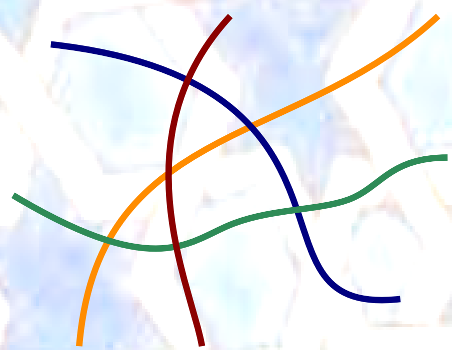
MATHINSIDE π-Day 2025

Technische Universität Berlin

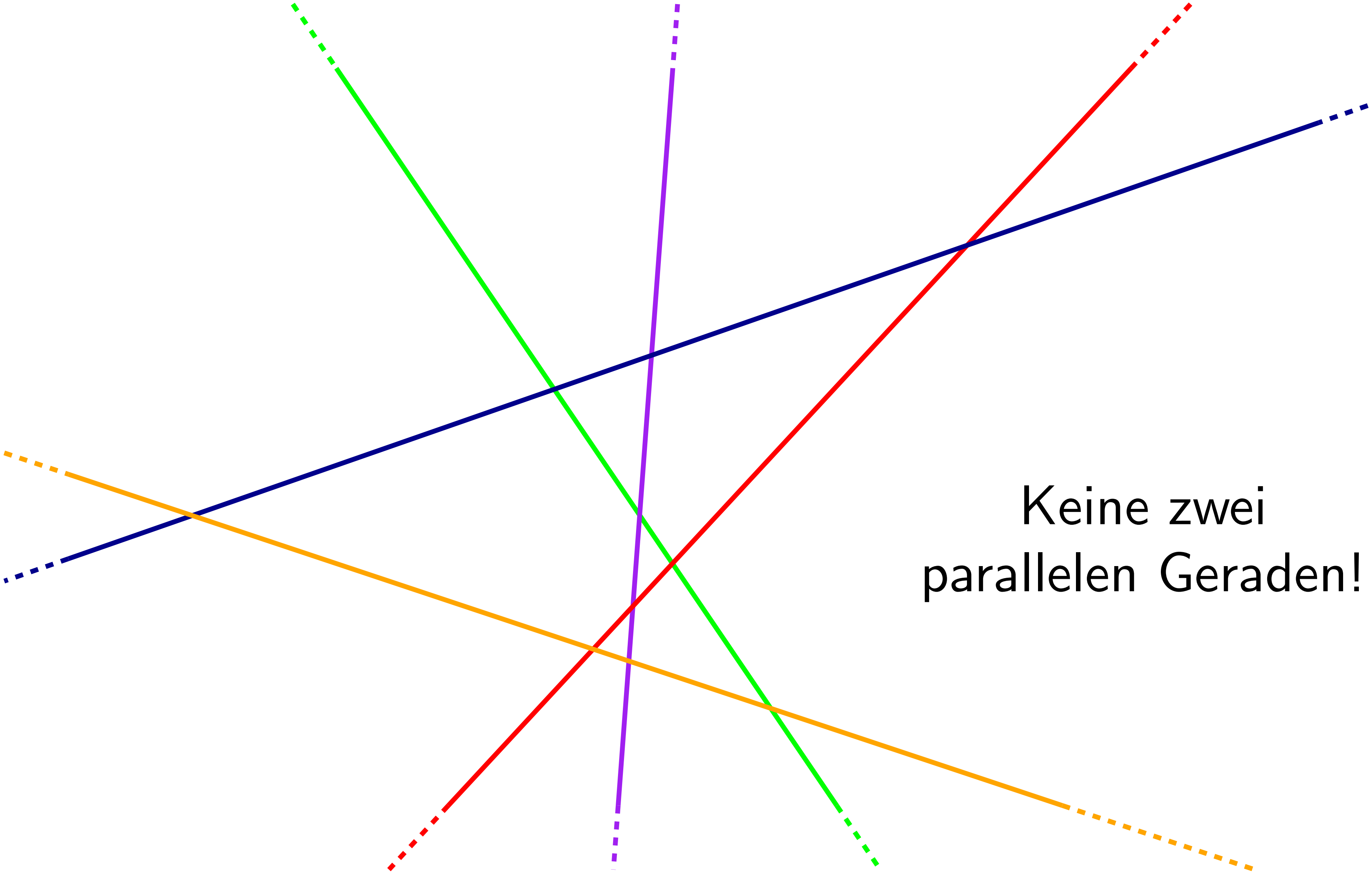
14. März 2025

Die Welt der Pseudogeraden



Sandro M. Roch

Geradenarrangements



Keine zwei
parallelen Geraden!

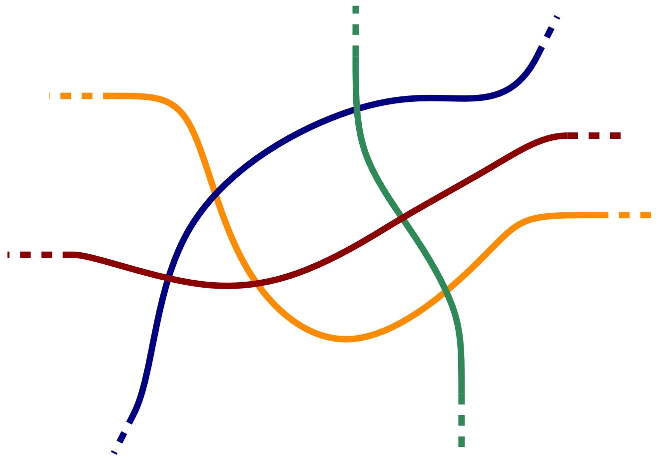
Pseudogeradenarrangement

Def: *Pseudogeradenarrangement*:

- Familie von Kurven $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}^2$ mit

$$\lim_{x\to\infty} \|f_i(x)\| = \lim_{x\to-\infty} \|f_i(x)\| = \infty$$

- Je zwei kreuzen sich in genau einem Punkt.



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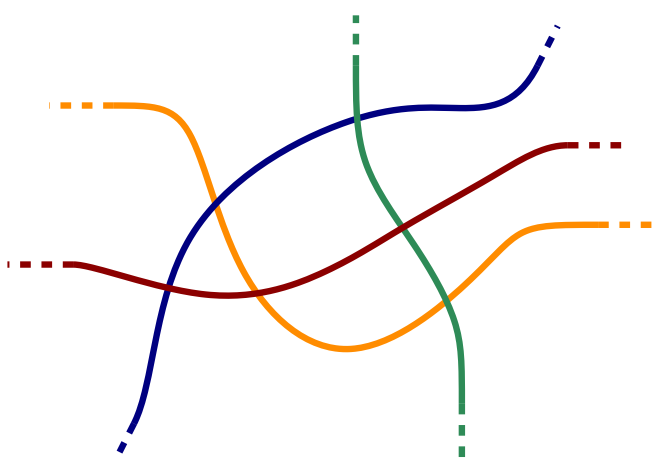
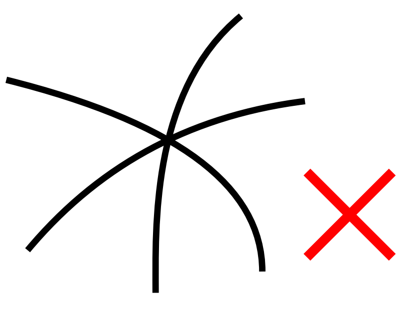
Einfaches Pseudogeradenarrangement:

Keine 3 Pseudogeraden kreuzen sich in einem Punkt.





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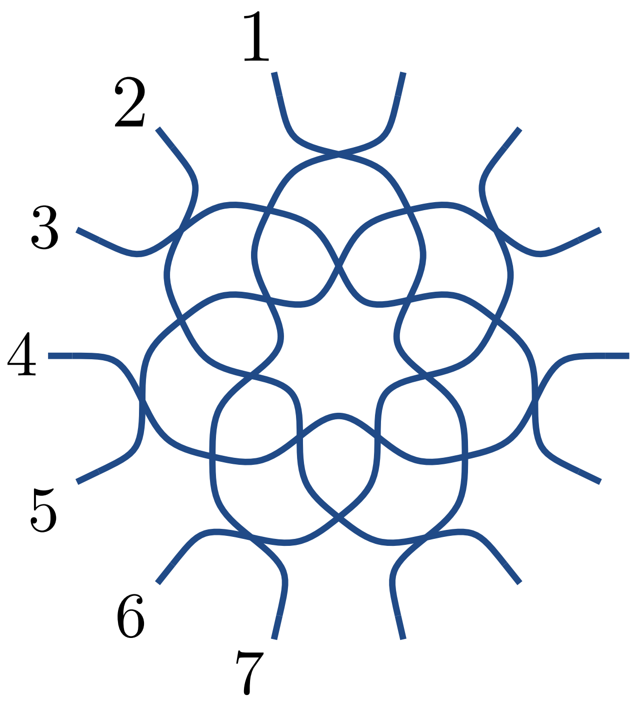

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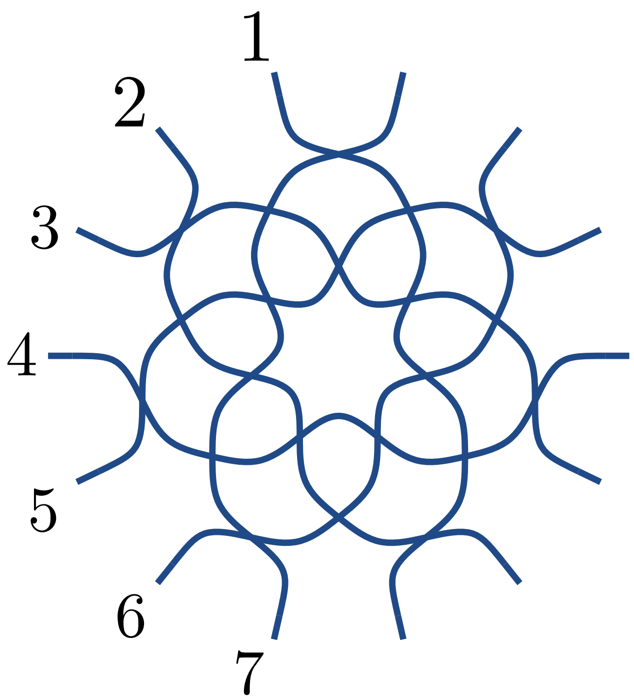

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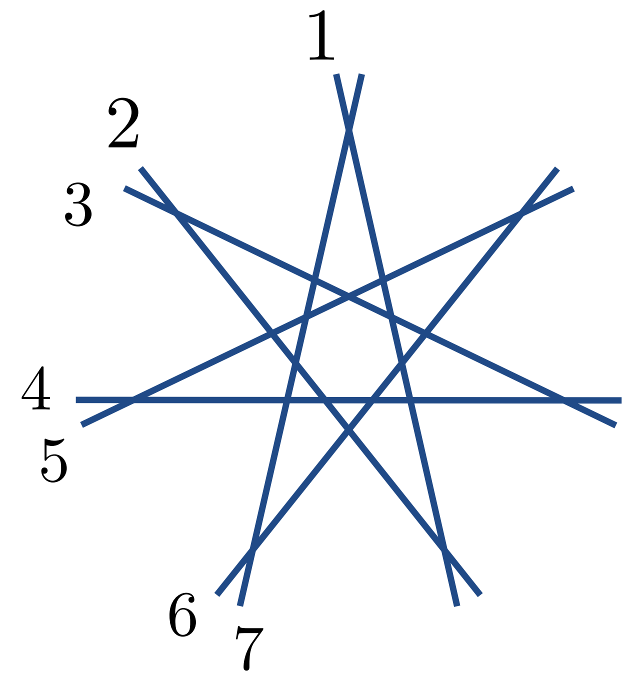

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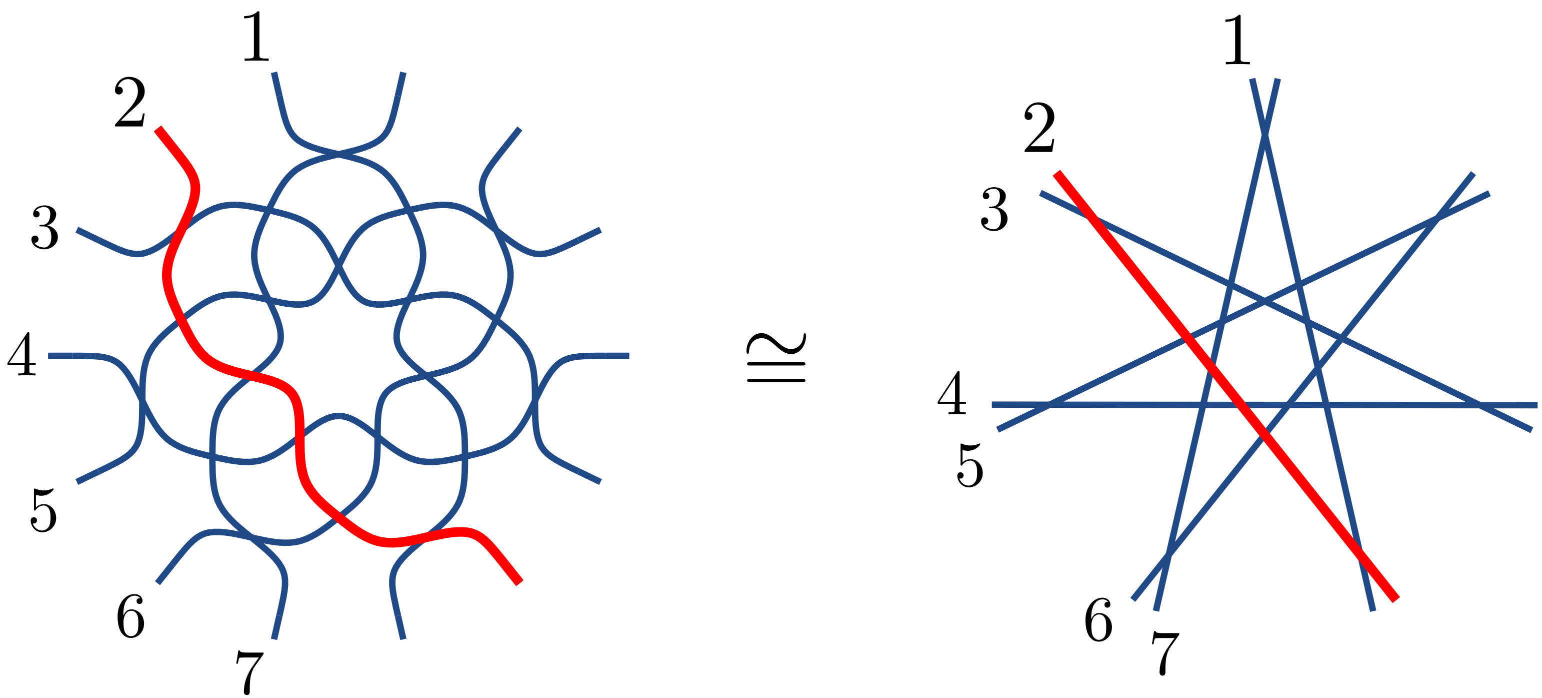

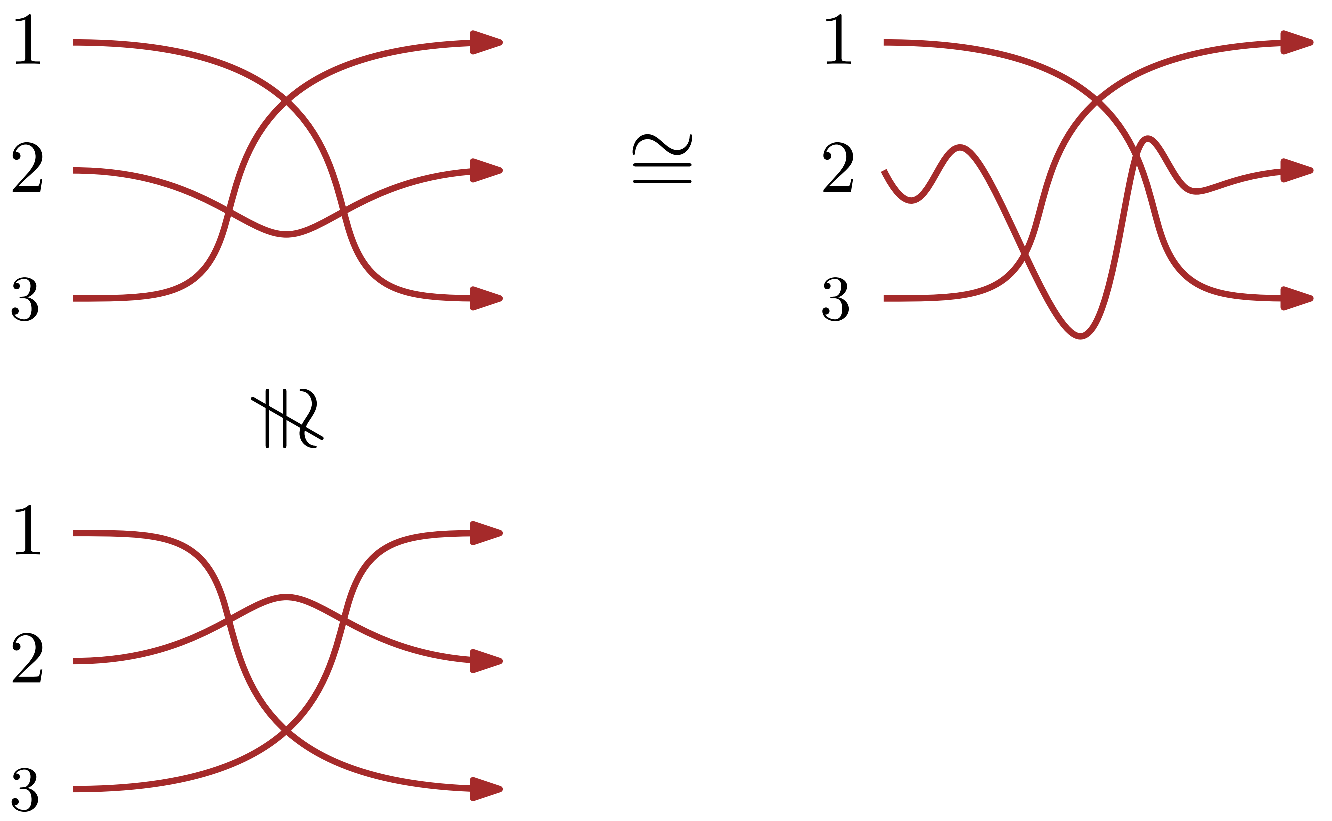
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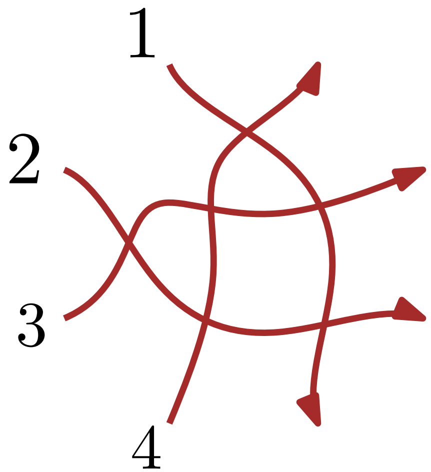
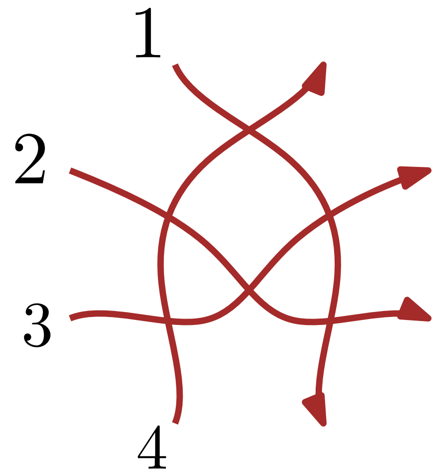
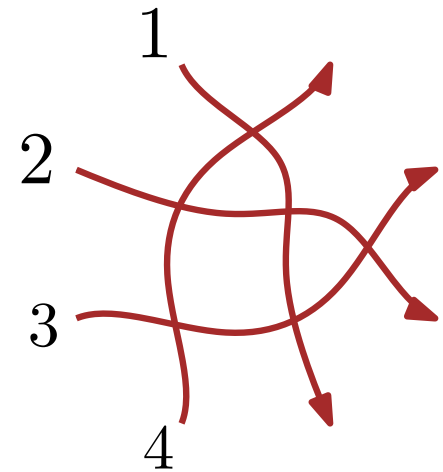
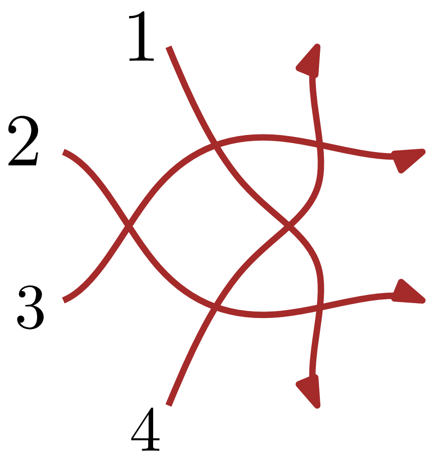
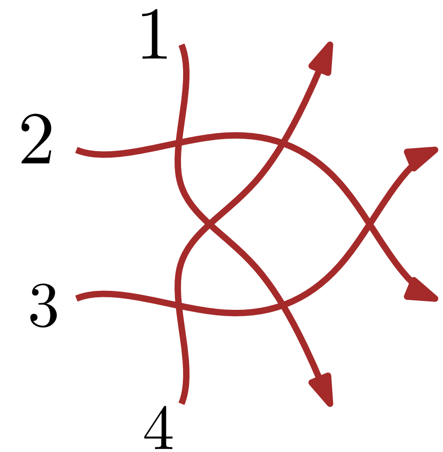


