Berlin Mathematical School

BMS

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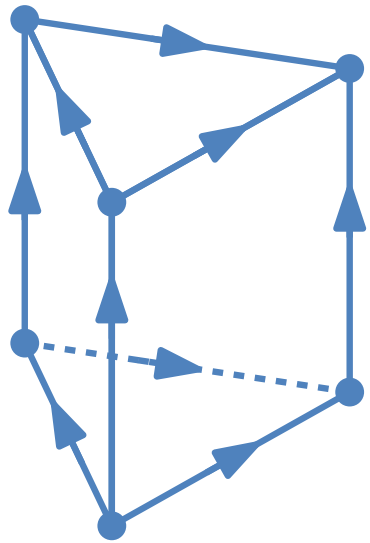
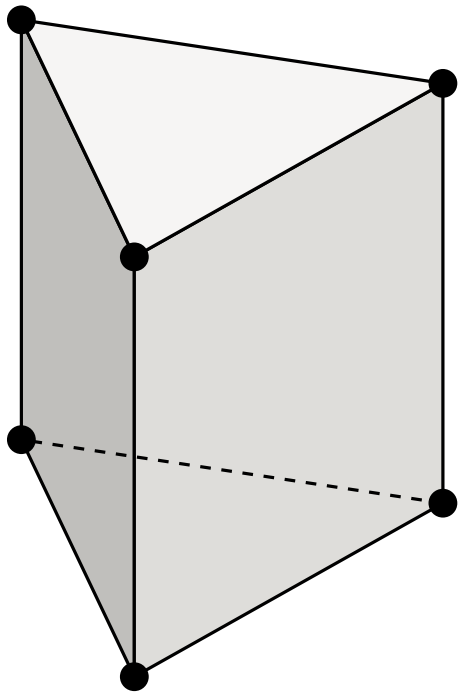
EuroCG 26

Signotopes Induce Unique Source Orientations on Grids

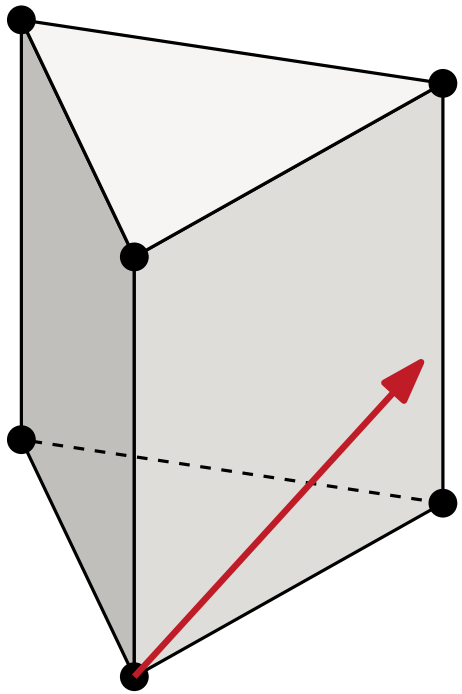


Sandro M. Roch

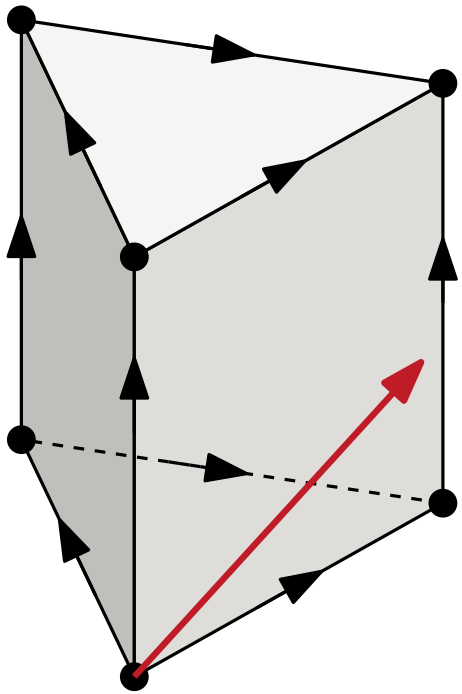
linear orientations



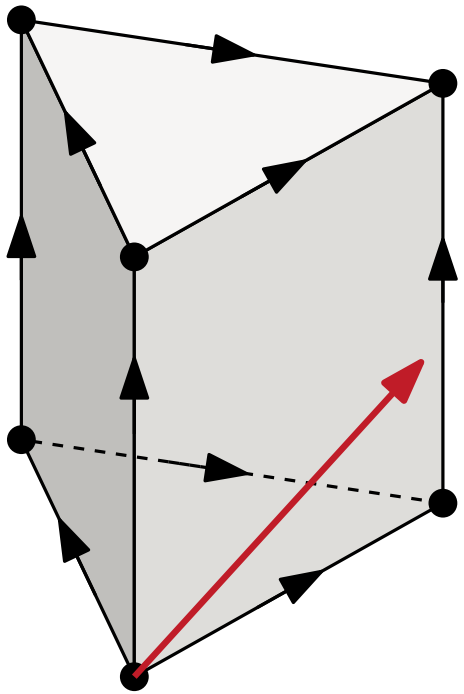
linear orientations



linear orientations



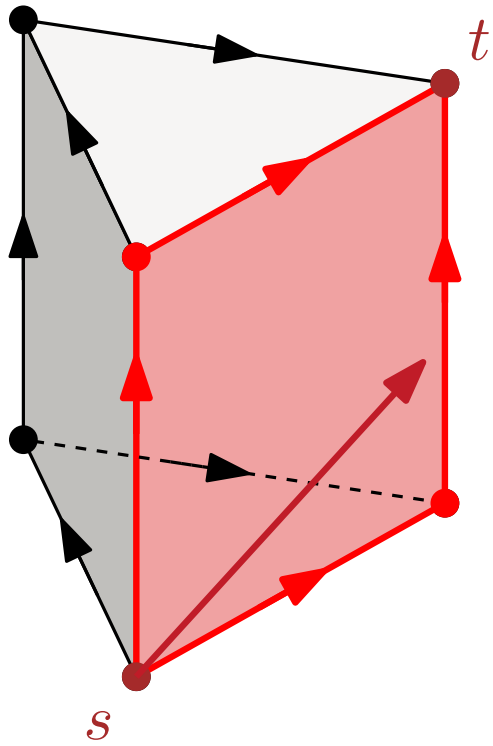
linear orientations



Properties of *linear orientations*:

- acyclic
- *Unique Sink Orientation USO*: Every face contains a unique source and a unique sink.
- *Holt-Klee-property:* On every face F, there exist $\dim F$ many internally disjoint paths from unique source to unique sink.

linear orientations



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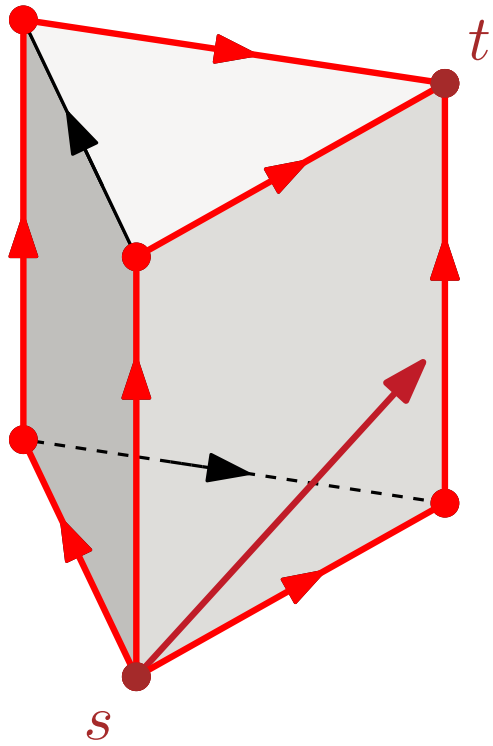

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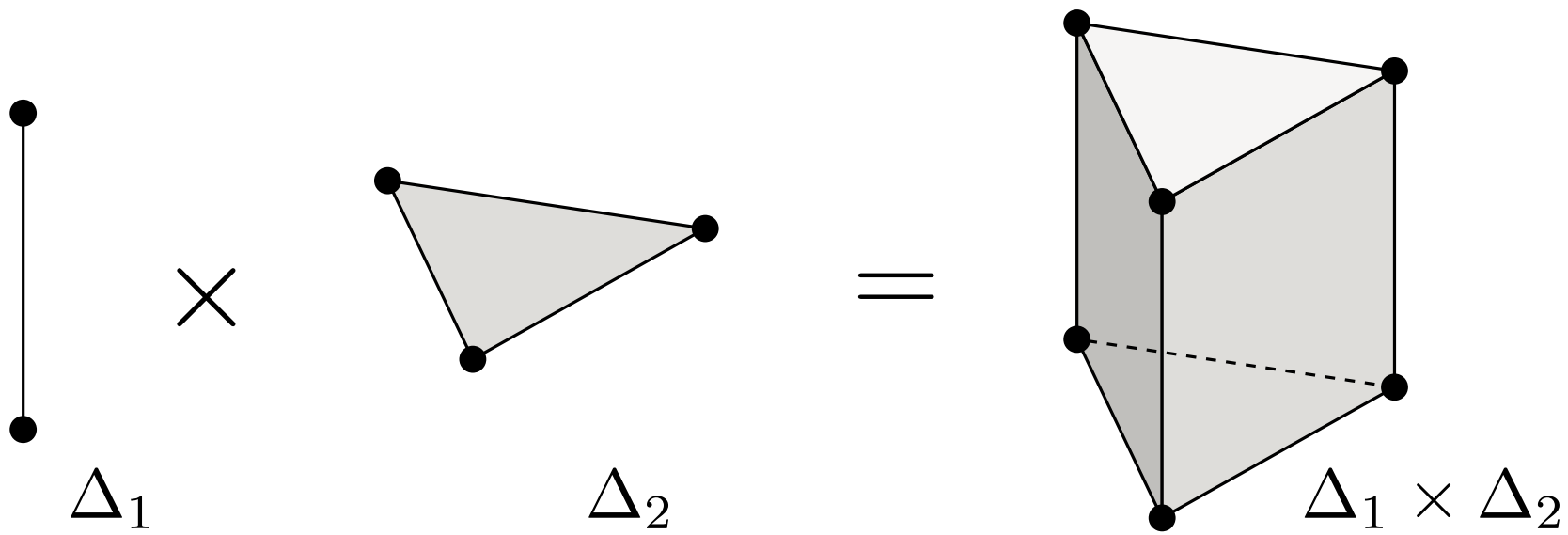
$(d, d+2)$-polytopes

Fact:
(Felsner, Gärtner, Tschirschnitz, 2005)

Every simple d-polytope with $d+2$ facets is combinatorially equivalent to a product of two simplices.



