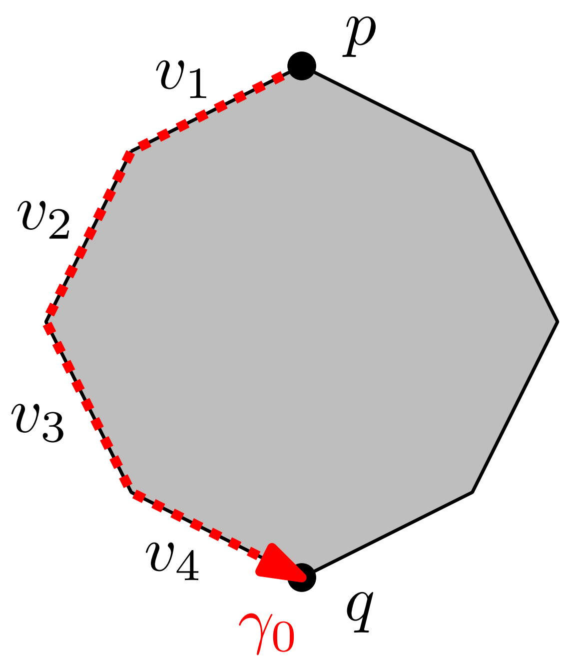

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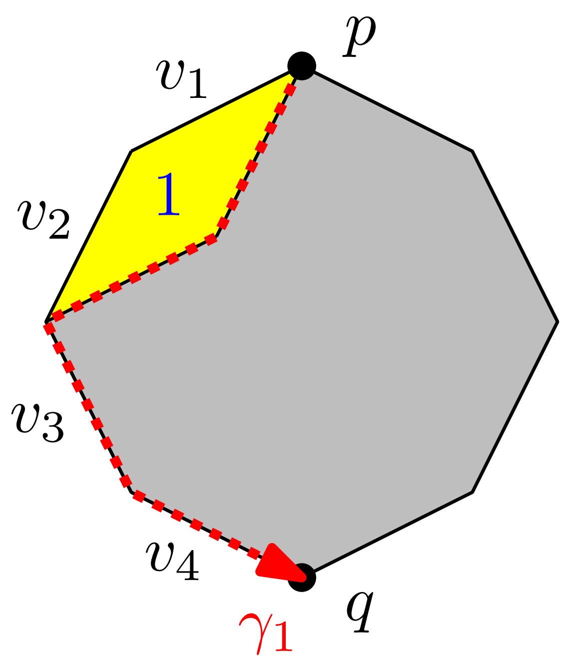

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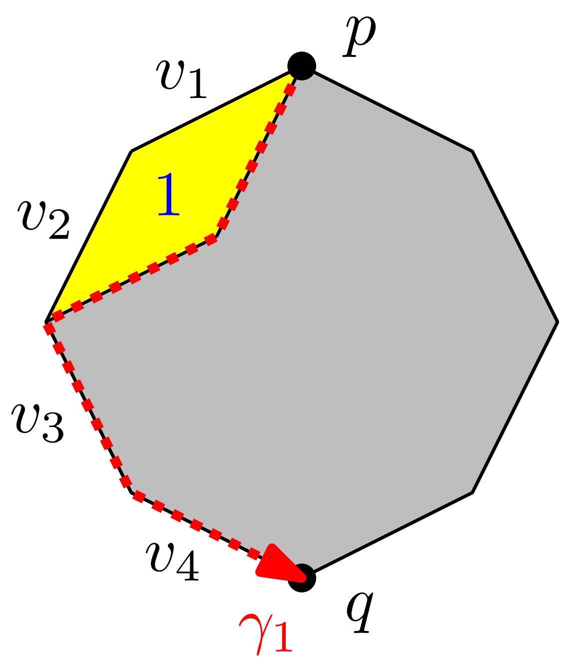

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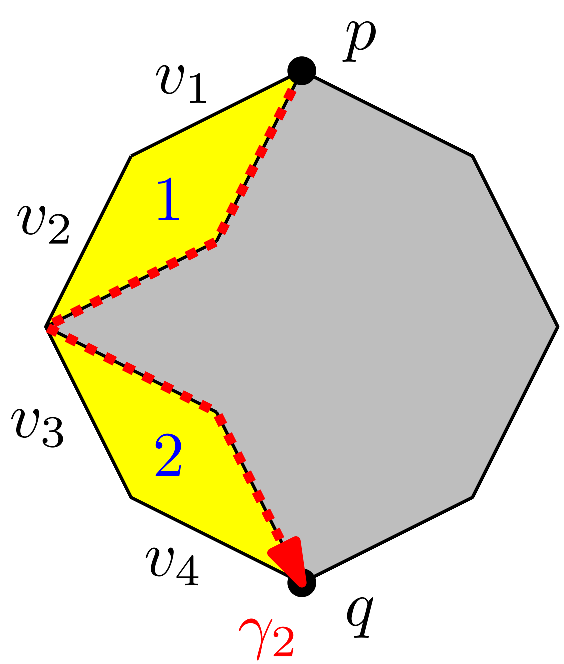

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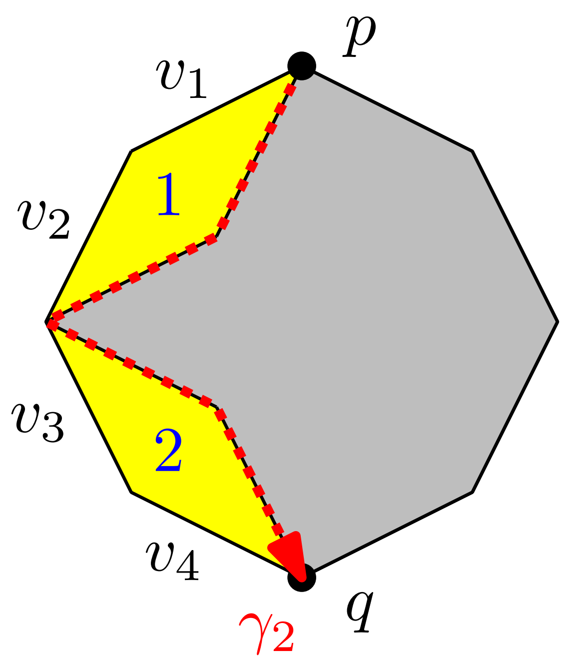

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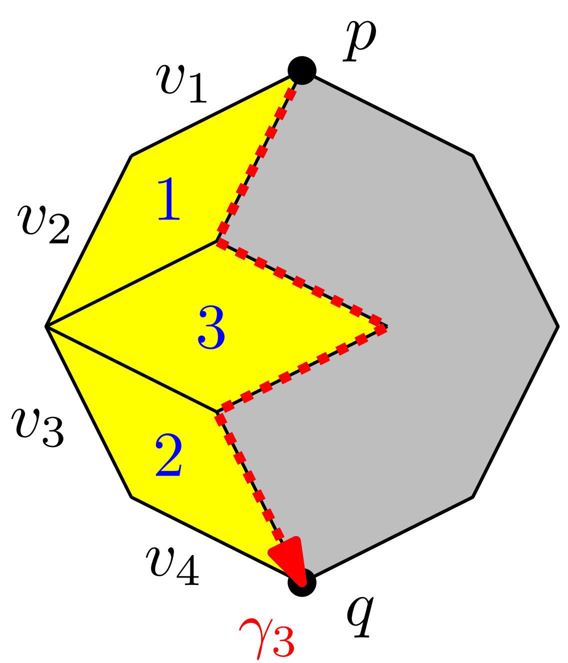

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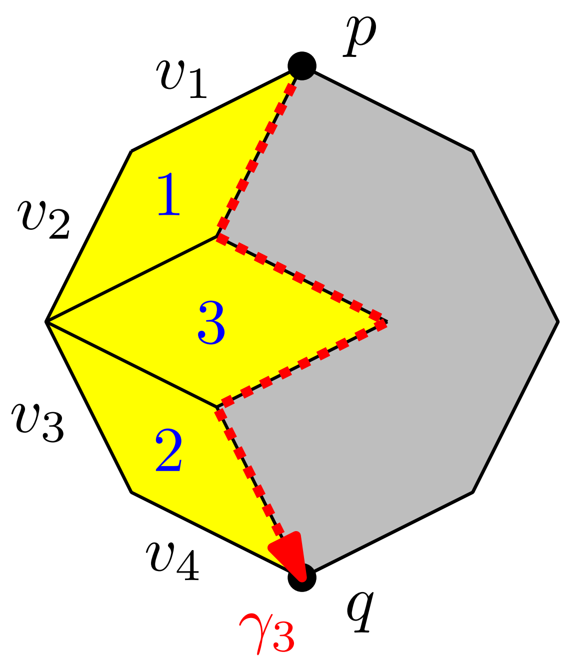

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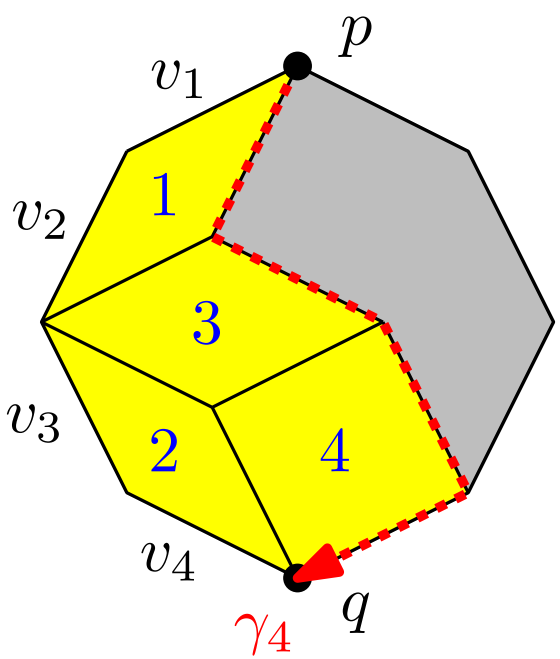

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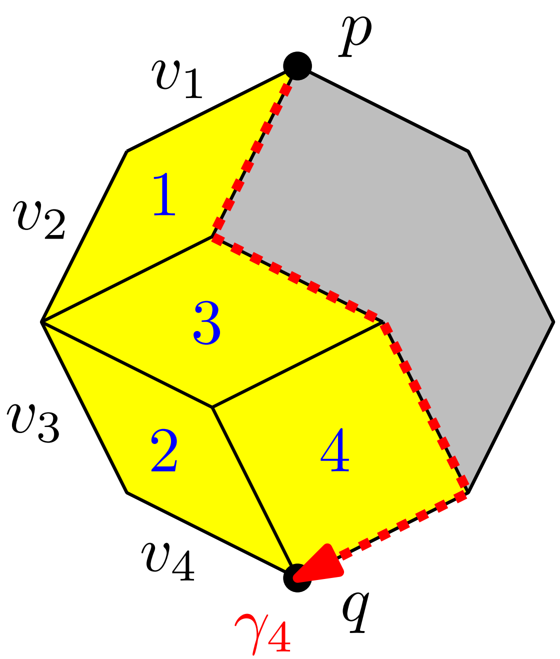

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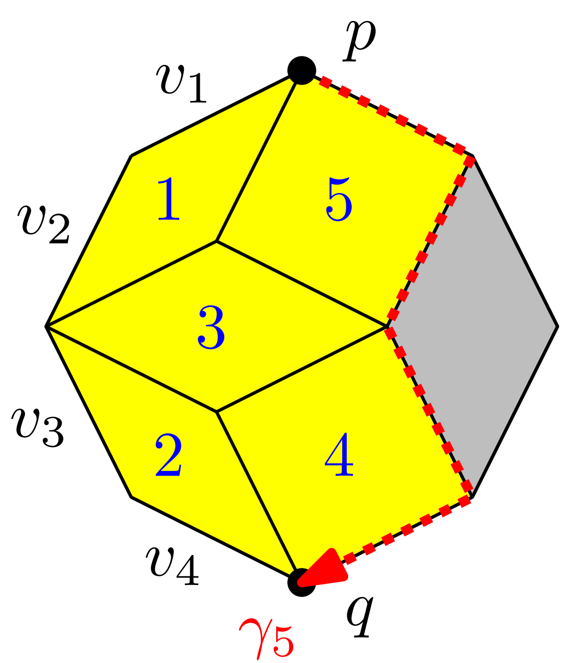

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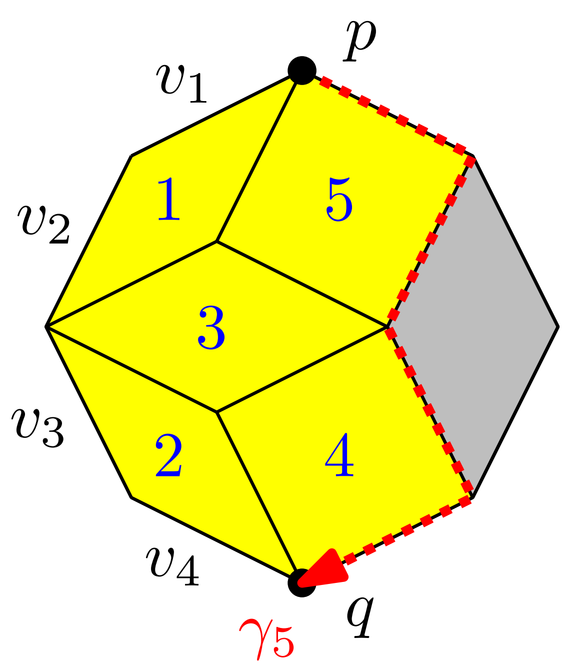

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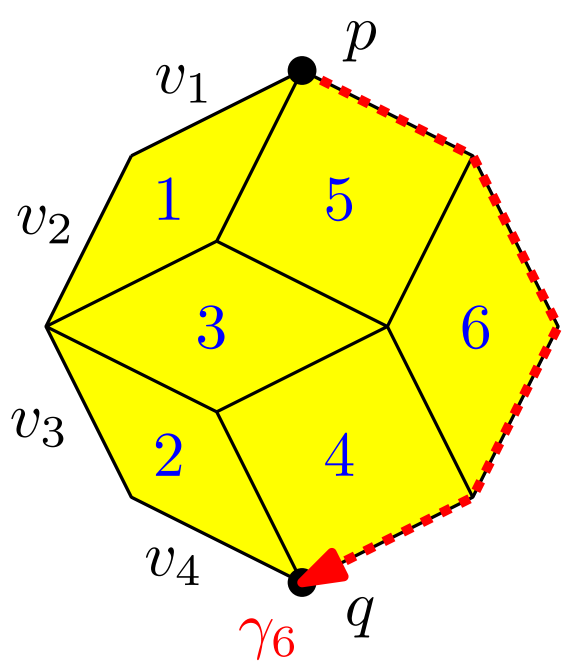

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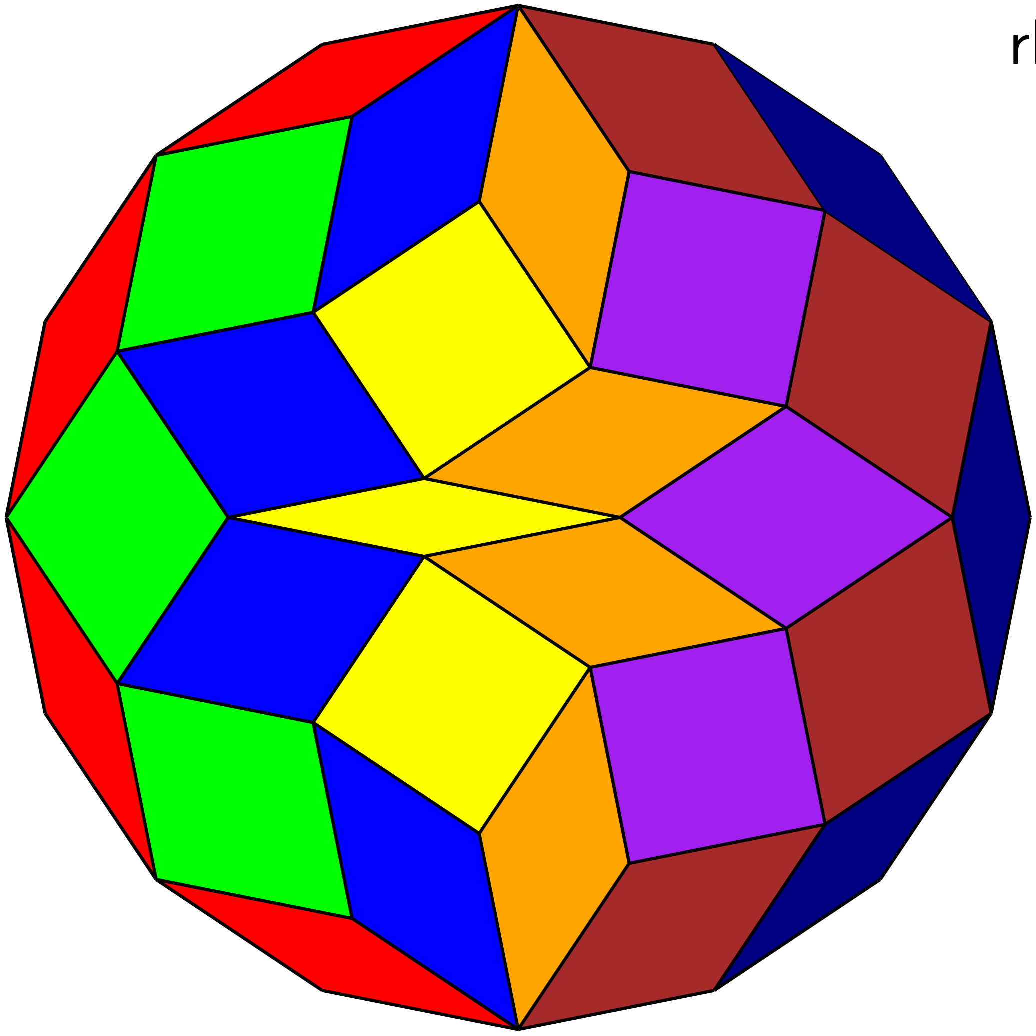
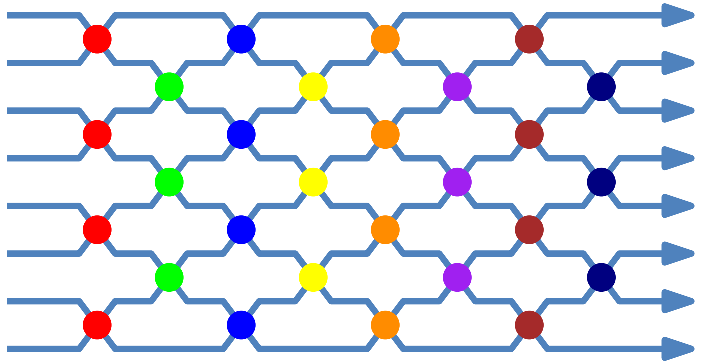
Claim: Under this construction, different top. sortings of $G_{\mathcal{T}}$ correspond exactly to equivalent allowable sequences.

$\square$

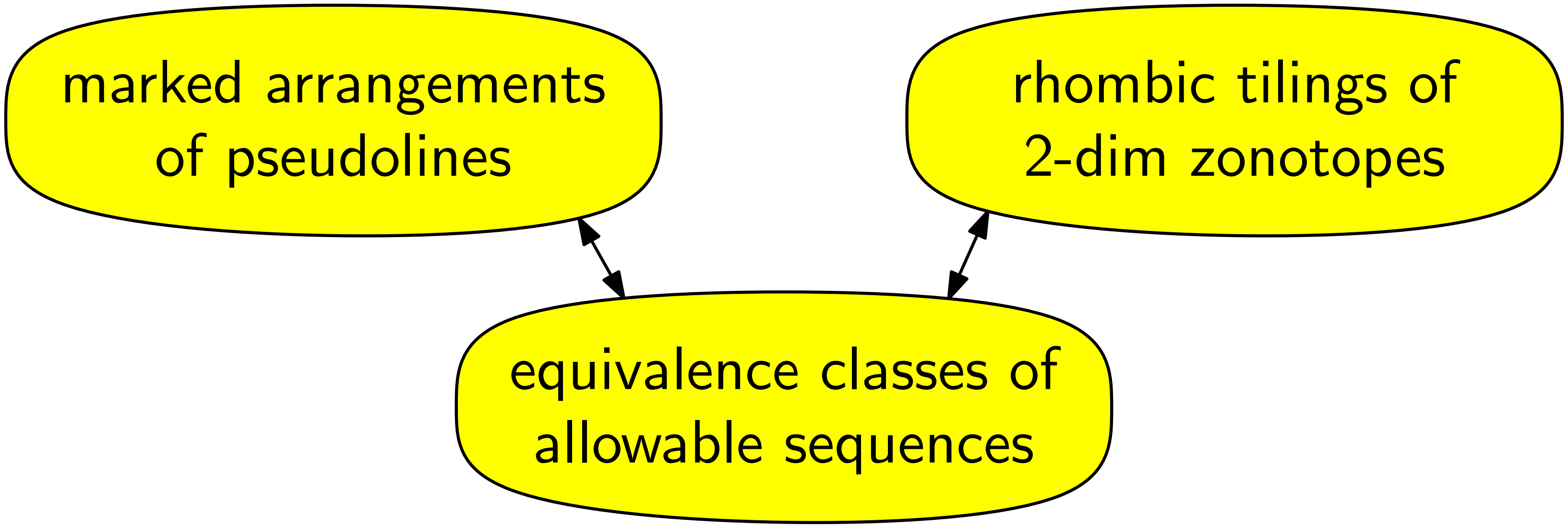
„brick wall conjecture“



rhombic tiling corresponding
to wall arrangement



standard Young tableaux



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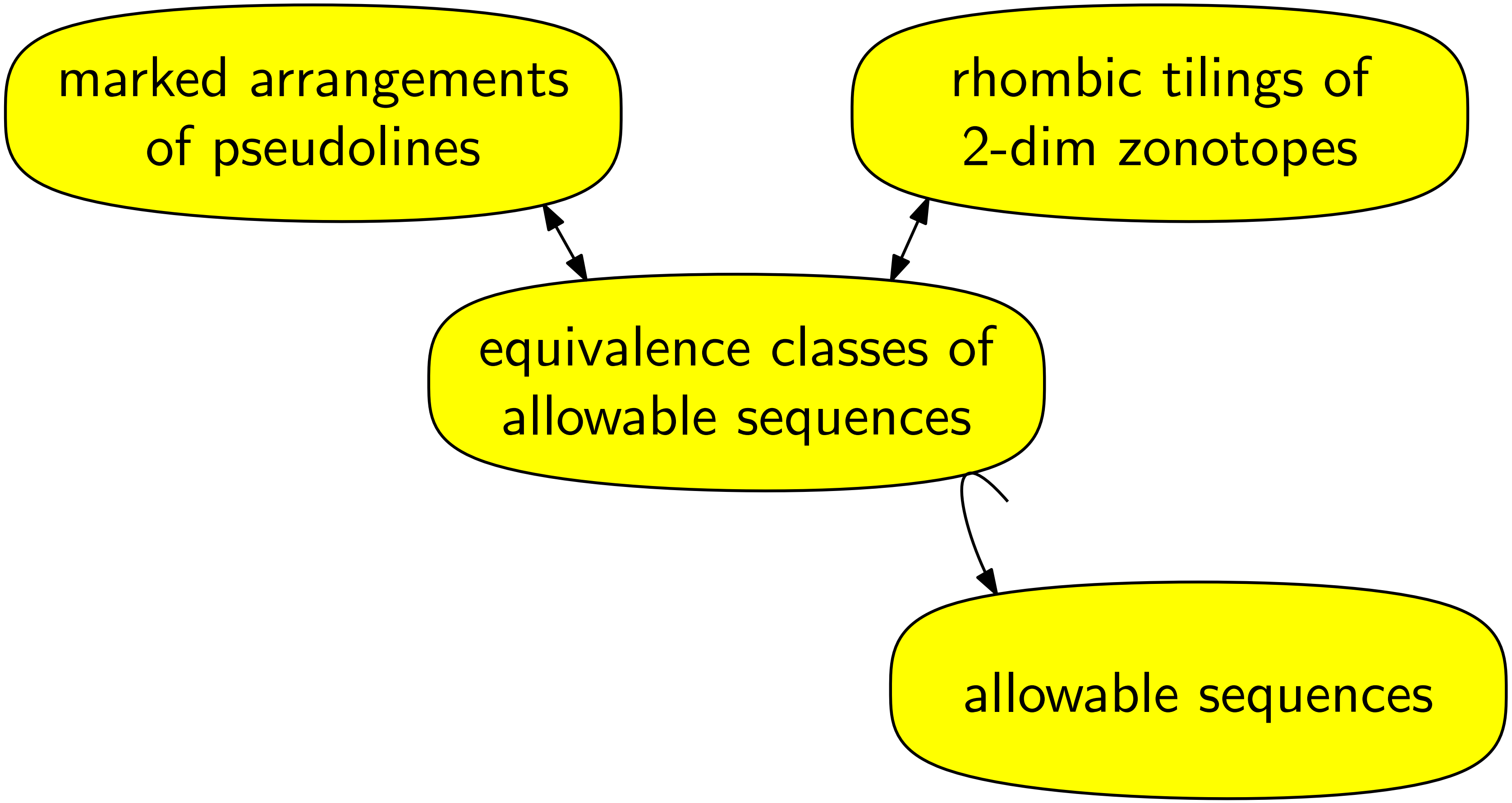