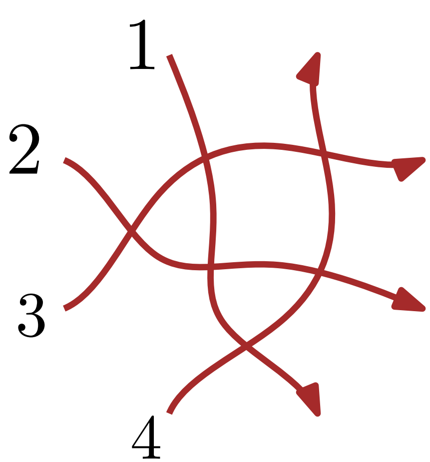


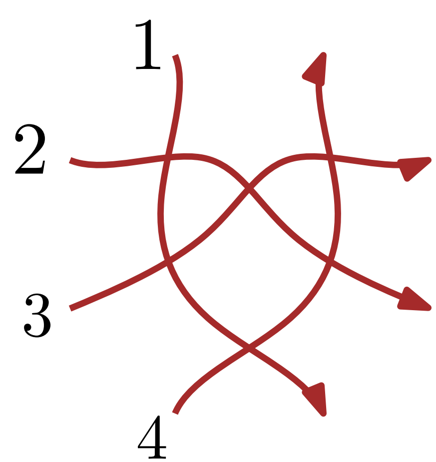


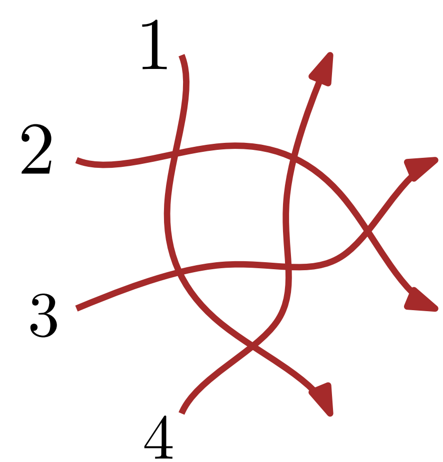


$n = 4$:
Genau 8
Arragnements.









Drahtdiagramme

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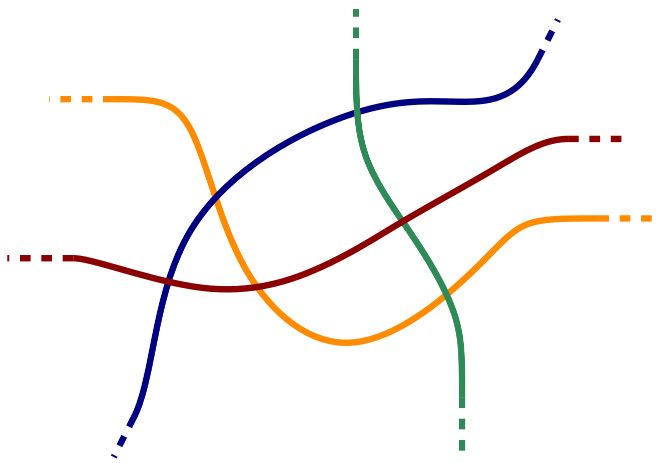

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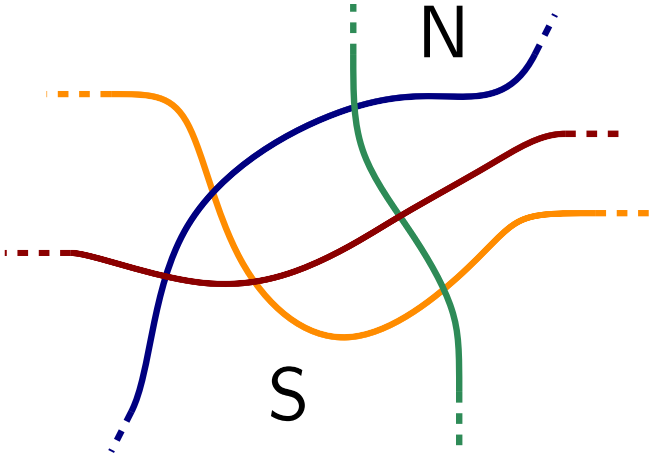

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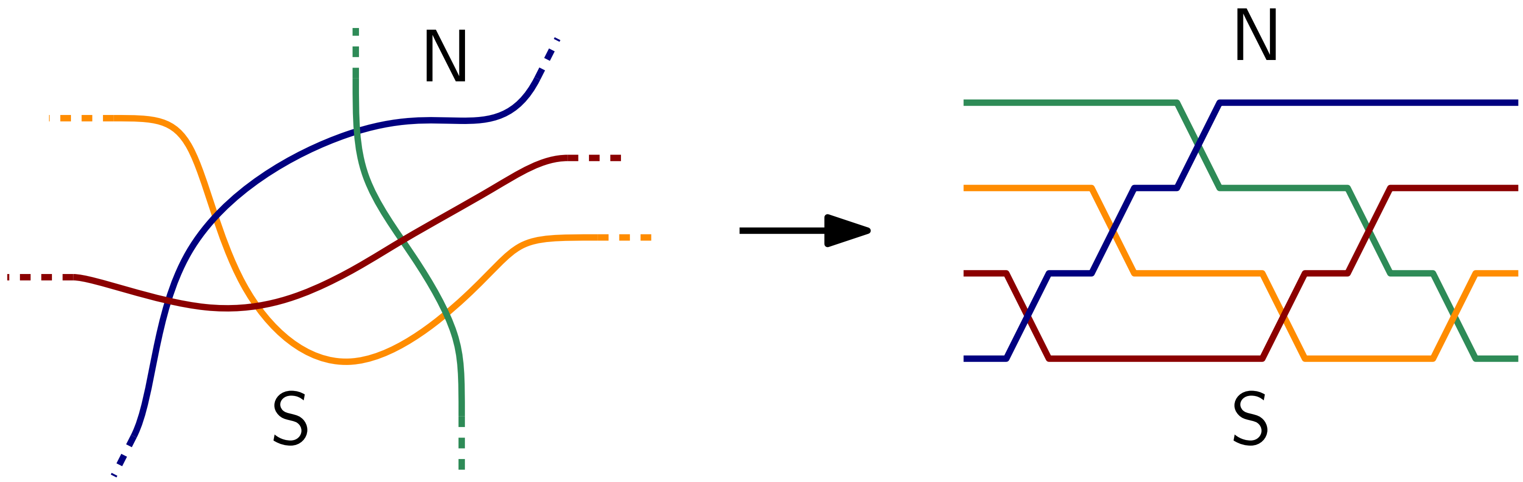

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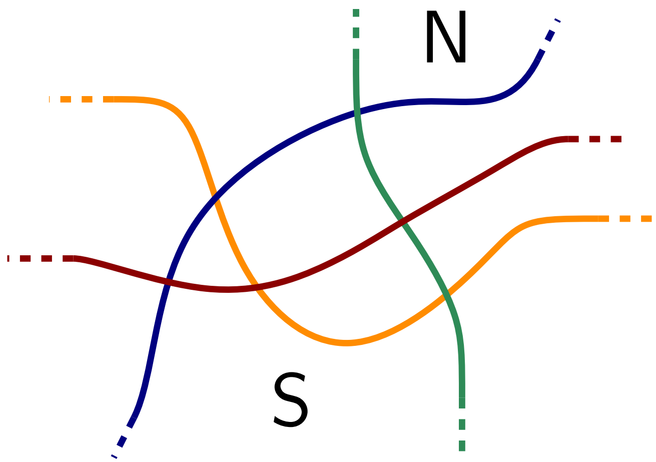

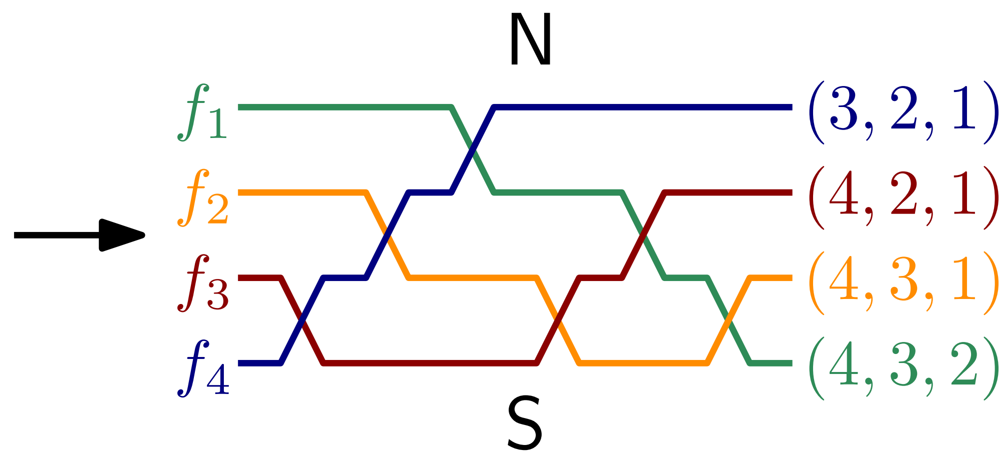

Kodierung mittels Permutationen:
Permutation $\pi_i \in S_{n-1}$ kodiert Schnittreihenfolge von f_i.

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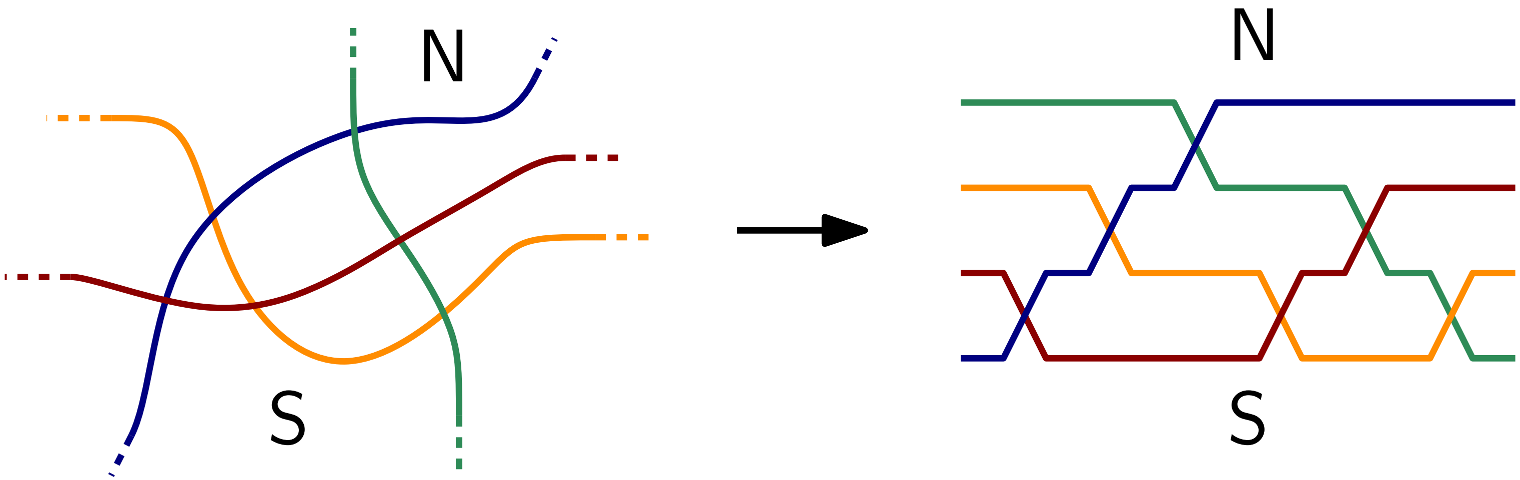
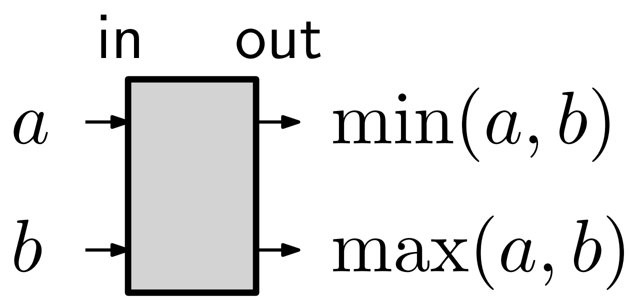


Drahtdiagramme als Sortiernetze:



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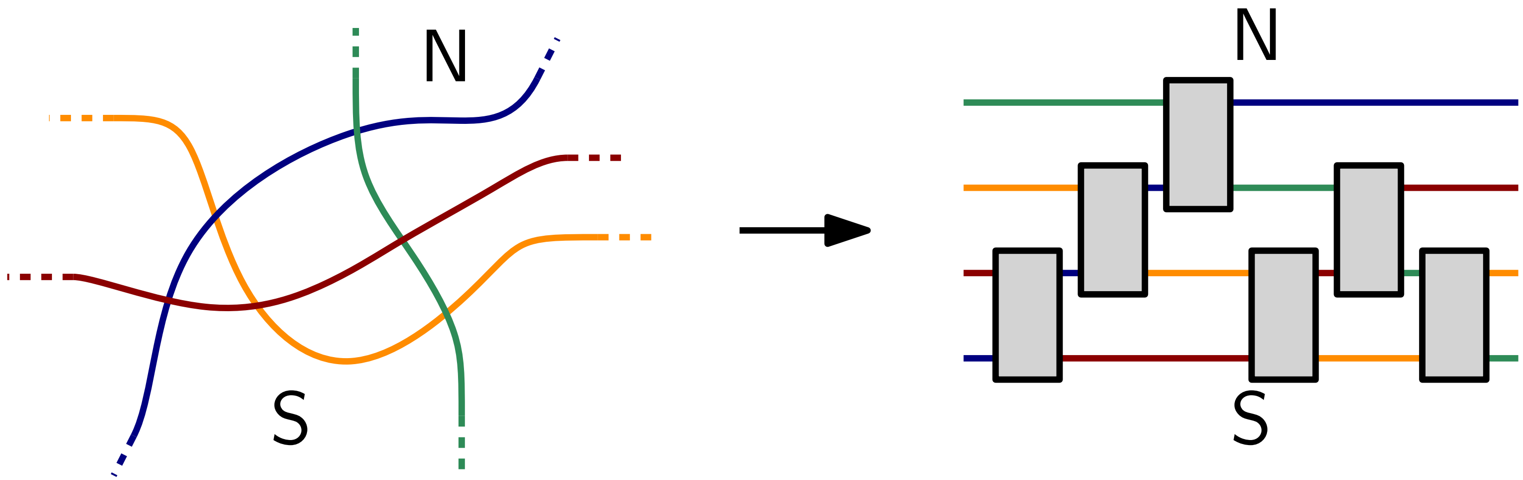

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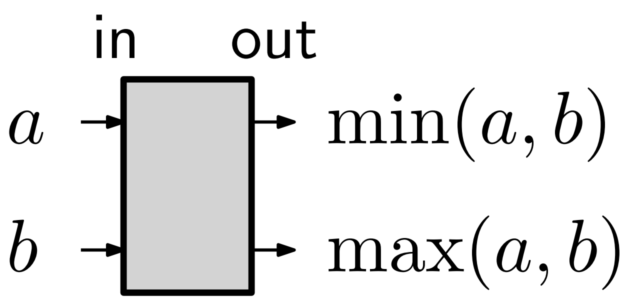