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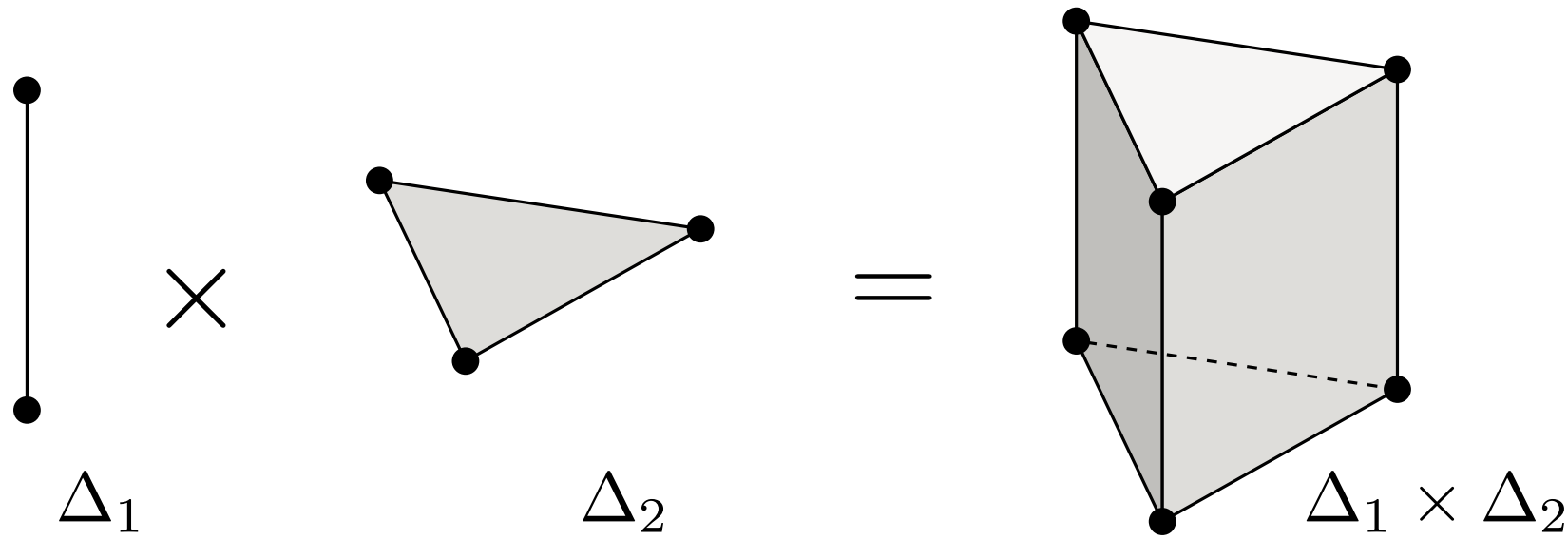

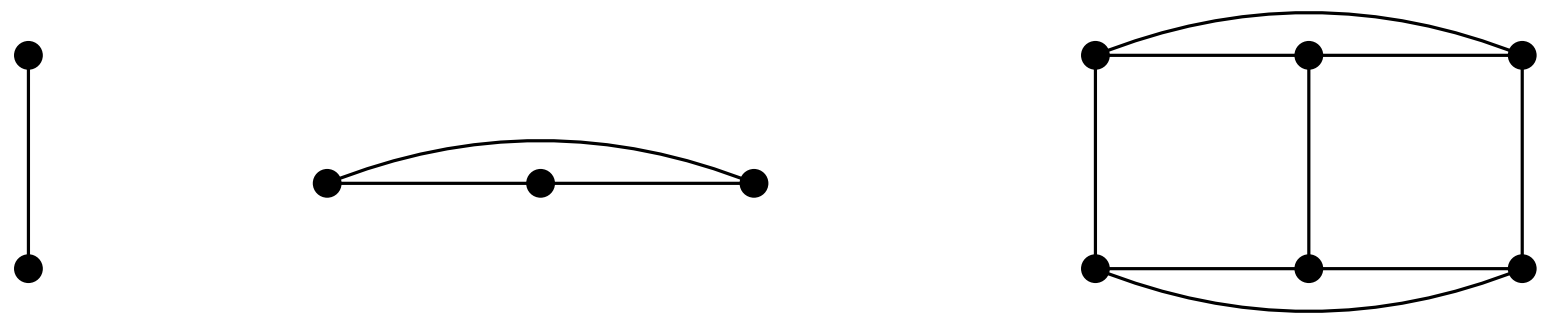

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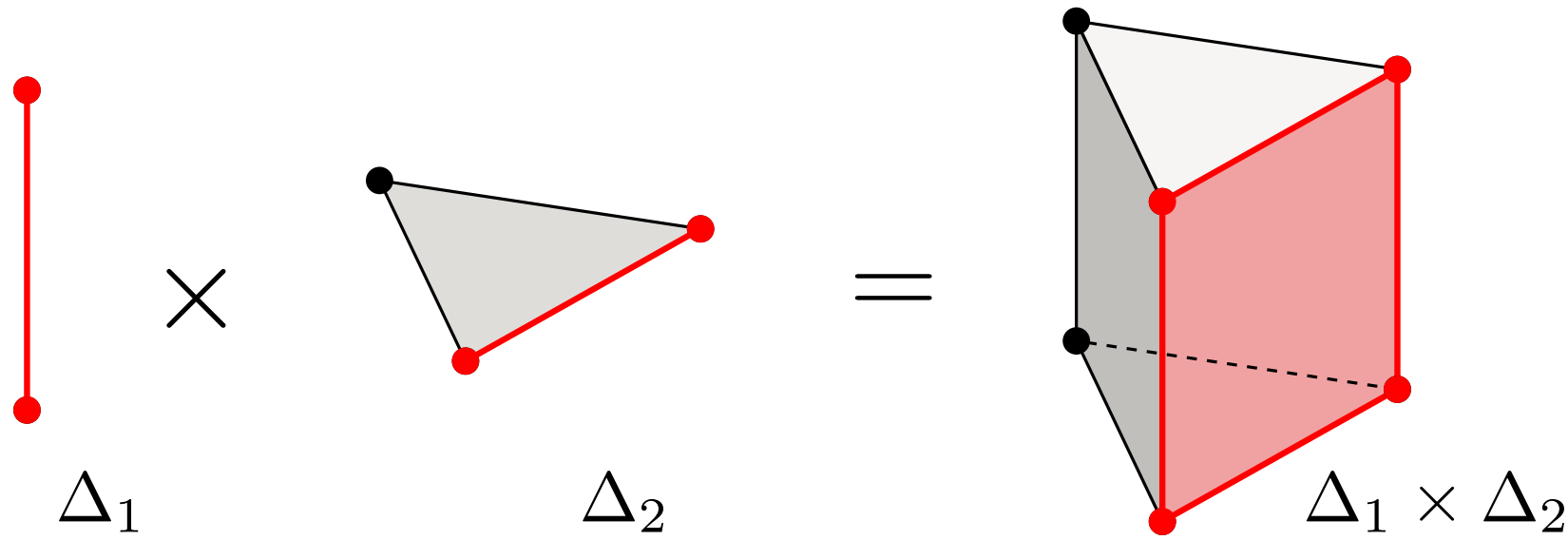
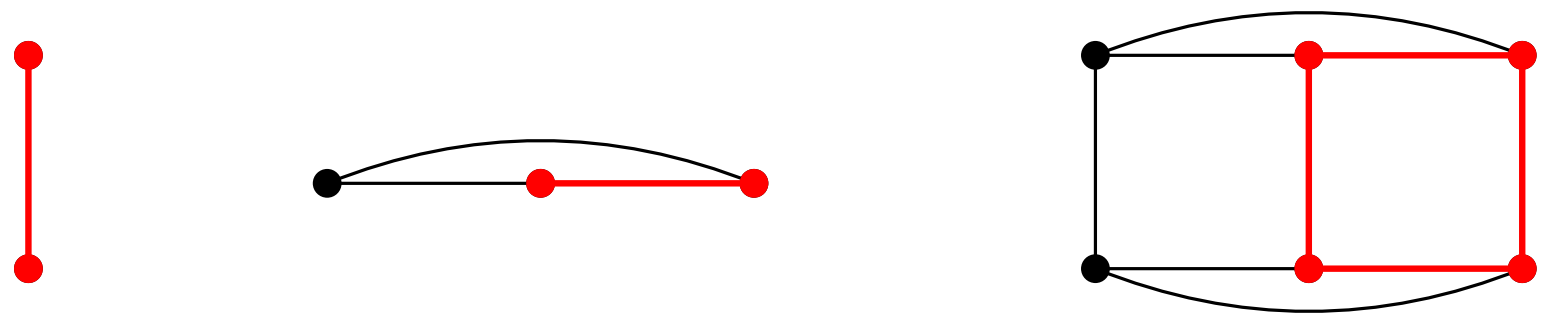
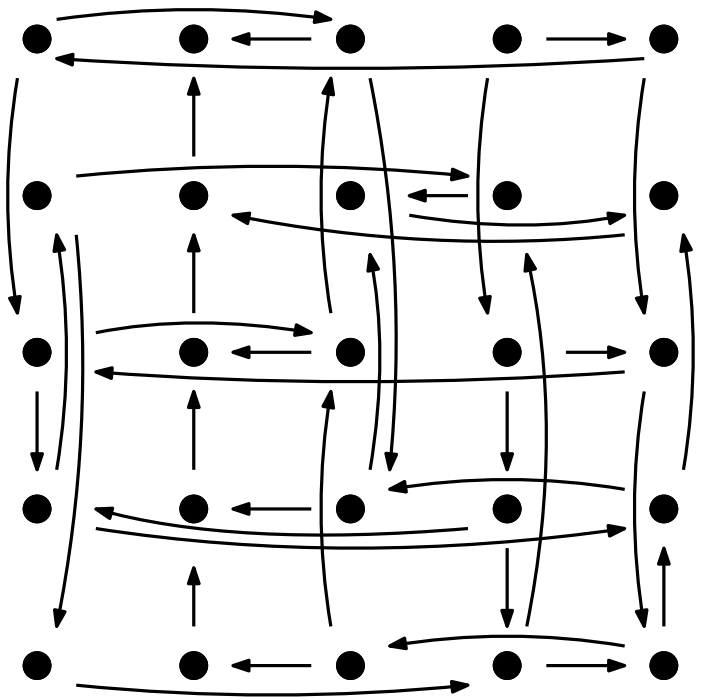




unique sink orientations (USOs)



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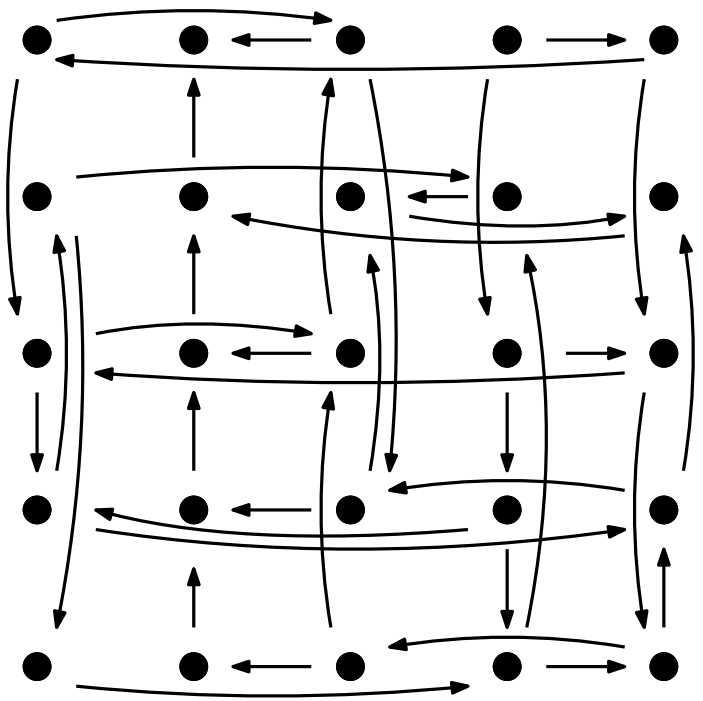