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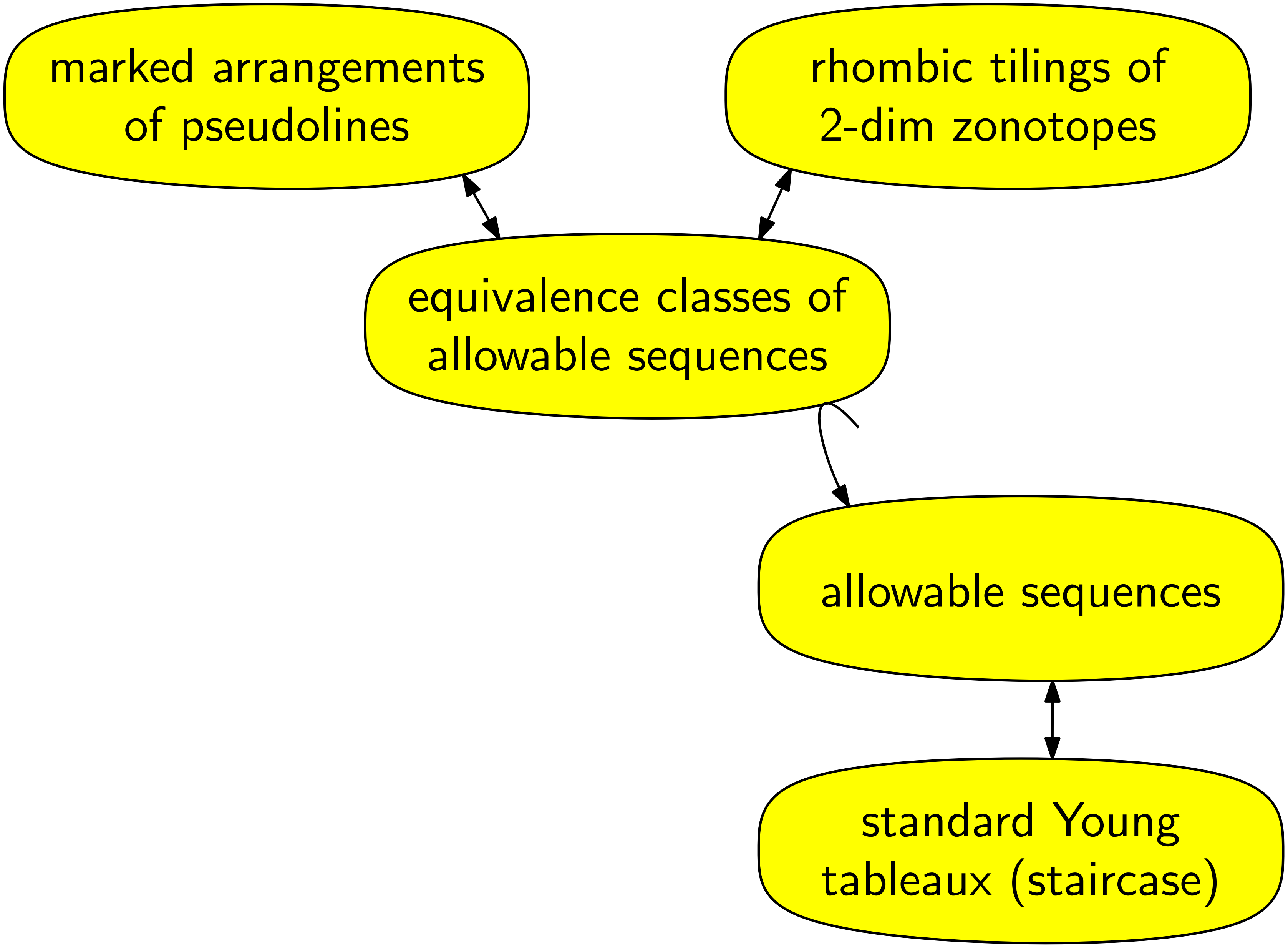

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Def: *Partition of* N: Integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, $\sum \lambda_i = N$.

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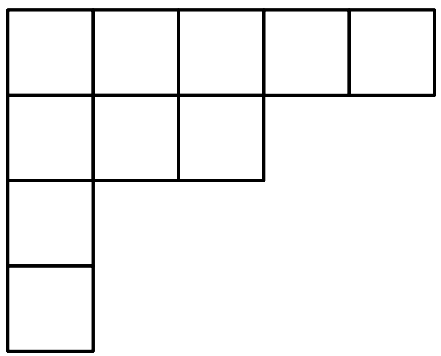
Ex: $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (4, 2, 2, 1, 1)$, $|\lambda| = 10$.



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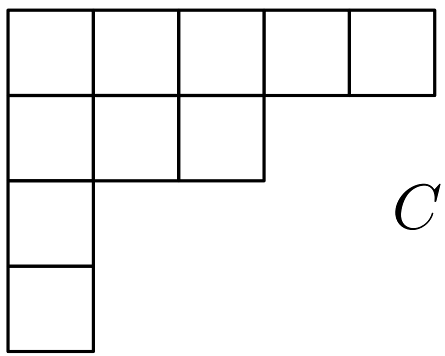

„Young diagram“

$$C(\lambda) = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (3,1), (4,1)\}$$

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1	2	3	5	9
4	6	8		
7				
10				

„Young diagram“

$$C(\lambda) = \{(1,1),(1,2),(1,3),(1,4),(1,5),\\ (2,1),(2,2),(2,3),(3,1),(4,1)\}$$

Def: *Standard Young tableau*: Assignment $C(\lambda) \to \{1, \cdots, |\lambda|\}$ of numbers to cells of Young diagram so that:

- Every number $1, \cdots, |\lambda|$ appears exactly once (bijective)
- Rows are monotonically increasing
- Columns are monotonically increasing

standard Young tableaux

Def: *Standard Young tableau of staircase shape*: Standard Young tableau for partition $\lambda = (n, n-1, \cdots, 1)$.

1	2	3	6
4	5	8	
7	10		
9			

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4	5	8	
7	10		
9			

Theorem (Edelman & Greene, 1987)

There is a bijection between allowable sequences of size n and standard Young tableaux of staircase shape $(n-1, n-2, \cdots, 1)$.

standard Young tableaux

Schensted insertion

Input: Original tableau $T : C(\lambda) \to \mathbb{N}$; insertion number $u \in \mathbb{N}$
Output: Enlarged tableau $T : C(\lambda') \to \mathbb{N}$ with $|\lambda'| = |\lambda| + 1$
Convention: For $(i,j) \notin C(\lambda)$ say $T(i,j) = \infty$

initialize: $i \leftarrow 1$; $q \leftarrow u$
while $q \neq \infty$
  $j_0 \leftarrow \min\{j \in \mathbb{N} \ : \ T(i,j) \geq q\}$
  if $T(i,j_0) = q$ **then** $q \leftarrow q+1$
  if $T(i,j_0) > q$ **then** $q' \leftarrow T(i,j_0)$; $T(i,j_0) \leftarrow q$; $q \leftarrow q'$
  $i \leftarrow i+1$
end

standard Young tableaux

Edelman-Greene bijection

Input: Allowable sequence $(s_1, \cdots, s_{\binom{n}{2}})$

Output: Standard Young tableau T of shape $(n-1, \cdots, 1)$

initialize Tableau $T \leftarrow \emptyset$; Tableau $R \leftarrow \emptyset$ (empty tableaux)

for $k = 1, \cdots, \binom{n}{2}$

$T' \leftarrow \mathrm{SchenstedInsertion}(T, s_k)$

Let (i, j) be the index of the new cell in $C(T') \setminus C(T)$.

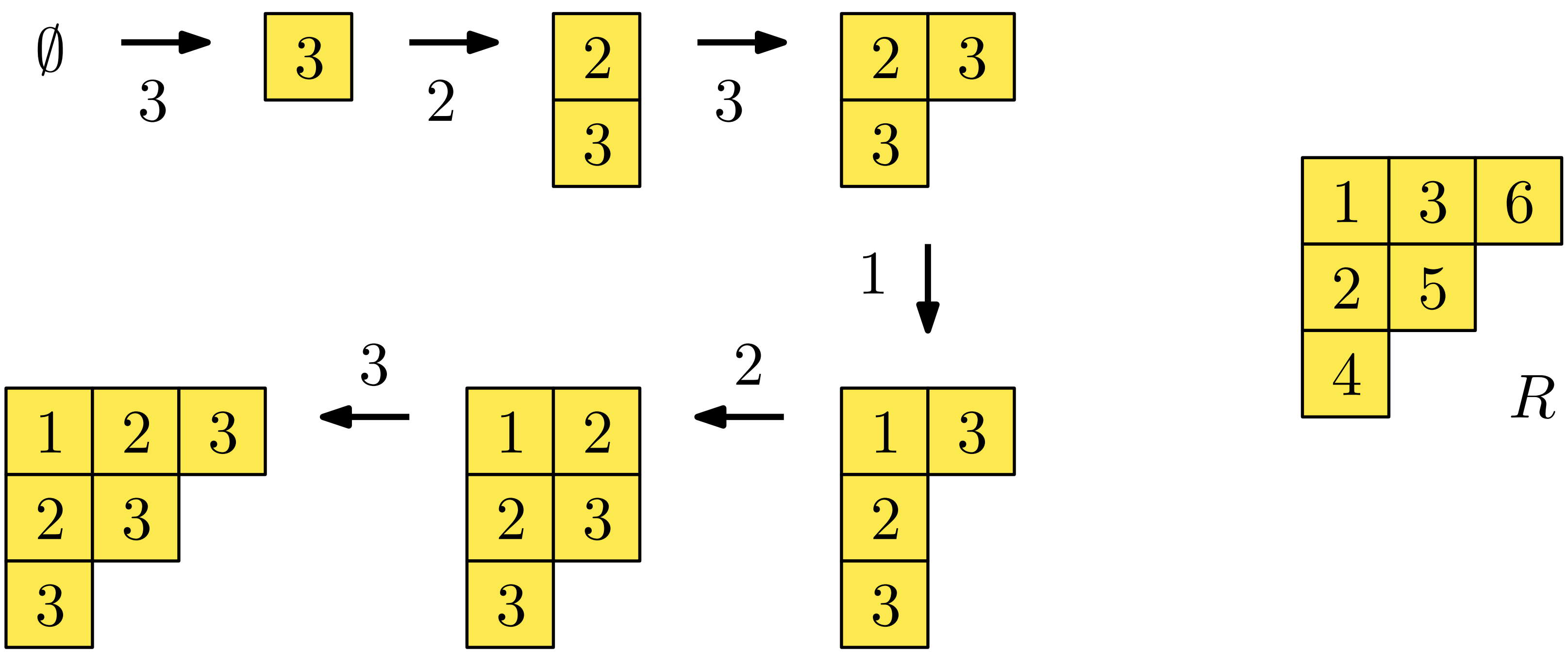
$T \leftarrow T'$

Add cell (i, j) with entry k to R.

output R

standard Young tableaux

Example: Edelman-Greene bijection applied on $(3, 2, 3, 1, 2, 3)$.



standard Young tableaux

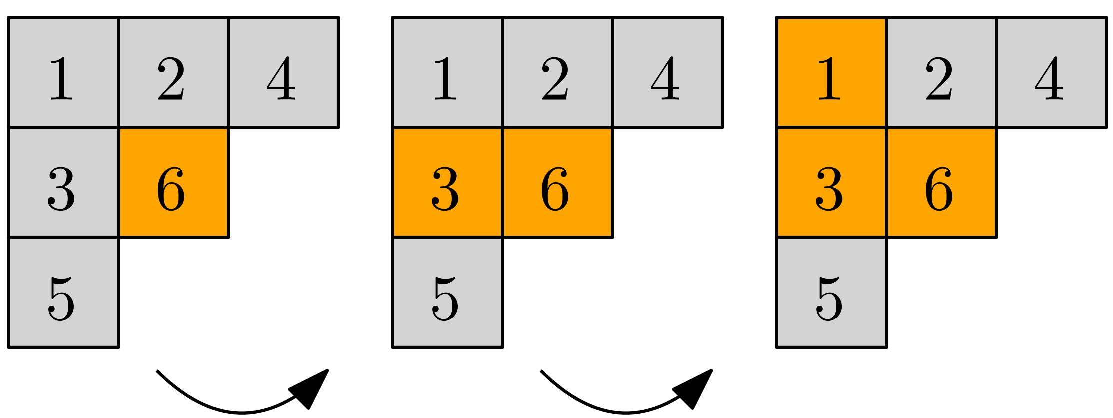
Schützenberger operator: Transforms standard Young tableau into new standard Young tableau of same shape.

standard Young tableaux

Schützenberger operator: Transforms standard Young tableau into new standard Young tableau of same shape.

Step I: Construct tableau path

- Start with cell that has largest entry
- Continue with top or left neighbor cell that has the larger entry
- Will end in cell $(1, 1)$ with entry 1.

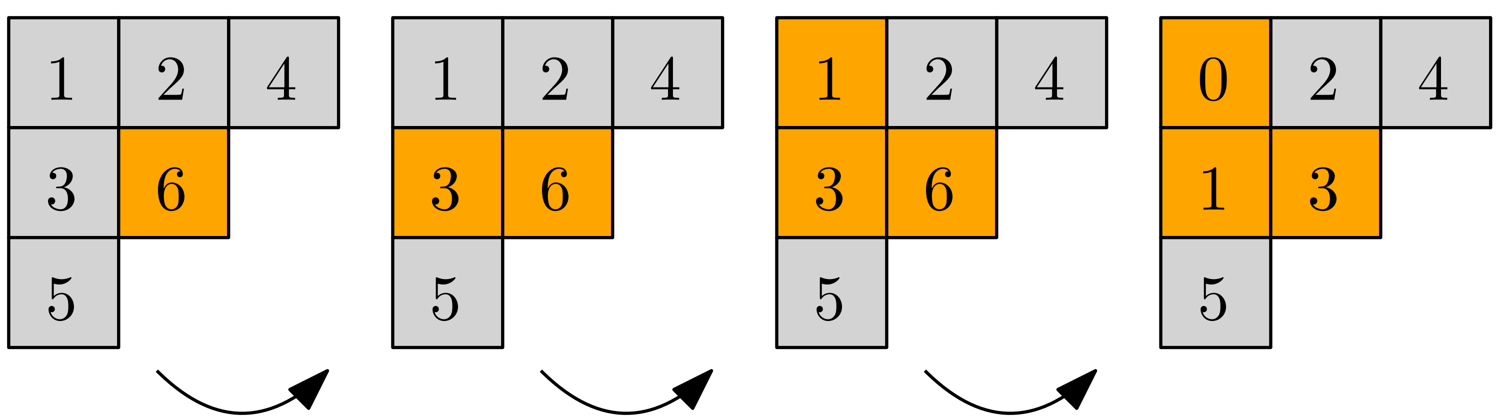


standard Young tableaux

Schützenberger operator: Transforms standard Young tableau into new standard Young tableau of same shape.

Step II: Shift path

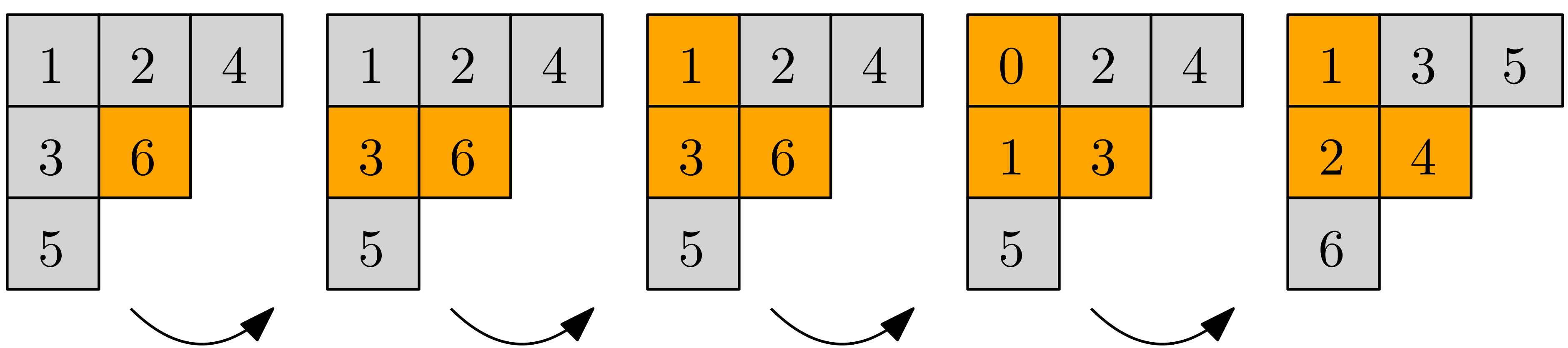
- Along path, move any entry one position further towards bottom or right
- At cell $(1,1)$ insert 0; on the other end drop out highest entry



standard Young tableaux

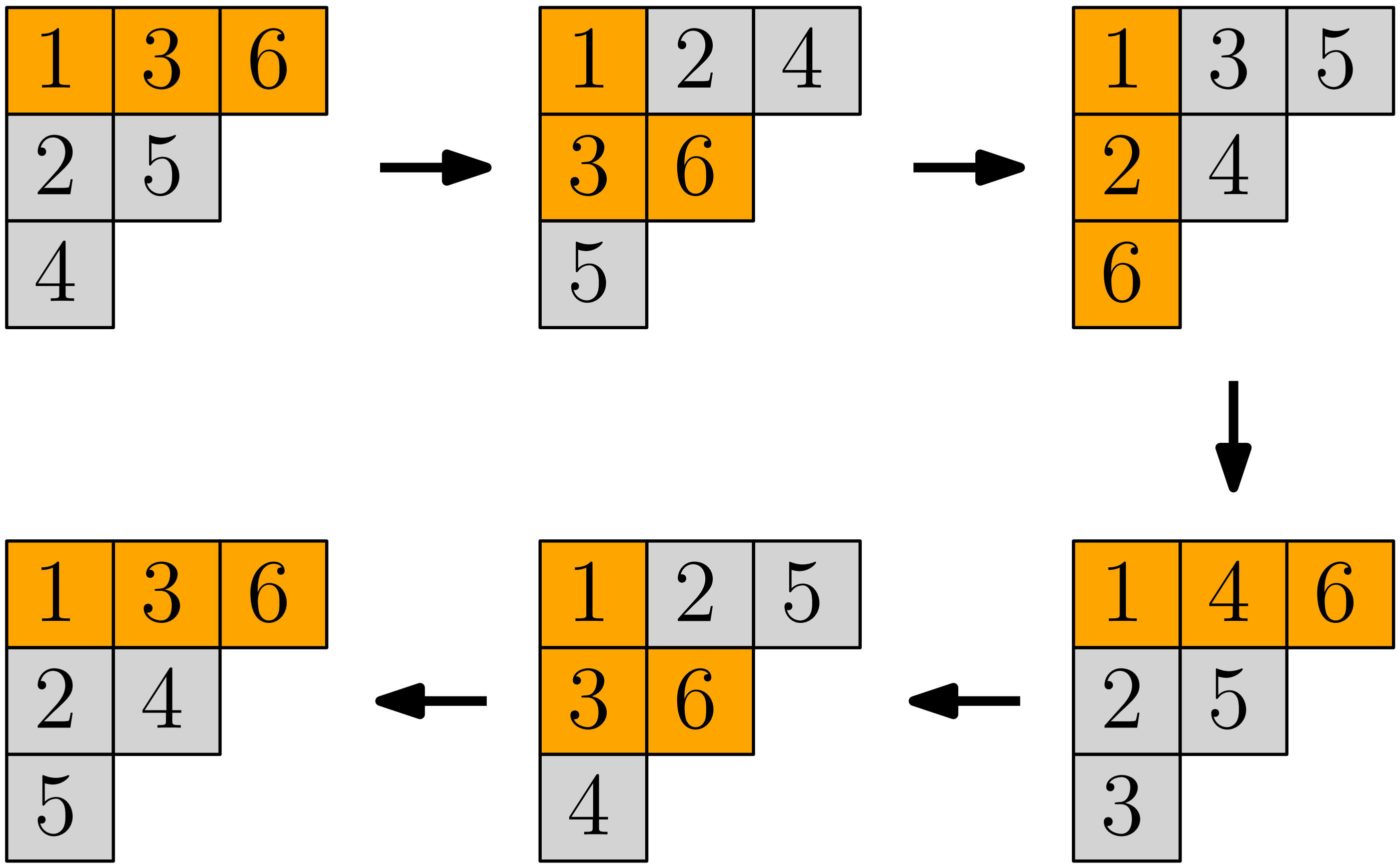
Schützenberger operator: Transforms standard Young tableau into new standard Young tableau of same shape.