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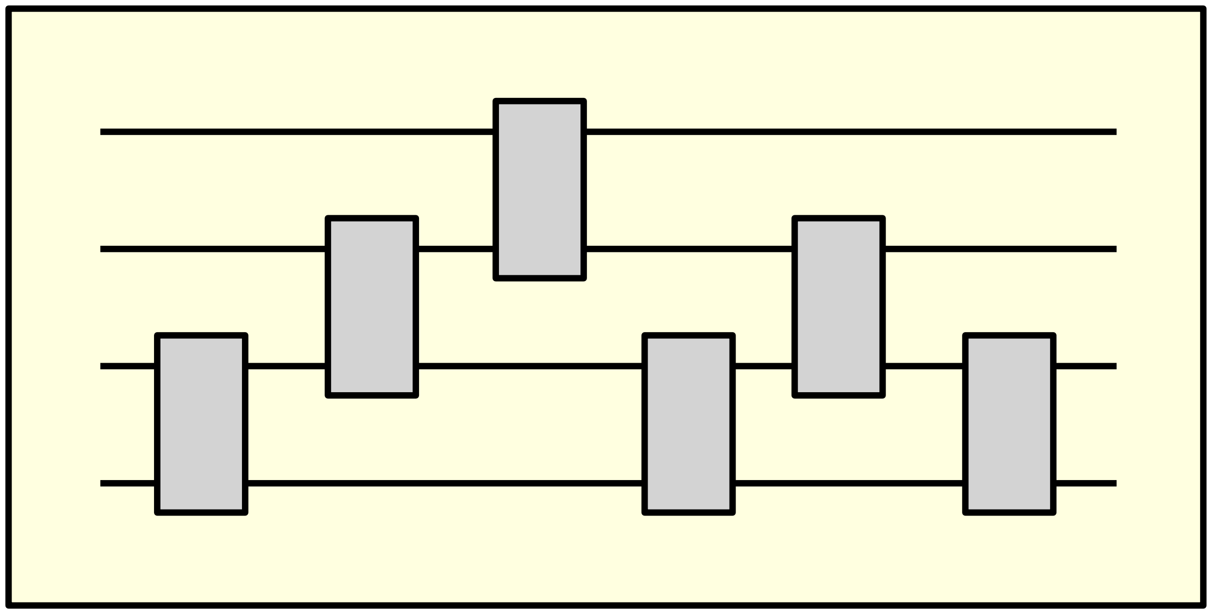

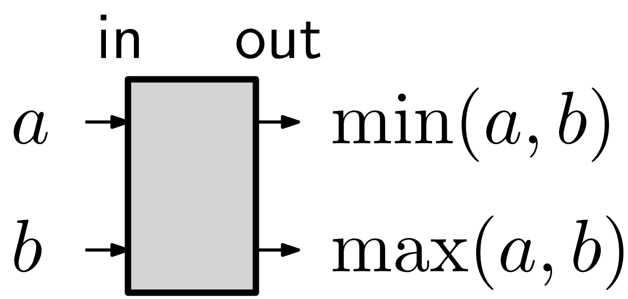

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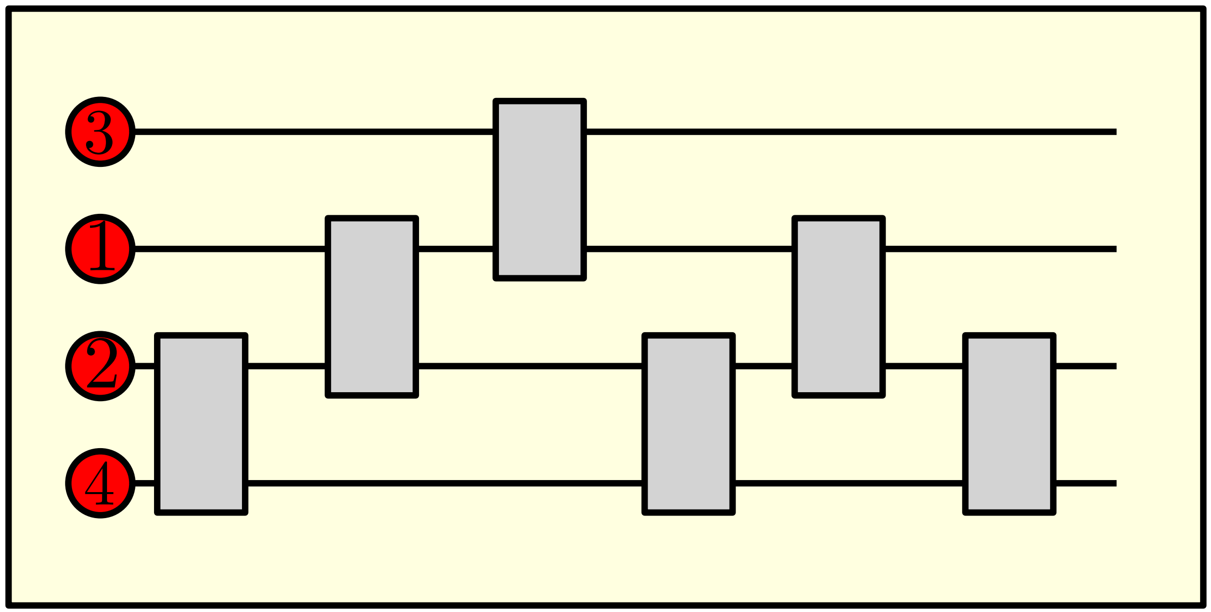

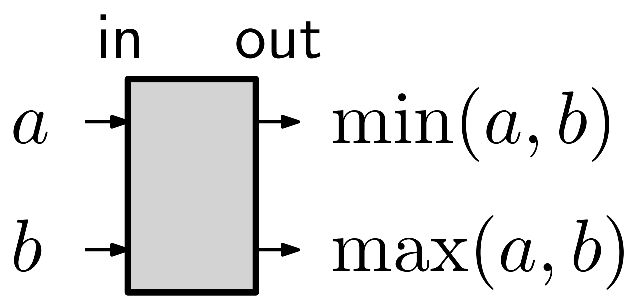

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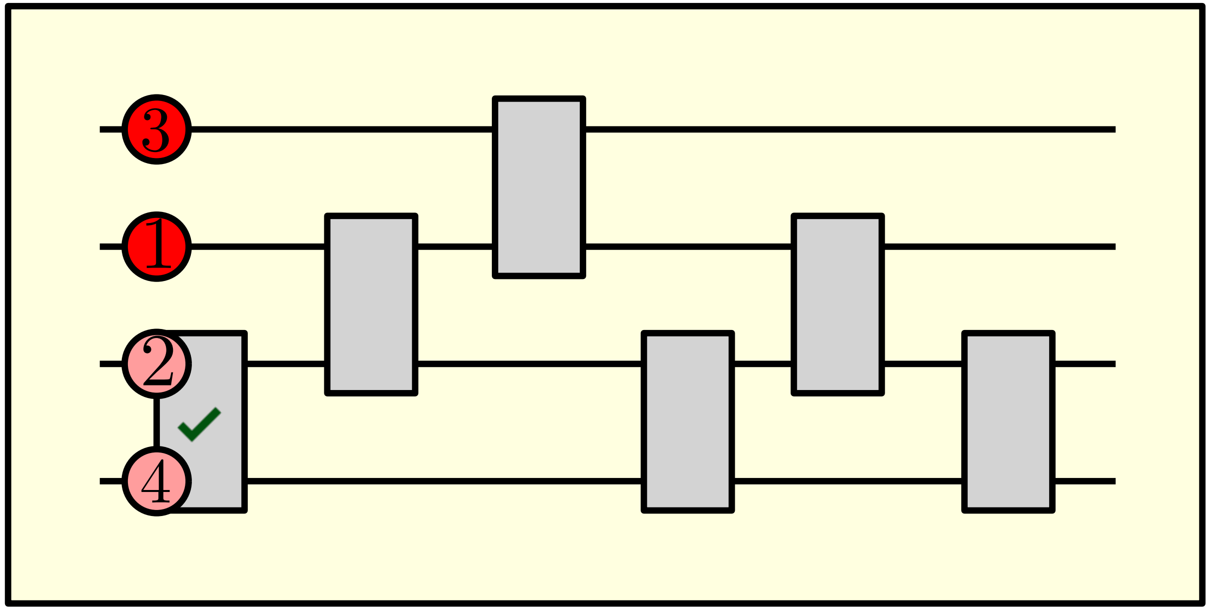

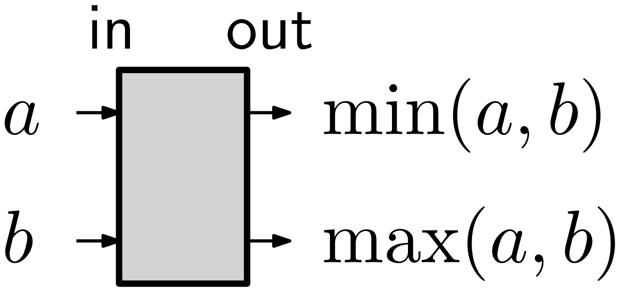

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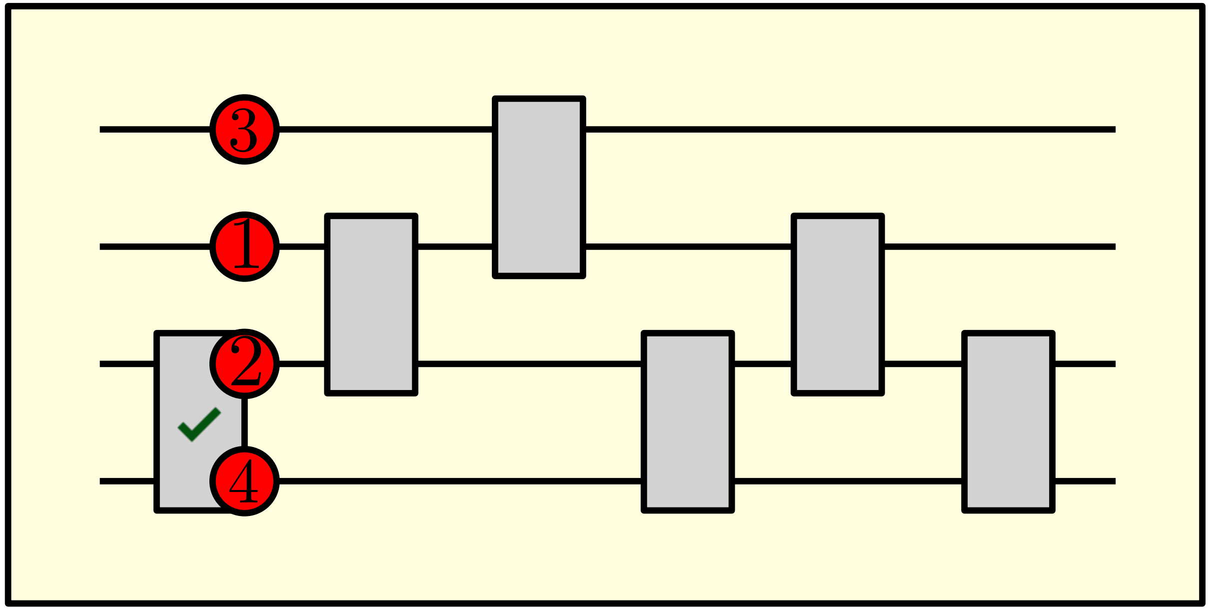

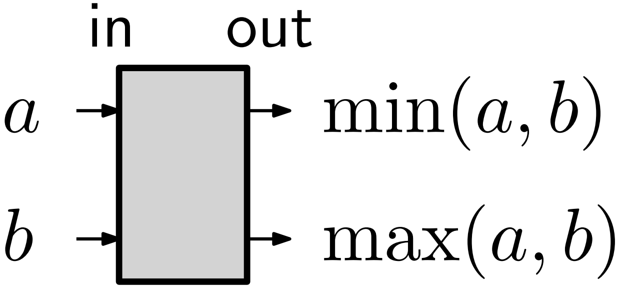

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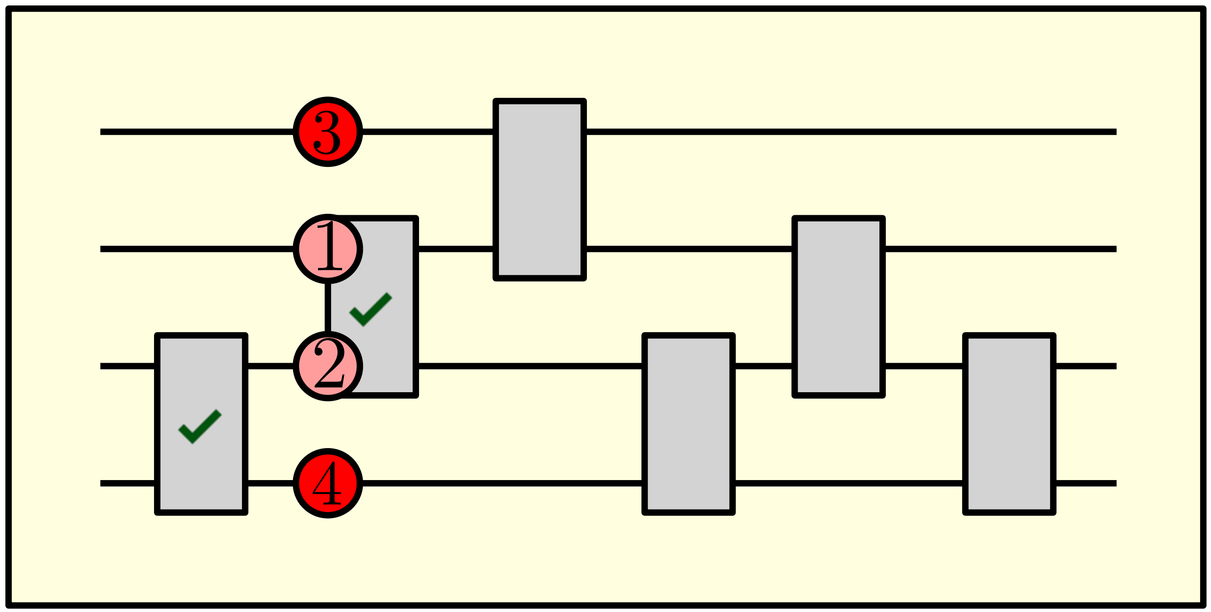

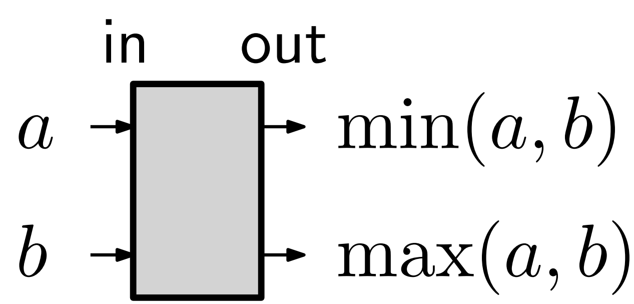

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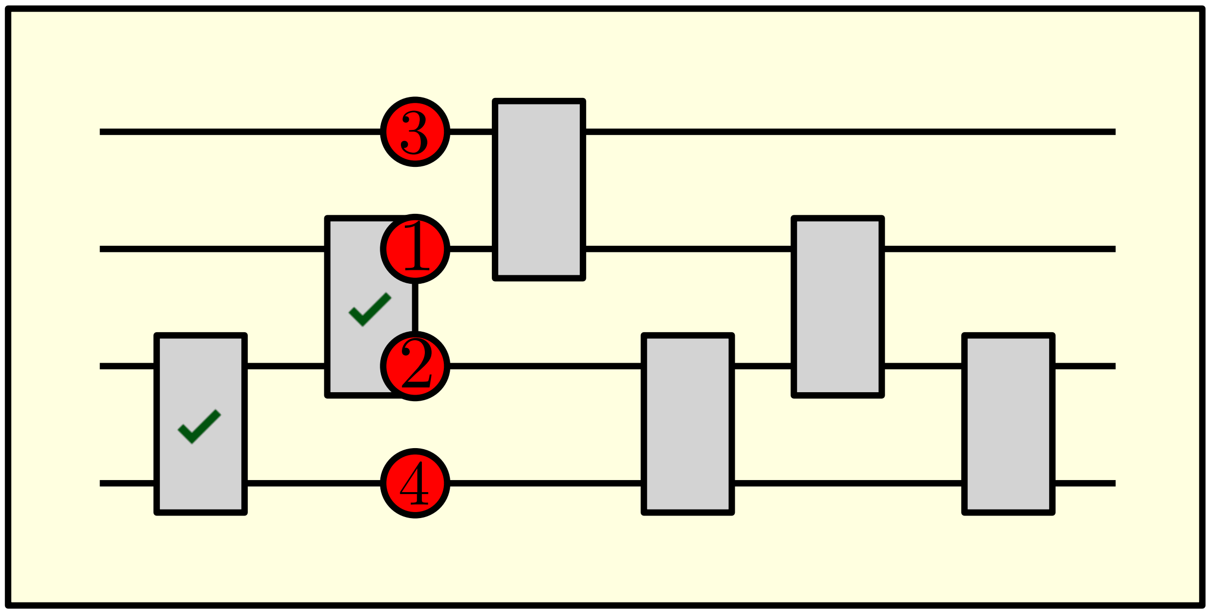

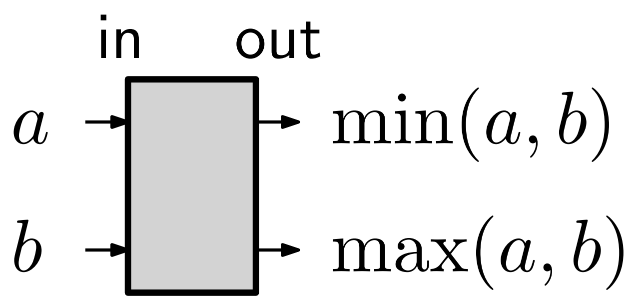

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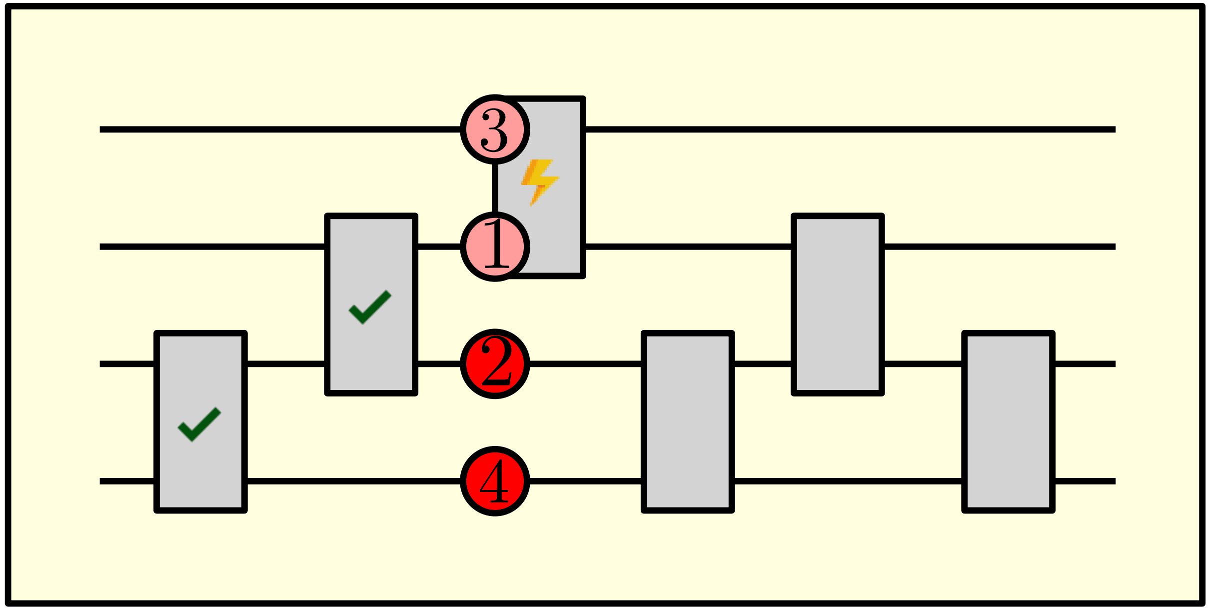