$(4,2)$ $(0,0)$ $(2,1)$ $(4,4)$ $(4,3)$

$(1,4)$ $(1,0)$ $(0,2)$ $(0,3)$ $(0,1)$

$(3,2)$ $(2,0)$ $(3,1)$ $(3,4)$ $(3,3)$

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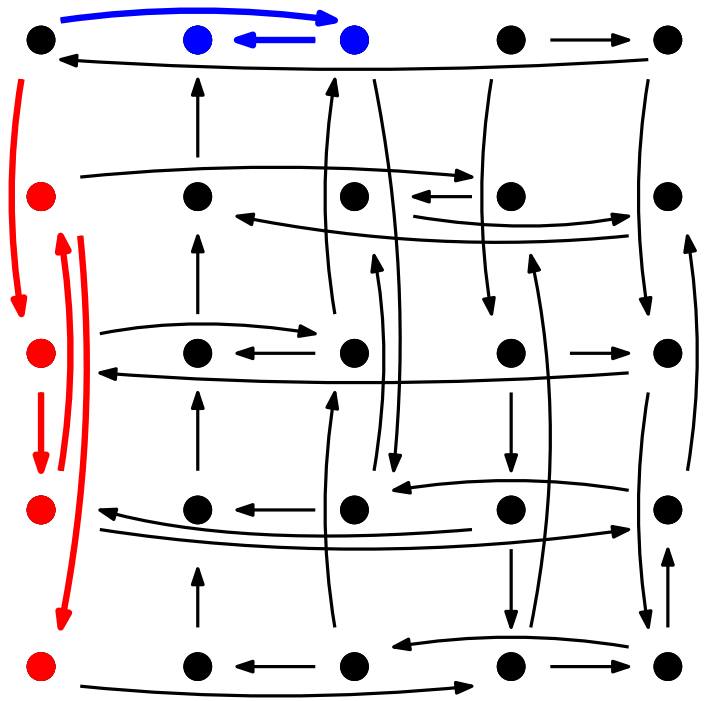

$\mathrm{rf}(i,j) := (\mathrm{outdeg_v}(i,j), \mathrm{outdeg_h}(i,j))$

$(4,2)\ (0,0)\ (2,1)\ (4,4)\ (4,3)$

$(1,4)\ (1,0)\ (0,2)\ (0,3)\ (0,1)$

$(3,2)\ (2,0)\ (3,1)\ (3,4)\ (3,3)$

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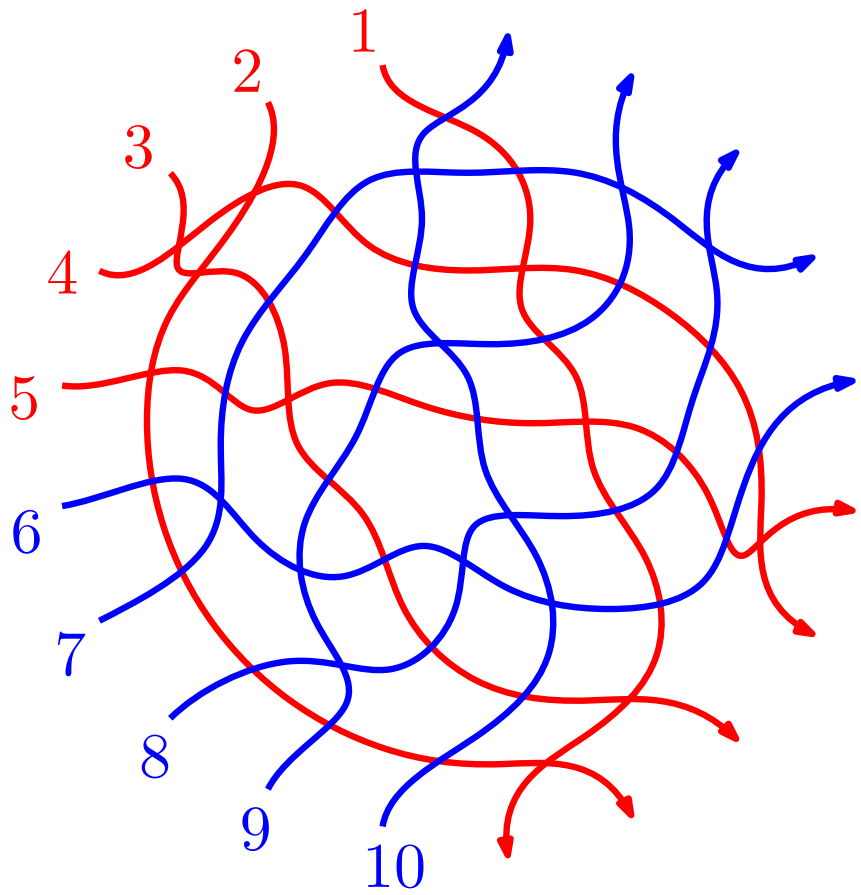
red-blue-arrangement $\rightarrow$ USO

red-blue-arrangement $\longrightarrow$ USO



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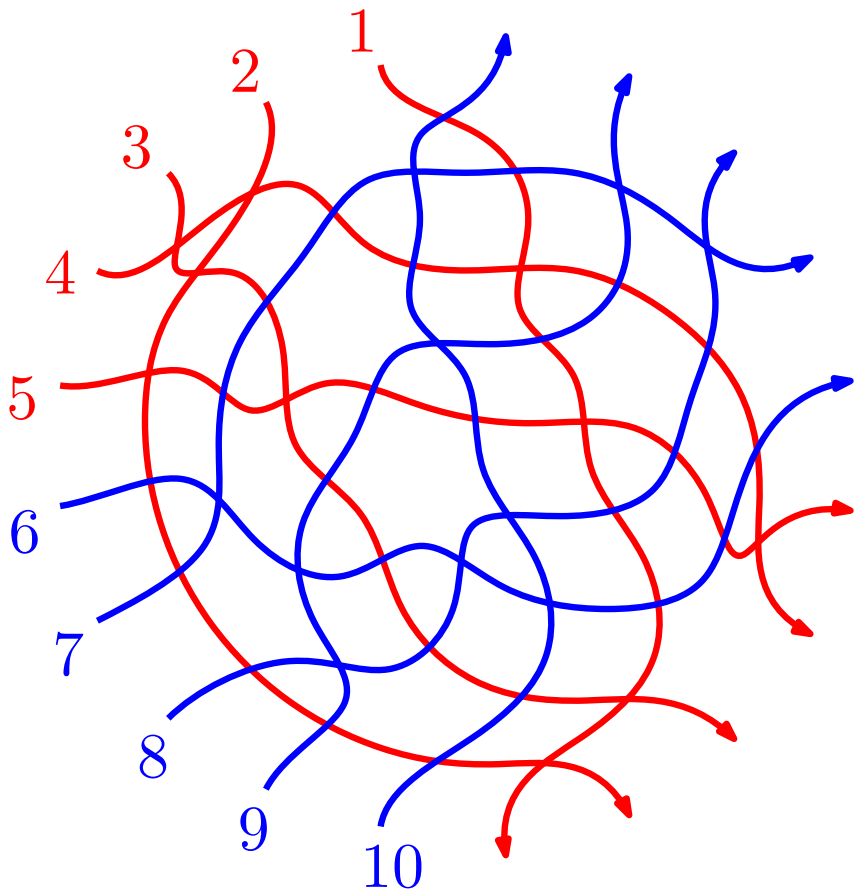

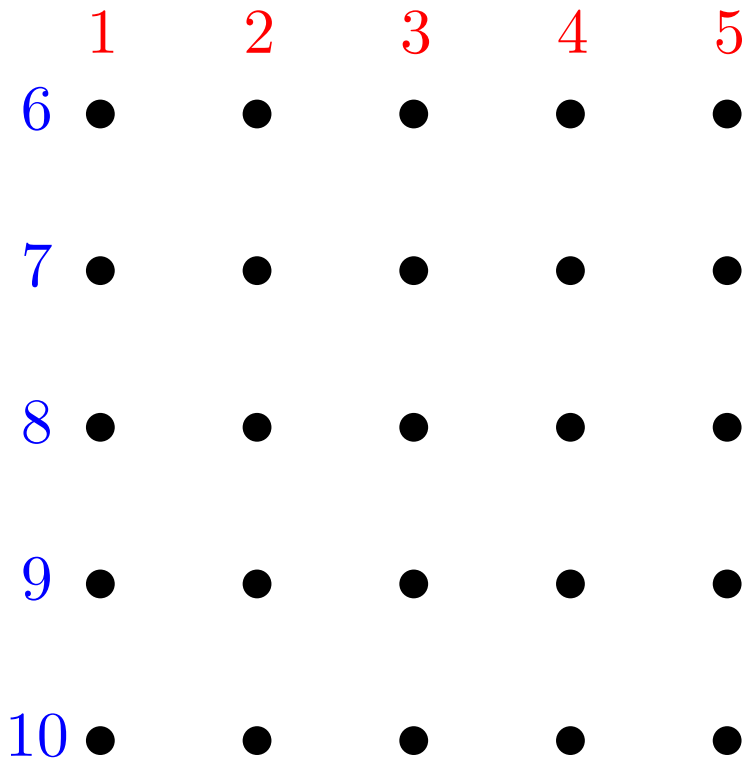

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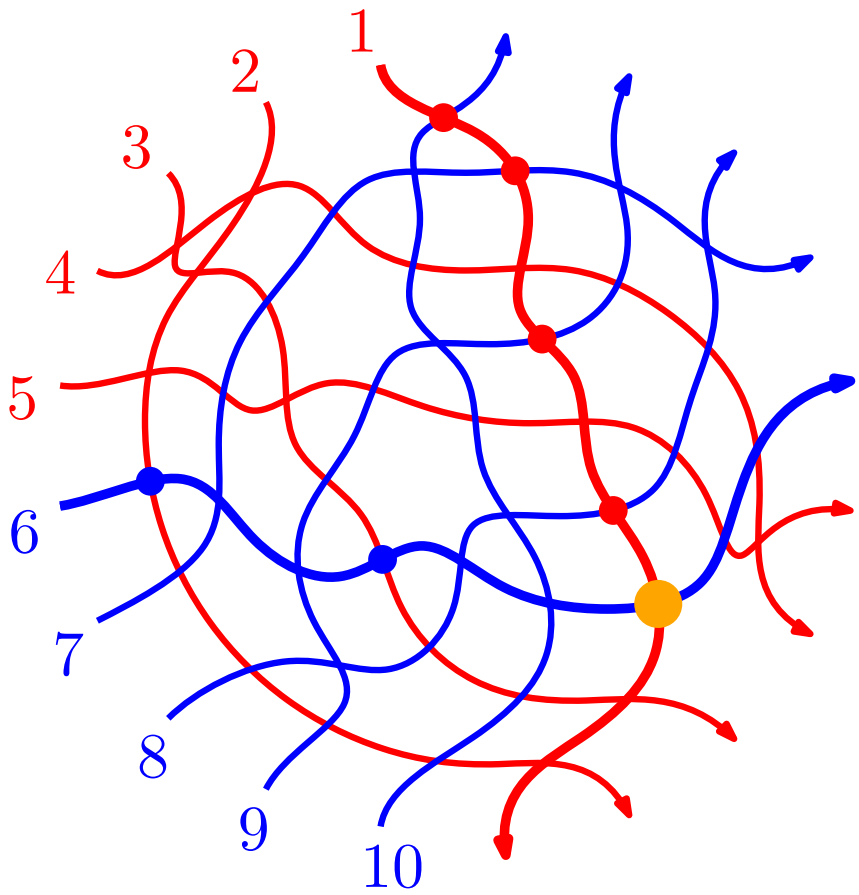