Step III: Add 1 to all entries.



standard Young tableaux

Applying the Schützenberg operater $\binom{n}{2}$ times:



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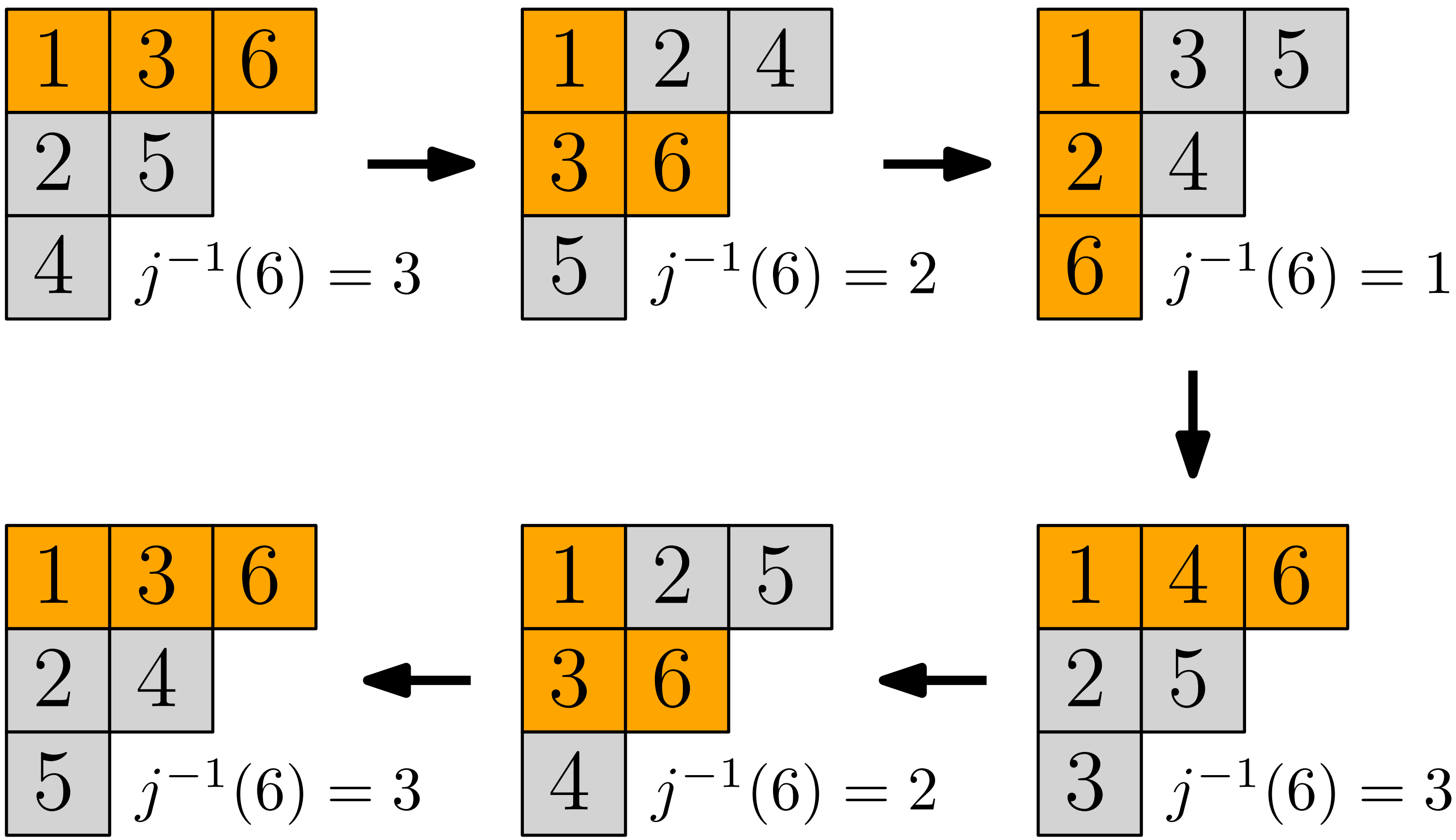
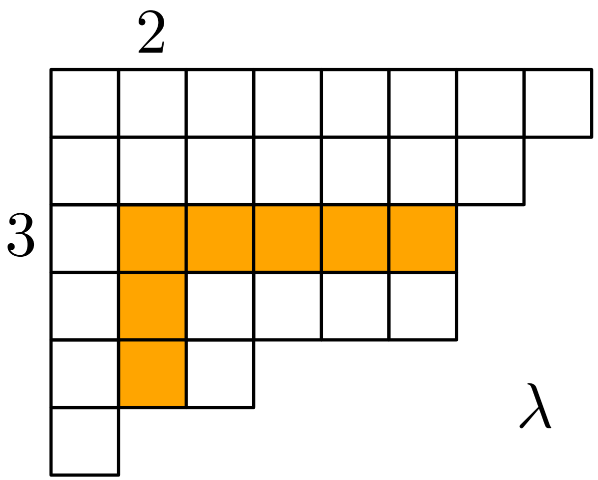


Observe: Recording the j-coordinate of largest entry gives back allowable sequence (reversed order): $(3, 2, 3, 1, 2, 3)$

standard Young tableaux

hook lengths in a Young diagram:



$$h_{3,2} = 7$$

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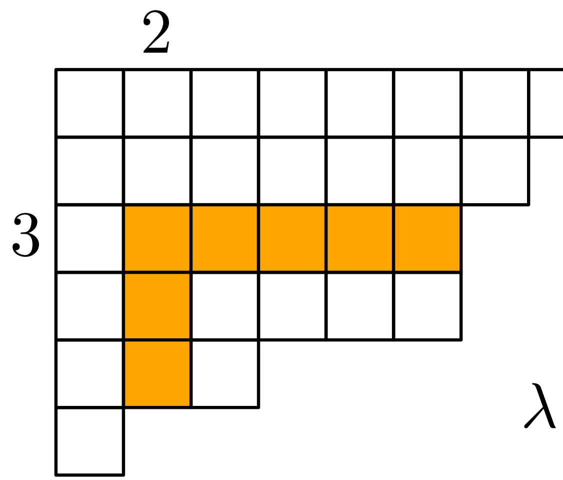

$$h_{3,2} = 7$$

hook length formula (Frame, Robinson, Thrall, 1953)

The number of standard Young tableaux of shape λ is given by

$$\frac{|\lambda|!}{\prod_{(i,j)\in C(\lambda)} h_{i,j}}.$$

standard Young tableaux

Corollary

The number of allowable sequences of size n is given by

$$\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2n-1-2i)^i}\ .$$

standard Young tableaux

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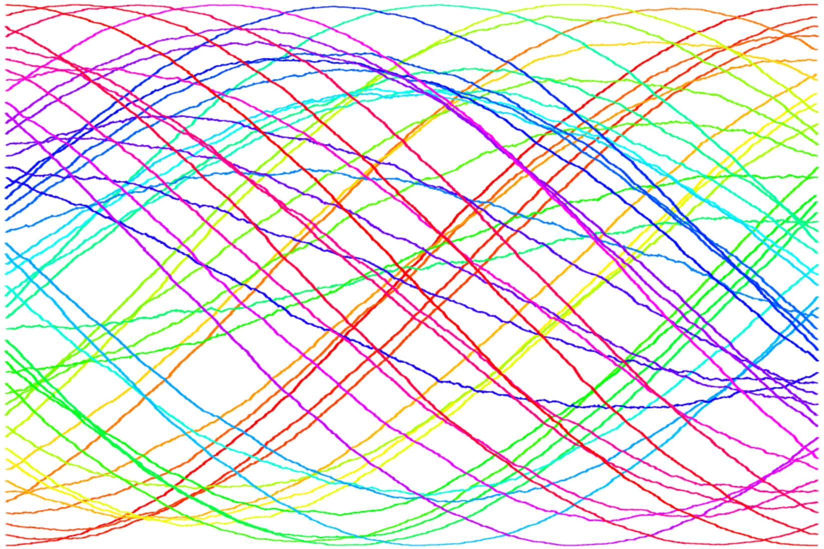
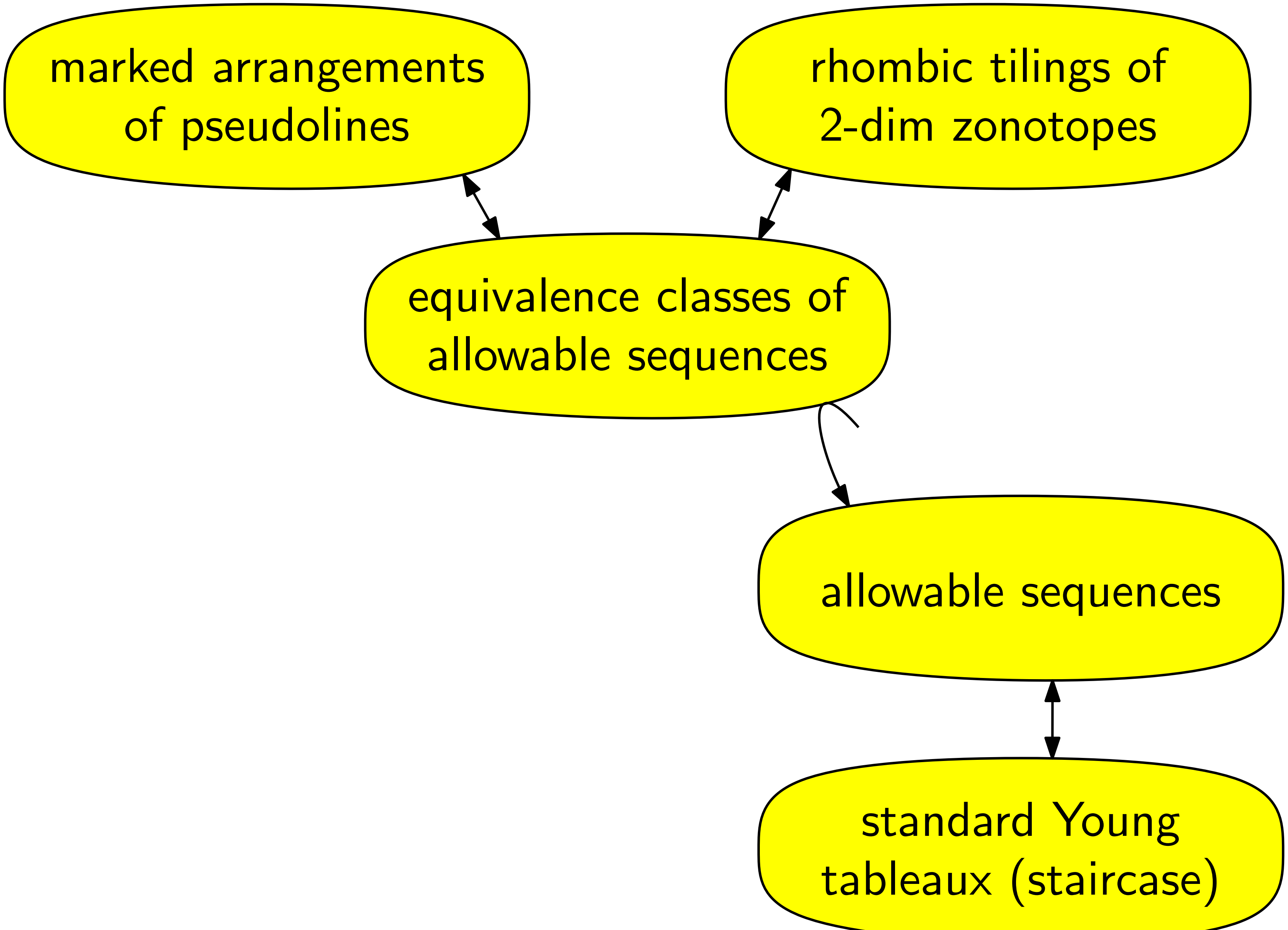
The number of allowable sequences of size n is given by

$$\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2n-1-2i)^{i}} .$$

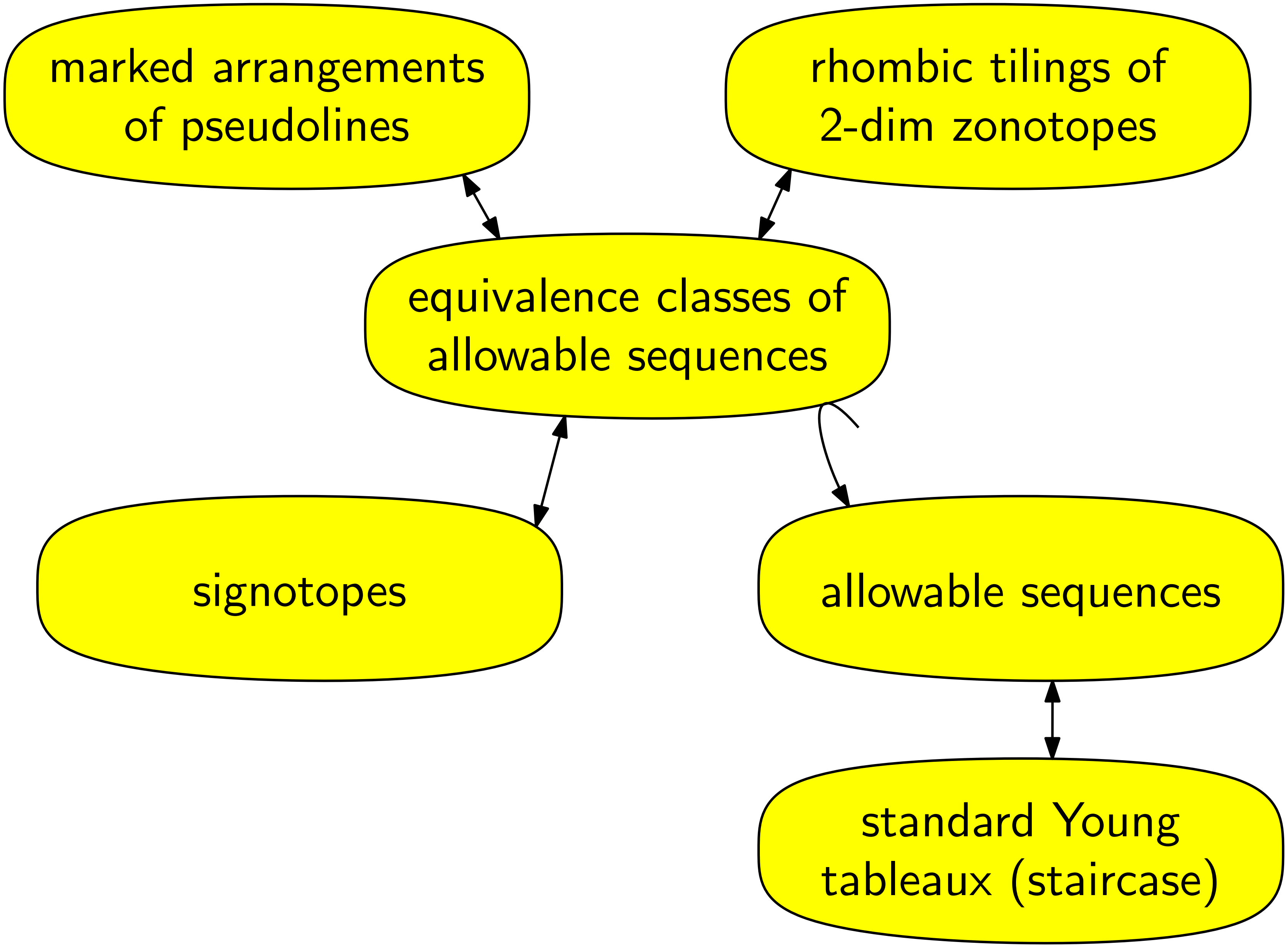


Uniformly sampled wiring diagram.
Taken from (Angel, Holroys, Romik, Virág, 2007)

signotopes


marked arrangements of pseudolines
rhombic tilings of 2-dim zonotopes
equivalence classes of allowable sequences
allowable sequences
standard Young tableaux (staircase)

signotopes


marked arrangements of pseudolines
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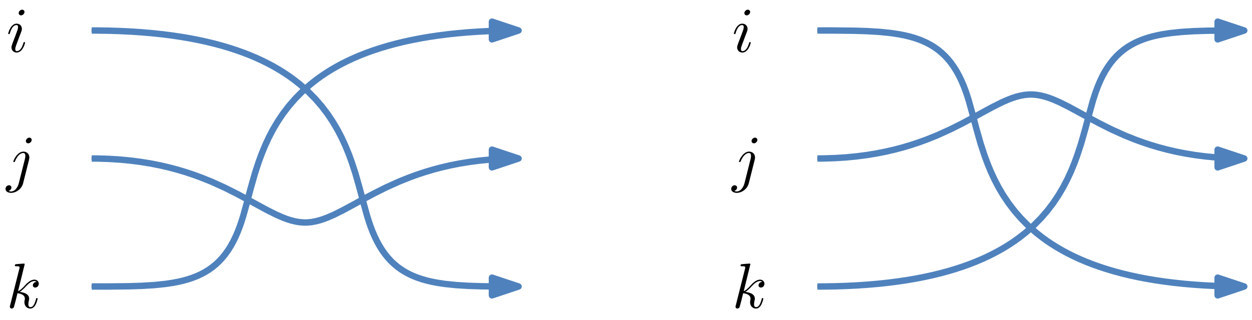
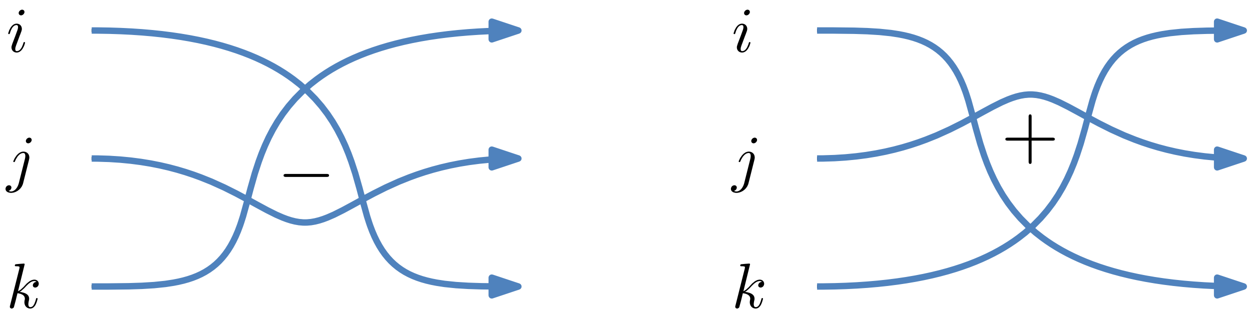
signotopes

Two cases for every triple $i < j < k$ in marked arrangement:



signotopes

Two cases for every triple $i < j < k$ in marked arrangement:

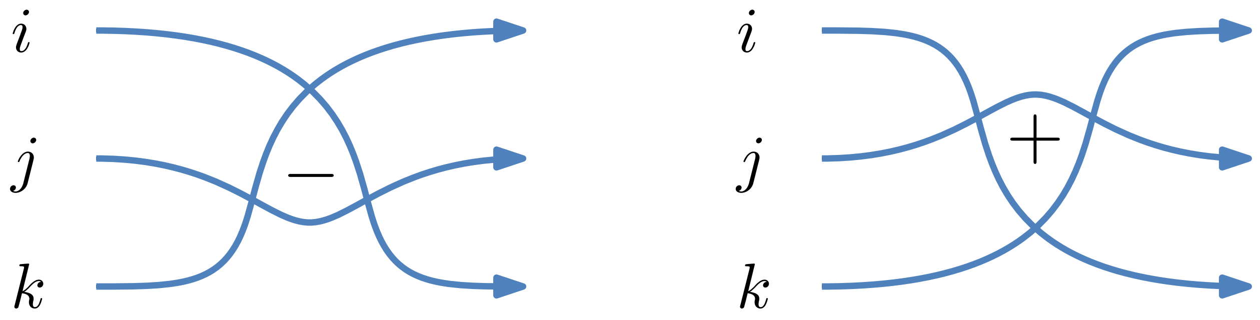


arrangement $\mathscr{A}$ defines map: $\chi_{\mathscr{A}} : \binom{[n]}{3} \to \{-, +\}$

„fingerprint“ of $\mathscr{A}$

signotopes

Two cases for every triple $i < j < k$ in marked arrangement:



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- Local intersection order σ_j of each pseudoline j determined by $\chi_{\mathscr{A}}$

signotopes

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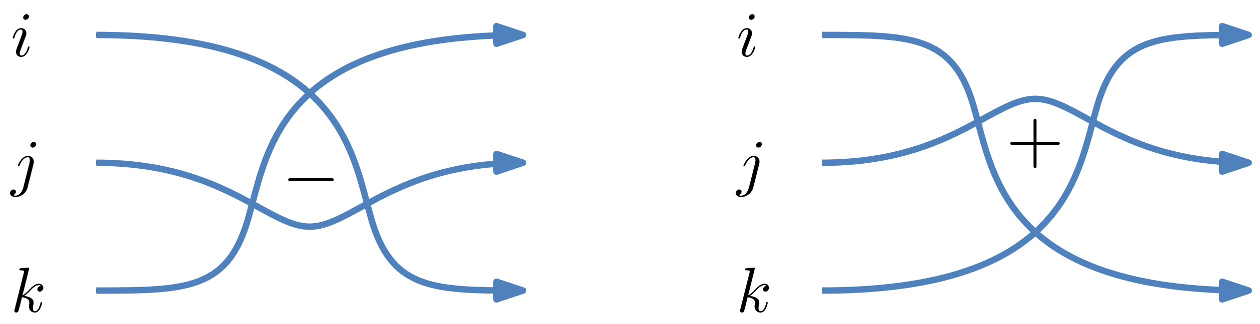

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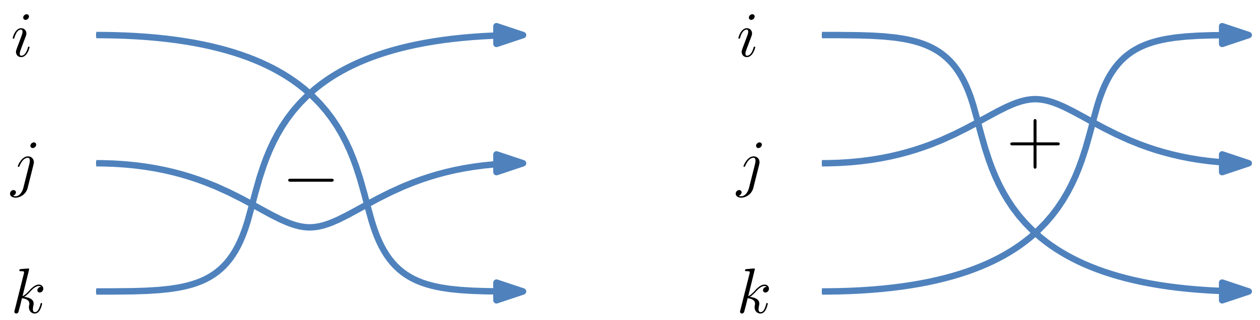
„fingerprint“ of $\mathscr{A}$

- Local intersection order σ_j of each pseudoline j determined by $\chi_{\mathscr{A}}$
- Hence, entire arrangement $\mathscr{A}$ uniquely determined by $\chi_{\mathscr{A}}$

signotopes

Two cases for every triple $i < j < k$ in marked arrangement:

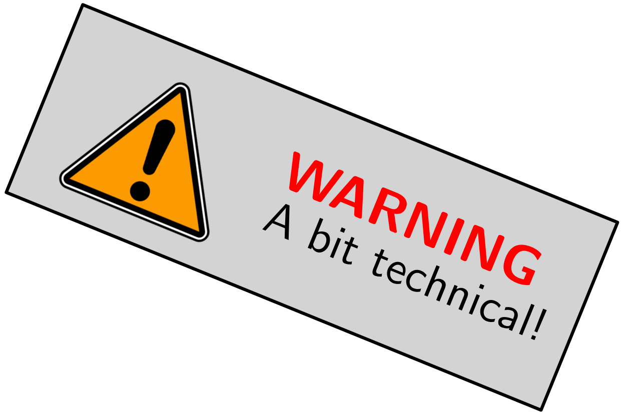


arrangement $\mathscr{A}$ defines map: $\chi_{\mathscr{A}} : \binom{[n]}{3} \to \{-,+\}$

„fingerprint“ of $\mathscr{A}$

- Local intersection order σ_j of each pseudoline j determined by $\chi_{\mathscr{A}}$
- Hence, entire arrangement $\mathscr{A}$ uniquely determined by $\chi_{\mathscr{A}}$
- Not all $2^{\binom{n}{3}}$ possible assignments are arrangements.

signotopes



Definition

For $1 \leq r \leq n$, a *signotope of rank r on n elements* is a sign function

$$\chi : \binom{[n]}{r} \to \{-, +\}$$

s.t. for every $(r+1)$-subset $X = \{x_1, \cdots, x_{r+1}\} \subseteq [n]$ with $x_1 < \cdots < x_{r+1}$ there is at most one sign change in the sequence

$$\chi(X \setminus \{x_1\}), \chi(X \setminus \{x_2\}), \cdots, \chi(X \setminus \{x_{r+1}\}) \ .$$

signotopes

Theorem

Signotopes of rank 3 are exactly the sign functions of marked arrangements of pseudolines.