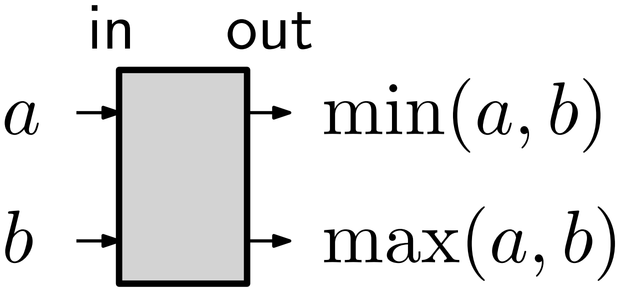

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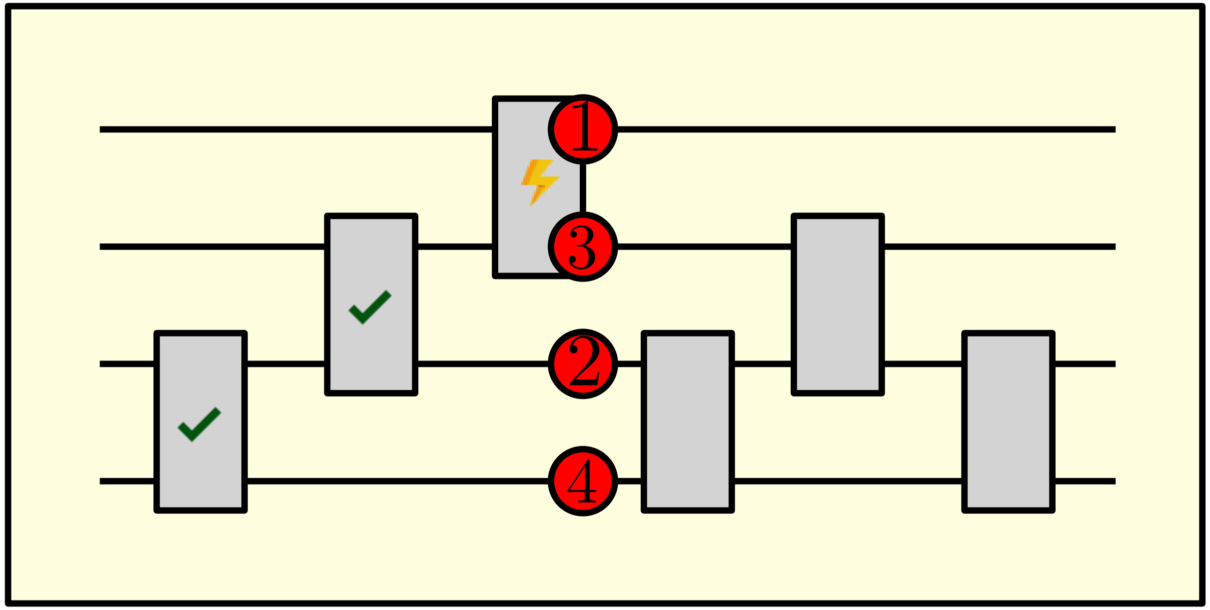

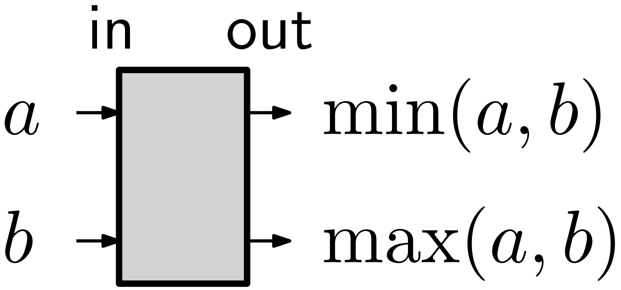

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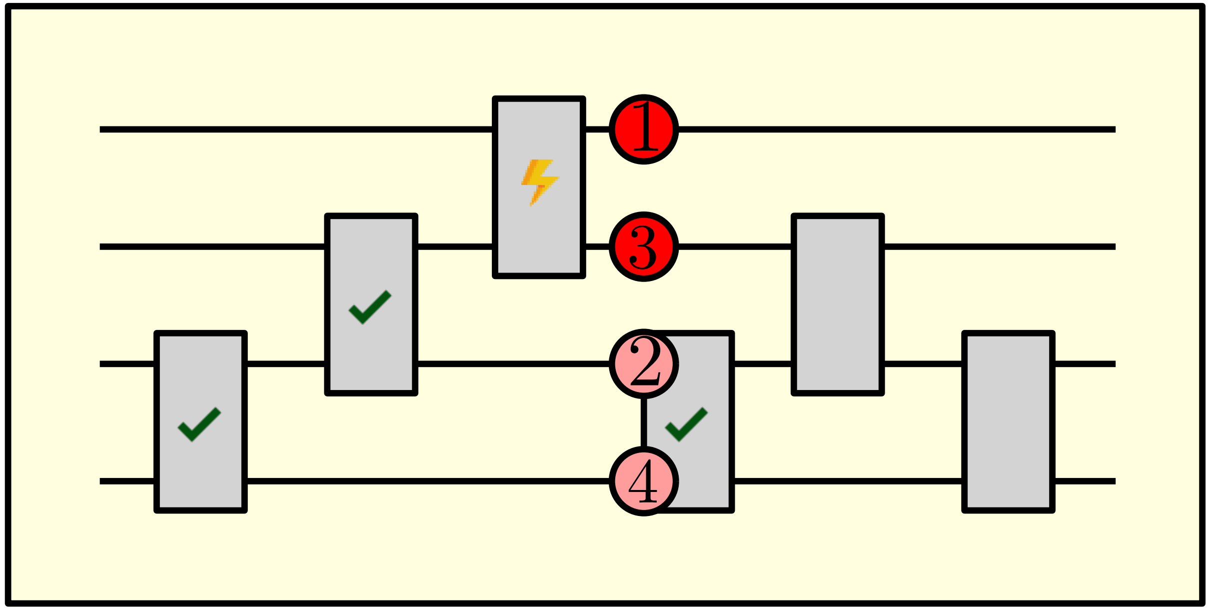

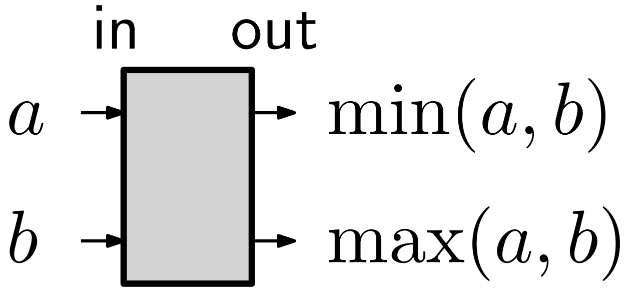

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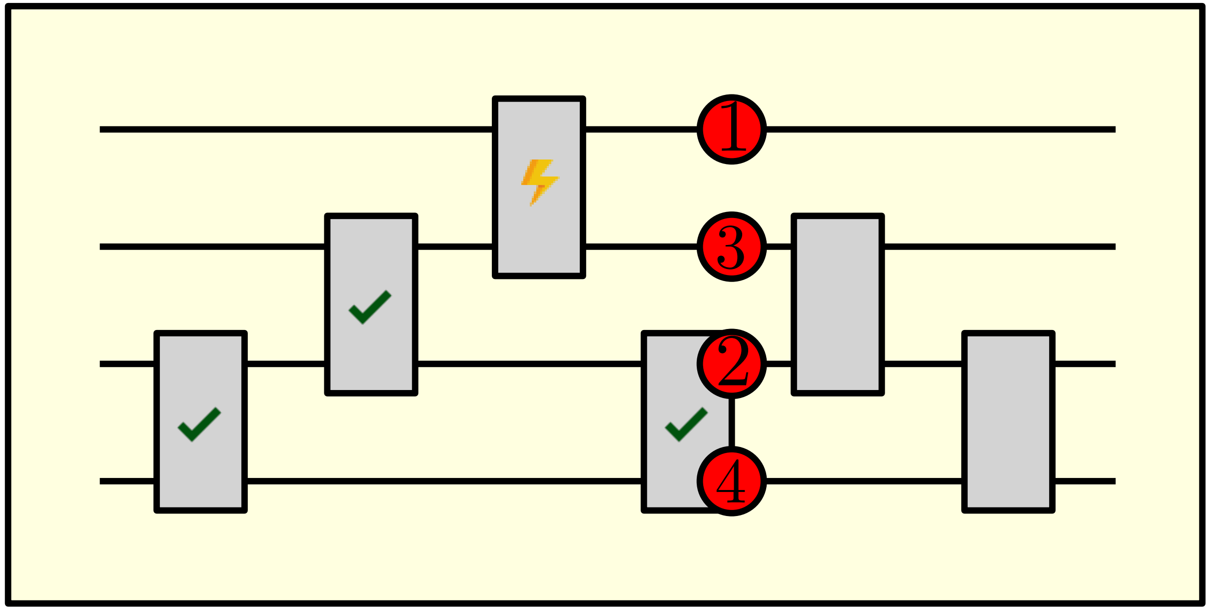

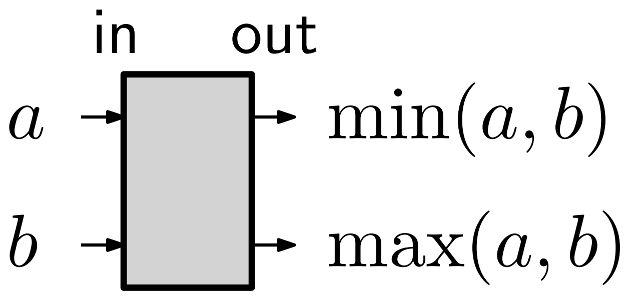

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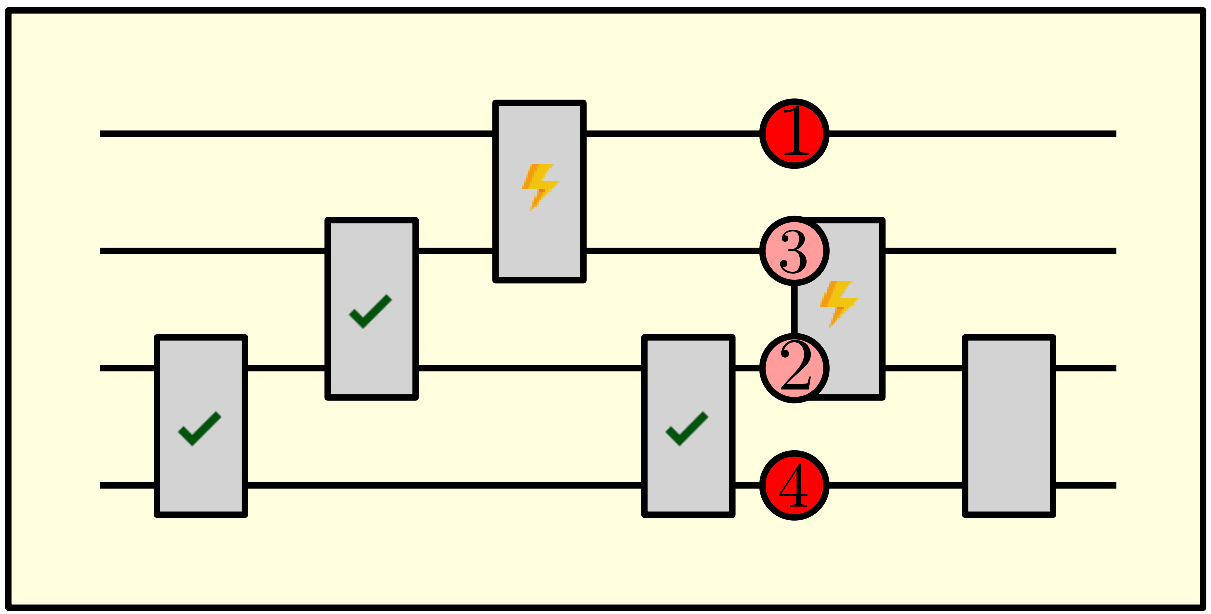

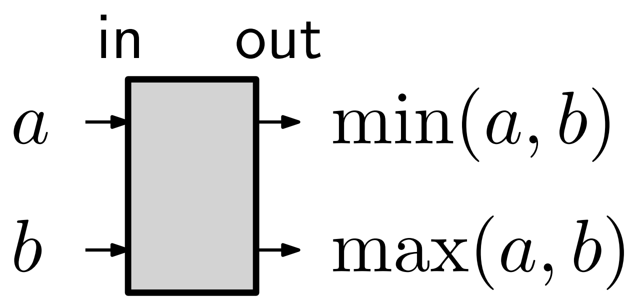

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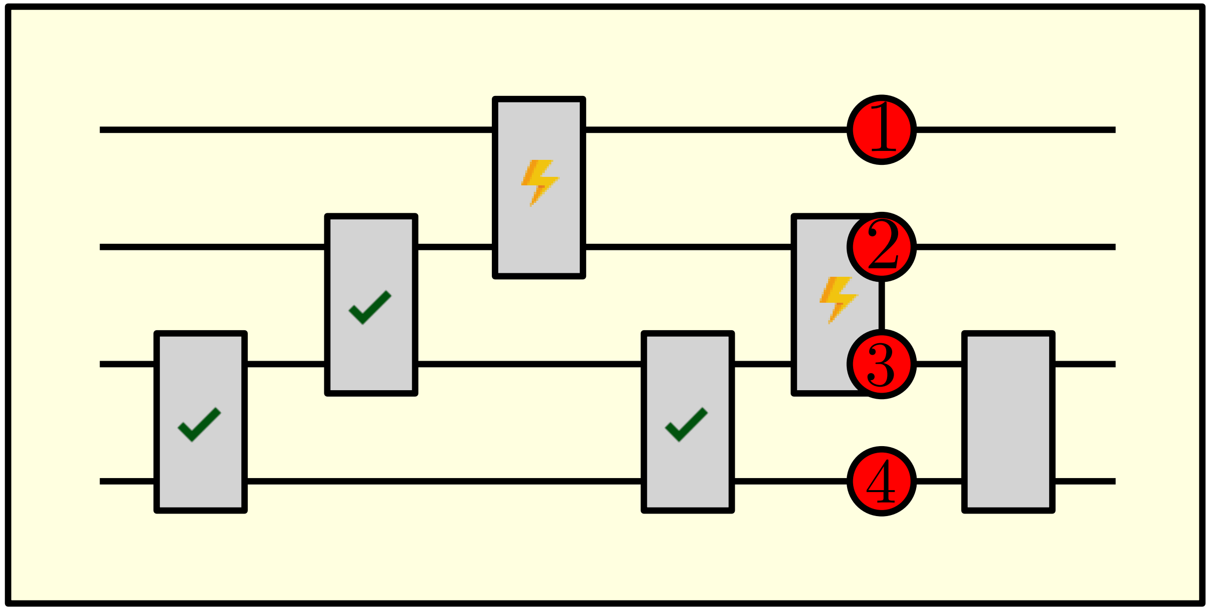

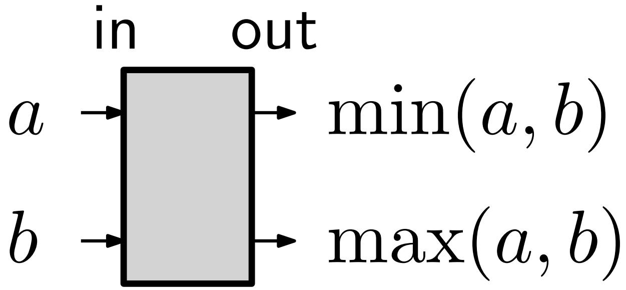

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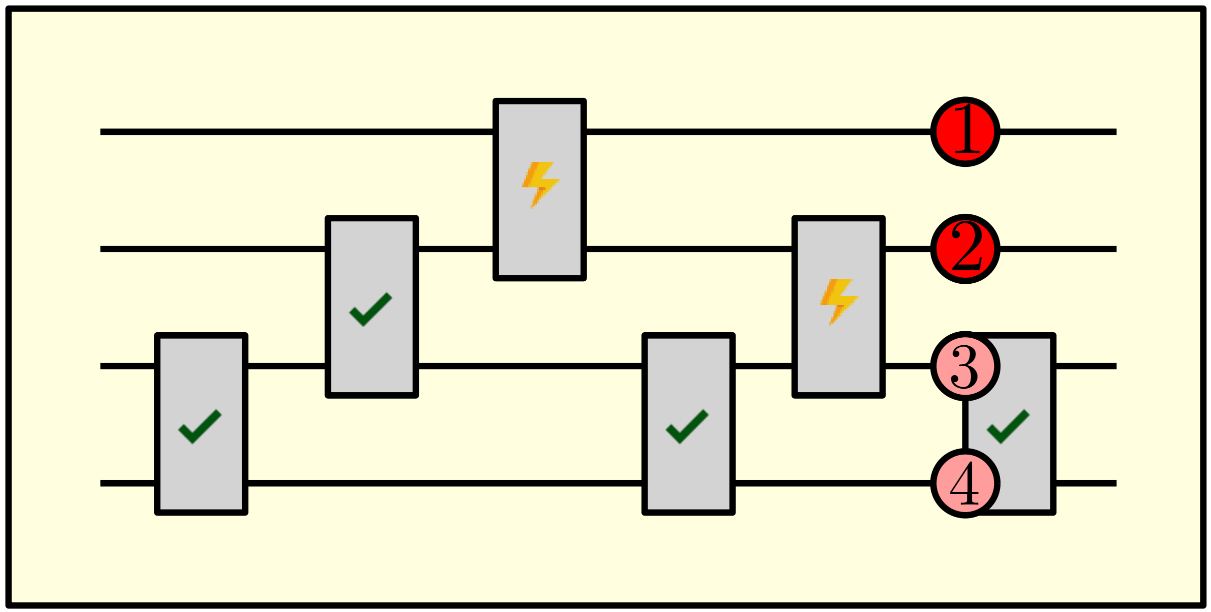