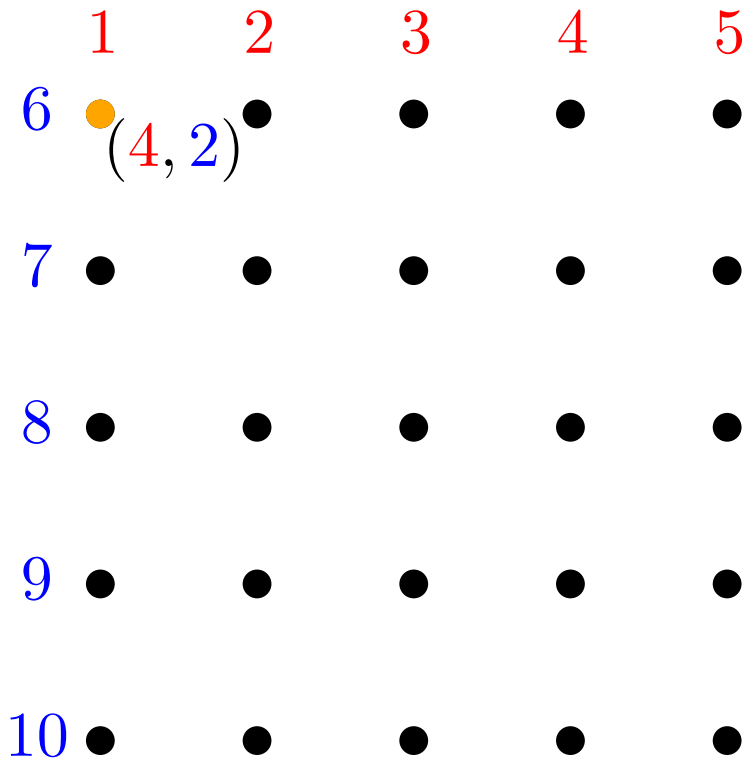

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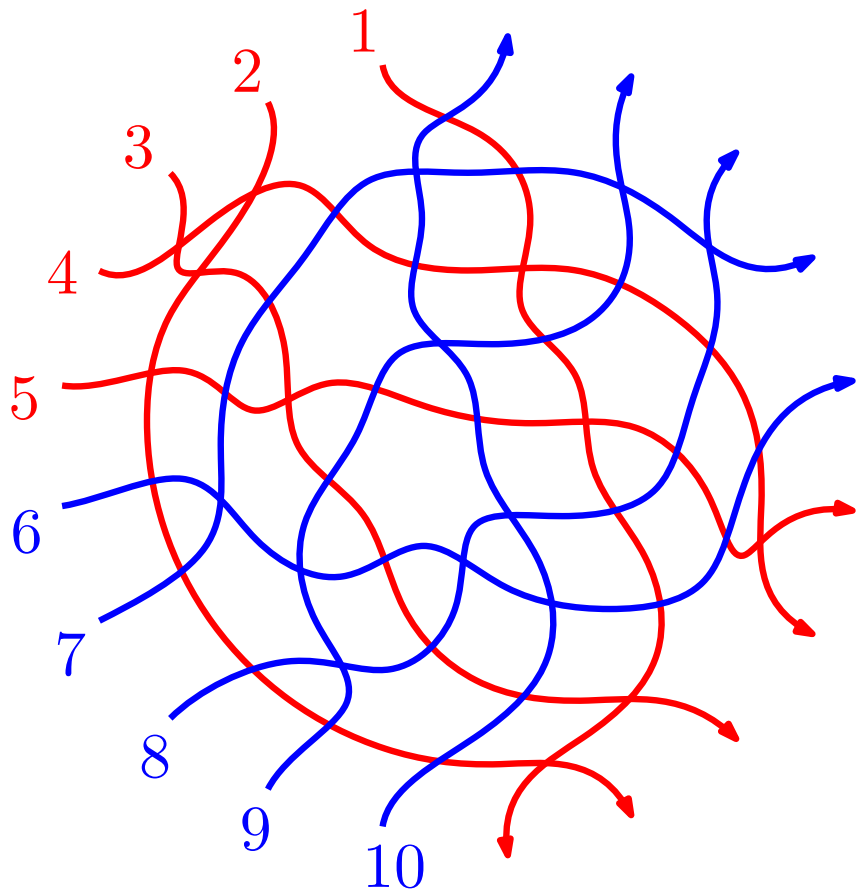

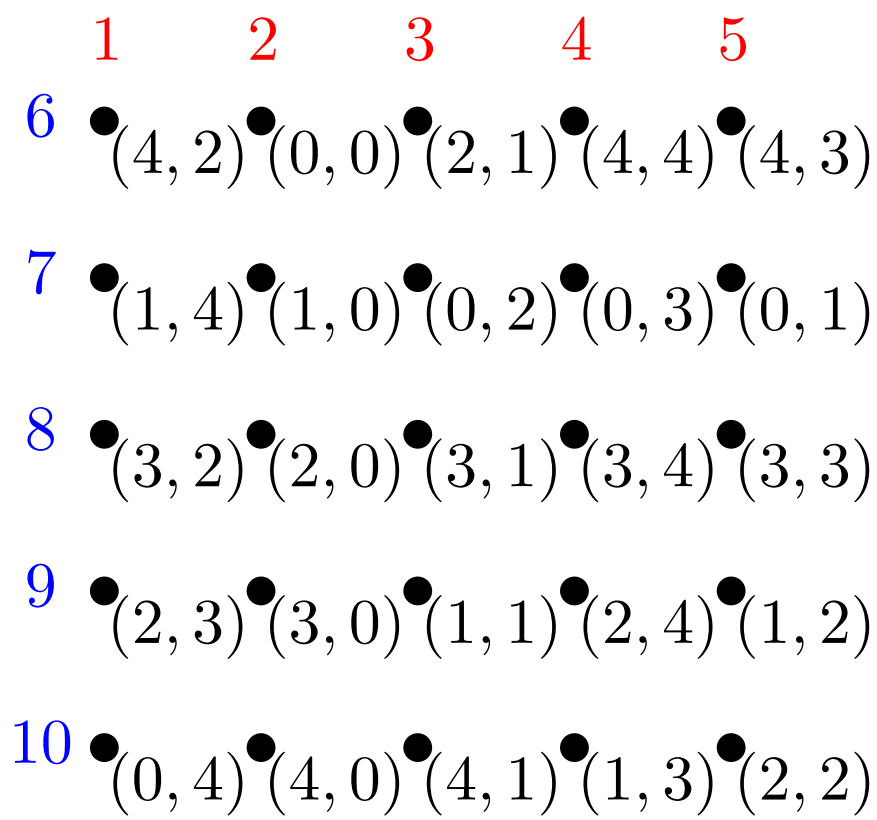

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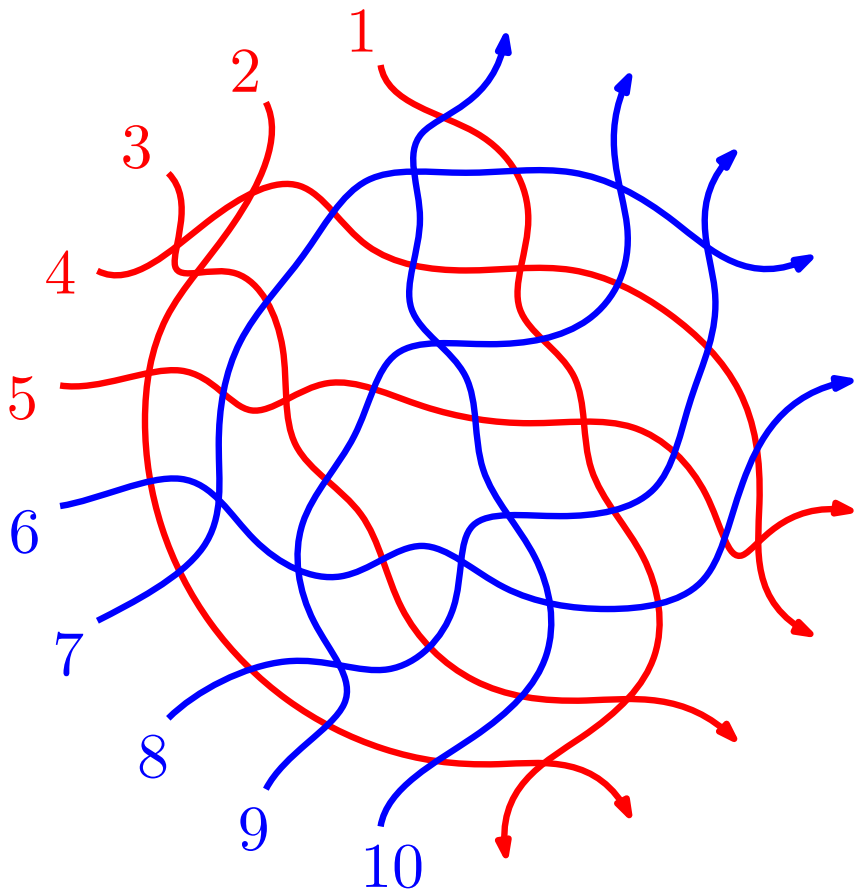

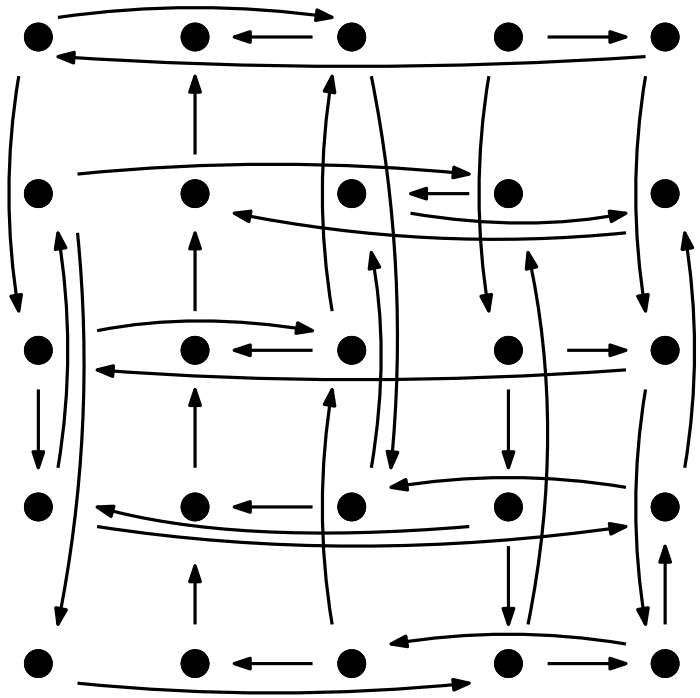

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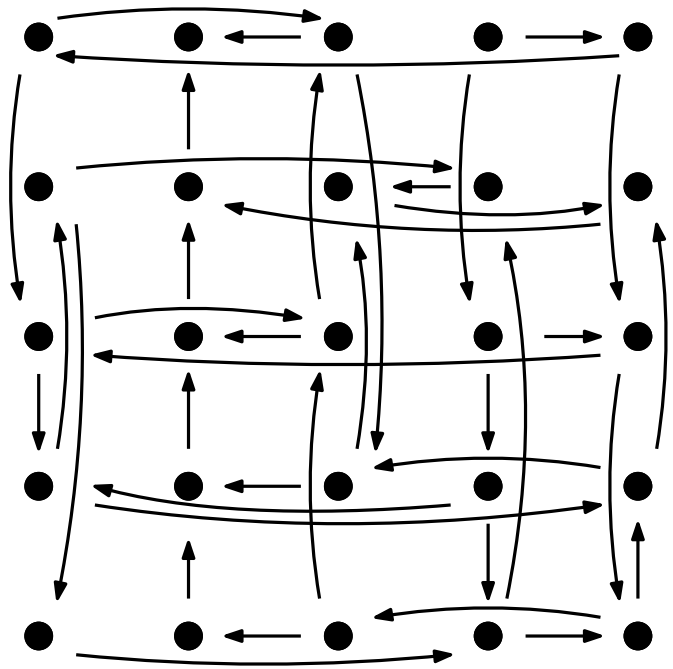