(without proof)

signotopes

Theorem
Signotopes of rank 3 are exactly the sign functions of marked arrangements of pseudolines.

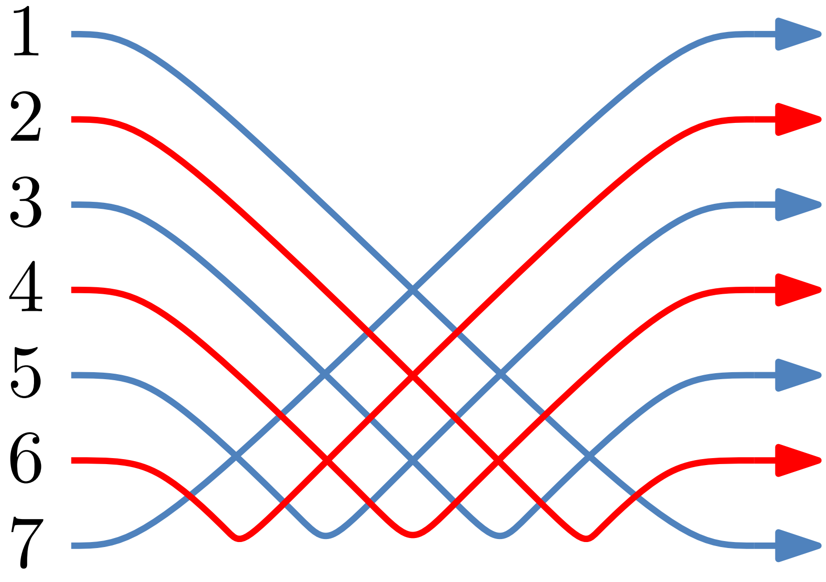
(without proof)

Consequence: For any arrangement $\mathscr{A}$ and pseudolines $1 \leq i < j < k < l \leq n$ we have:

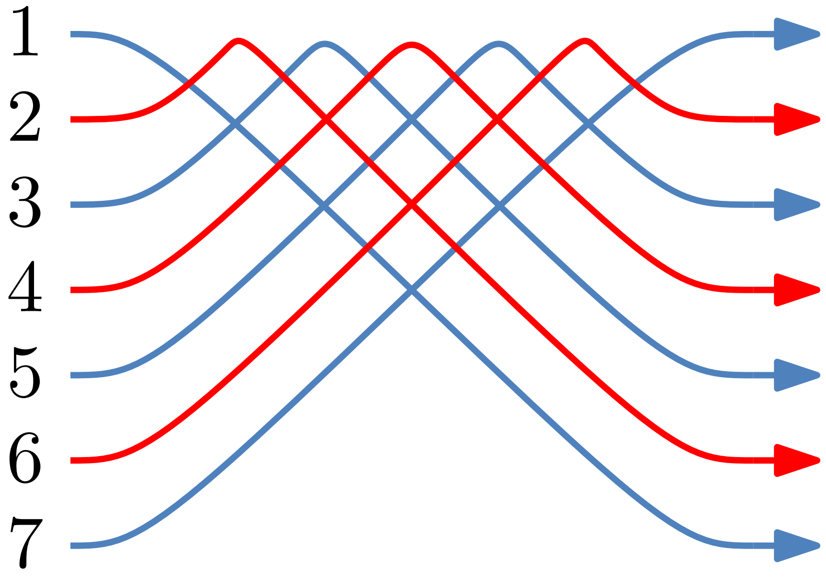
$$(\chi_{\mathscr{A}}(jkl), \chi_{\mathscr{A}}(ikl), \chi_{\mathscr{A}}(ijl), \chi_{\mathscr{A}}(jkl)) \in \Big\{(++++), (+++-), (++--), (+---), (----), (---+), (--++), (-+++)\Big\}$$

signotopes

all-minus-arrangement: $\chi_{\mathscr{A}} = -$

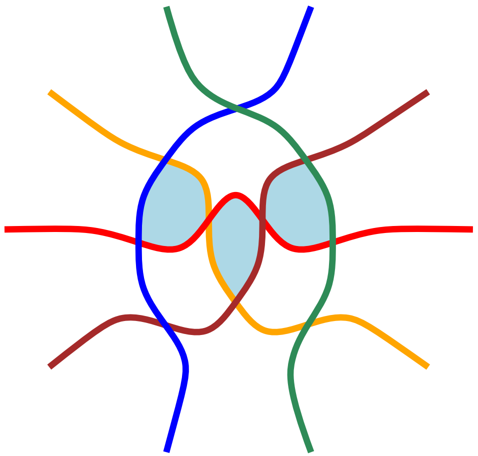


all-plus-arrangement: $\chi_{\mathscr{A}} = +$

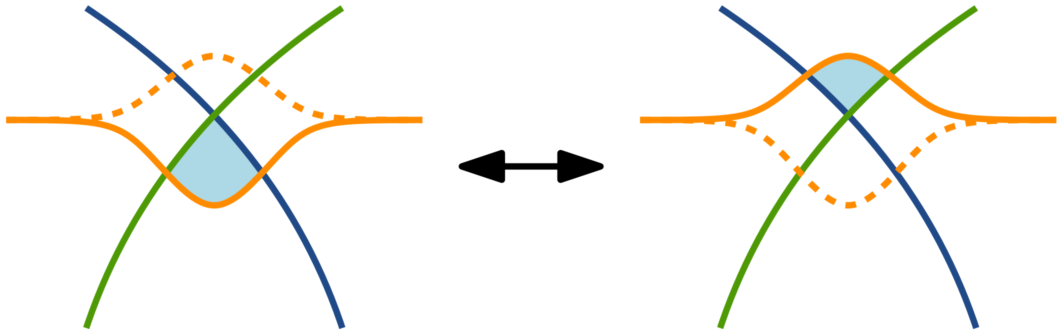


triangle flip

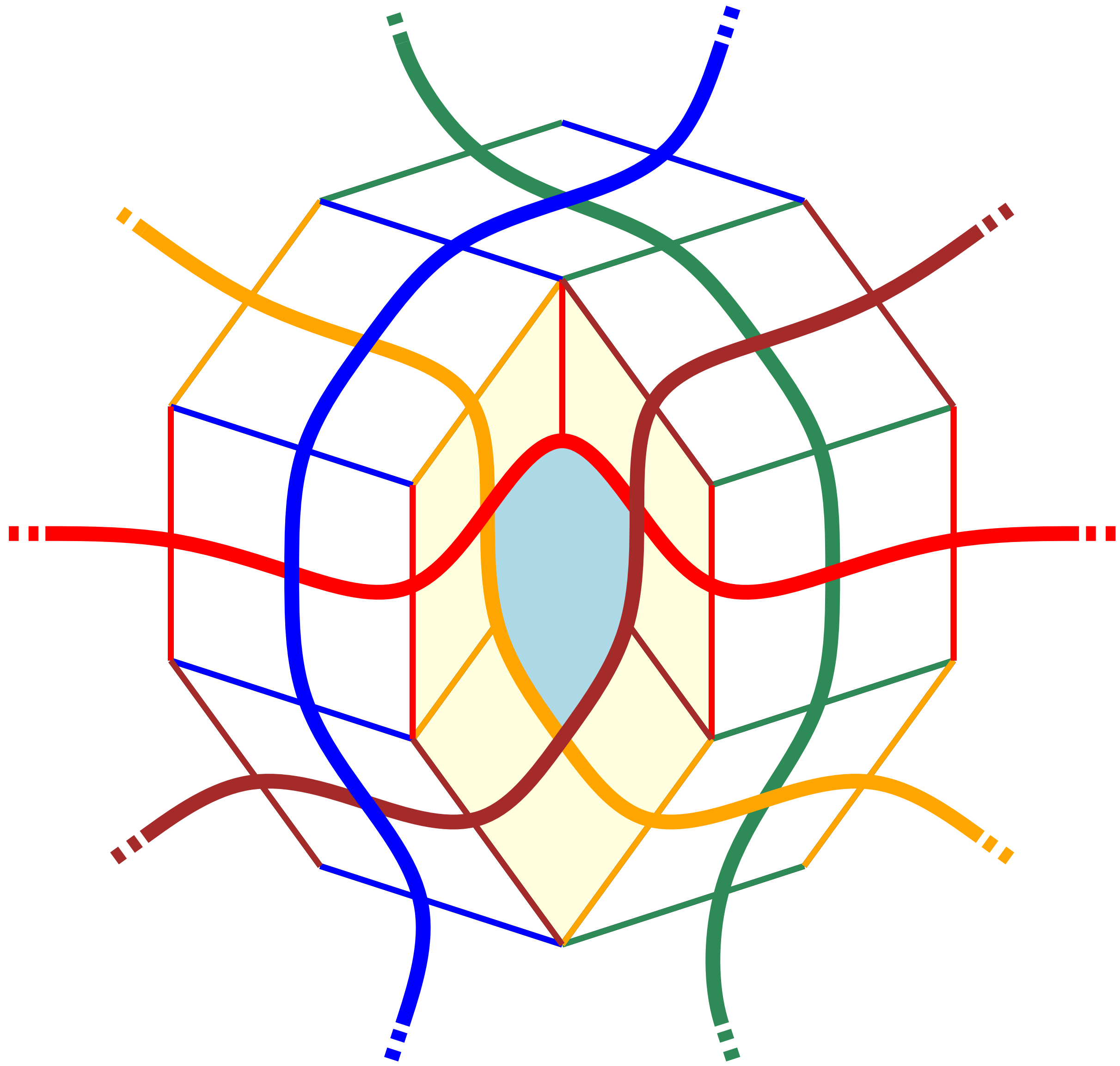
triangle: cell bounded by exactly three pseudolines.



triangle flip: move any involved pseudoline over opposite crossing.



triangle flip



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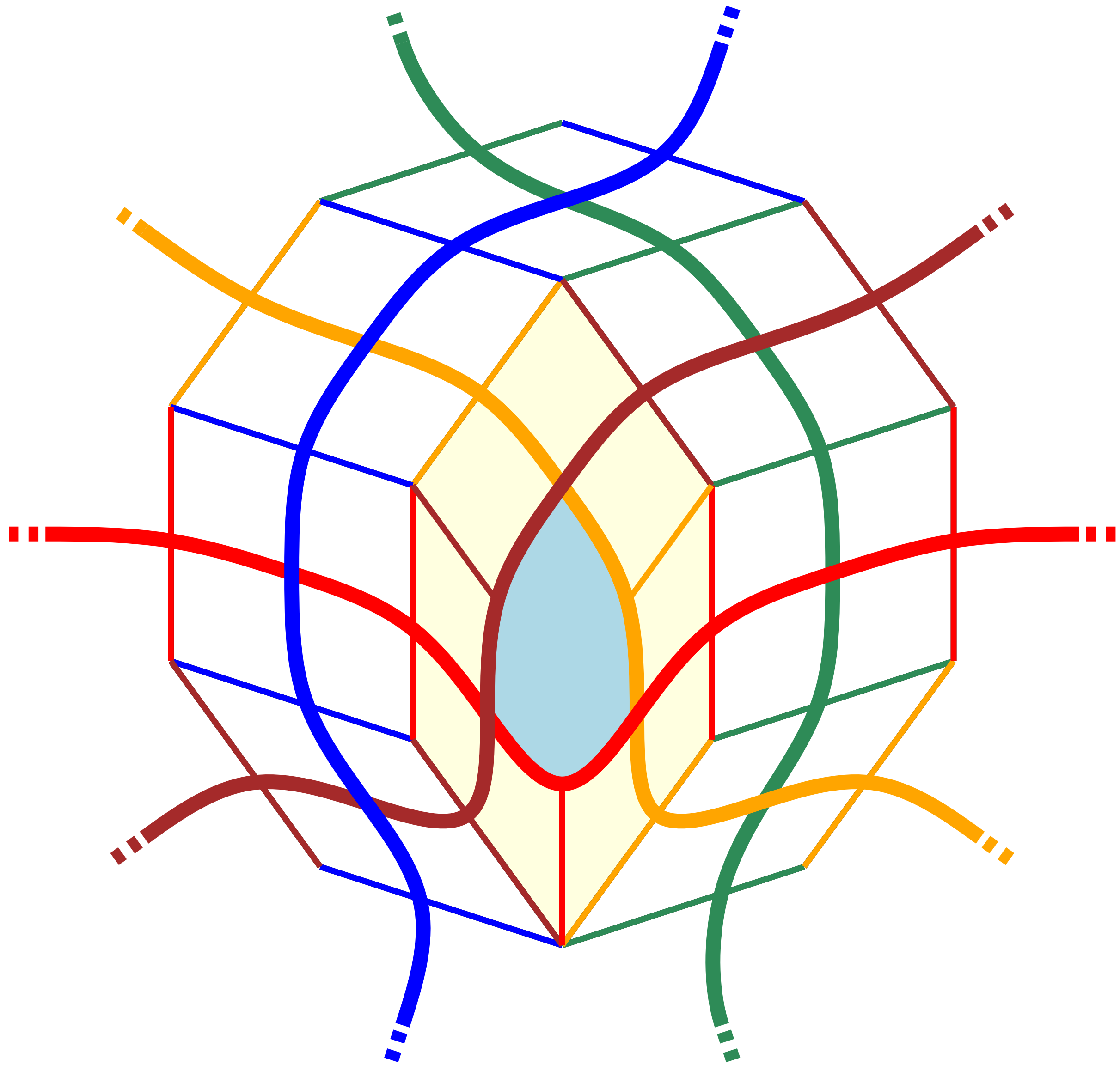
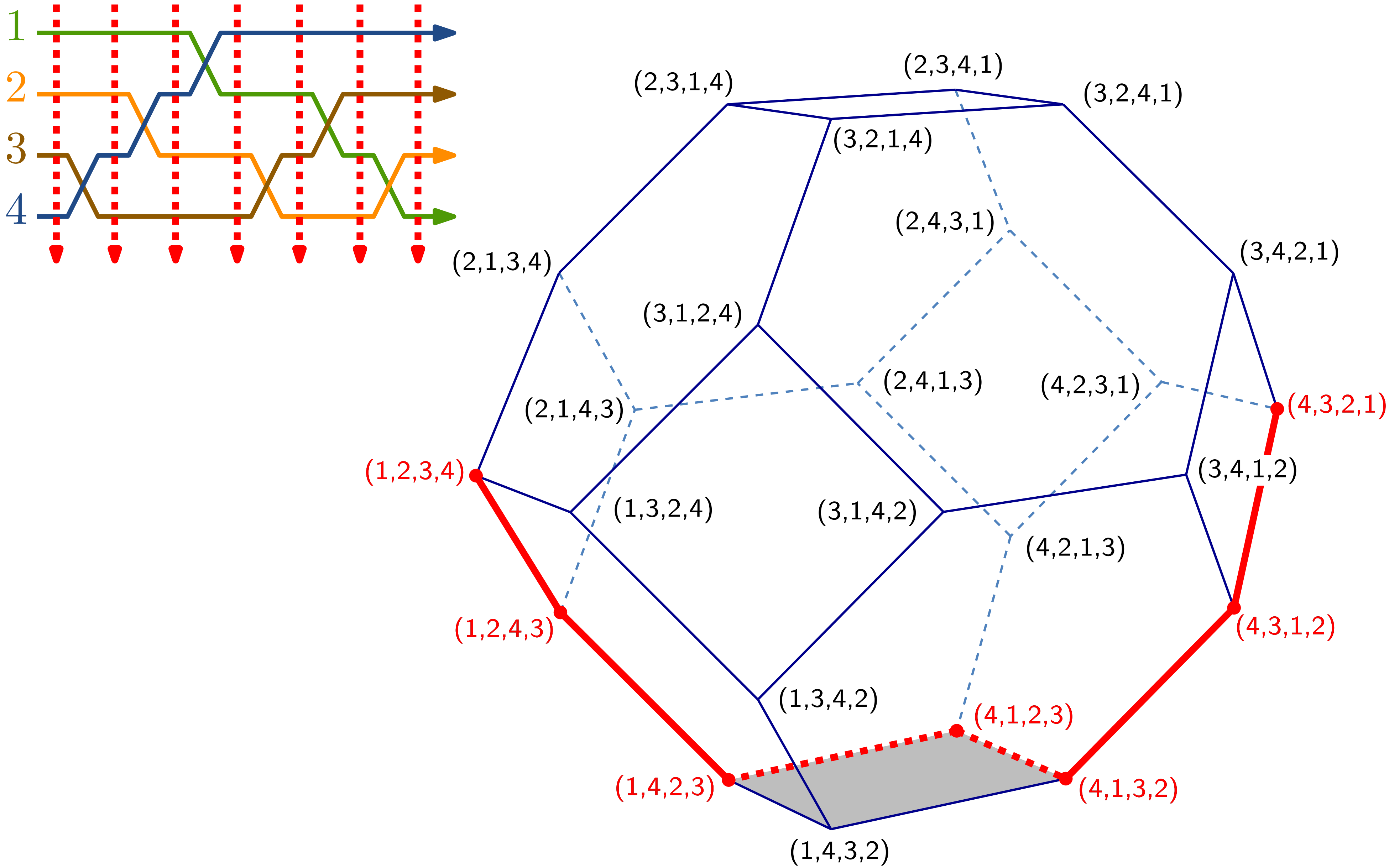


triangle flip



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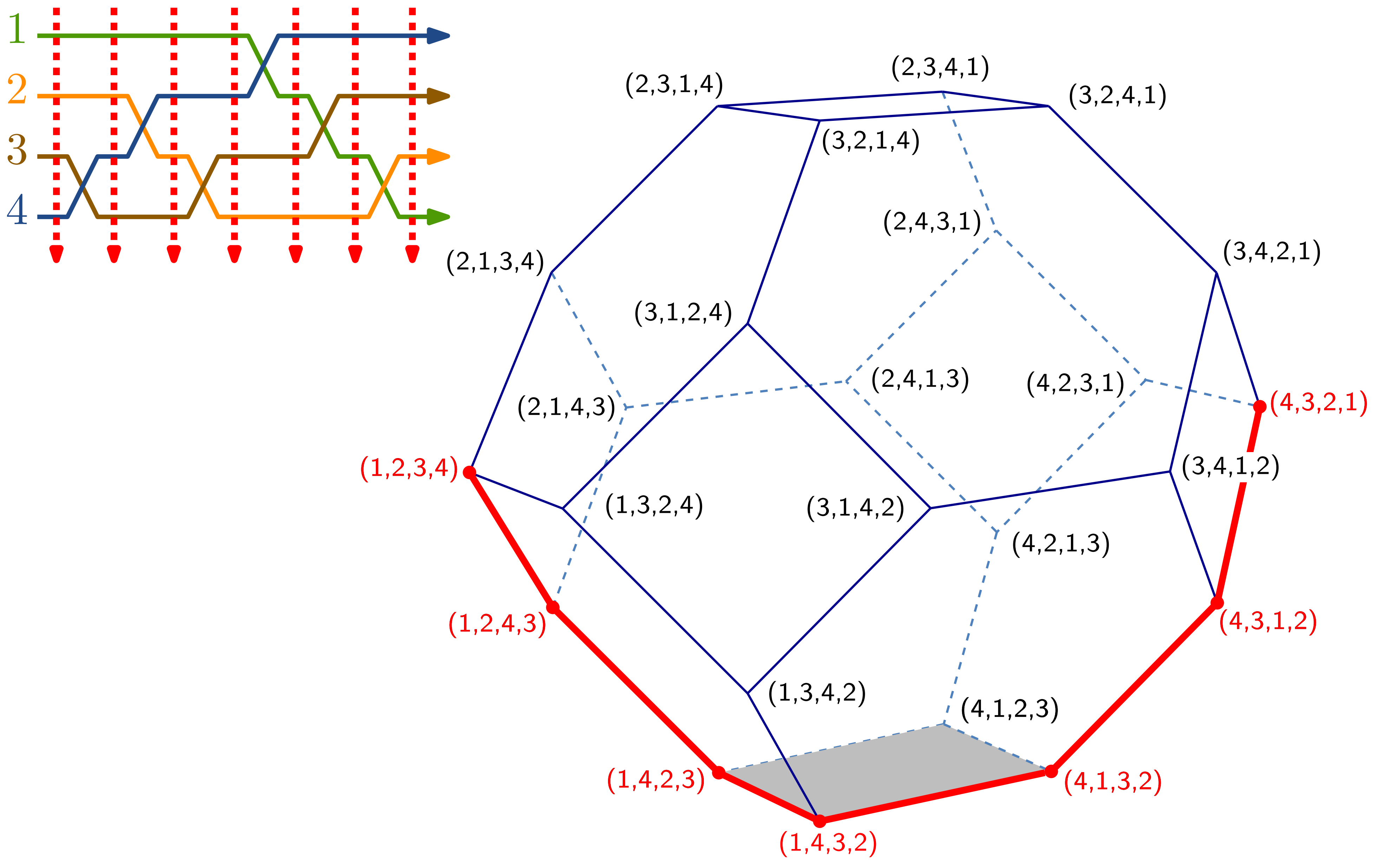