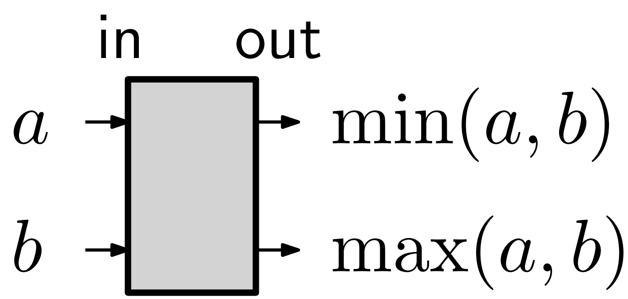

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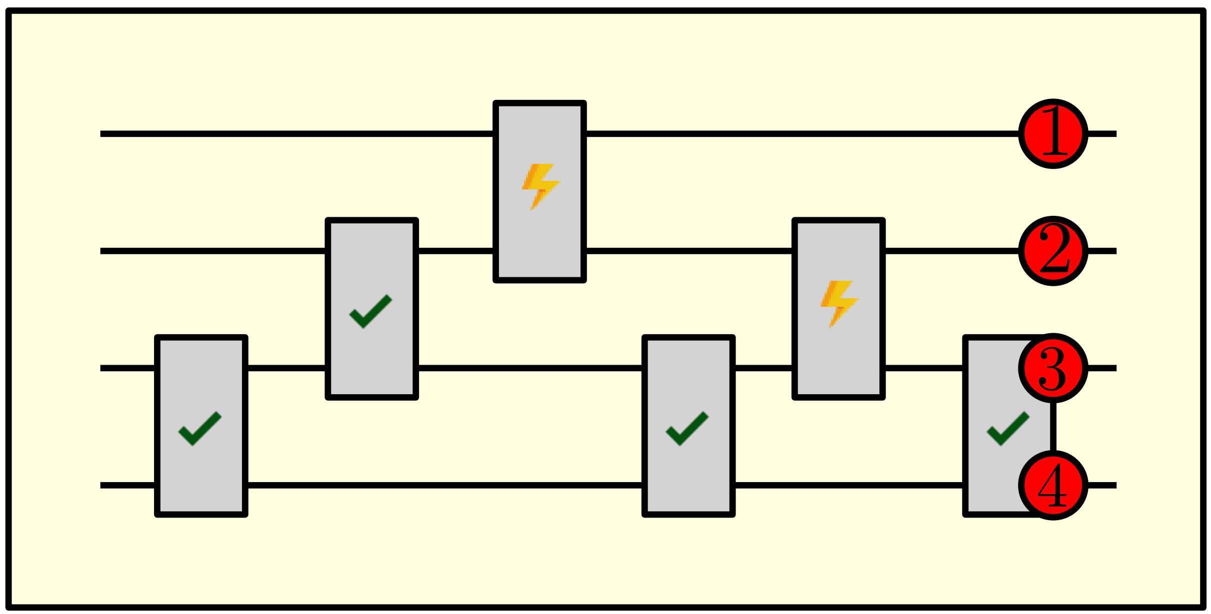

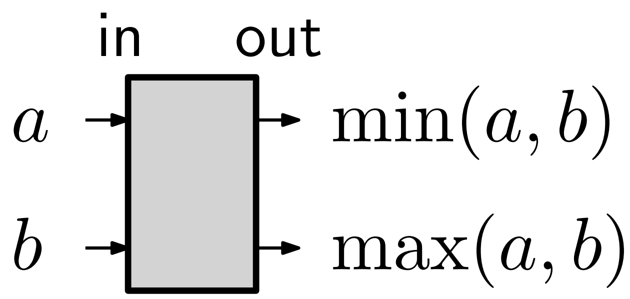

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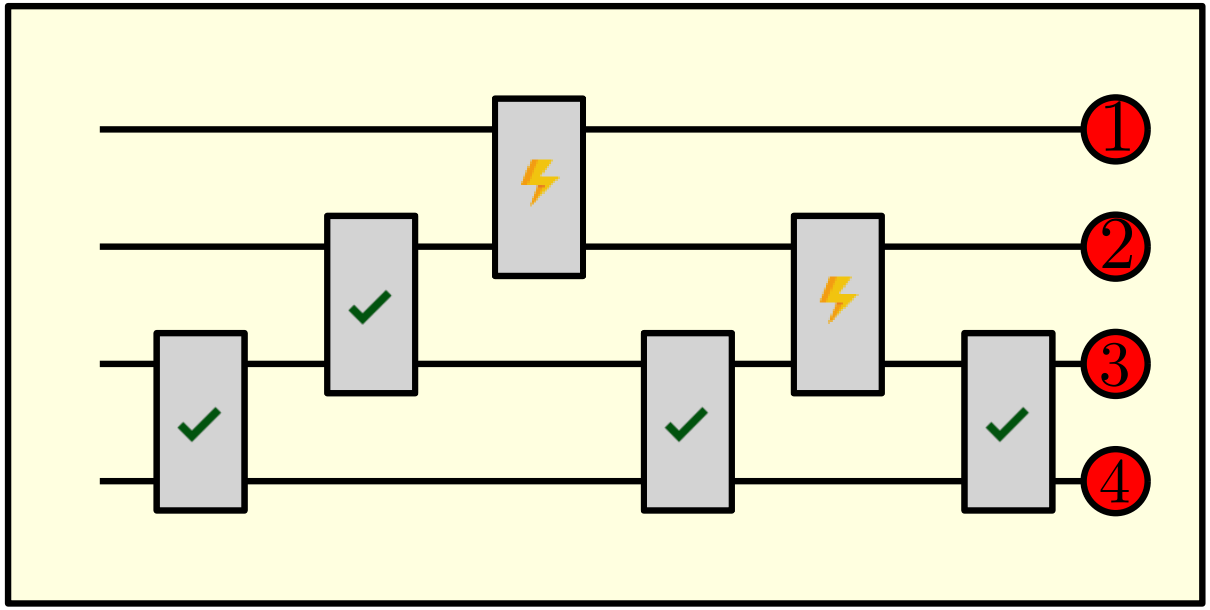

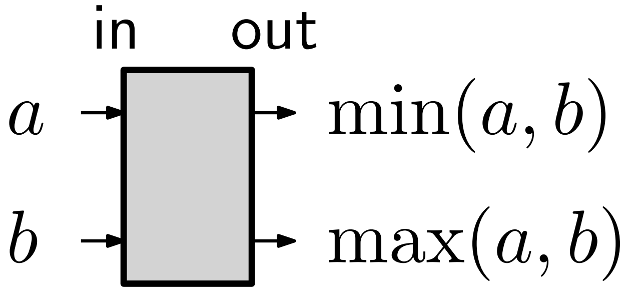

Permutaeder

Permutaeder der Ordnung n:

$$P_n := \operatorname{conv}\left(\left\{\begin{pmatrix} \pi(1) \\ \vdots \\ \pi(n) \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \pi \in S_n\right\}\right)$$

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Beispiel: $n = 3$

Permutationen $\pi \in S_3$:

$(1,2,3)$, $(1,3,2)$, $(2,1,3)$,
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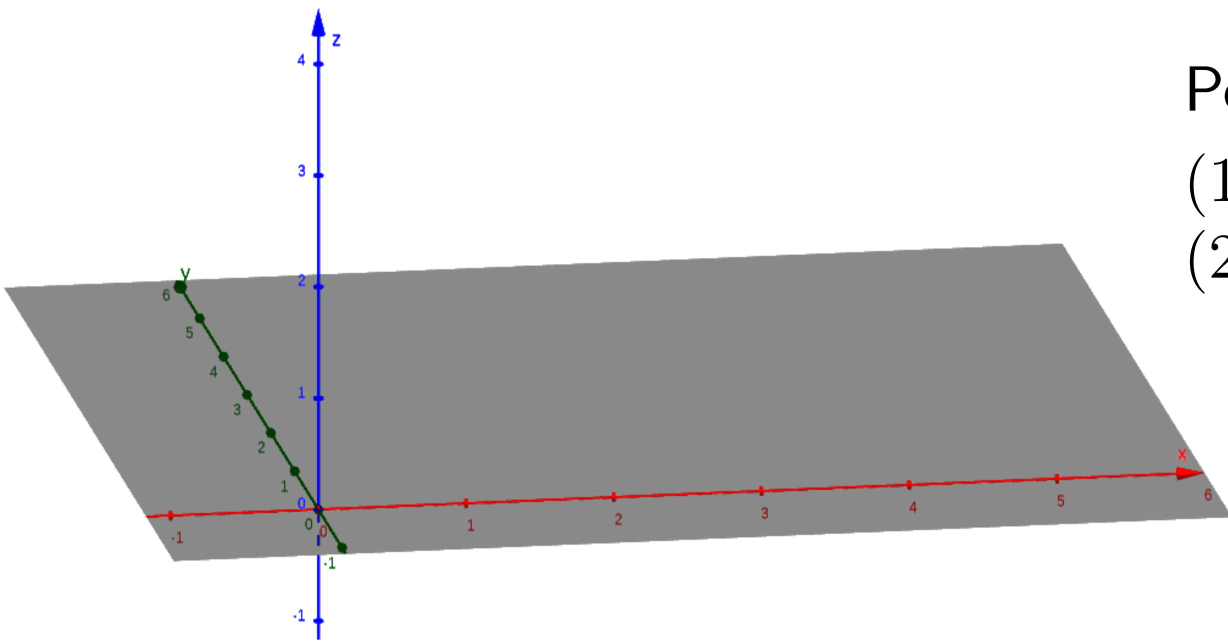

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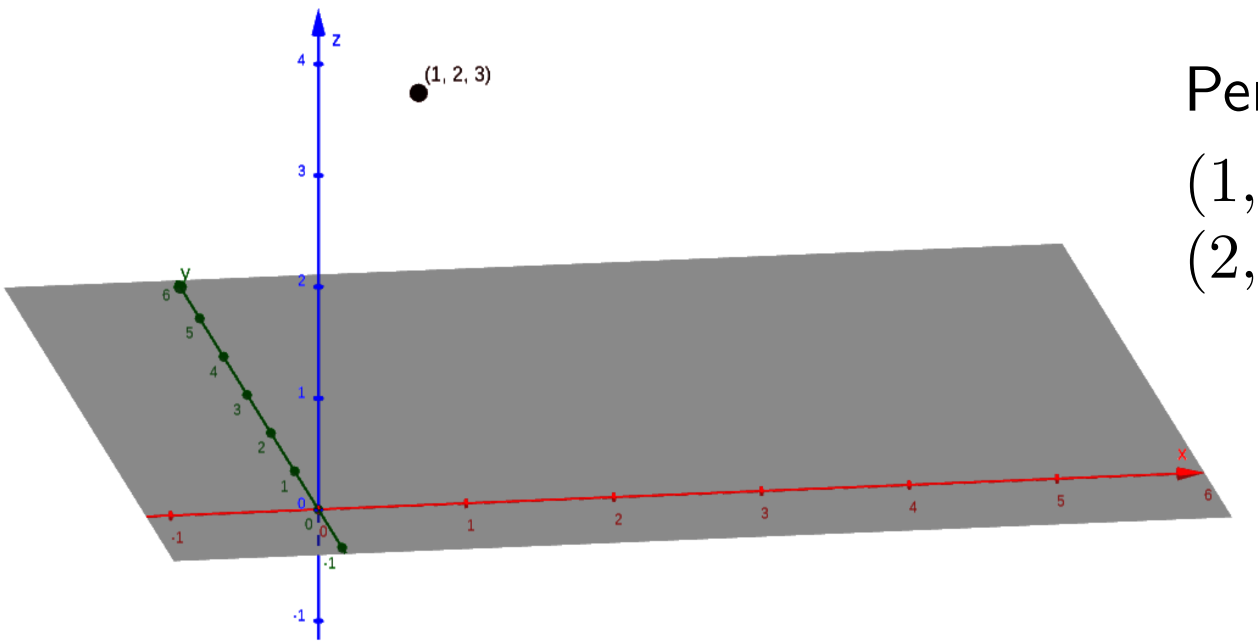

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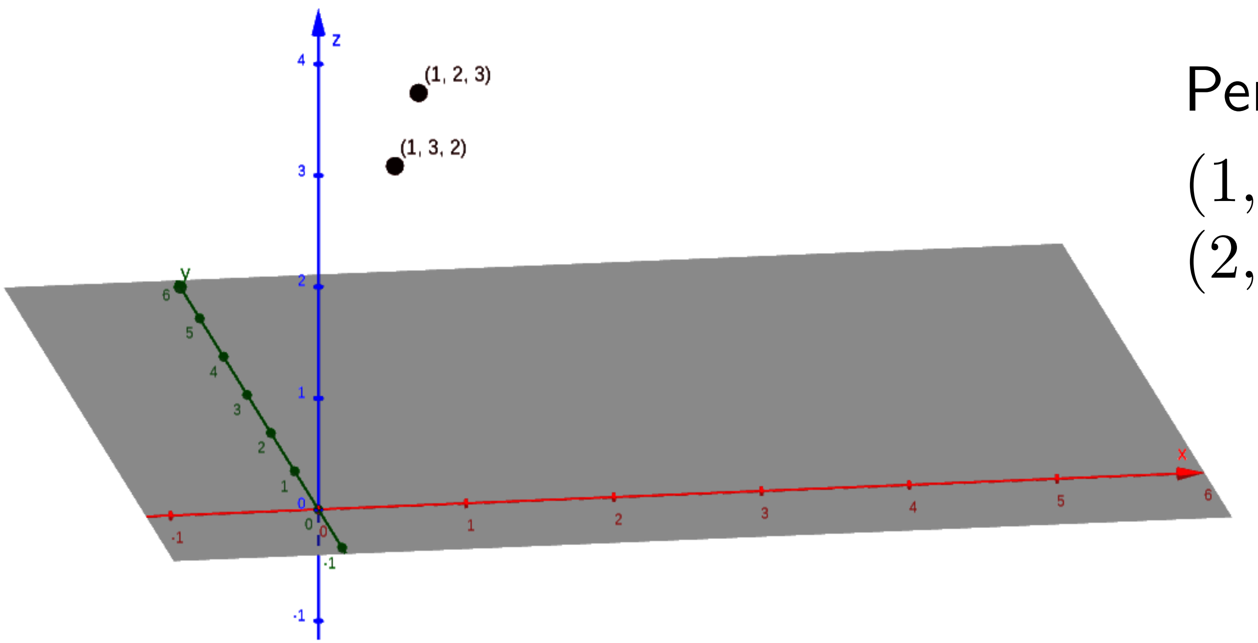

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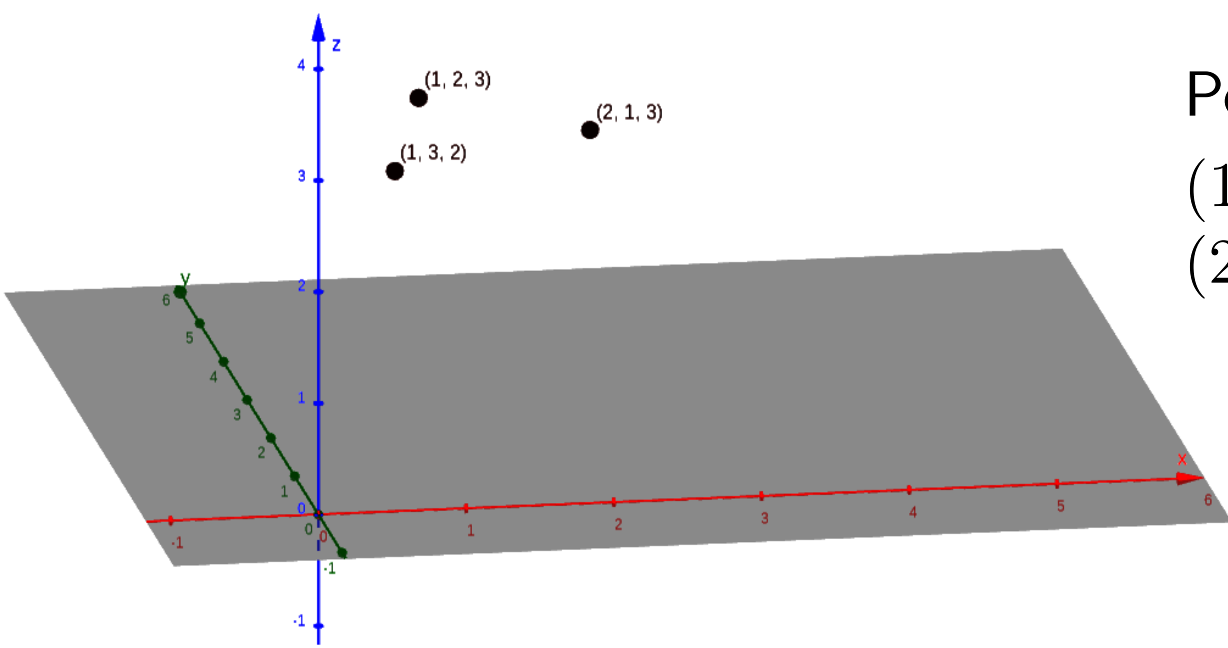