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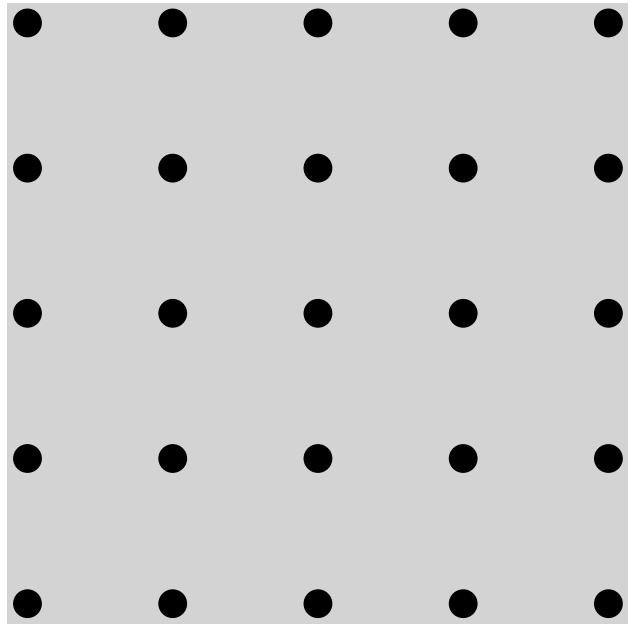

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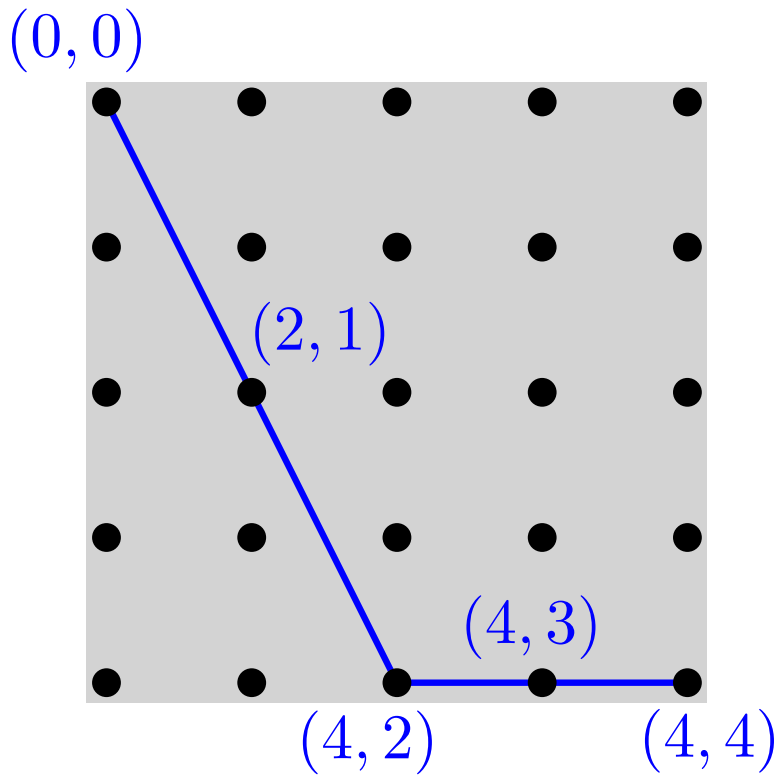

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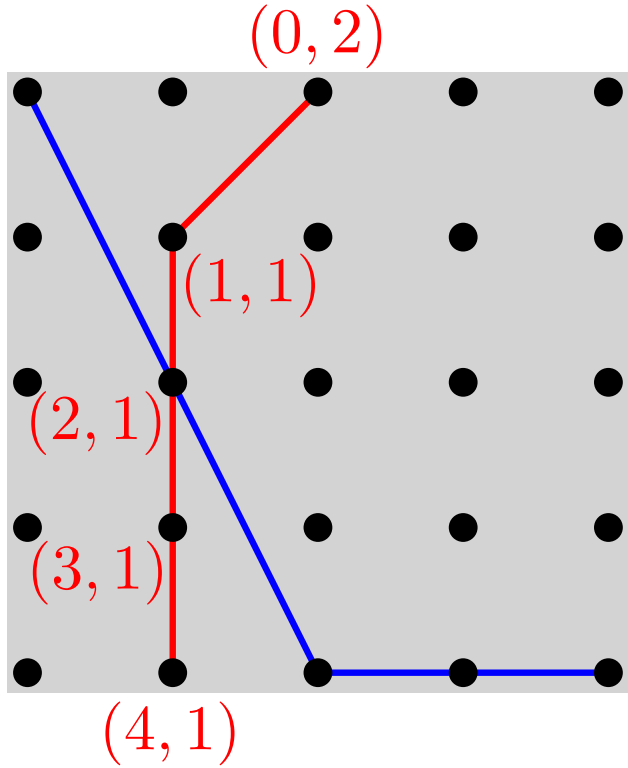

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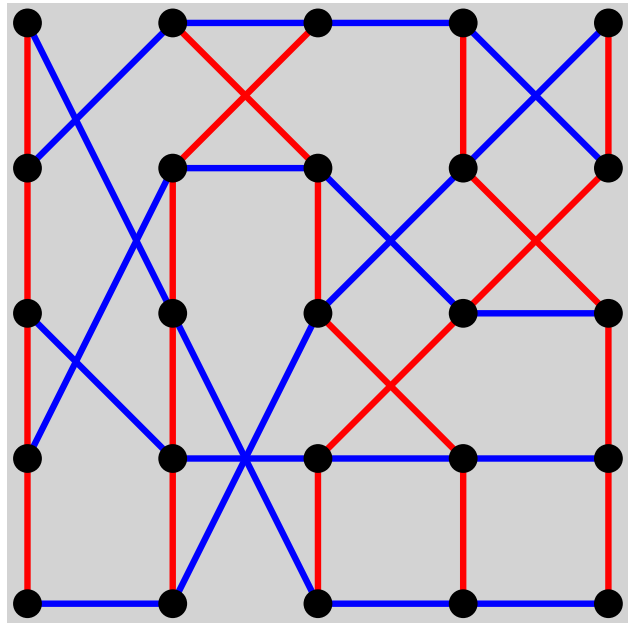

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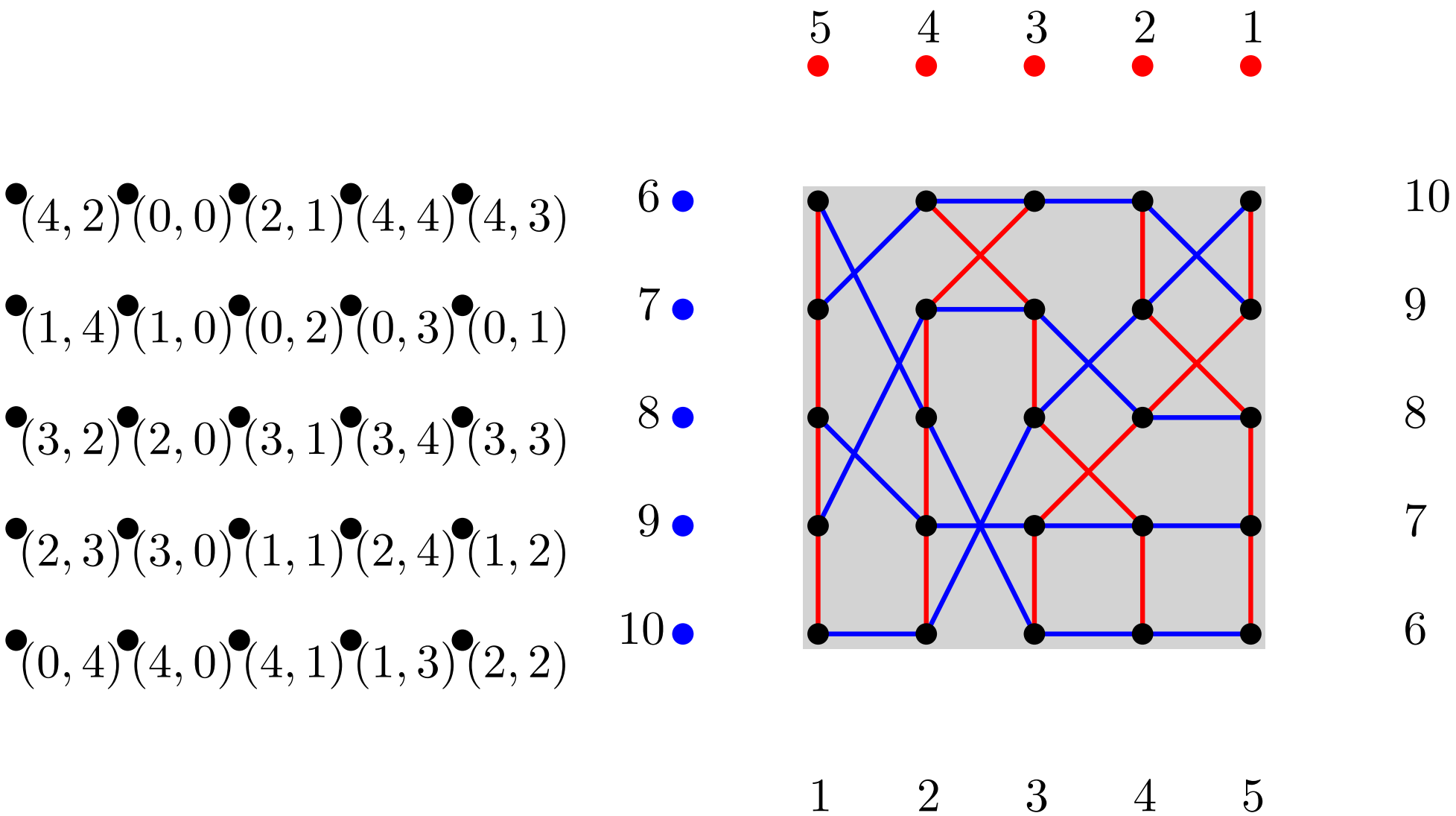