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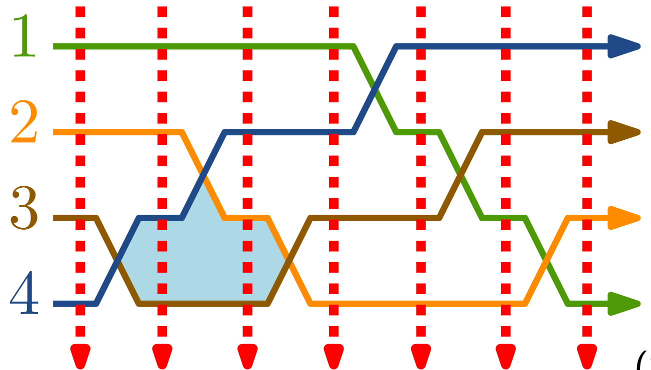
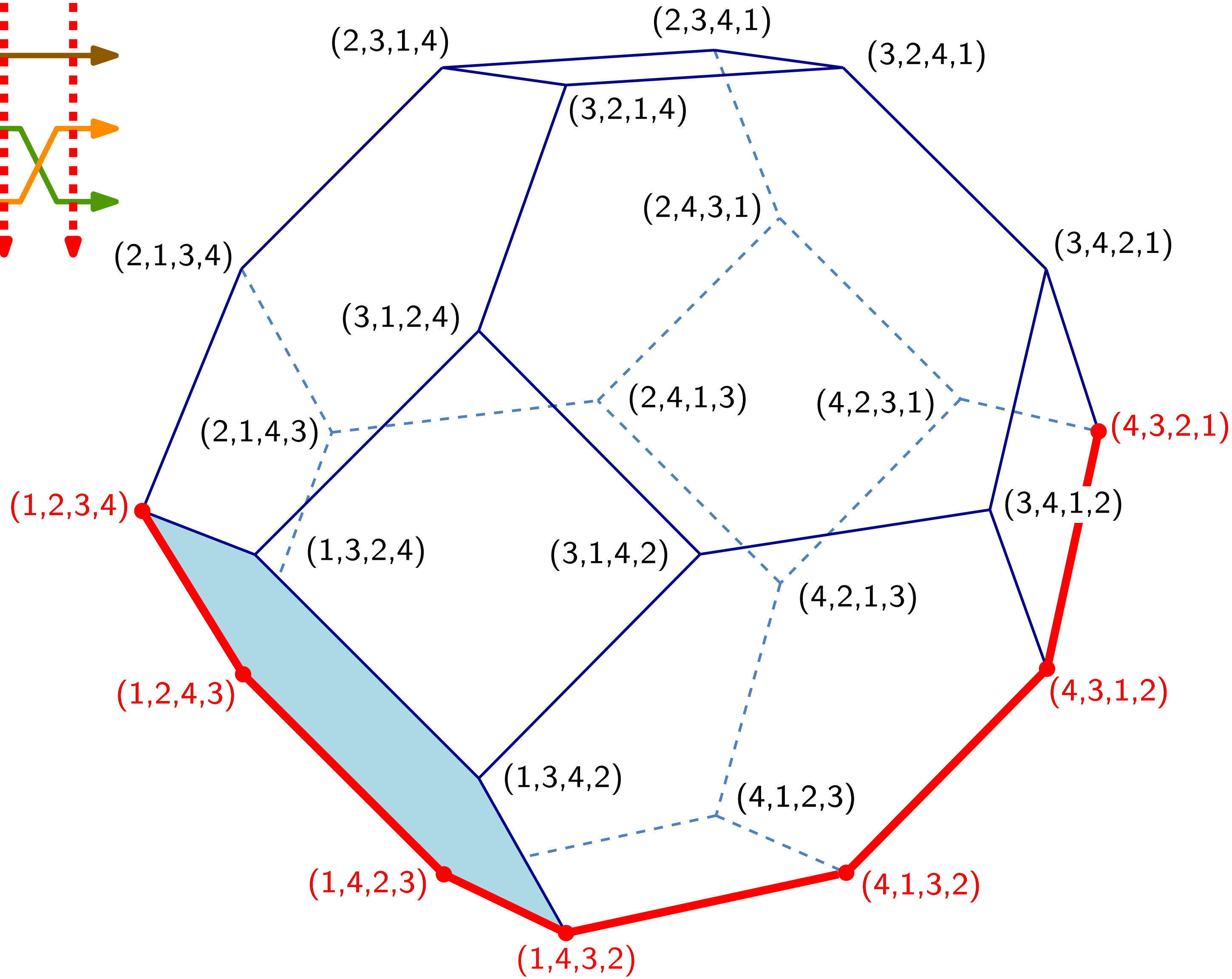
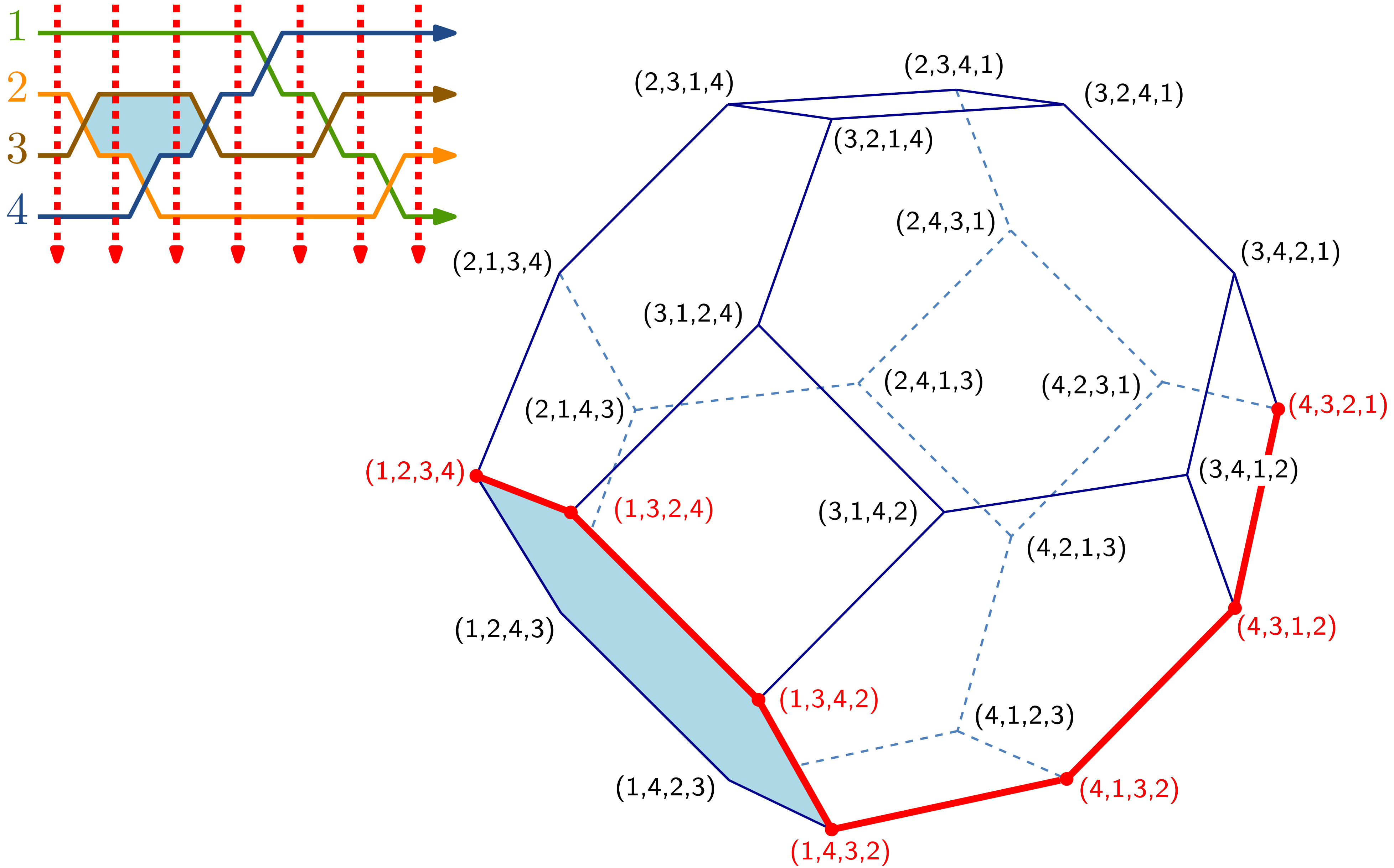




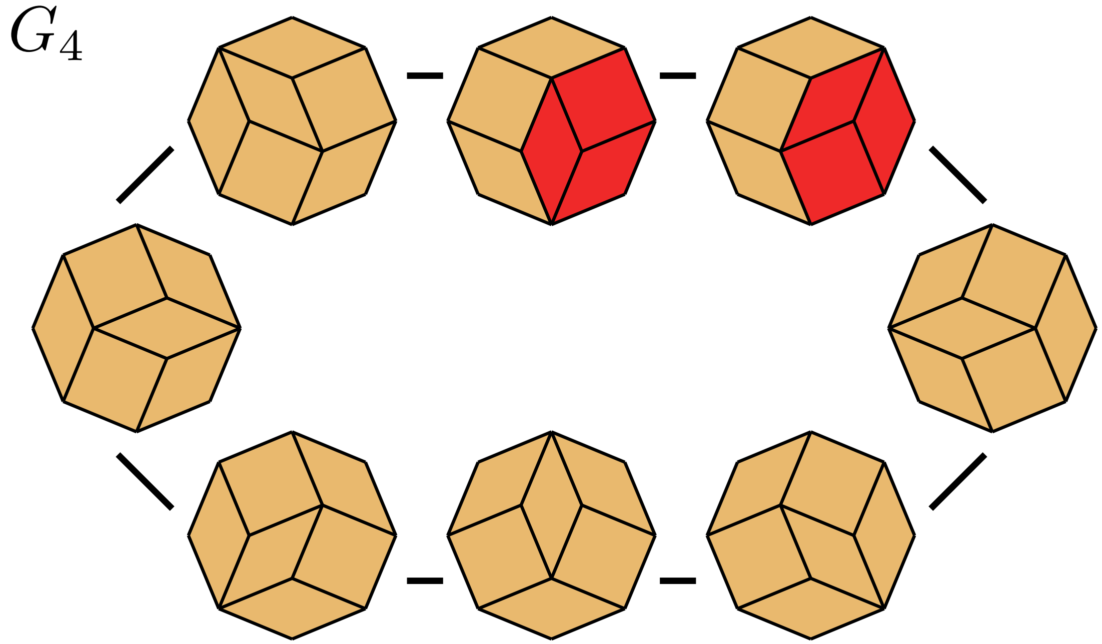
triangle flip



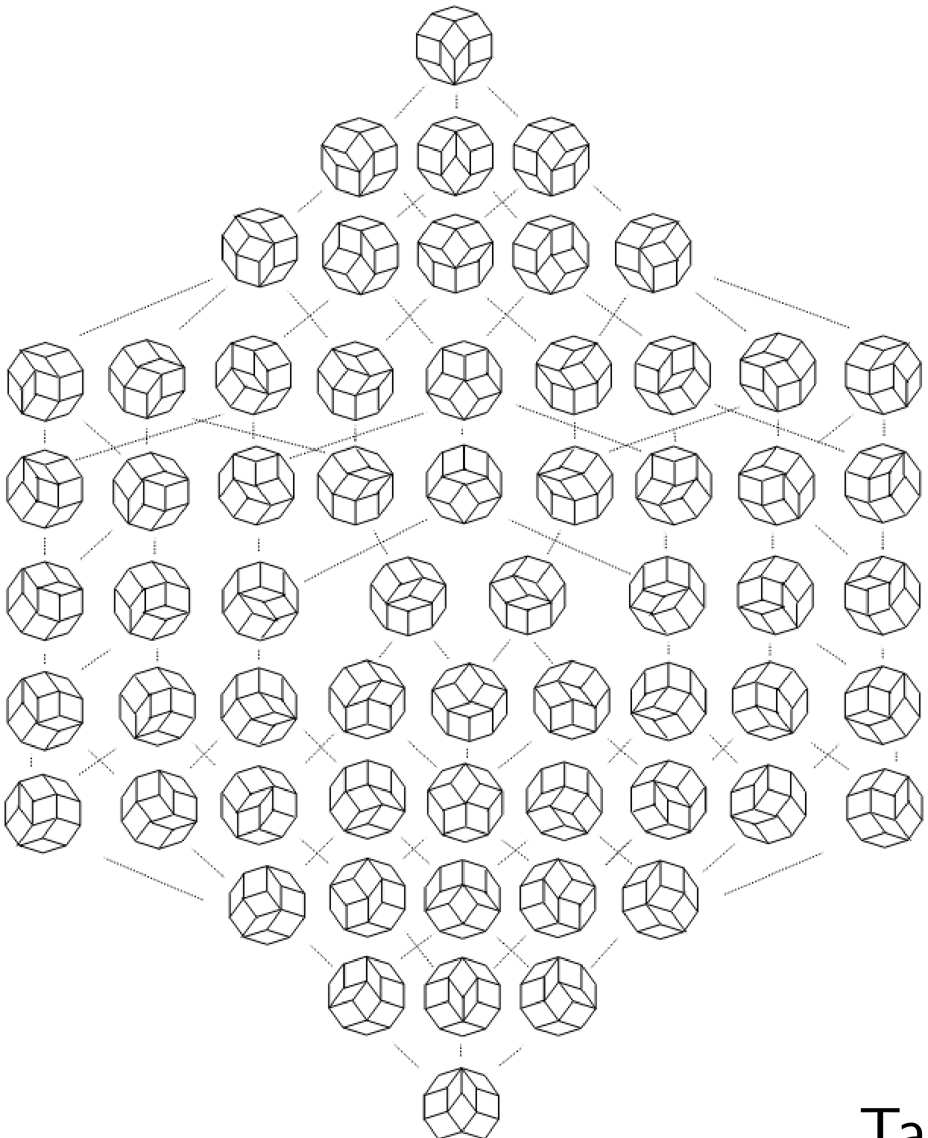
triangle flip

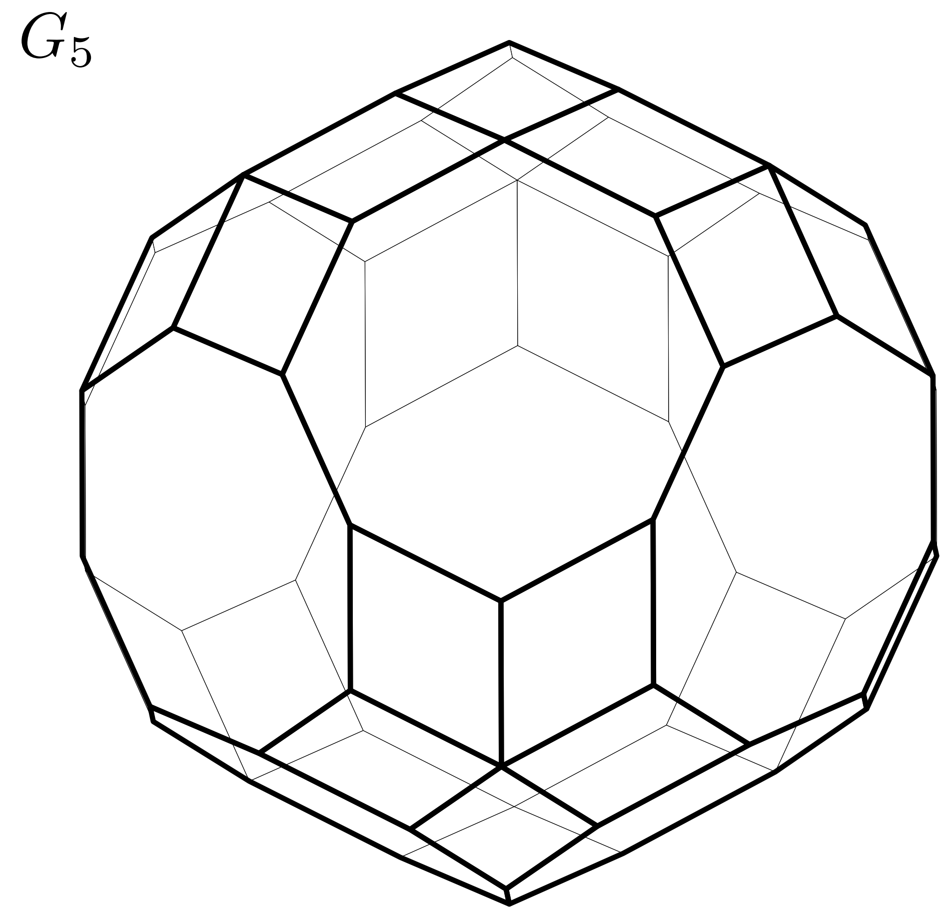
flip graph G_n:

- vetices $\sim$ marked arrangements of fixed size n
- edges $\sim$ triangle flips



triangle flip





Taken from (Felsner & Ziegler, 1999)

triangle flip

Lemma

Unless $\mathscr{A}$ is the all-plus-arrangement, there exists a triple of pseudolines $i < j < k$ with $\chi_{\mathscr{A}}(\{ijk\}) = -$ that form a triangle.

triangle flip

Lemma

Unless $\mathscr{A}$ is the all-plus-arrangement, there exists a triple of pseudolines $i < j < k$ with $\chi_{\mathscr{A}}(\{ijk\}) = -$ that form a triangle.

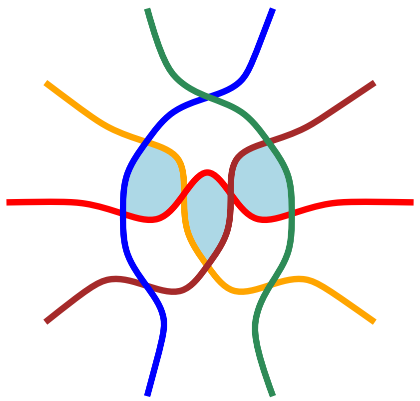
Theorem (Ringel, 1957)

The triangle flip graph G_n consisting of marked pseudoline arrangements of size n is connected and has diameter $\binom{n}{3}$.

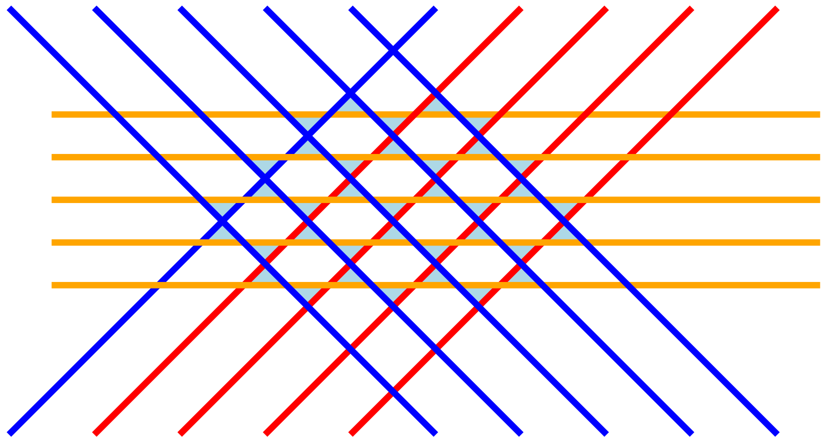
triangles

Theorem (Felsner & Kriegel, 1991)

Every Euclidean arrangement of n pseudolines contains at least $n-2$ triangles. This bound is tight.



#triangles $= n-2$



#triangles $\in \Omega(n^2)$

triangles

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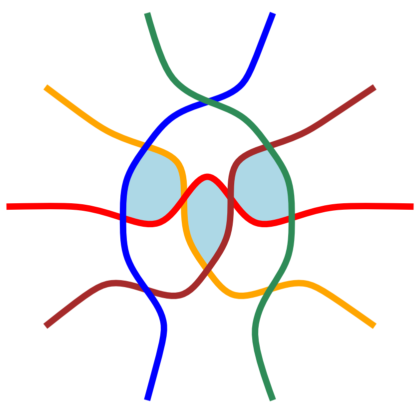
Every Euclidean arrangement of n pseudolines contains at least $n-2$ triangles. This bound is tight.



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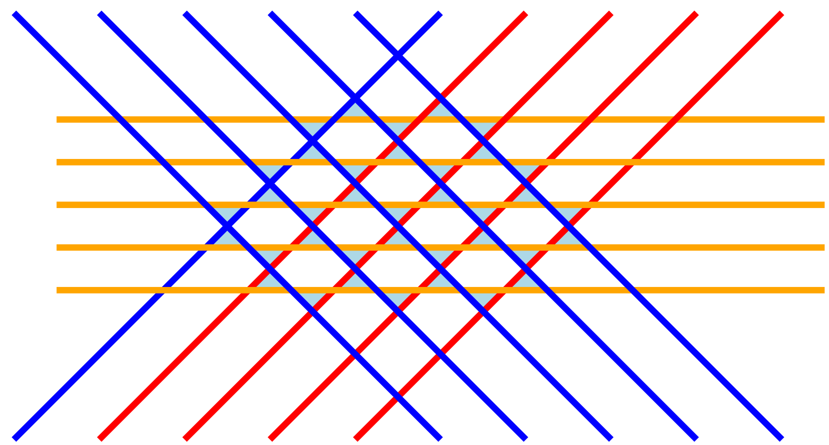
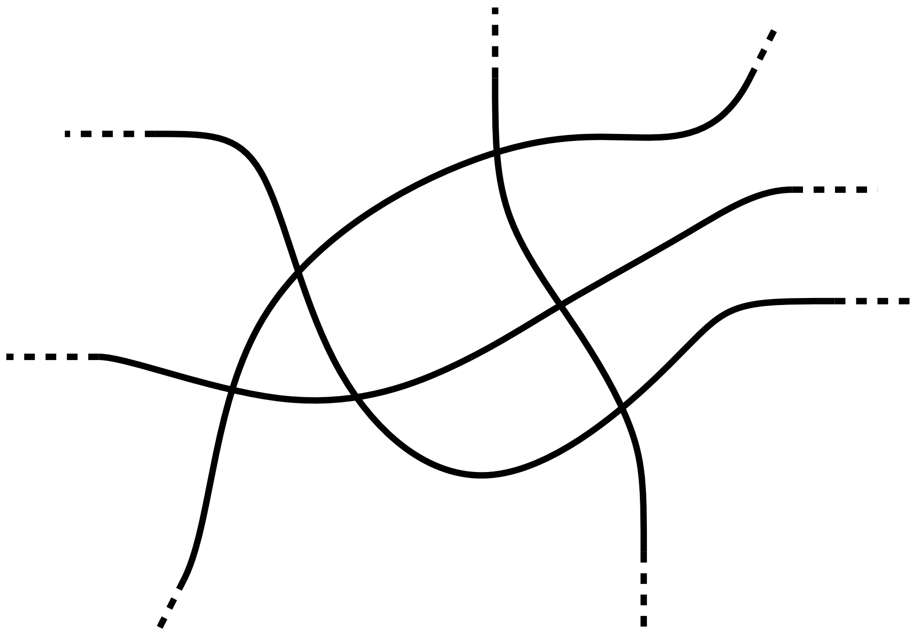