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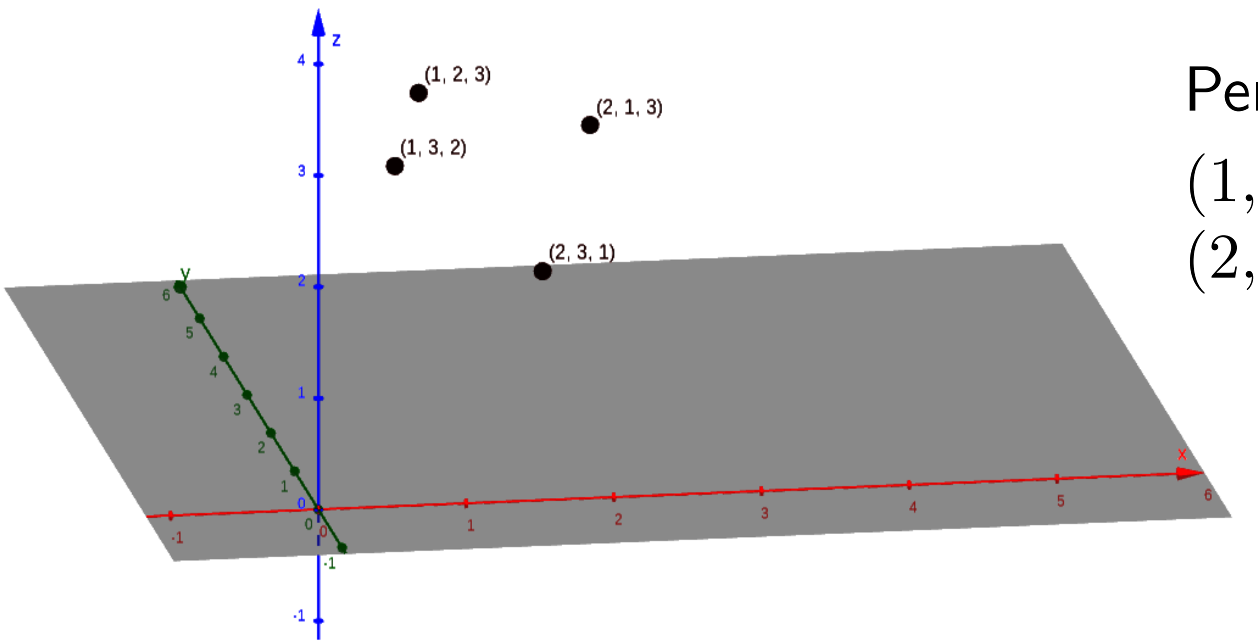

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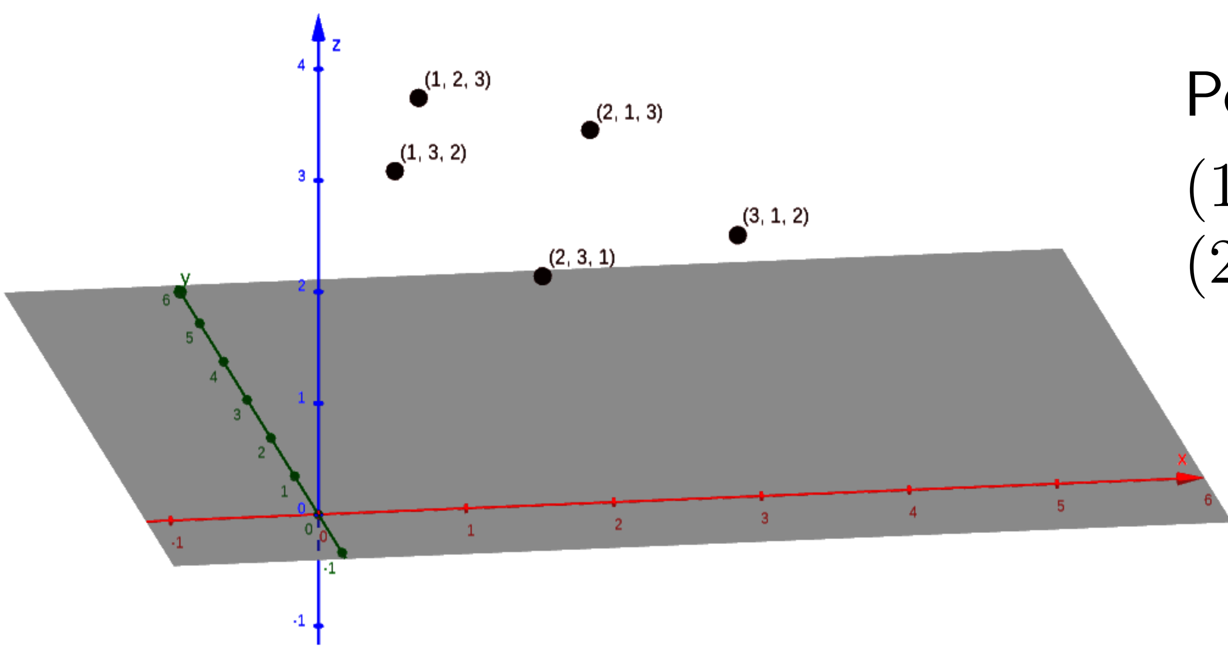

Beispiel: $n = 3$

Permutationen $\pi \in S_3$:

$(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$,
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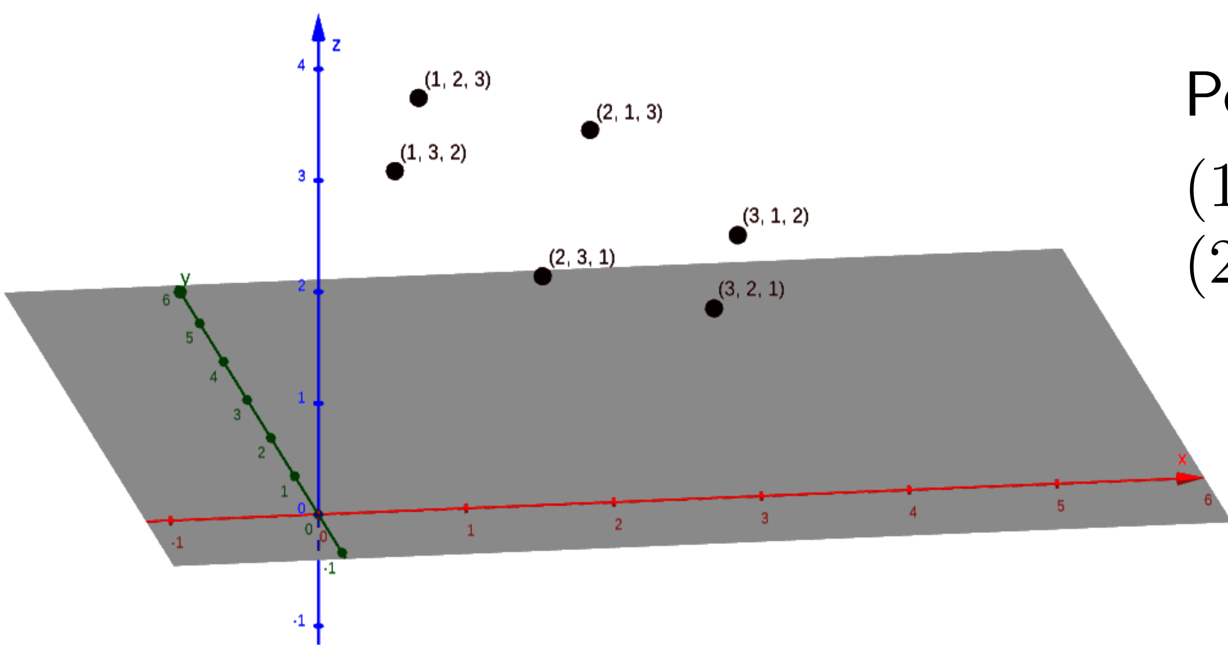

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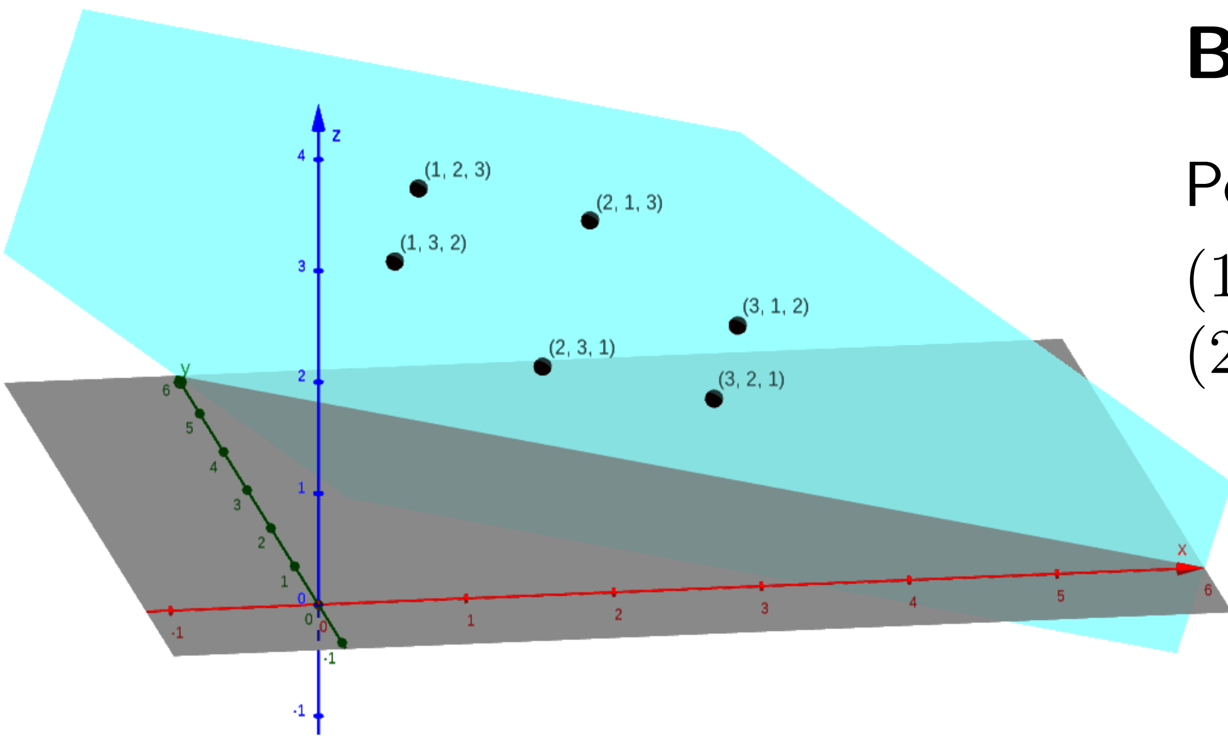

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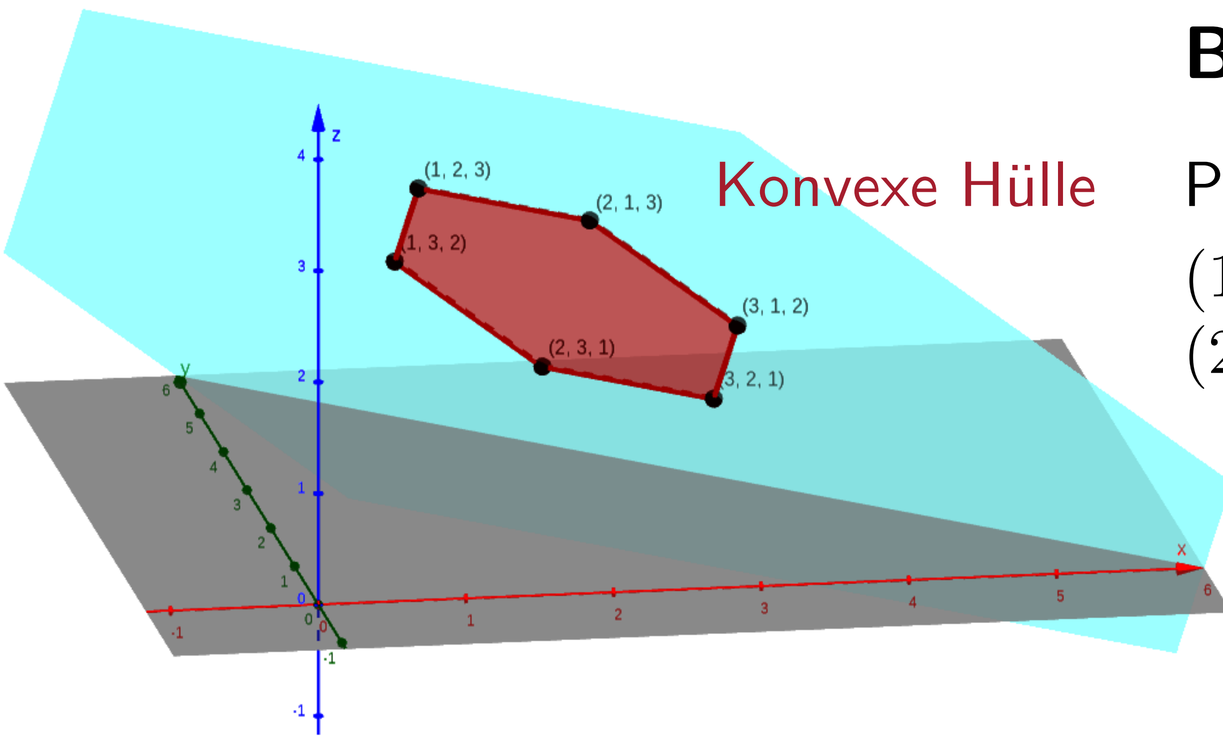

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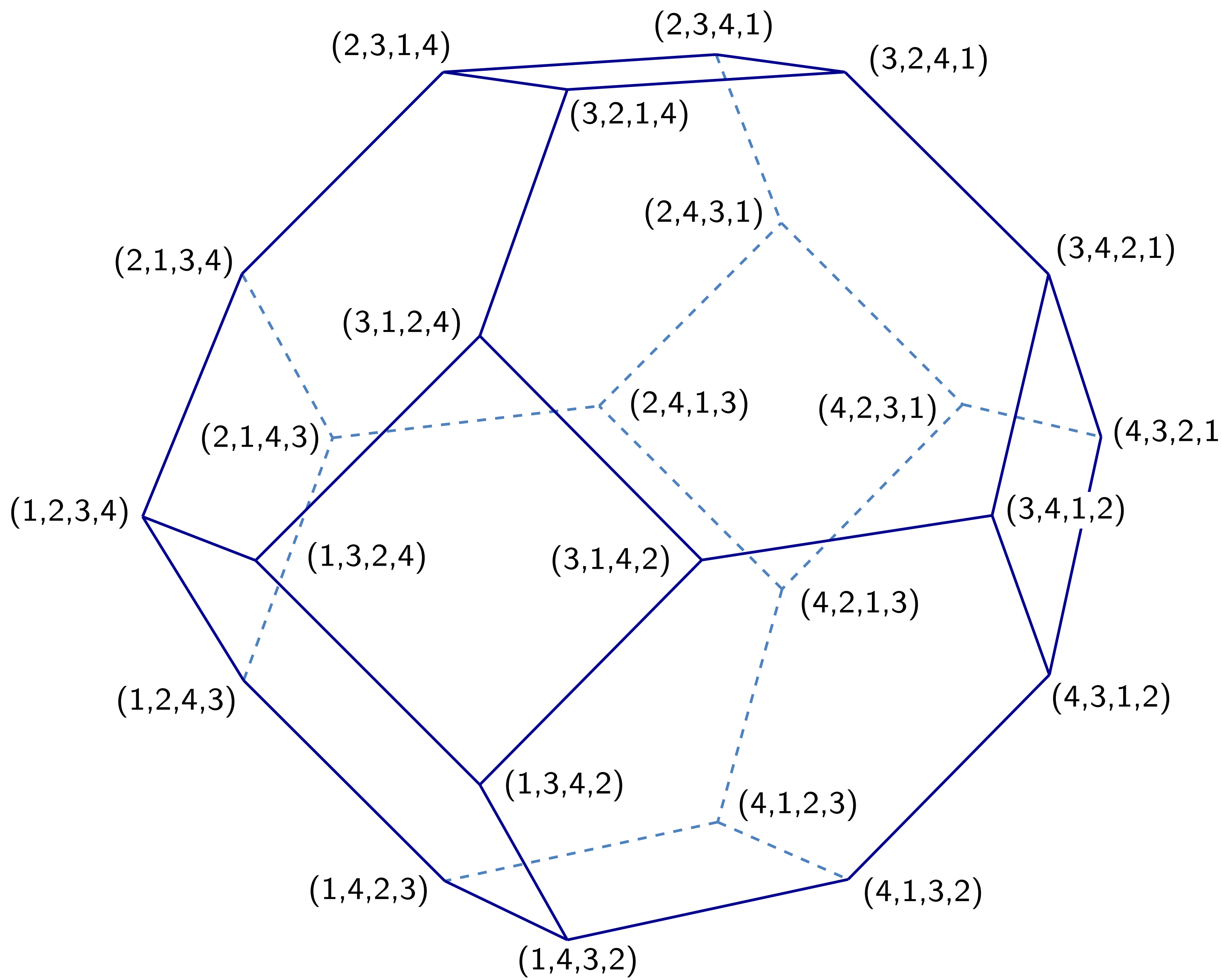

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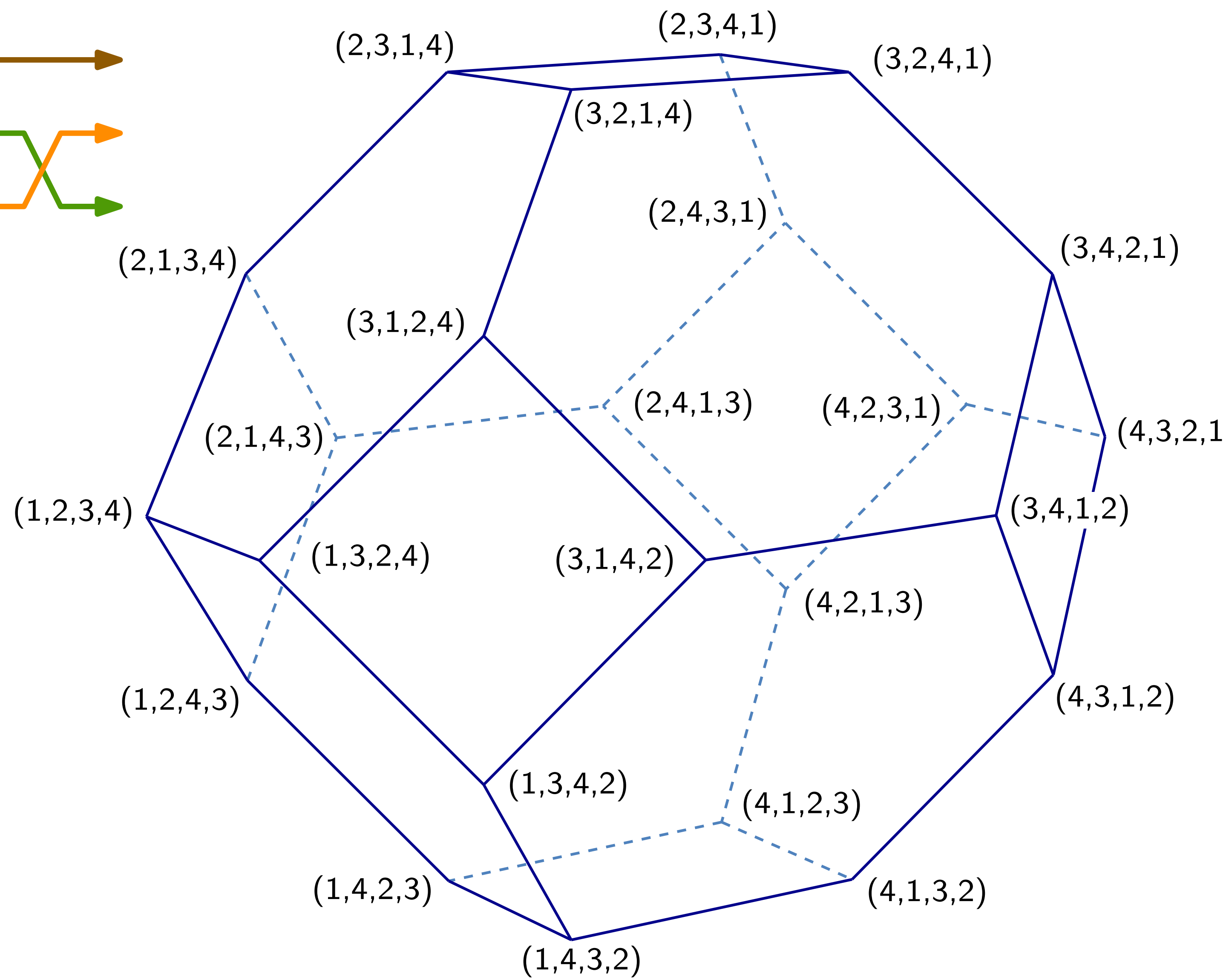