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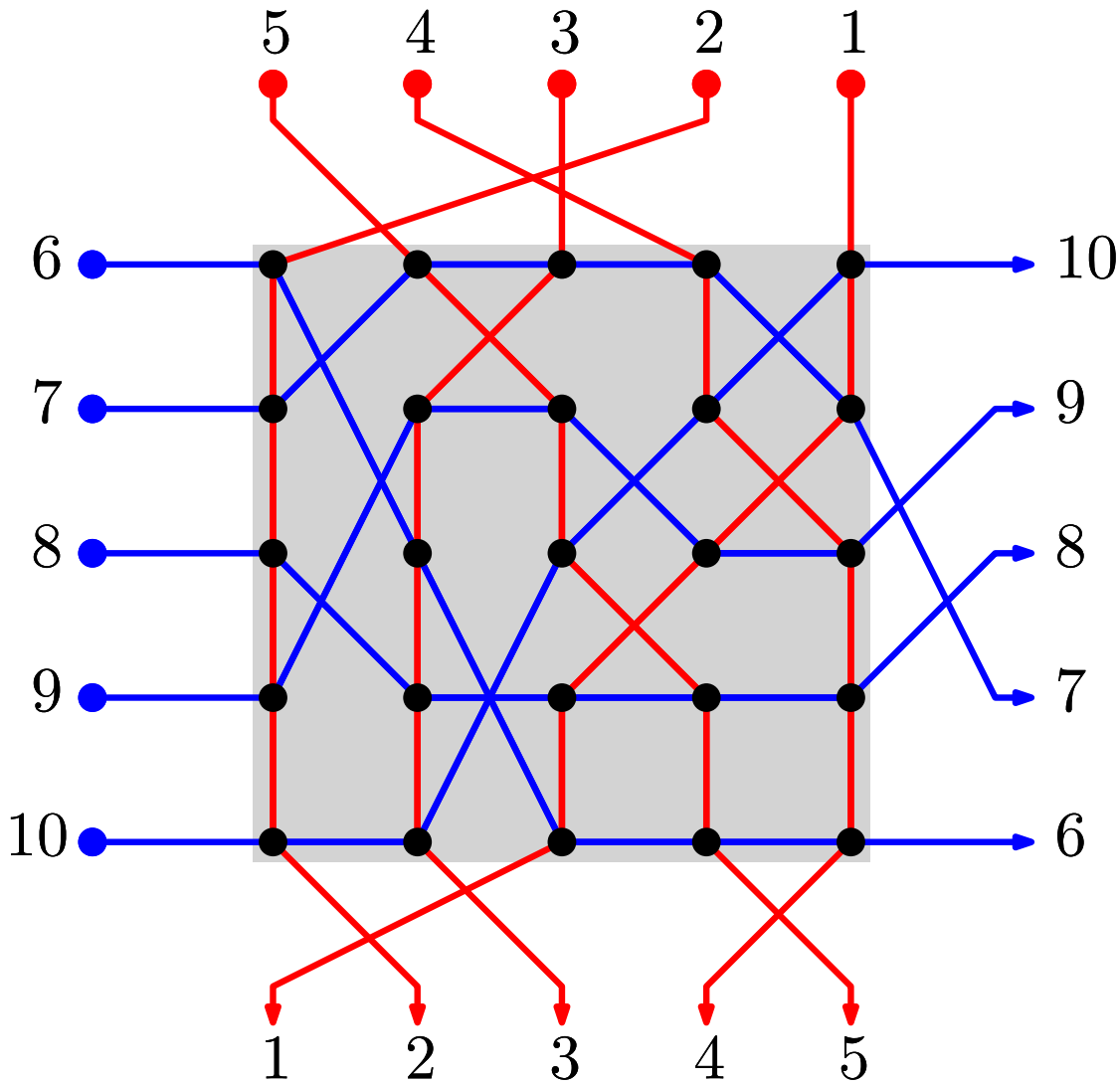

USOs with Holt-Klee $\leftrightarrow$ red-blue-arrangements

Theorem
(Felsner, Gärtner, Tschirschnitz, 2005)

1) USOs with Holt-Klee-property are exactly the orientations induced by red-blue-arrangements.
2) They are linear orientations if and only if the arrangement is *stretchable*.

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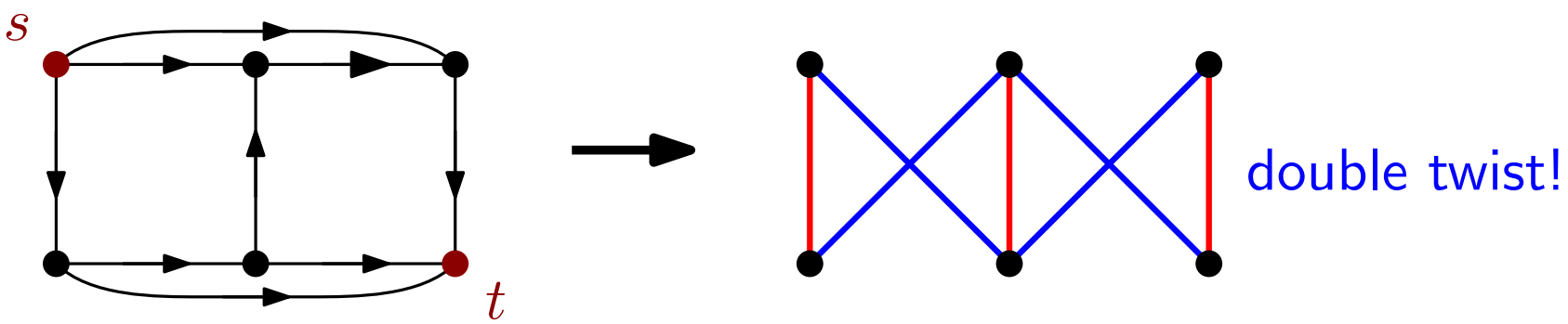
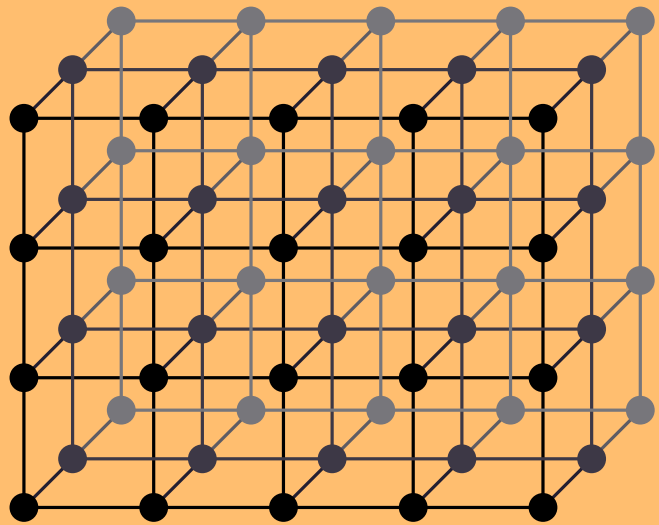
Acyclic USO which violates Holt-Klee-property:



USOs with Holt-Klee $\leftrightarrow$ red-blue-arrangements

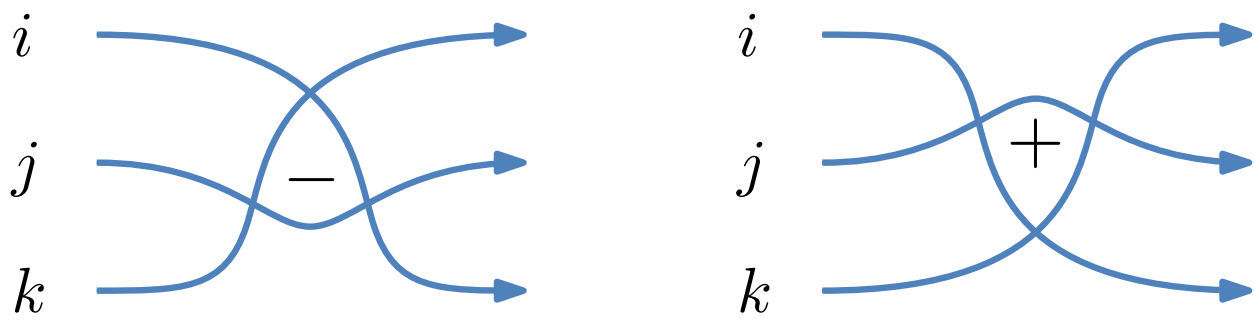
Generalize this to
higher dimensional grids ???



$\cong \quad \Delta_2 \times \Delta_4 \times \Delta_2$

arrangements $\leftrightarrow$ 3-signotopes

Two cases for pseudolines $i < j < k$:



Pseudoline arrangement $\mathscr{A}$ defines map: $\chi_{\mathscr{A}} : \binom{[n]}{3} \to \{-, +\}$

arrangements $\leftrightarrow$ 3-signotopes

Definition:
A map $\chi : \binom{[n]}{3} \to \{-,+\}$ is called 3-*signotope* if for all 4-tuples $1 \le i < j < k < l \le n$ we have:

$$(\chi(jkl), \chi(ikl), \chi(ijl), \chi(jkl)) \in \Big\{(+++{+}), (+++{-}),$$
$$(++{--}), (+{--}{-}),$$
$$({-}{-}{--}), ({-}{-}{-}{+}),$$
$$({-}{-}{+}{+}), ({-}{+}{+}{+})\Big\}$$

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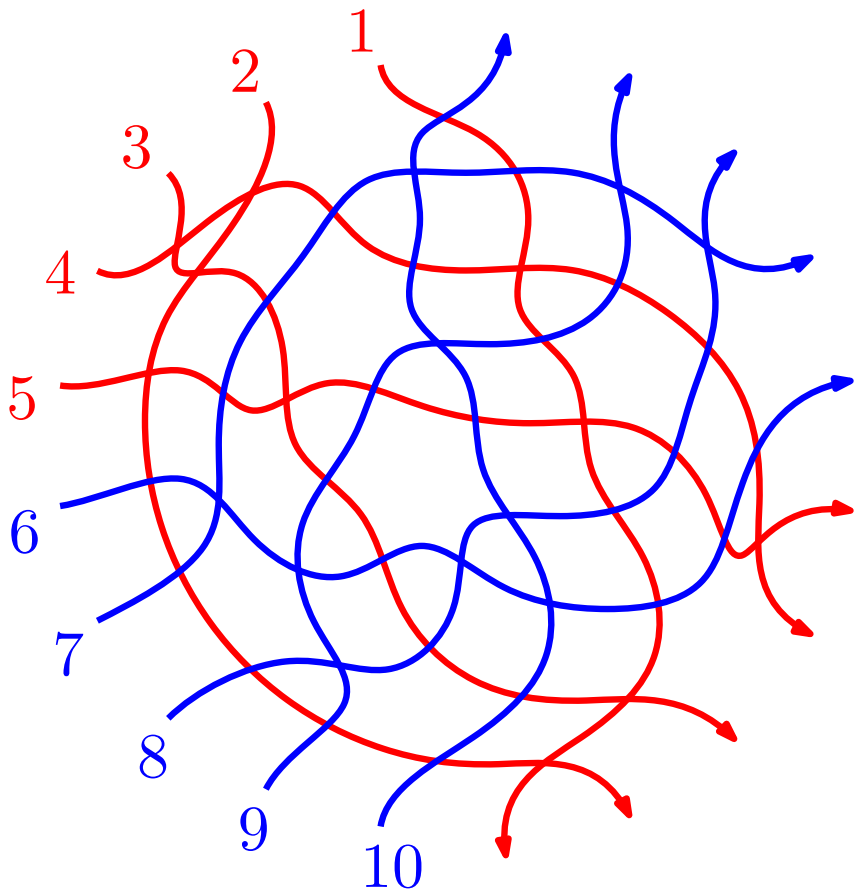
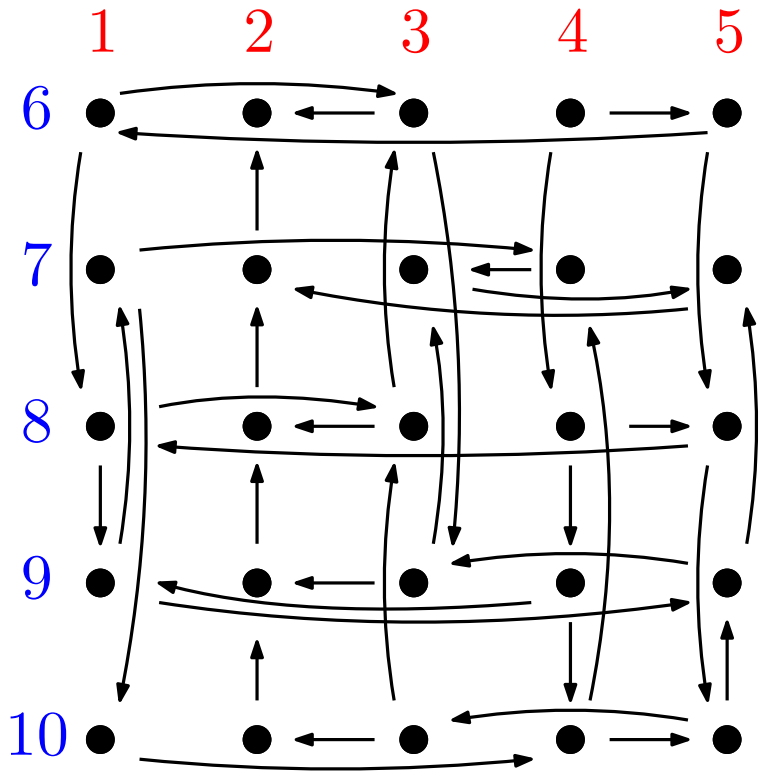
$$(++{--}), (+{--}{-}),$$
$$({---}{-}), ({---}{+}),$$
$$({--}{++}), ({-}{+}{++})\Big\}$$