#triangles $\in \Omega(n^2)$

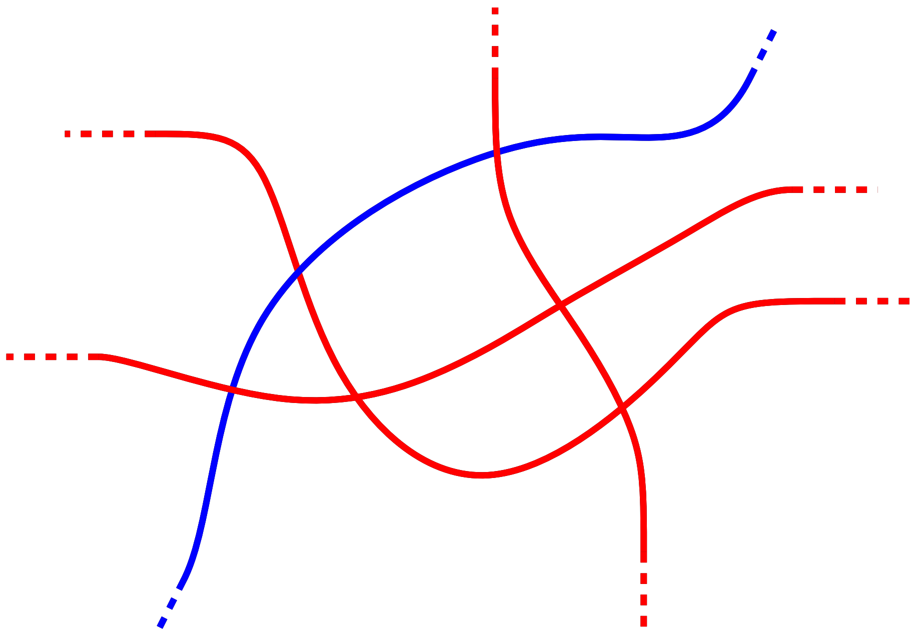
Theorem
The flip graph G_n is $(n-2)$-connected.
(A. Radtke, Felsner, Obenaus, R., Scheucher, Vogtenhuber, 2024)

bichromatic triangle conjecture

bichromatic triangle conjecture



bichromatic triangle conjecture



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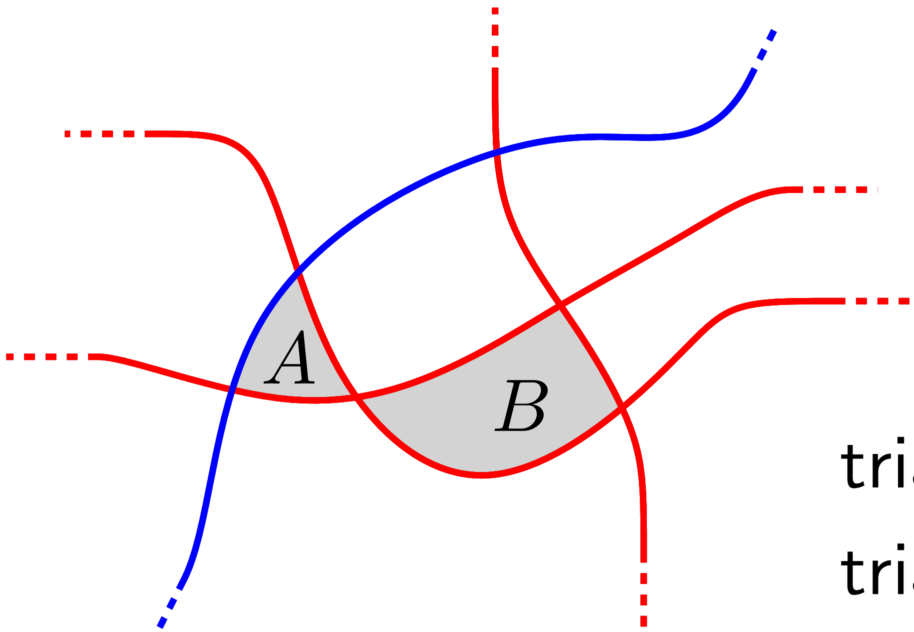
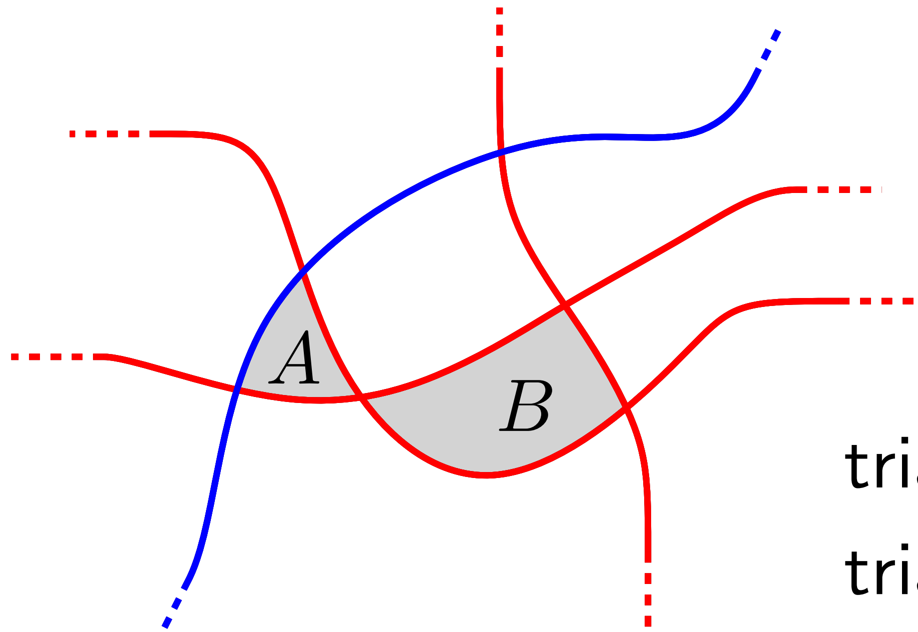


triangle A bichromatic

triangle B monochromatic

bichromatic triangle conjecture



triangle A bichromatic

triangle B monochromatic

Conjecture

Every truly two-colored arrangement of at least three pseudolines contains a bichromatic triangle.

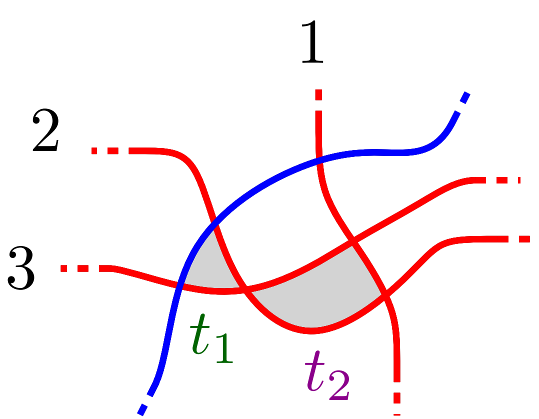
(Björner, Las Vergnas, Sturmfels, White, Ziegler, 1999)

bichromatic triangle conjecture

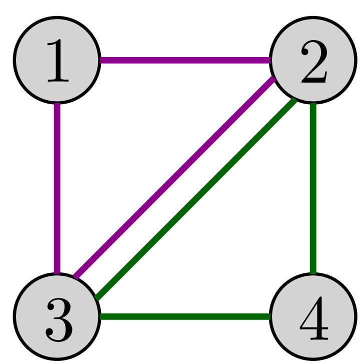
Proposition

Let $\mathscr{A}$ be an arrangement of $n \geq 3$ pseudolines.
The following are equivalent:

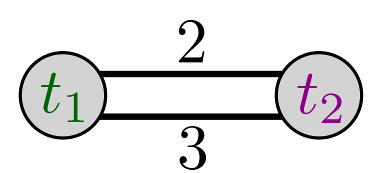
- Every coloring of the pseudolines using exactly two colors produces a bichromatic triangle (Conjecture).
- The pseudoline-triangle-graph $G_l(\mathscr{A})$ is connected.
- The triangle-pseudoline-graph $G_\Delta(\mathscr{A})$ is connected.



$\mathscr{A}$

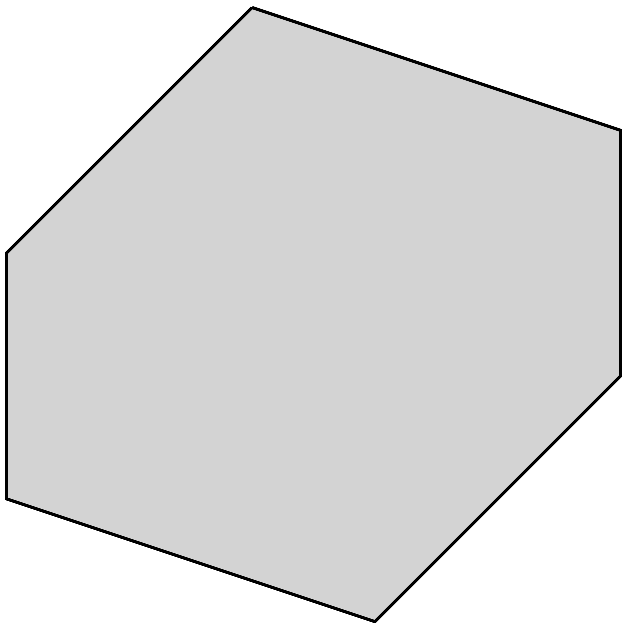


$G_l(\mathscr{A})$

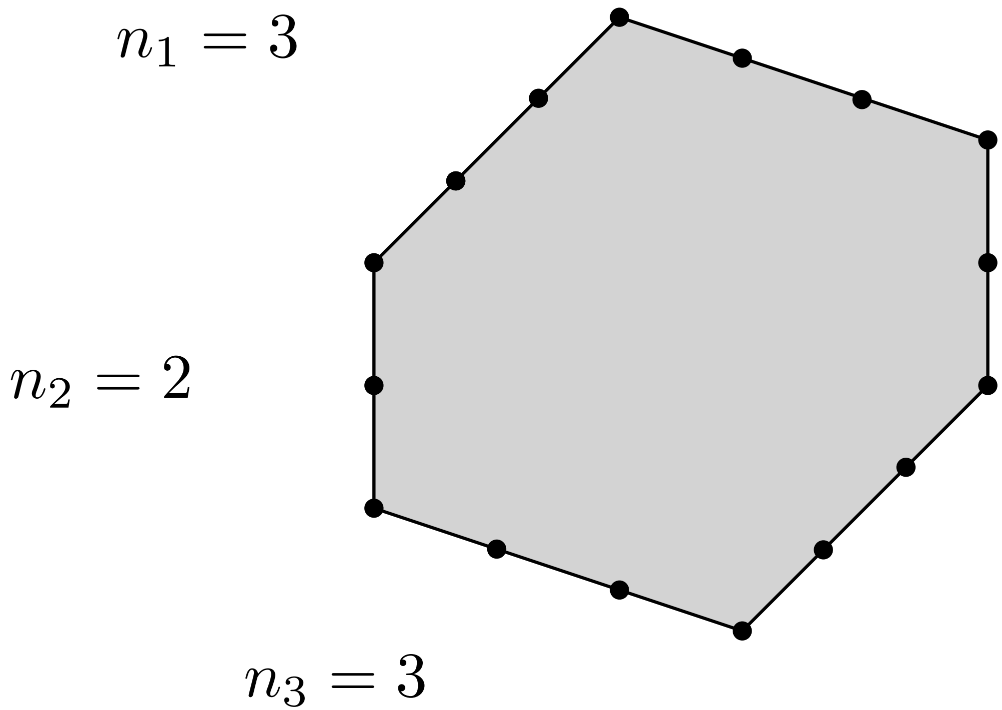


$G_\Delta(\mathscr{A})$

generalized arrangements



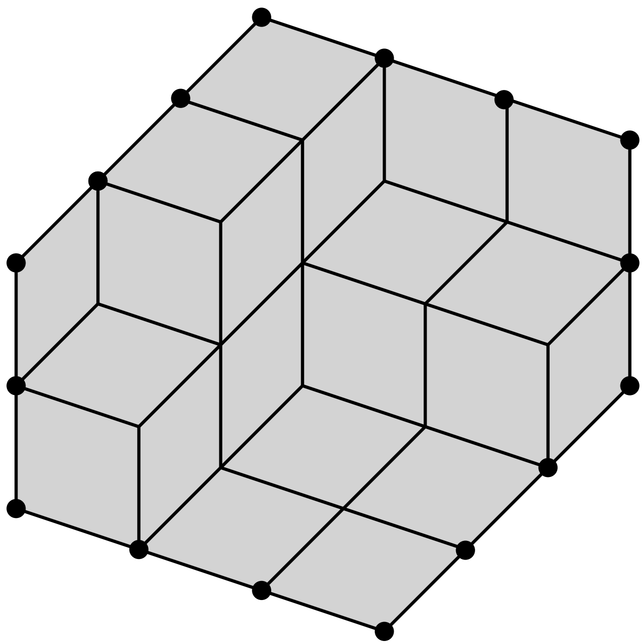
generalized arrangements



generalized arrangements

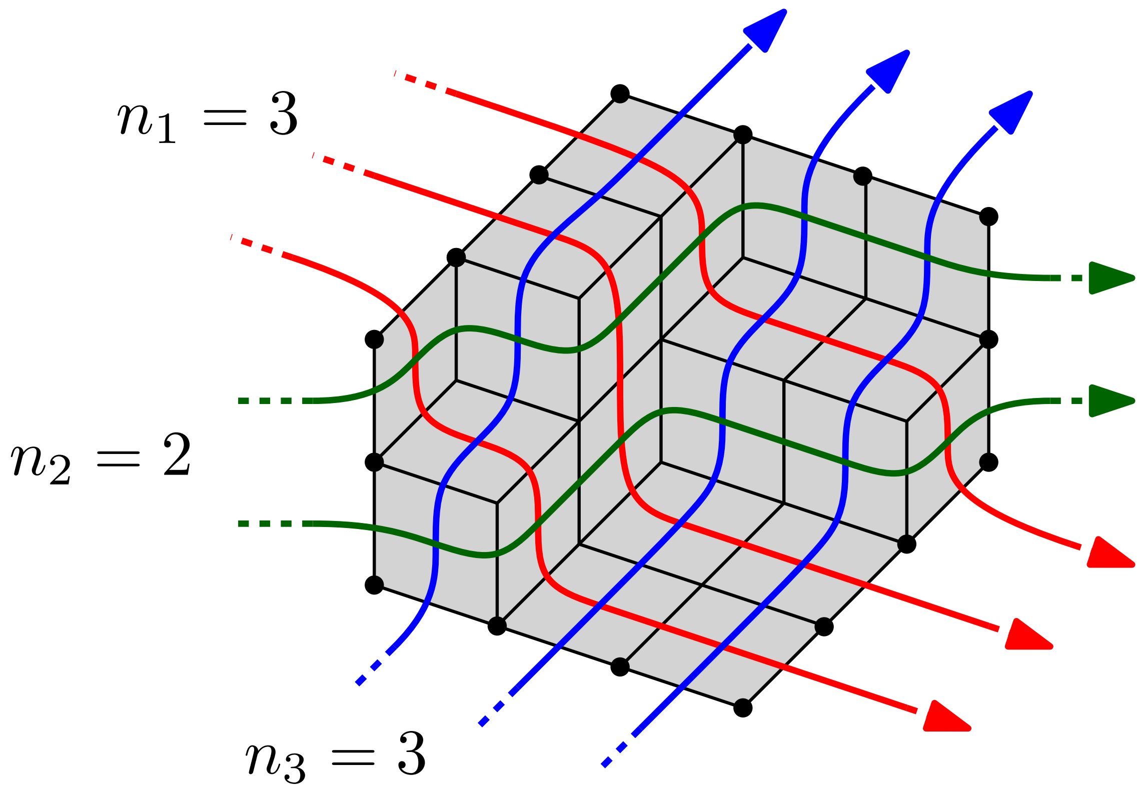
$n_1 = 3$

$n_2 = 2$

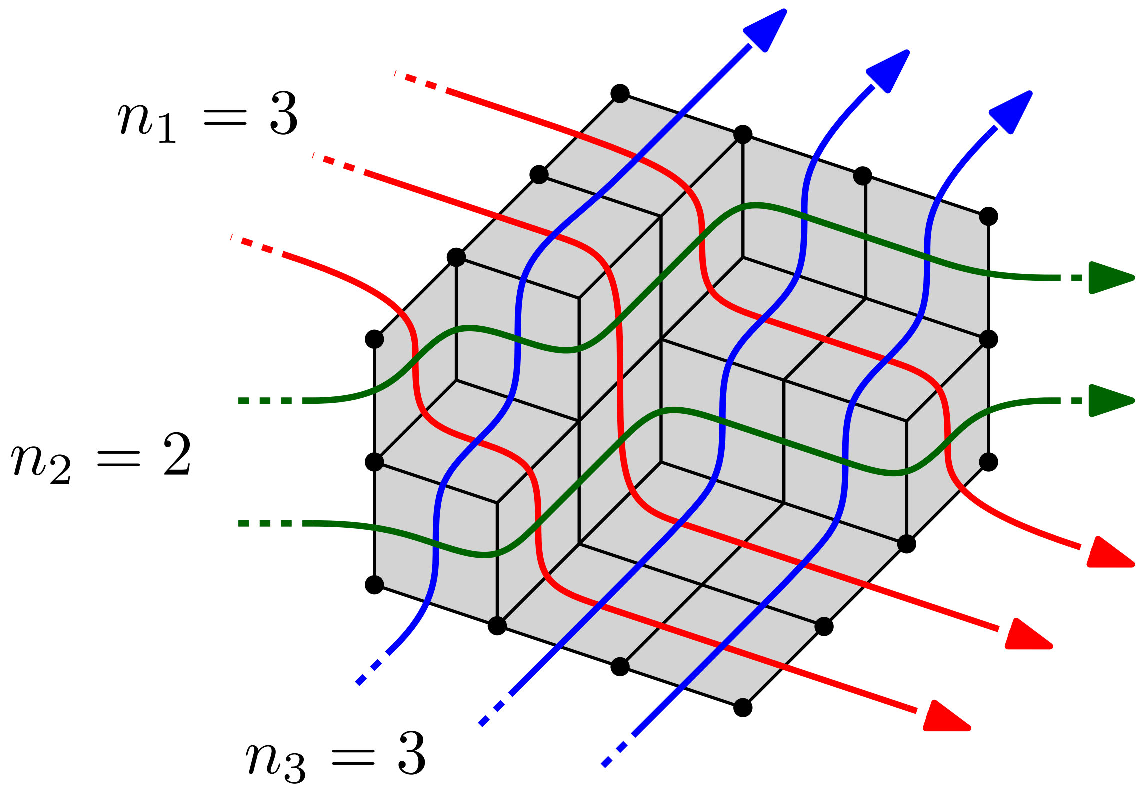


$n_3 = 3$

generalized arrangements



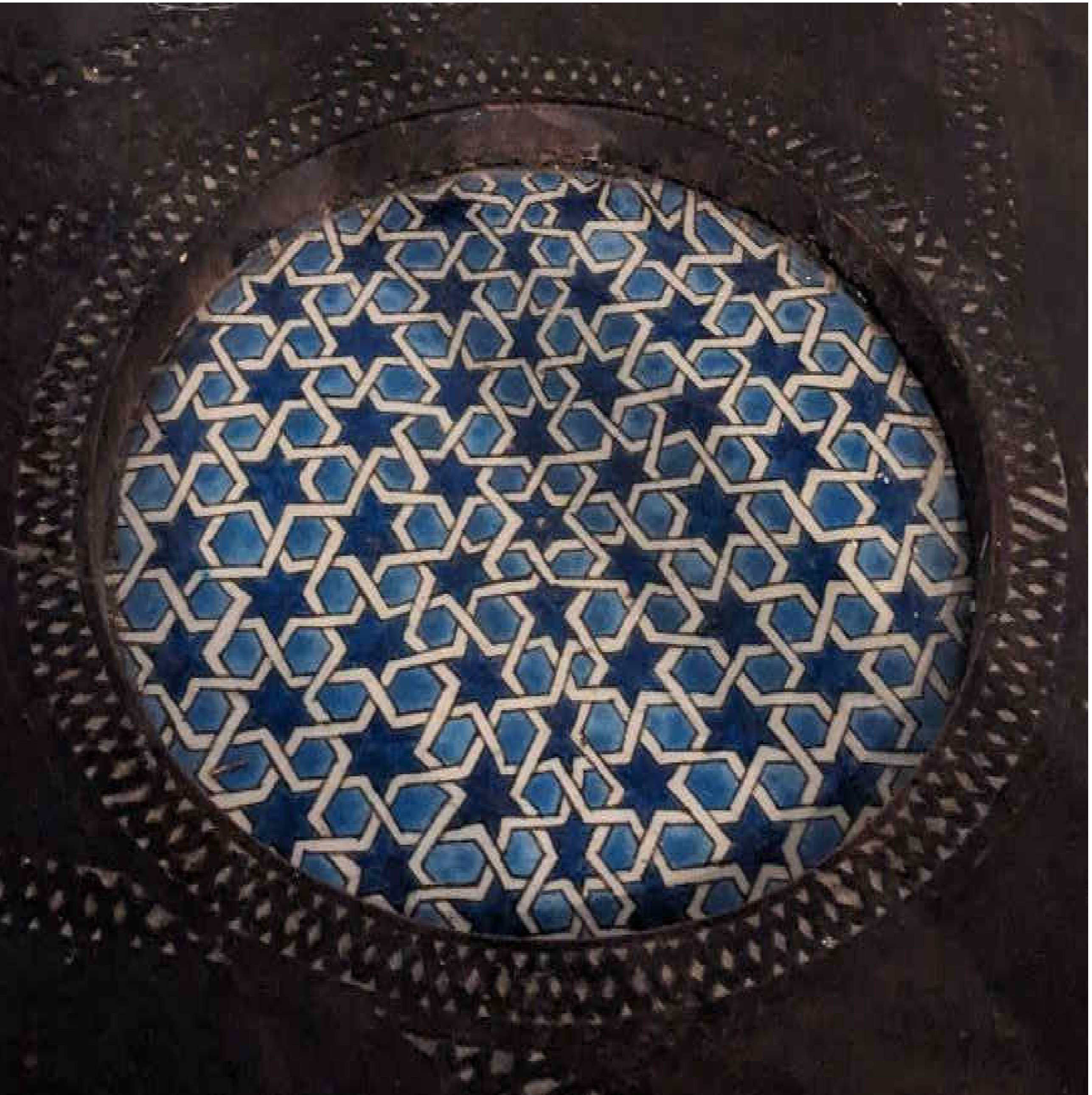
generalized arrangements

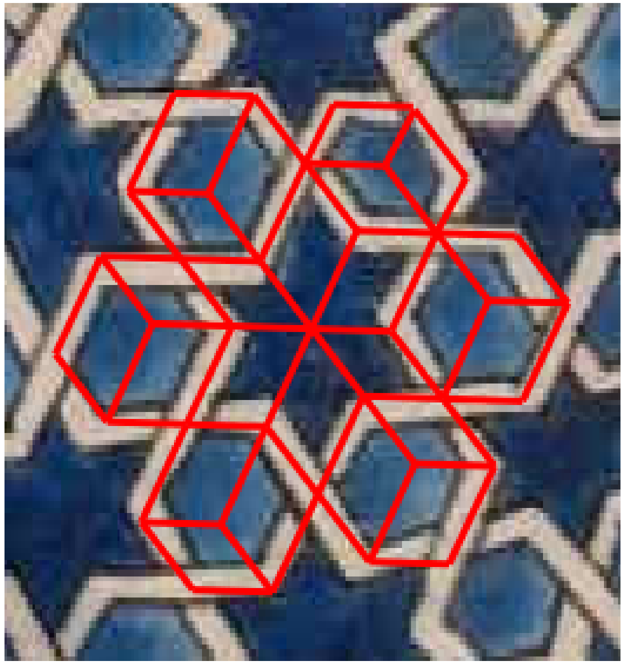


$\Rightarrow$ *generalized pseudoline arrangement*:

- *parallel class* of $n_1, \ldots, n_r$ pseudolines
- (Only) pseudolines of different classes cross