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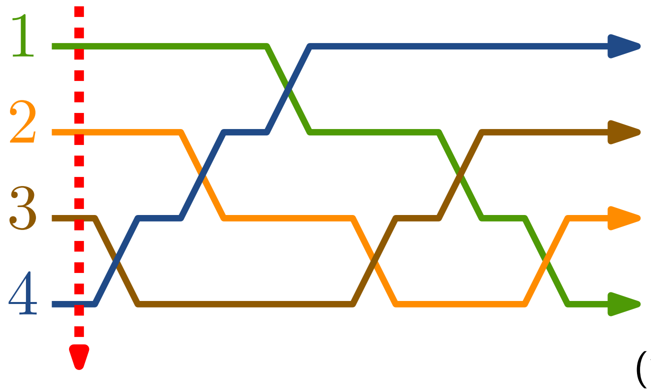

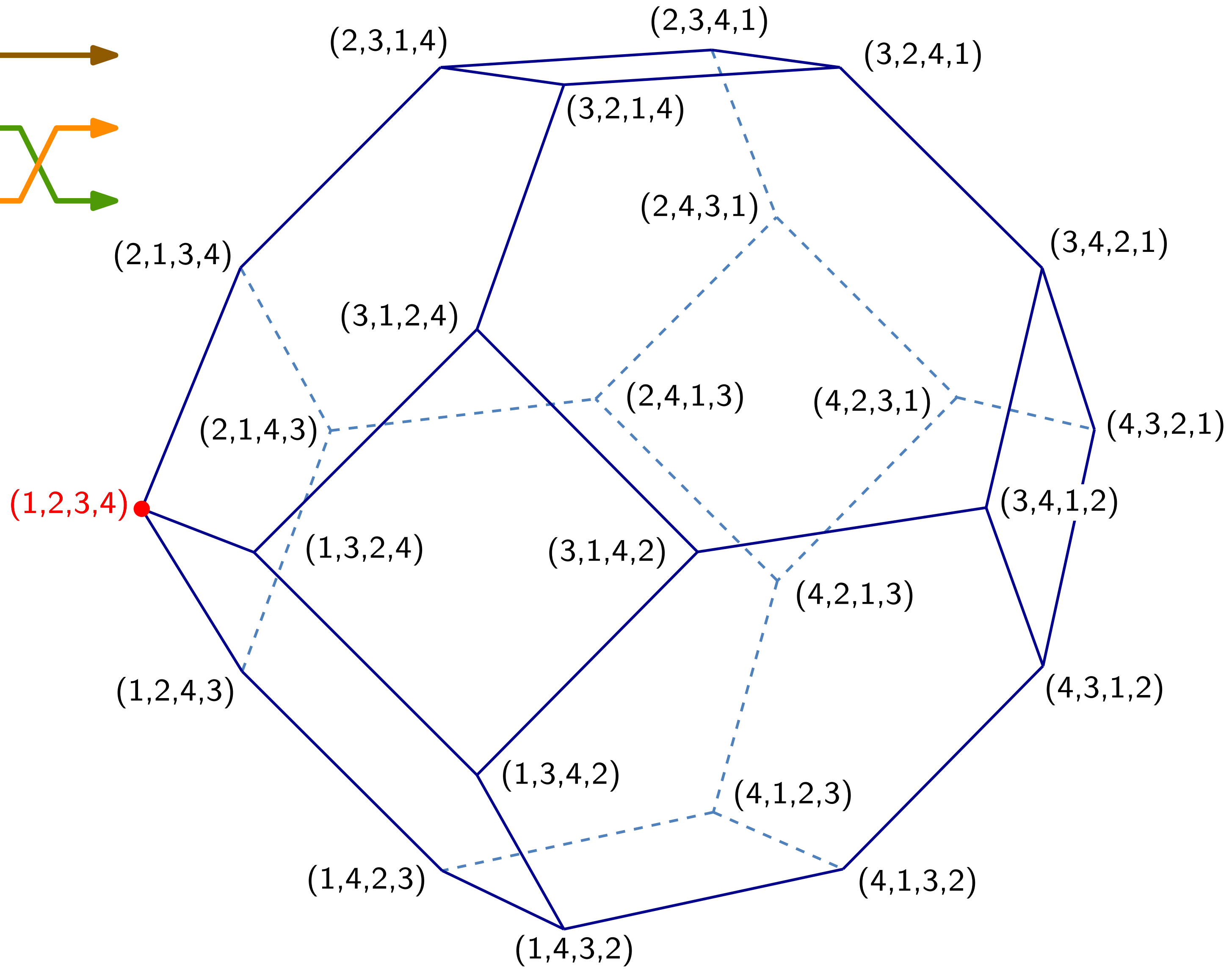

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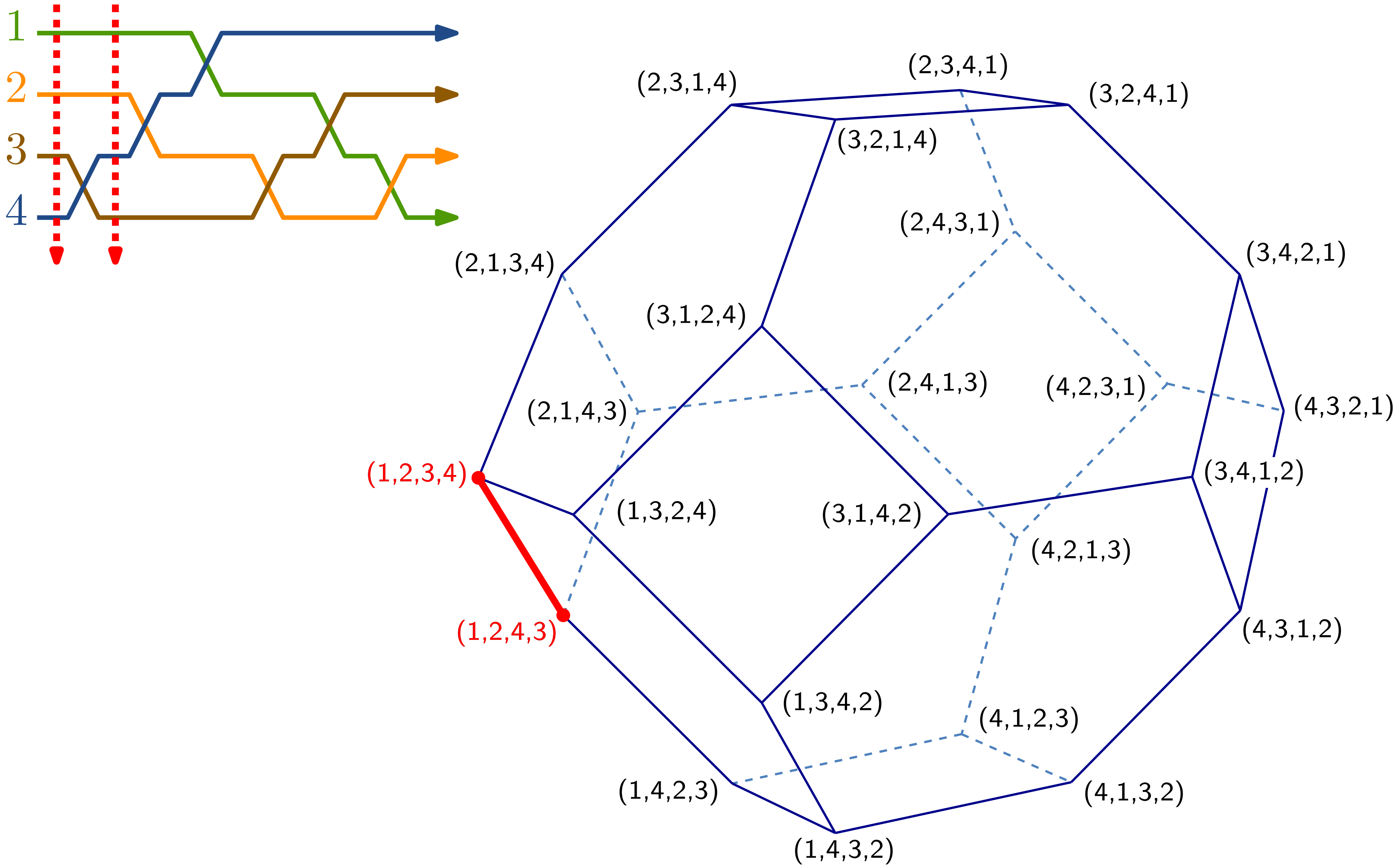

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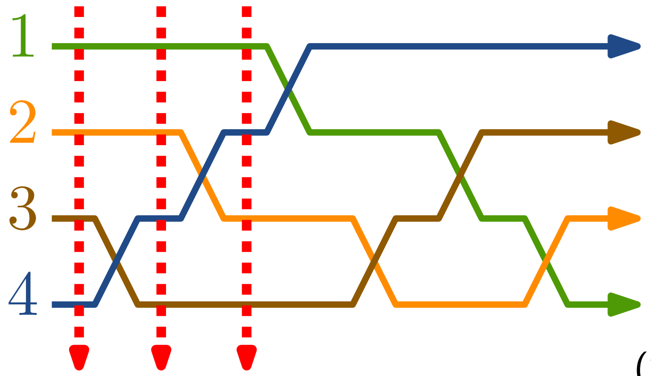

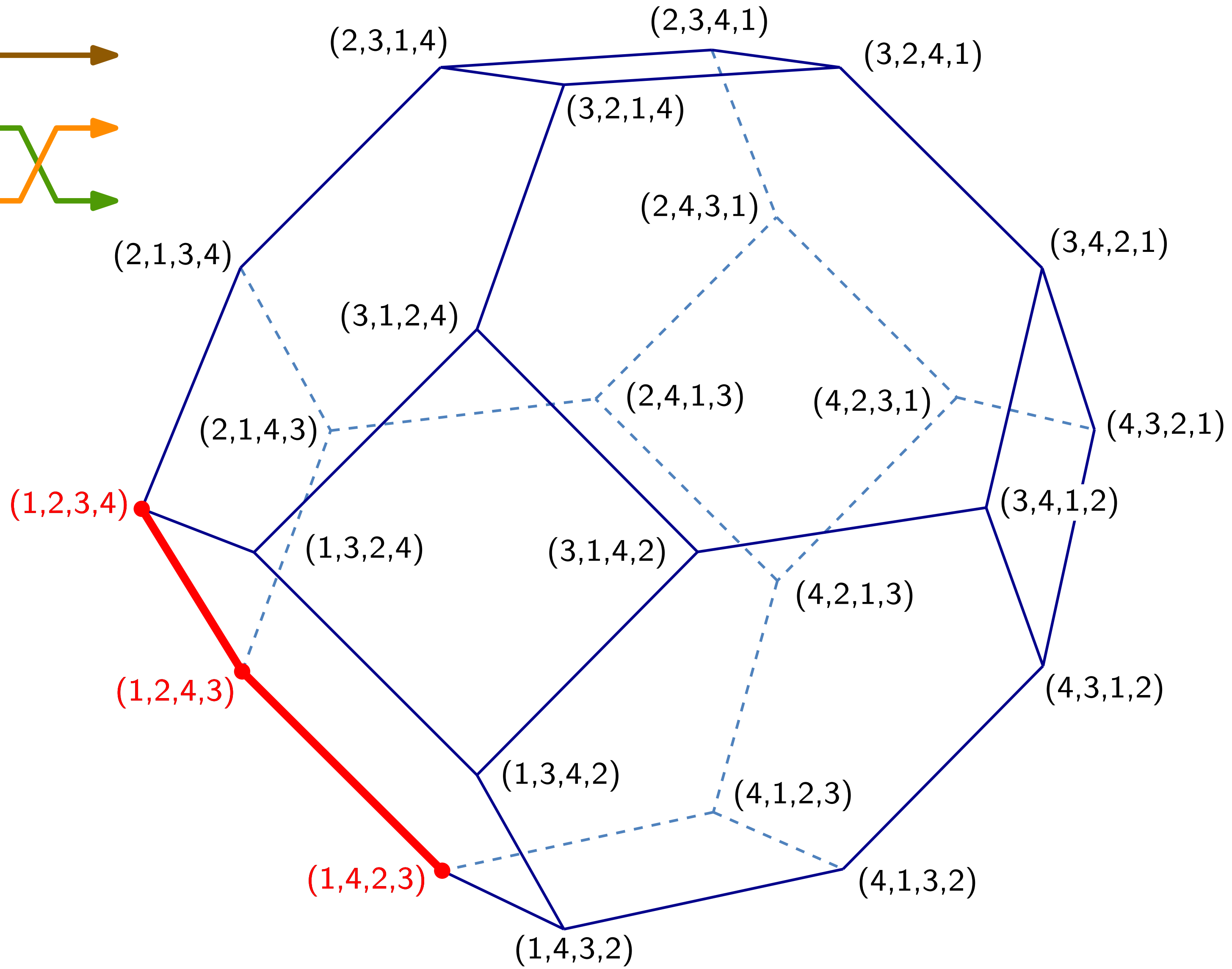

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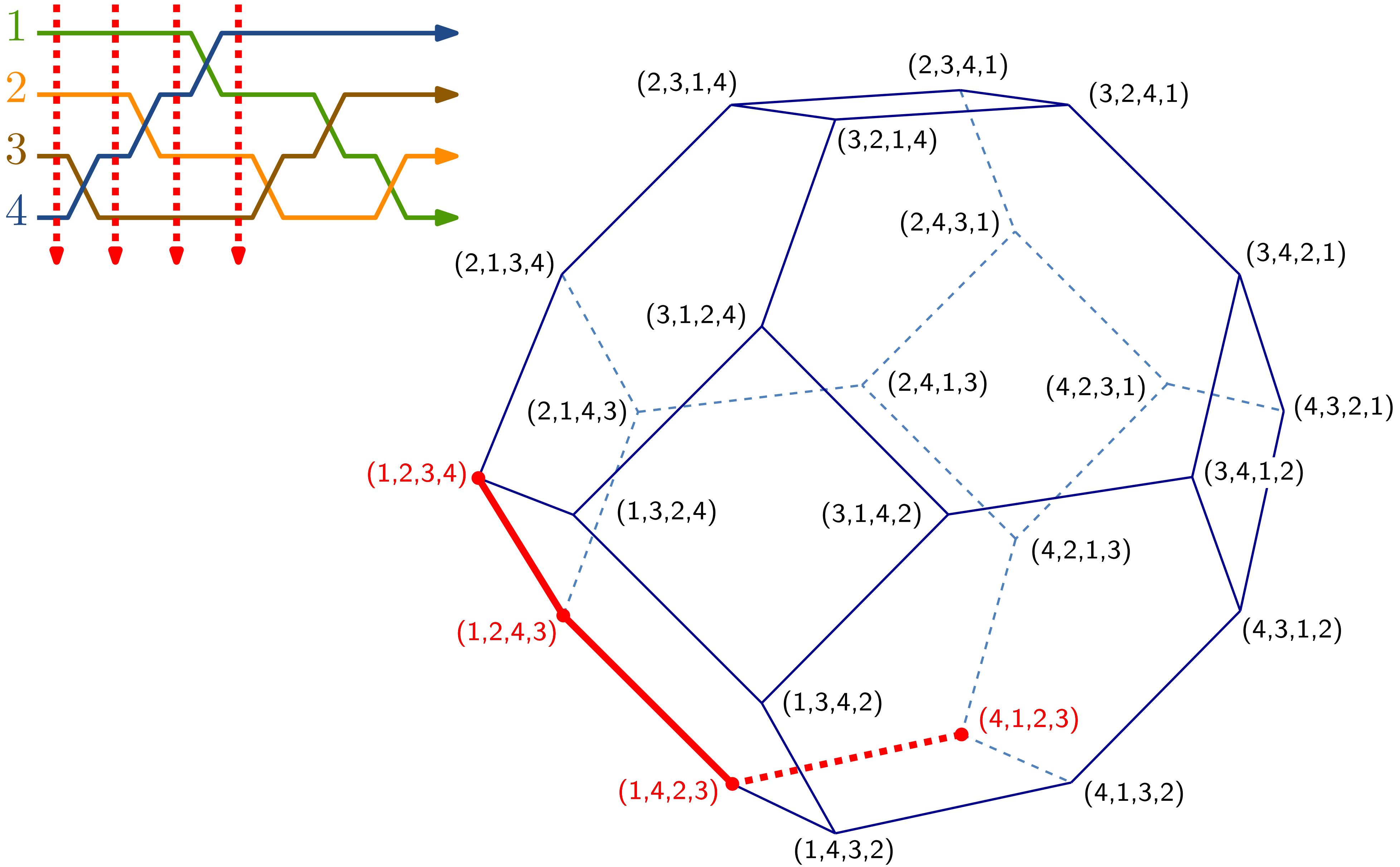

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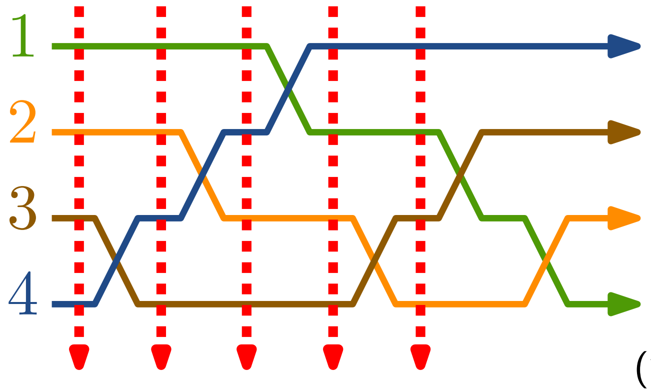