- Bijection: pseudoline arrangements $\leftrightarrow$ 3-signotopes
- Define more generally r-*signotopes* $\chi : \binom{[n]}{r} \to \{-,+\}$

3-signotopes $\leftrightarrow$ USOs





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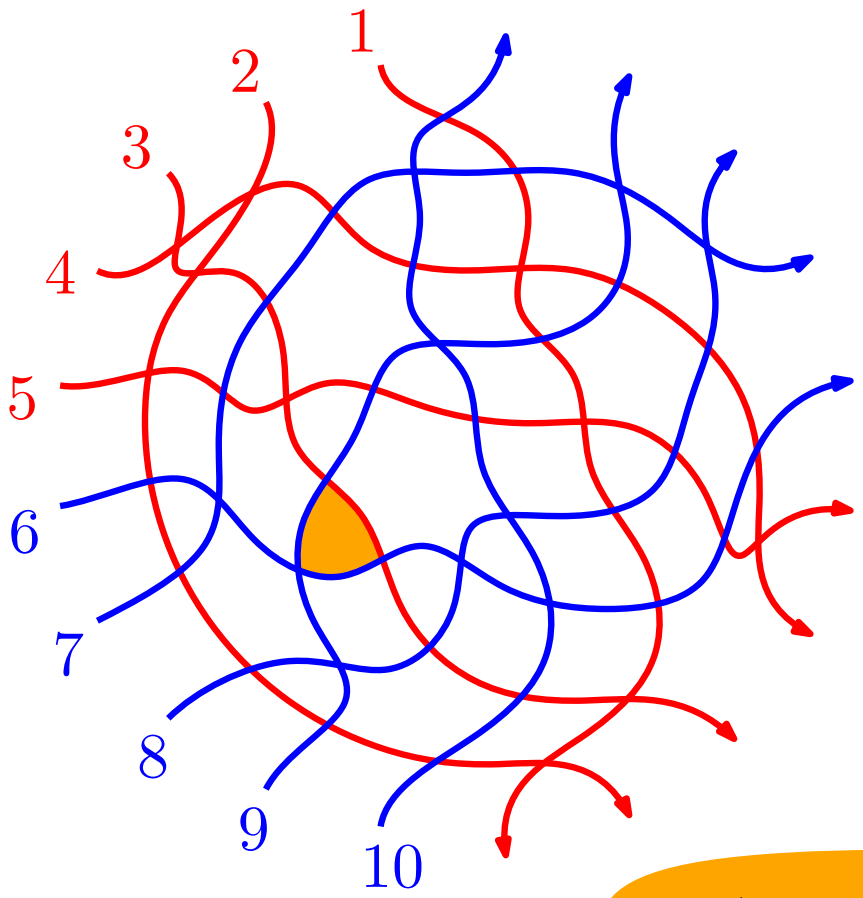
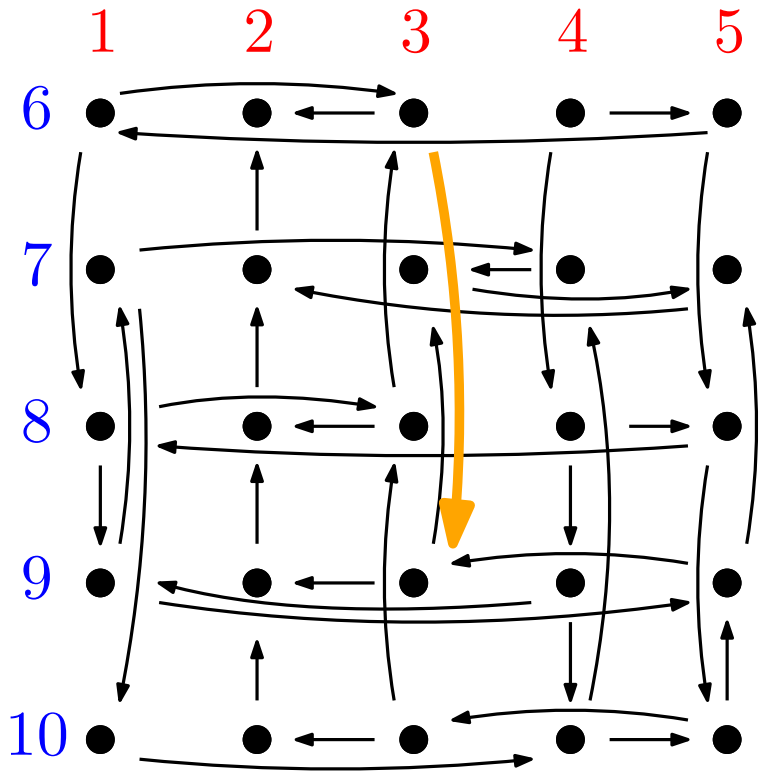
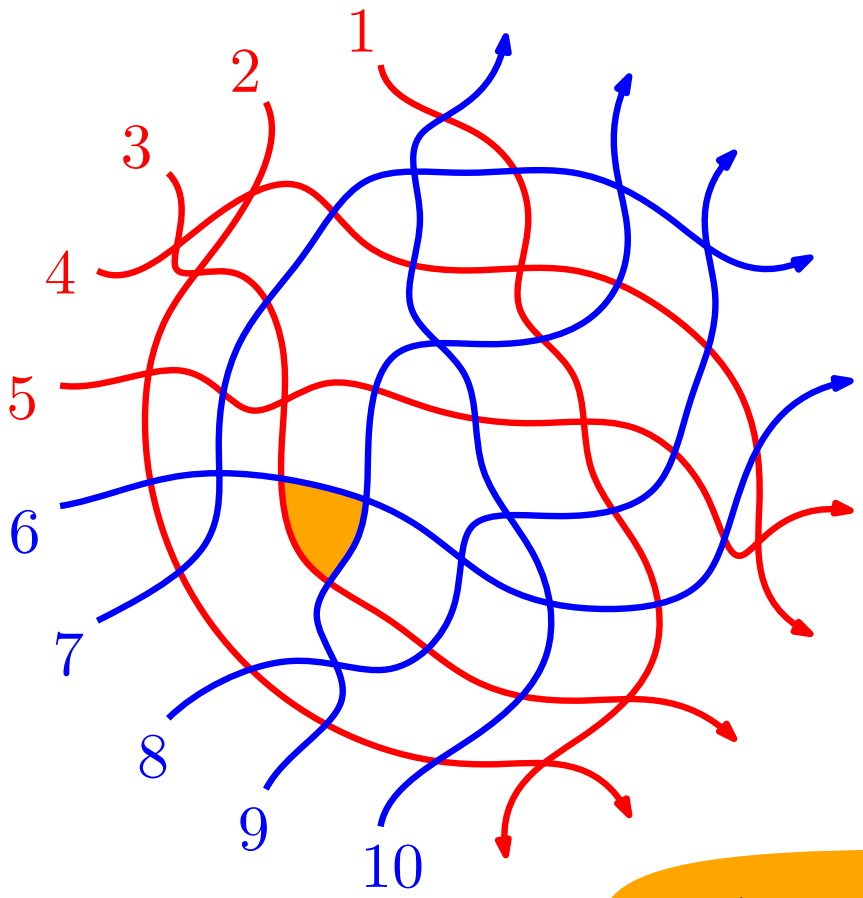
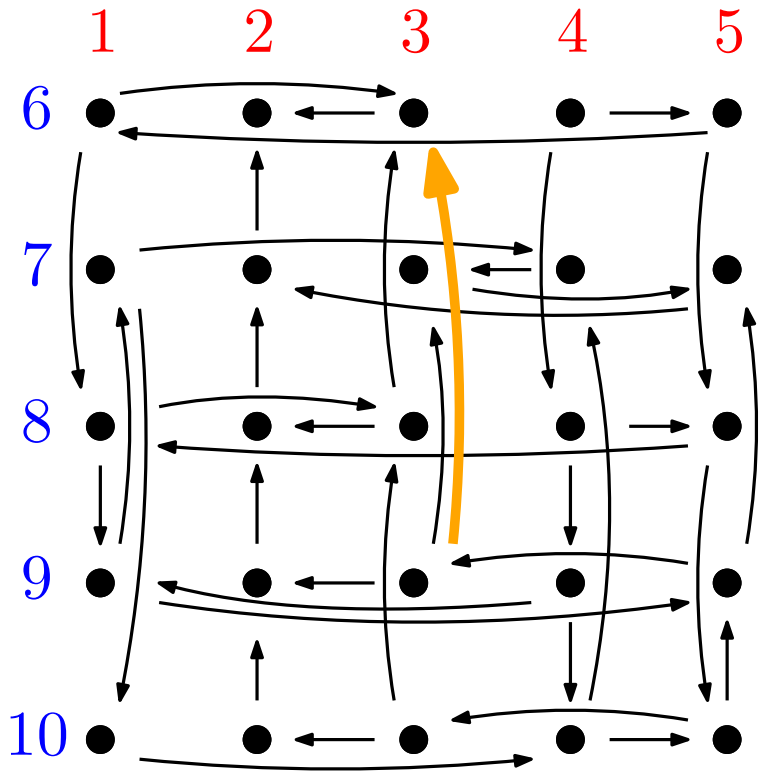




$\chi(3, 6, 9) = -$

3-signotopes $\leftrightarrow$ USOs





$\chi(3, 6, 9) = +$

4-signotopes $\leftrightarrow$ 3-dimensional USOs

Assume 4 signotope $\chi : \binom{[n]}{4} \to \{-, +\}$
and *block partition* $[n] = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

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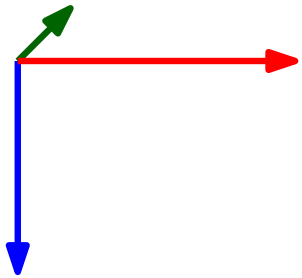