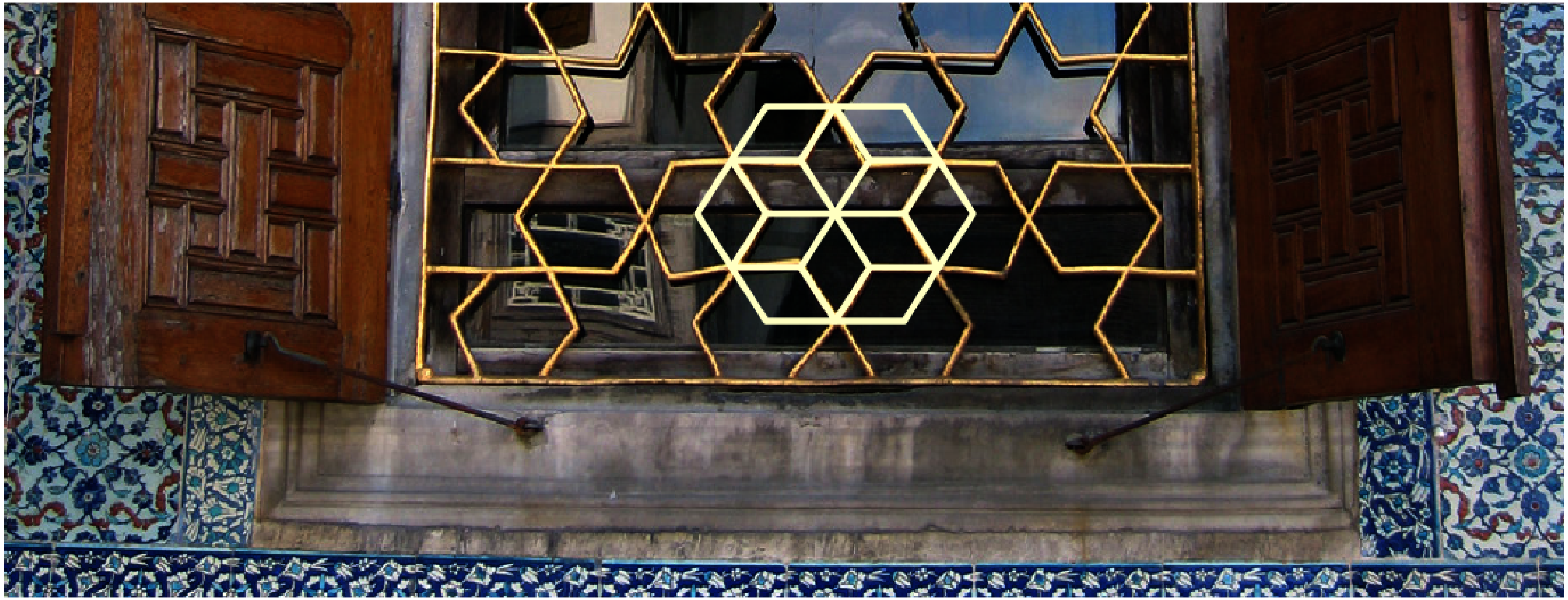
generalized arrangements





Aslan Pasha Mosque
Ioannina, Greece

generalized arrangements



Topkapı Palace, Istanbul, Turkey

plane partitions and grid paths

plane partitions and grid paths

Def: matrix $[h_{i,j}] \in {\mathbb{N}_0}^{r \times s}$ is called *plane partition*, if rows and columns are monotonic increasing.

plane partitions and grid paths

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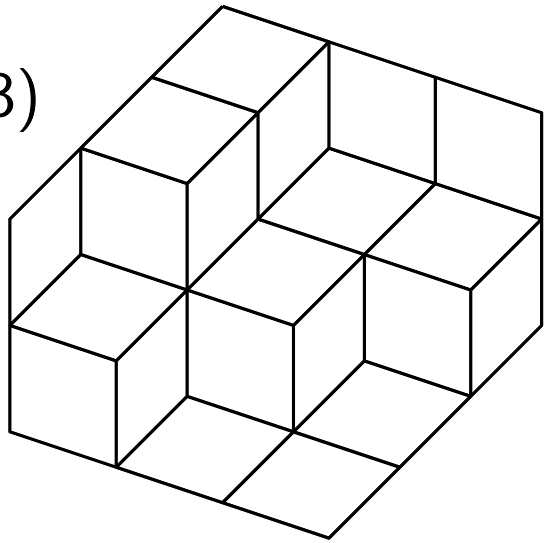
rhombic tiling
of size (3, 2, 3)



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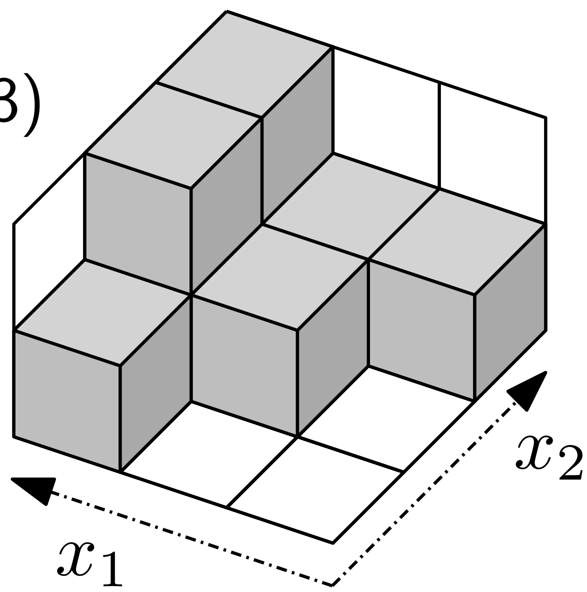

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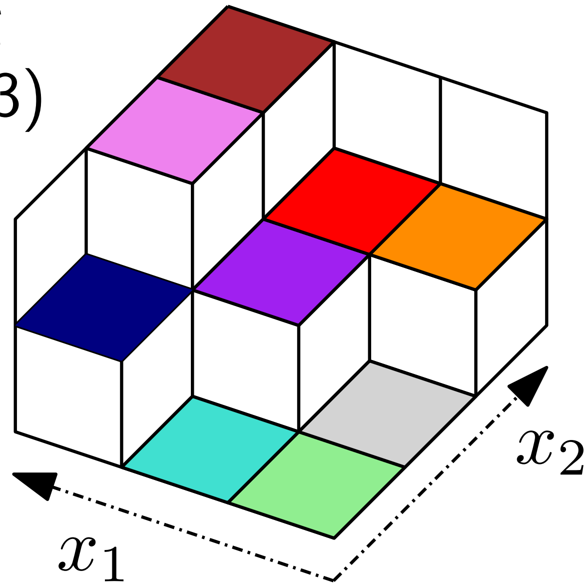

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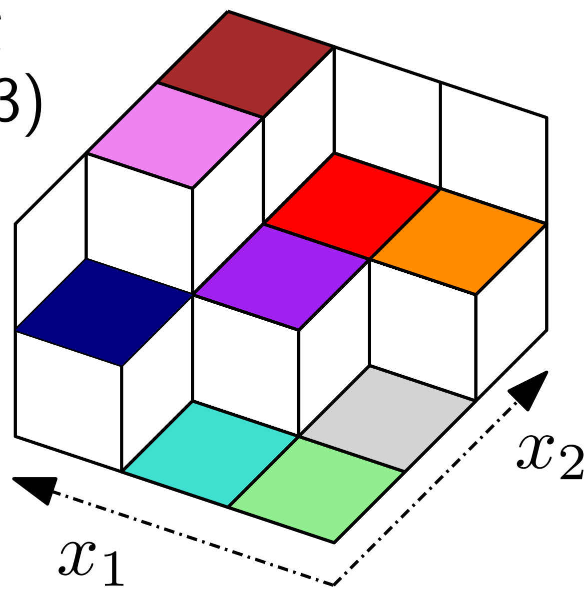

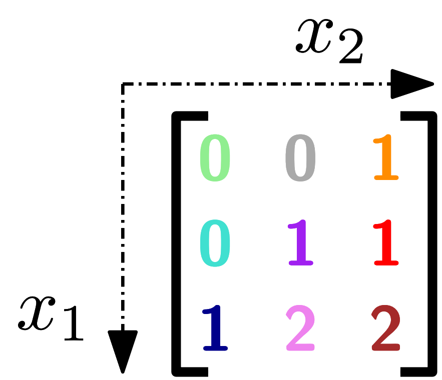

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

plane partition with entries $h_{i,j} \leq 2$

plane partitions and grid paths

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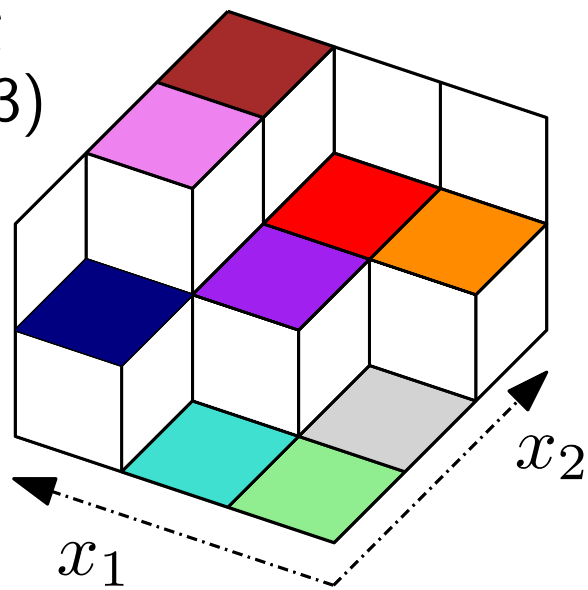

$$x_1 \downarrow \quad \overset{x_2 \rightarrow}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}}$$

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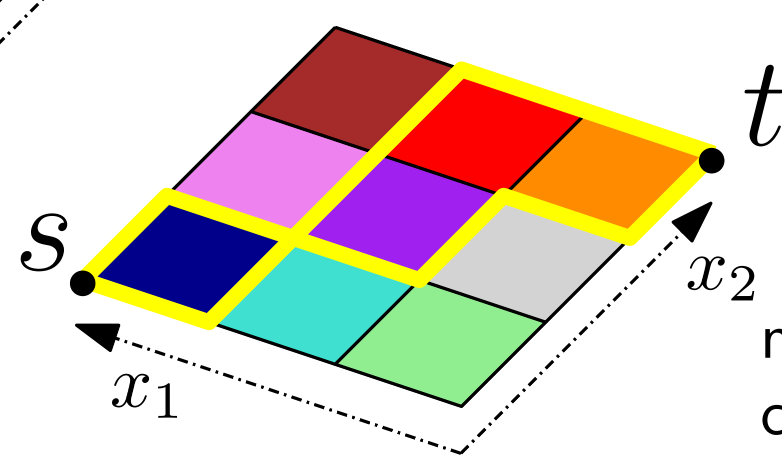
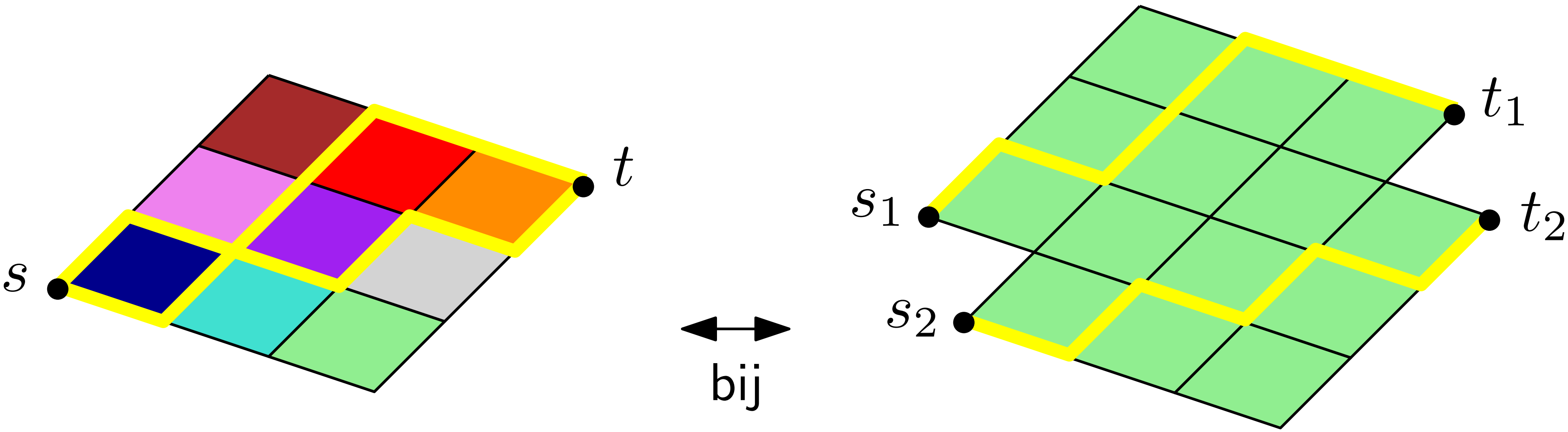


monotone, non-crossing grid paths

plane partitions and grid paths



systems of monotonic non-crossing $s \to t$ paths in grid

systems of monotonic vertex-disjoint $s_i \to t_i$ paths in lifted grid

plane partitions and grid graphs

Lindström-Gessel-Viennot Lemma

- Let $G = (V, E)$ acyclic directed graph, edge weights $\{w_e\}_{e\in E}$.
- Let $A = \{a_1, \cdots, a_n\} \subset V$ (start points)
- Let $B = \{b_1, \cdots, b_n\} \subset V$ (end points)
- For any path P define its weight $w(P) := \prod_{e\in P} w_e$.
- For any pair $a, b \in V$ define $e(a, b) := \sum_{P:a\to b} w(P)$.
- Any system of n vertex-disjoint paths $P_1, \cdots, P_n : A \to B$ from A to B defines permutation $\sigma(P_1, \cdots, P_n)$.

Then:

$$\det([e(a_i, b_j)]_{i,j}) = \sum_{(P_1,\cdots,P_n):A\to B} \operatorname{sgn}(\sigma(P_1, \cdots, P_n)) \prod_{i=1}^{n} w(P_i)$$

plane partitions and grid graphs

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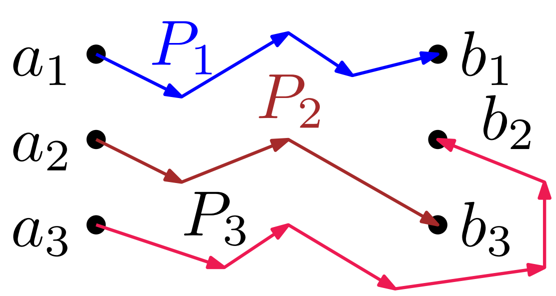
Ex.



$\sigma(P_1, P_2, P_3) = (1, 3, 2)$

plane partitions and grid graphs

Theorem [MacMahon, 1916]

The number of plane partitions of size $a \times b$ with entries at most n equals

$$\det_{1\leq i,j\leq n}\left(\left[\binom{a+b}{a-i+j}\right]\right) = \prod_{i=1}^{n}\prod_{j=1}^{a}\prod_{k=1}^{b}\frac{i+j+k-1}{i+j+k-2}\,.$$

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expression symmetric in a, b, n!

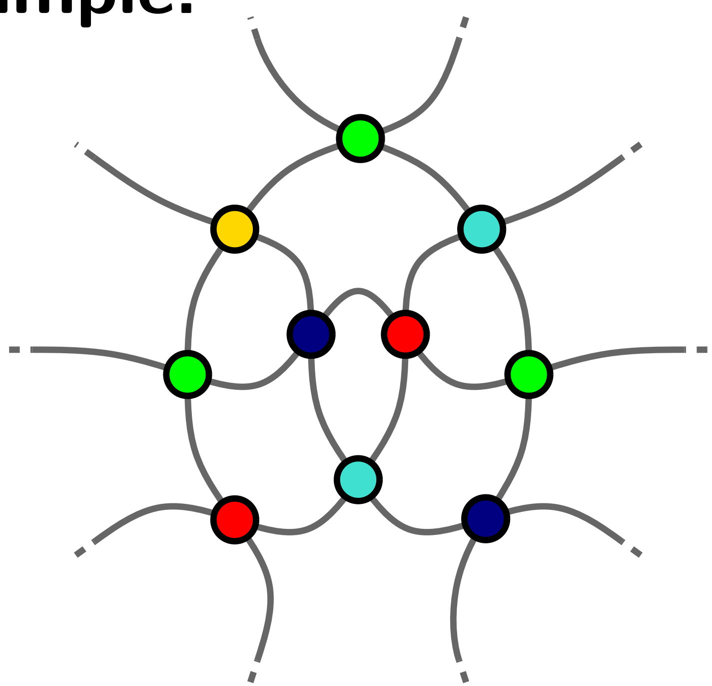
This formula also counts:

- Number of ways to tile a hexagon of side lengths a, b, n with rhombics of unit length.
- Number of generalized arrangements of three classes with a, b and n pseudolines.

face respecting colorings

Theorem: Let $\mathscr{A}$ be an arrangement of n pseudolines.
The crossings of $\mathscr{A}$ can be colored using n colors so that no color appears twice **on the boundary of any cell**.

Example:



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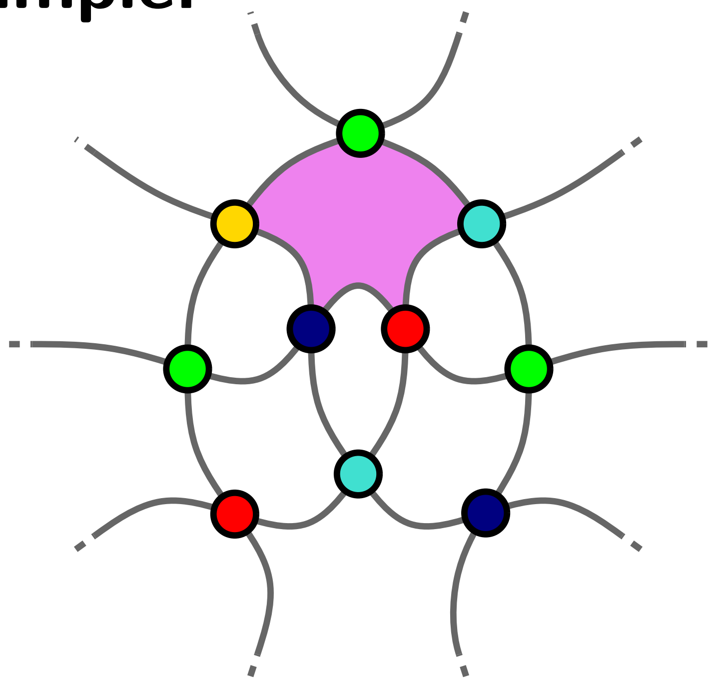

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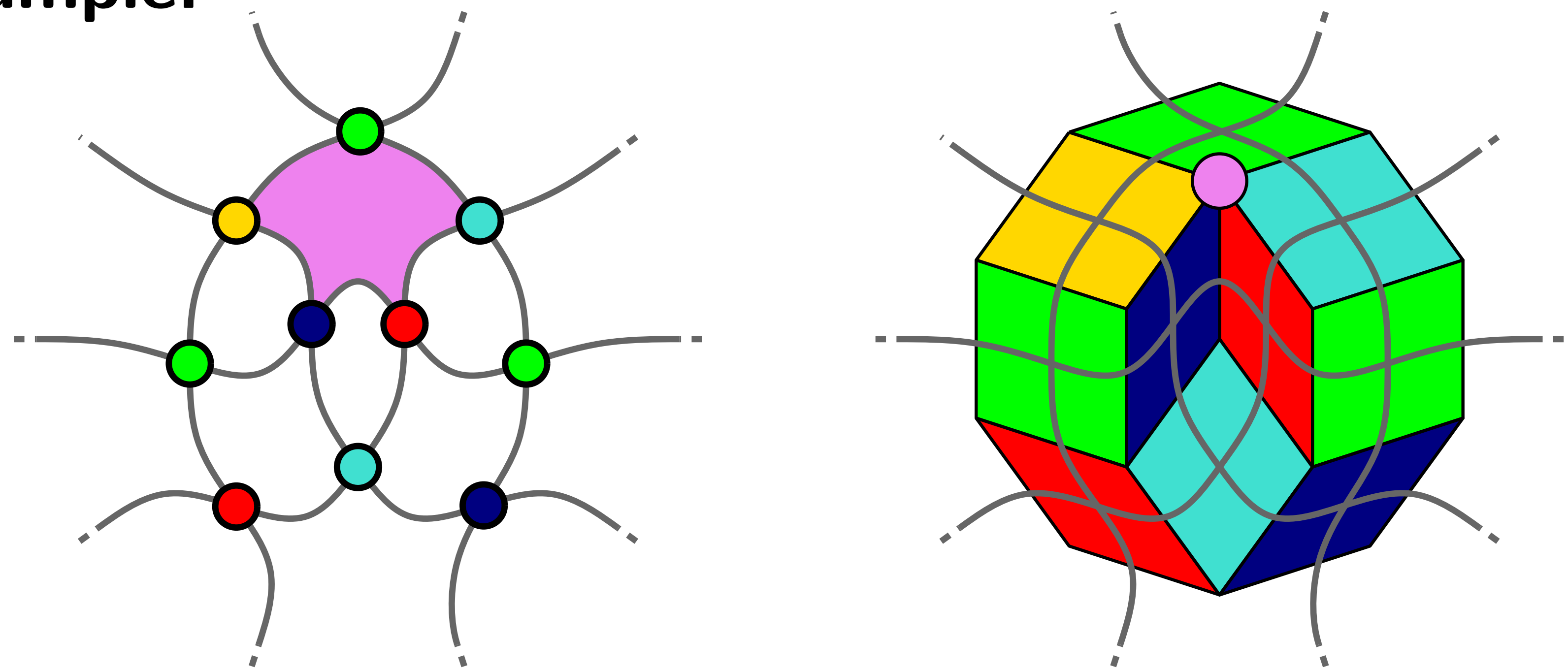

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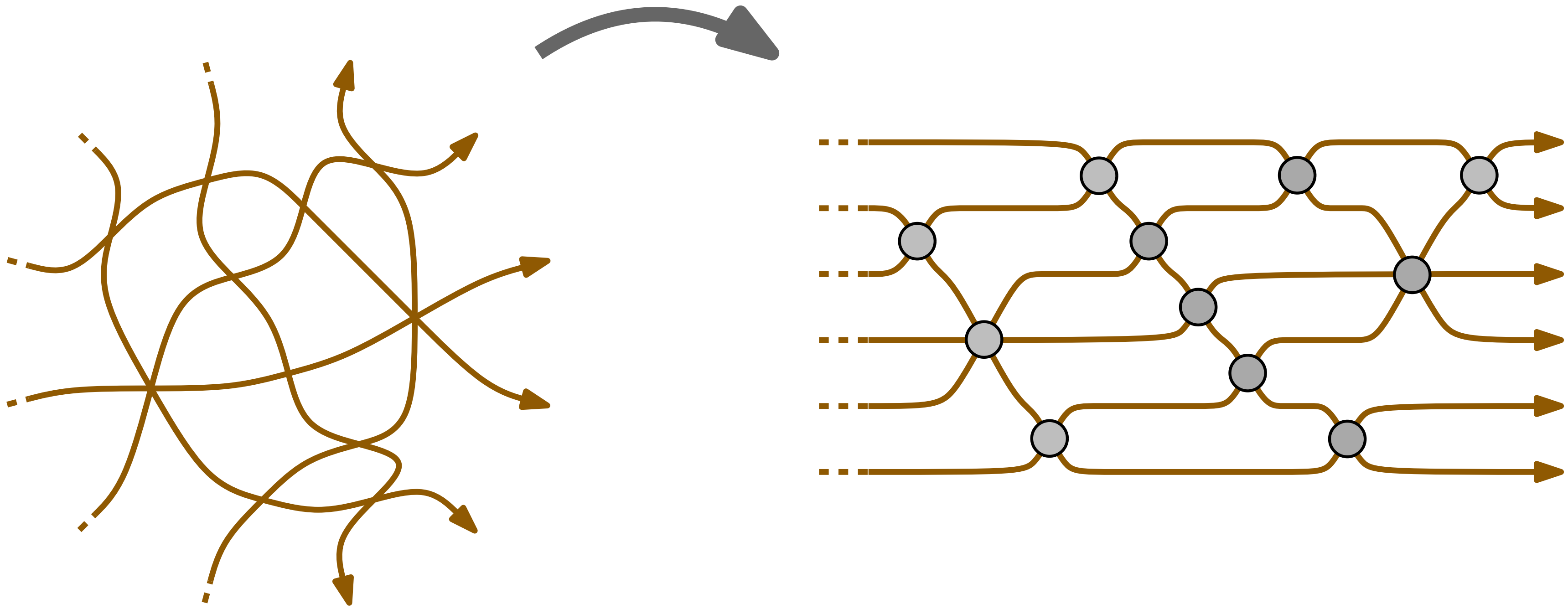
Proof idea: Greedily color the wiring diagram!