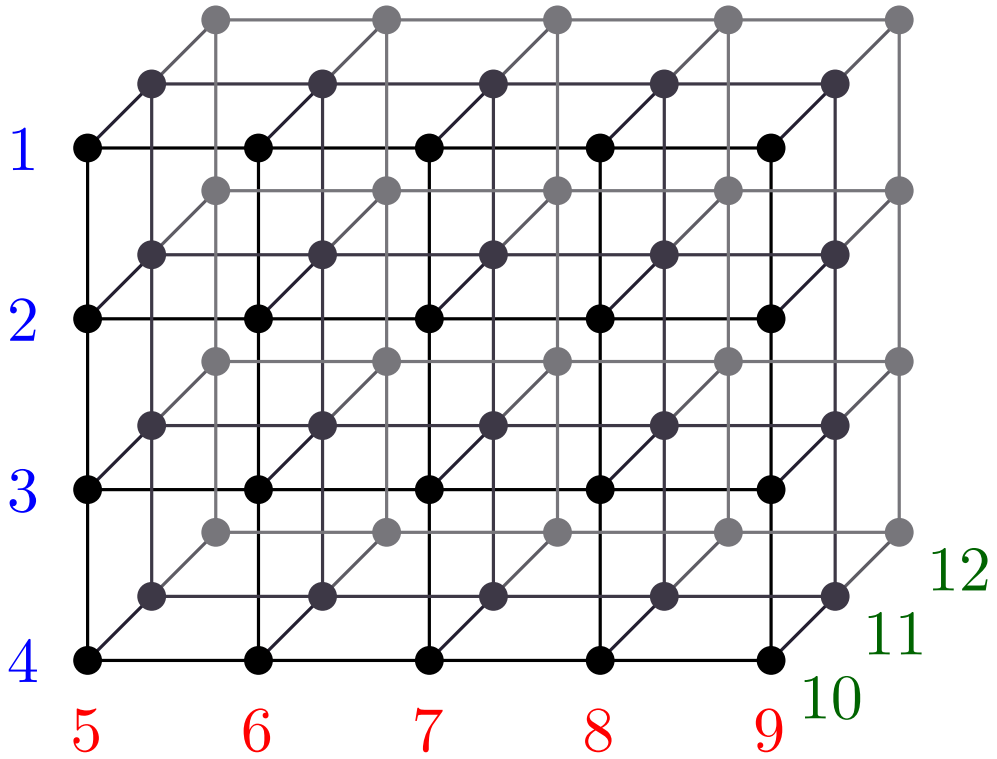

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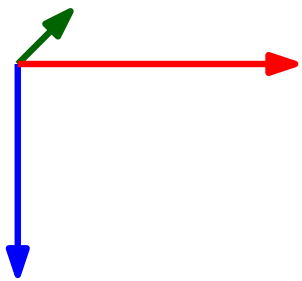
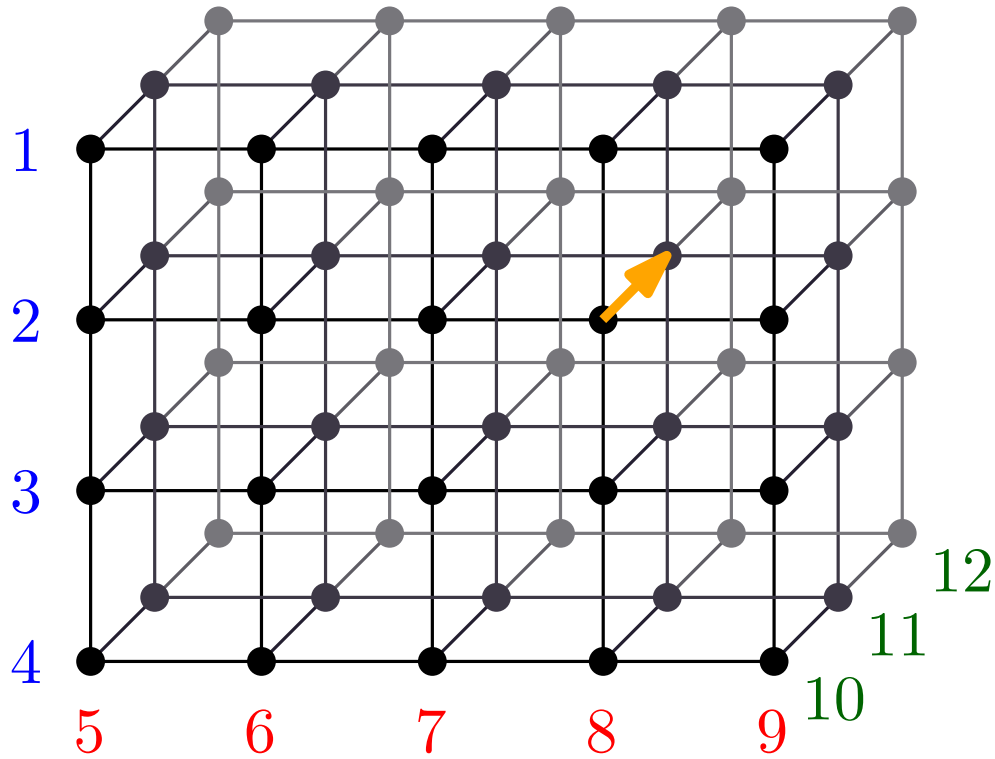
and *block partition* $[n] = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

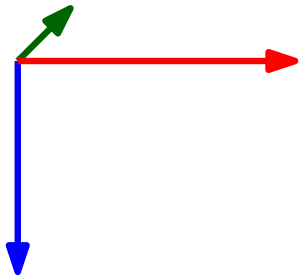


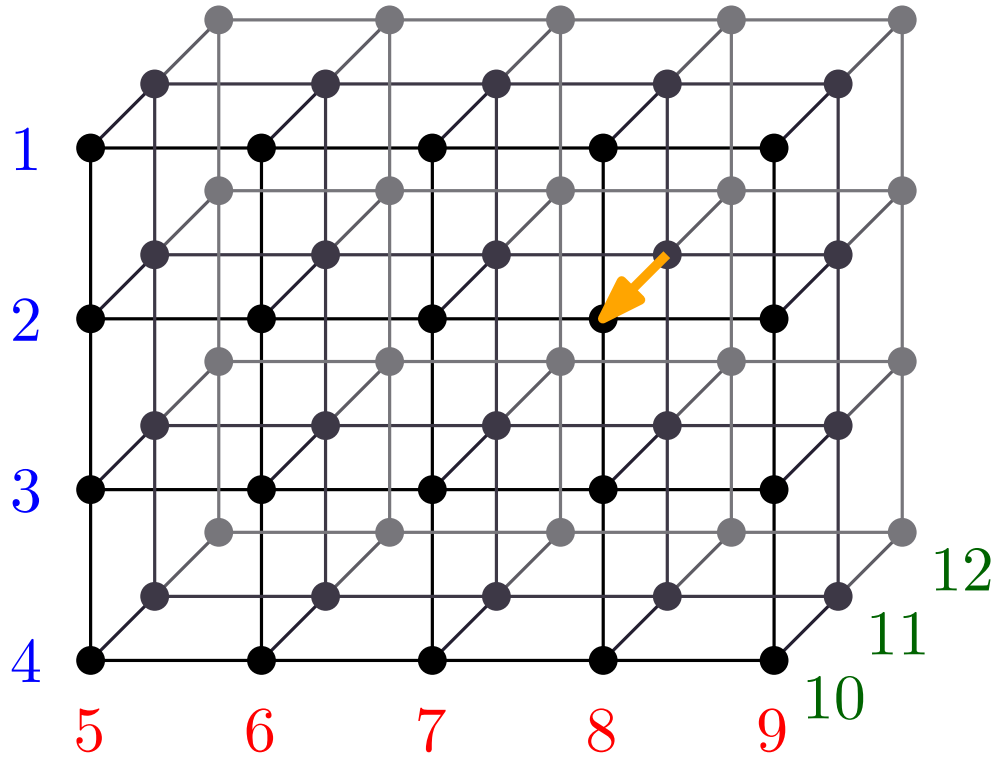


... if $\chi(2,8,10,11) = +$

4-signotopes $\leftrightarrow$ 3-dimensional USOs

Assume 4 signotope $\chi : \binom{[n]}{4} \to \{-,+\}$
and *block partition* $[n] = \{1,2,3,4,5,6,7,8,9,10,11,12\}$



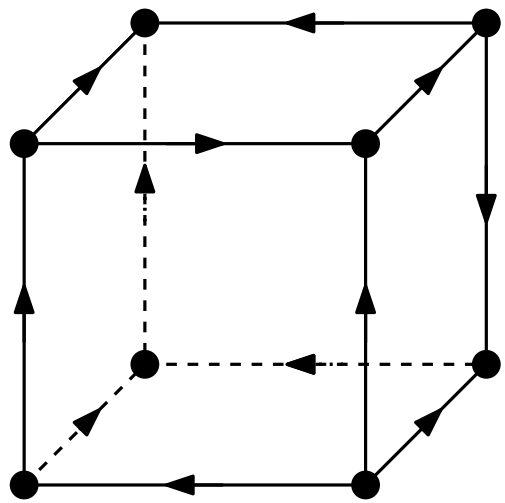


... if $\chi(2,8,10,11) = -$

4-signotopes $\leftrightarrow$ 3-dimensional USOs

Theorem (R. 2025)

1) $(r+1)$-signotopes induce acyclic USOs on r-dimensional grid.
2) For $r \in \{2, 3\}$, they fulfill the Holt-Klee-property.
3) There exist 3-dimensional acyclic USOs with Holt-Klee-property that are **not** induced by 4-signotopes.

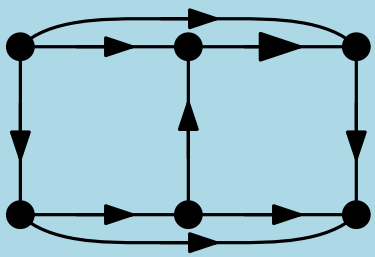


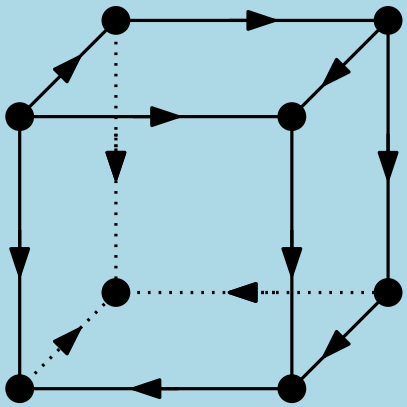
acyclic USO with
Holt-Klee. But not induced
by signotope):

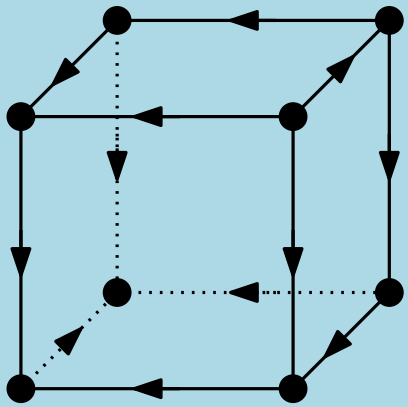
3-dim USOS that violate Holt-Klee-property

Theorem (Gärtner 2002)

An acyclic USO of dimension at most 3 satisfies the Holt-Klee-property if and only if it does not contain any of:







Questions?