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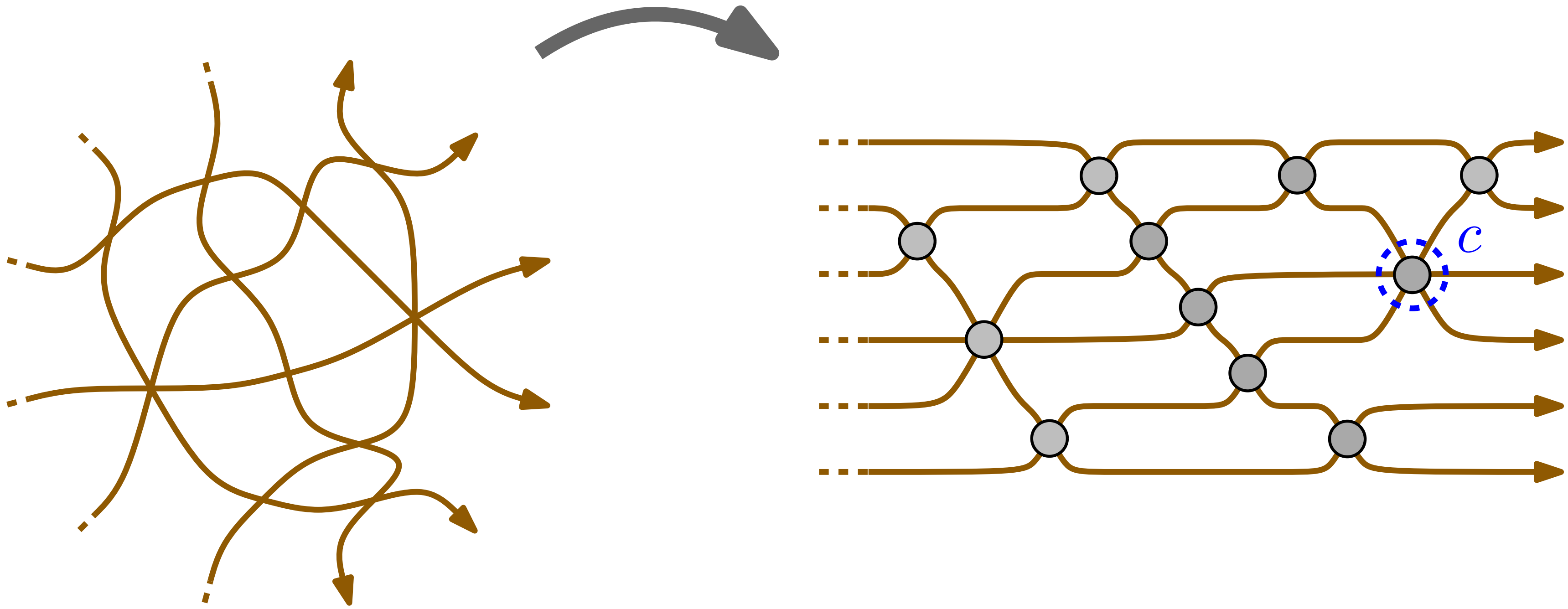

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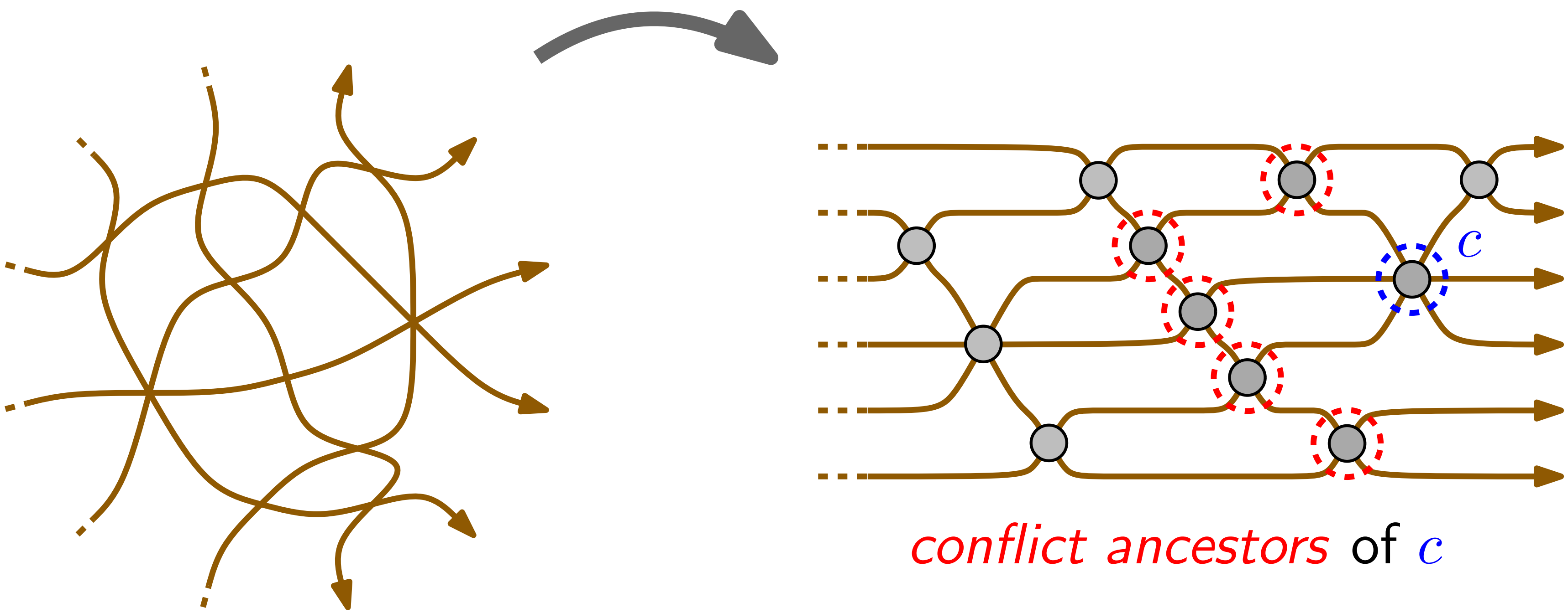
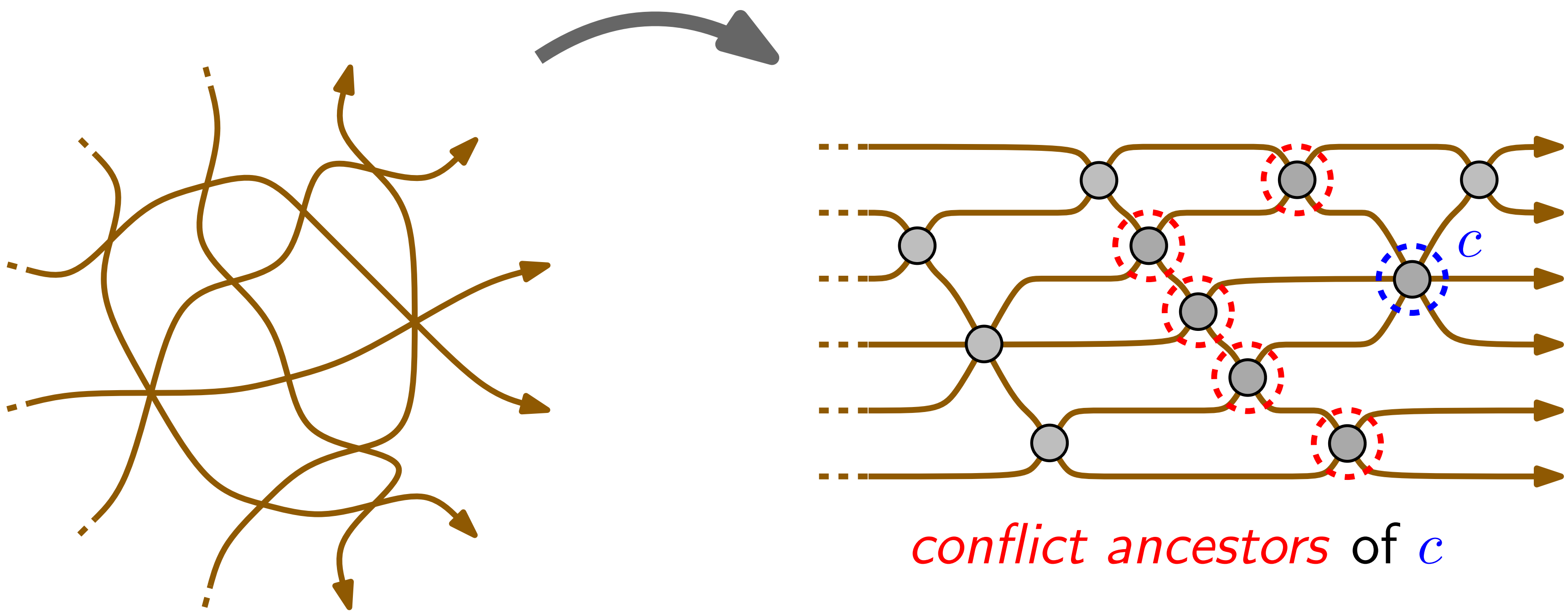


conflict ancestors of c

face respecting colorings

Proof idea: Greedily color the wiring diagram!



Claim: Every crossing has at most $n-1$ conflict ancestors.

$\square$

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Theorem: Let $\mathscr{A}$ be an arrangement of n pseudolines.

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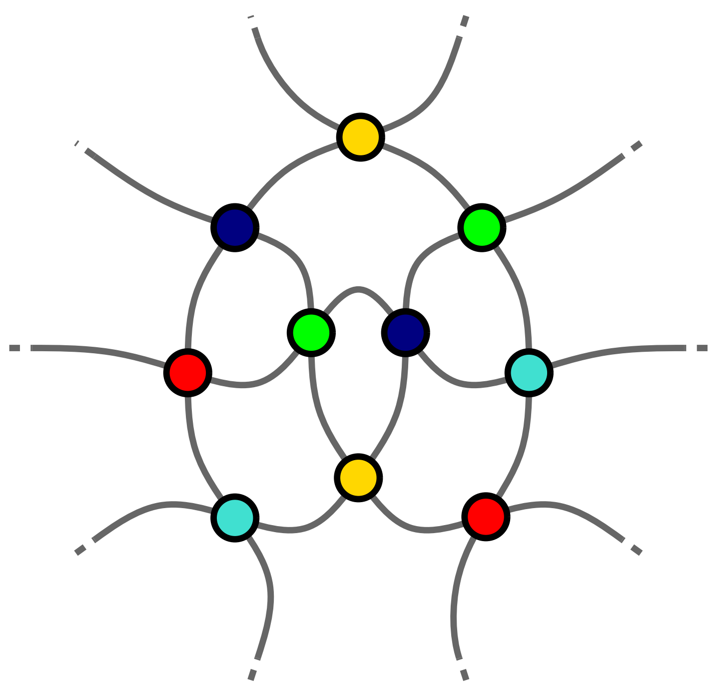

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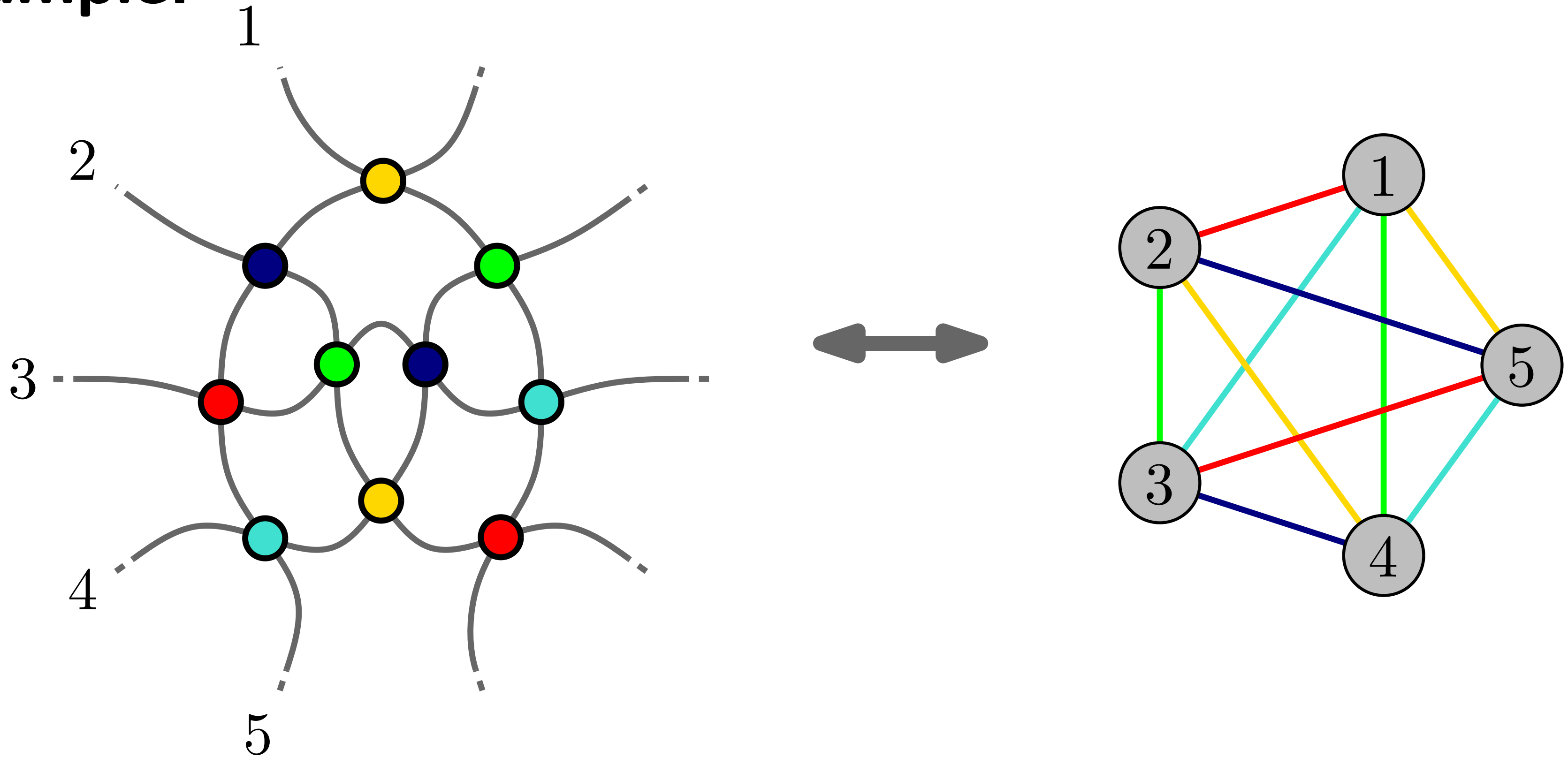

line respecting colorings

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line respecting colorings

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Hypergraph $\mathscr{H}(\mathscr{A})$:

- vertices $\sim$ pseudolines
- hyperedges $\sim$ crossings

line respecting colorings

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Theorem (Kang, Kelly, Kühn, Methuku, Osthus, 2023)

Every simple hypergraph on n vertices can be edge-colored using n colors.

Recent breakthrough in hypergraph coloring!!!

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line respecting colorings

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direct proof?

deterministic algorithm?

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Def:

$\mathrm{mx}(\mathscr{A}) :=$ max. number of crossings per pseudoline in $\mathscr{A}$

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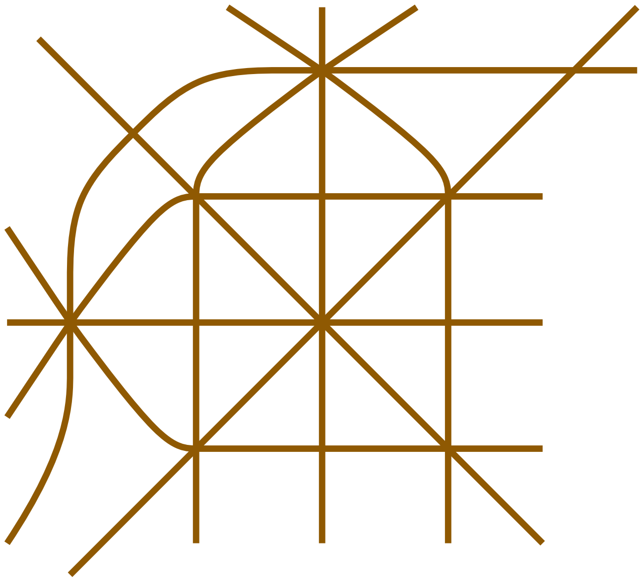

$\mathrm{mx}(\mathscr{A}) = 4$

line respecting colorings

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Fact: number of pseudolines $n \leq 845 \cdot \mathrm{mx}(\mathscr{A})$

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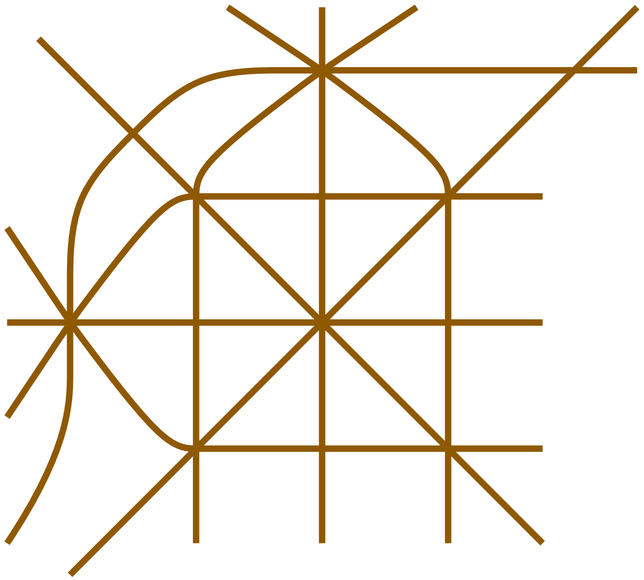

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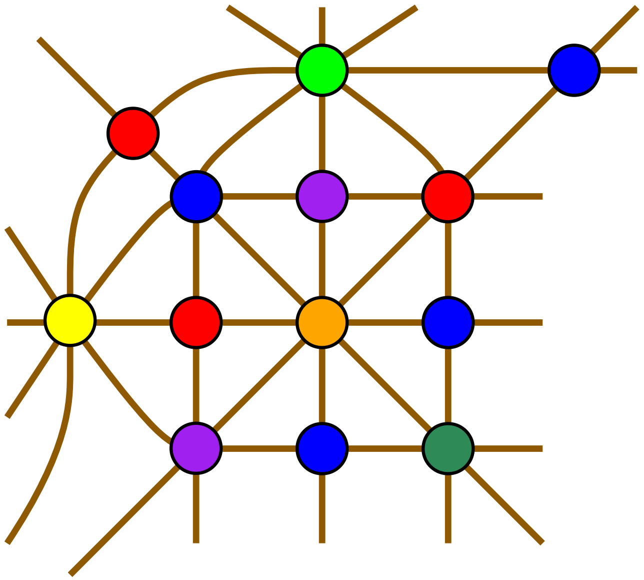

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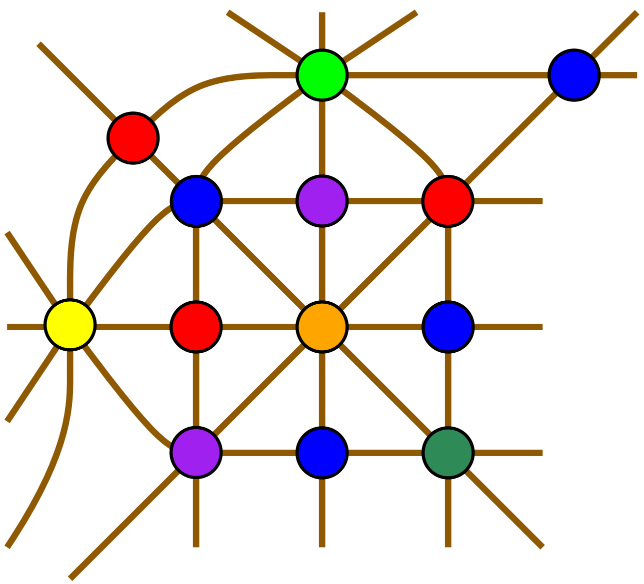
need $\mathrm{mx}(\mathscr{A}) + 3 = 7$ colors

line respecting colorings

Conjecture:
There exists some constant c so that one can color the crossings of every arrangement using $\mathrm{mx}(\mathscr{A}) + c$ colors.

Example:



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pseudoline coloring

Def: *pseudoline coloring* of arrangement $\mathscr{A}$:

- color the pseudolines of $\mathscr{A}$
- avoiding monochromatic crossings

$\chi_{pl}(\mathscr{A})$: minimal number of colors in pseudoline coloring

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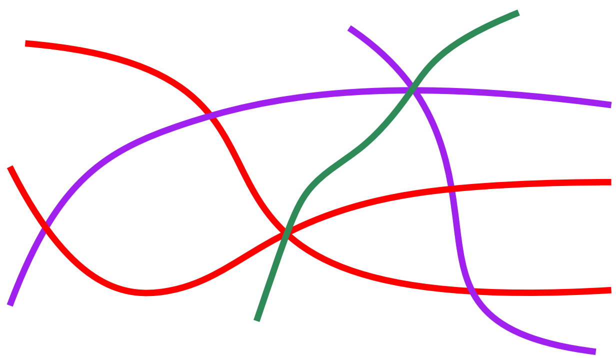

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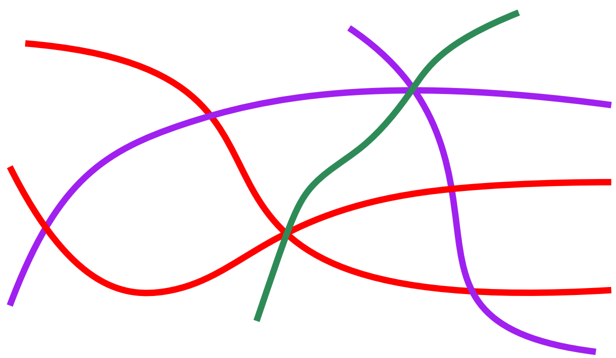

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First observations:

- $2 \leq \chi_{pl}(\mathscr{A}) \leq n$ (unless $n < 2$)

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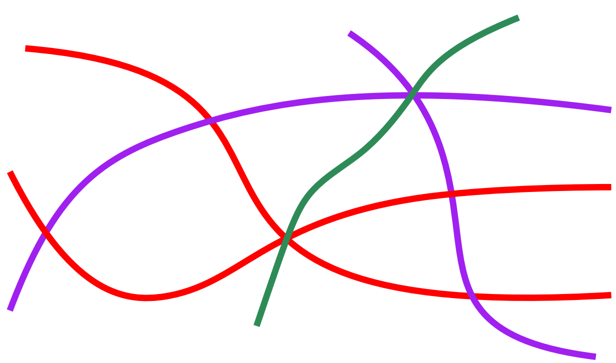

$\chi_{pl}(\mathscr{A}) = 3$

First observations:

- $2 \leq \chi_{pl}(\mathscr{A}) \leq n$ (unless $n < 2$)
- $\mathscr{A}$ simple $\Leftrightarrow \chi_{pl}(\mathscr{A}) = n$

pseudoline coloring

Theorem:

Let $\mathscr{A}$ be an arrangement of n pseudolines.

The pseudolines of $\mathscr{A}$ can be colored using $\mathscr{O}(\sqrt{n})$ colors avoiding monochromatic crossings of degree at least 4.

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Proposition:

Given an arrangement $\mathscr{A}$ of n pseudolines, it is NP-hard to compute $\chi_{pl}(\mathscr{A})$.

Questions?