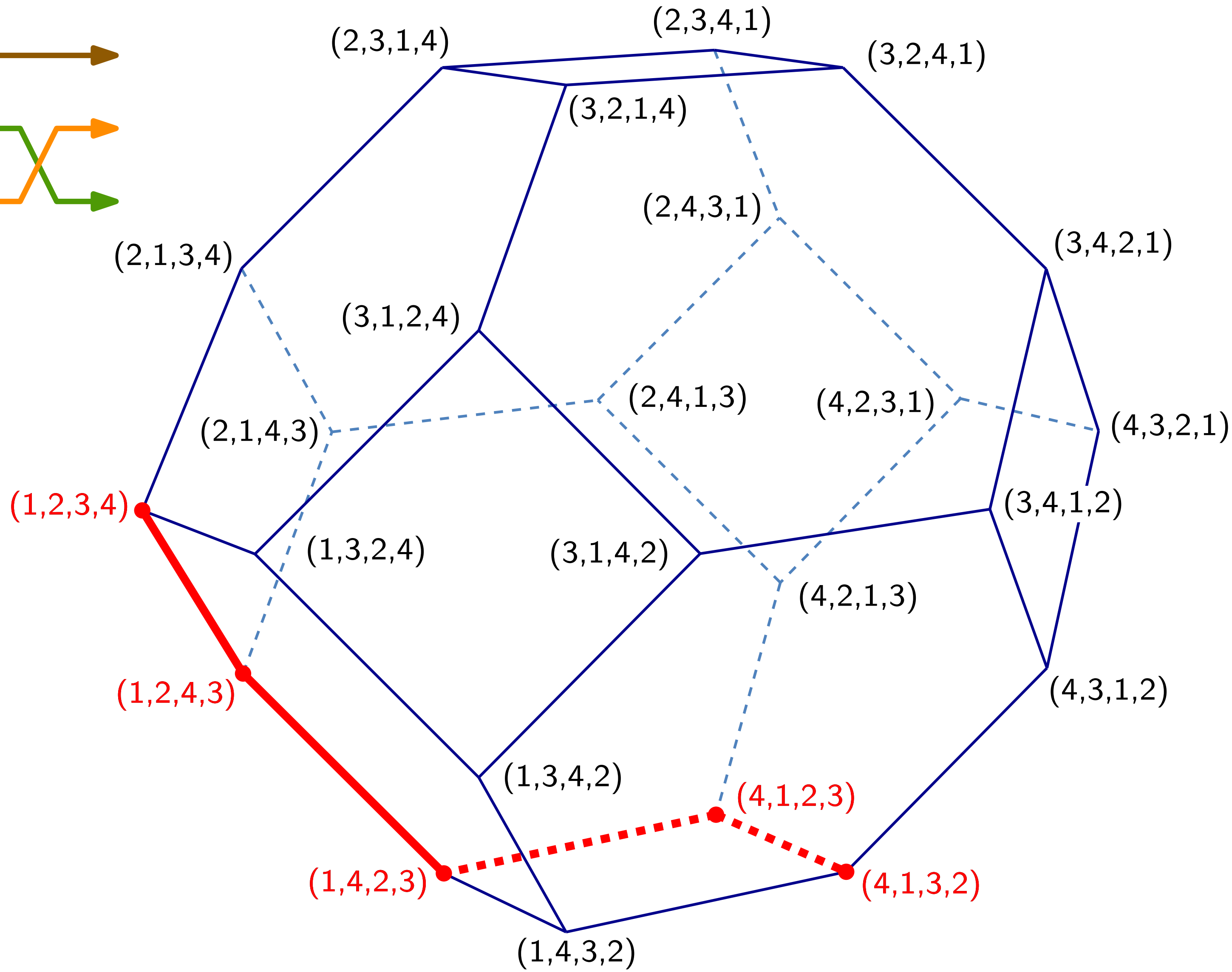

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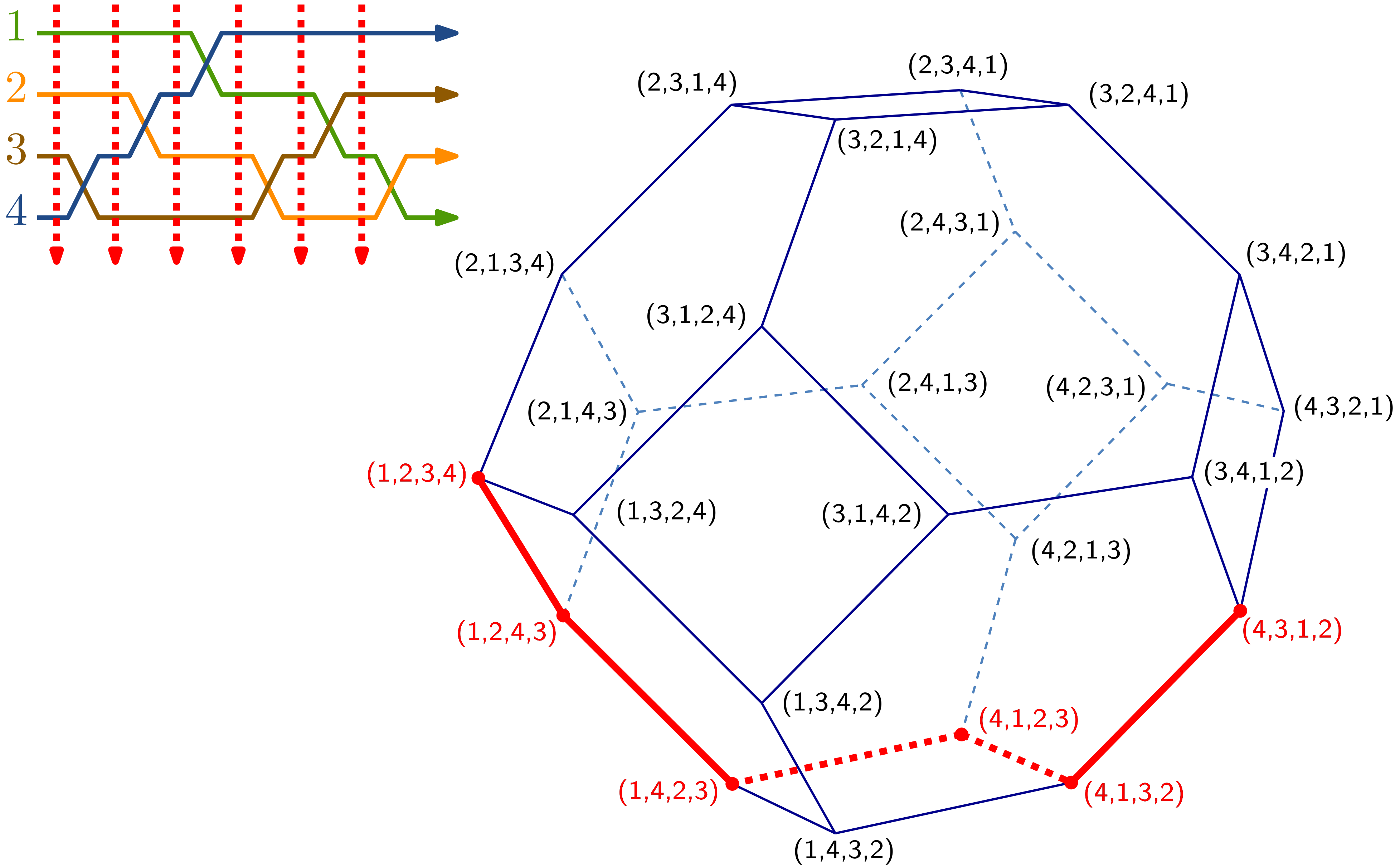

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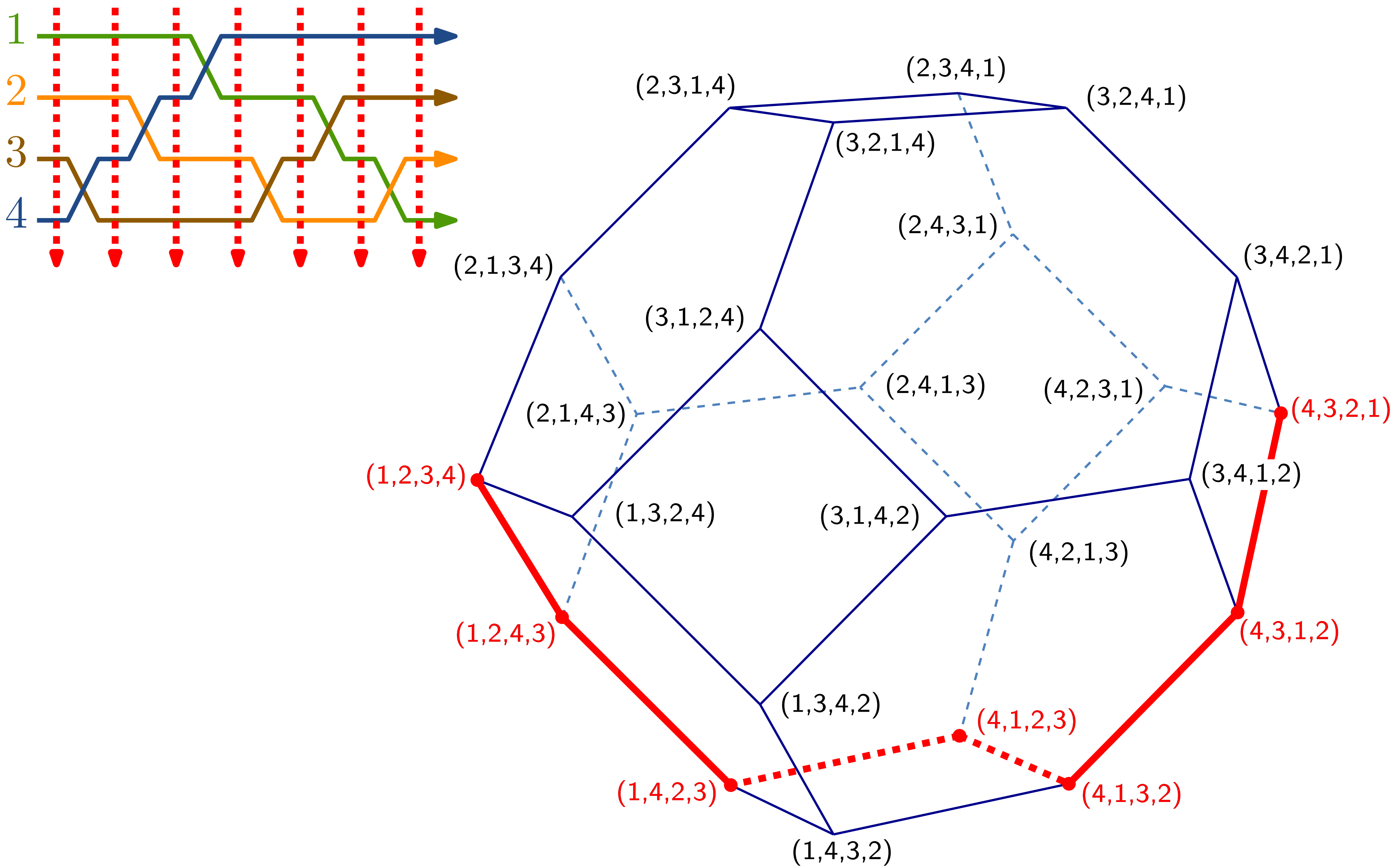

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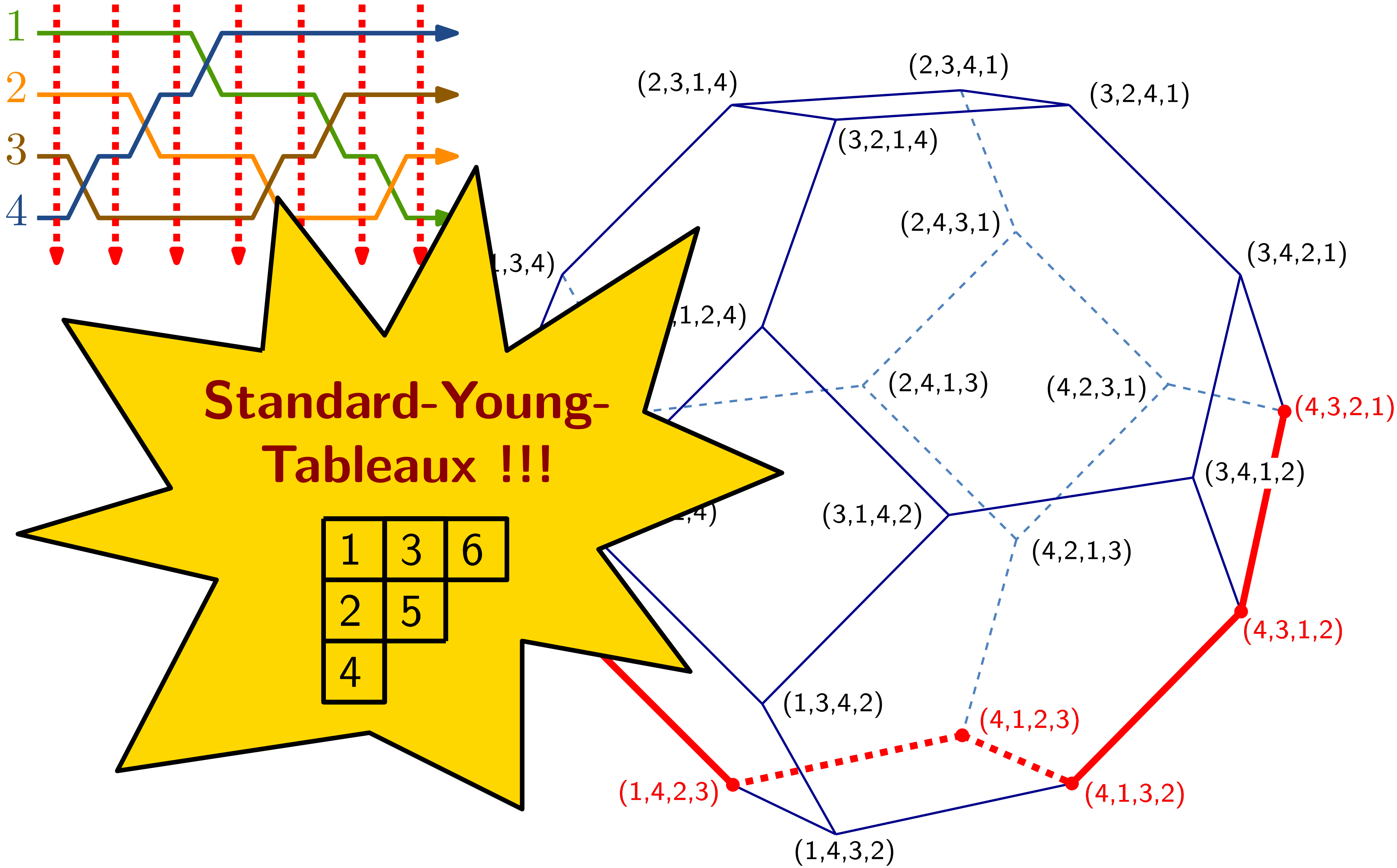
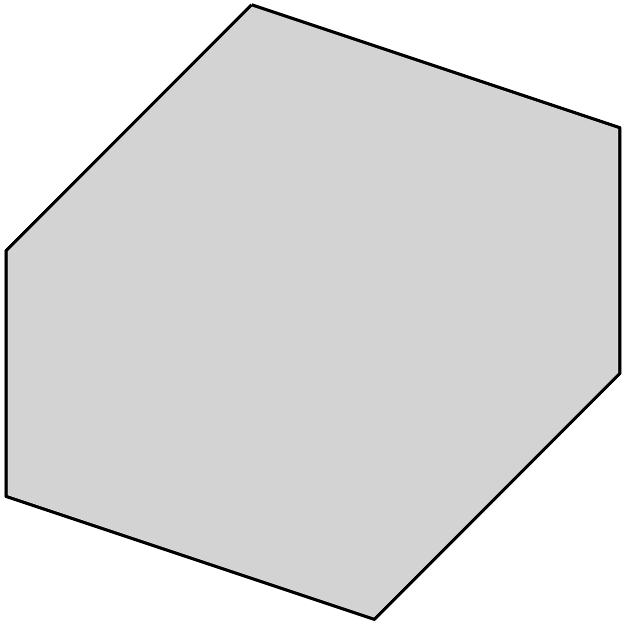


Rhombenpflasterungen



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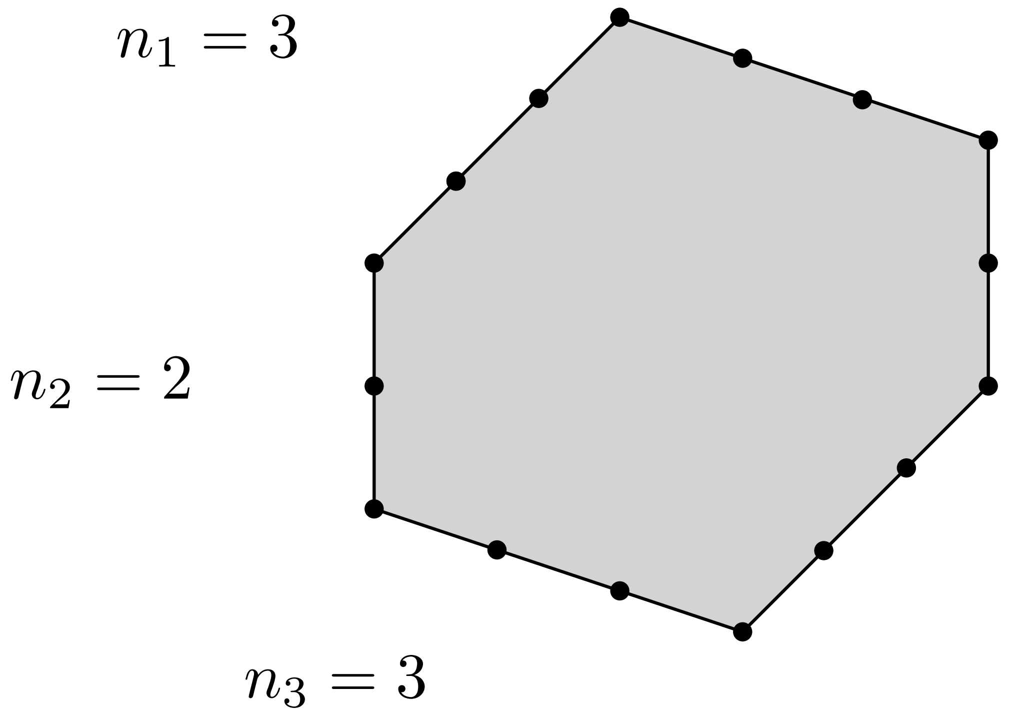

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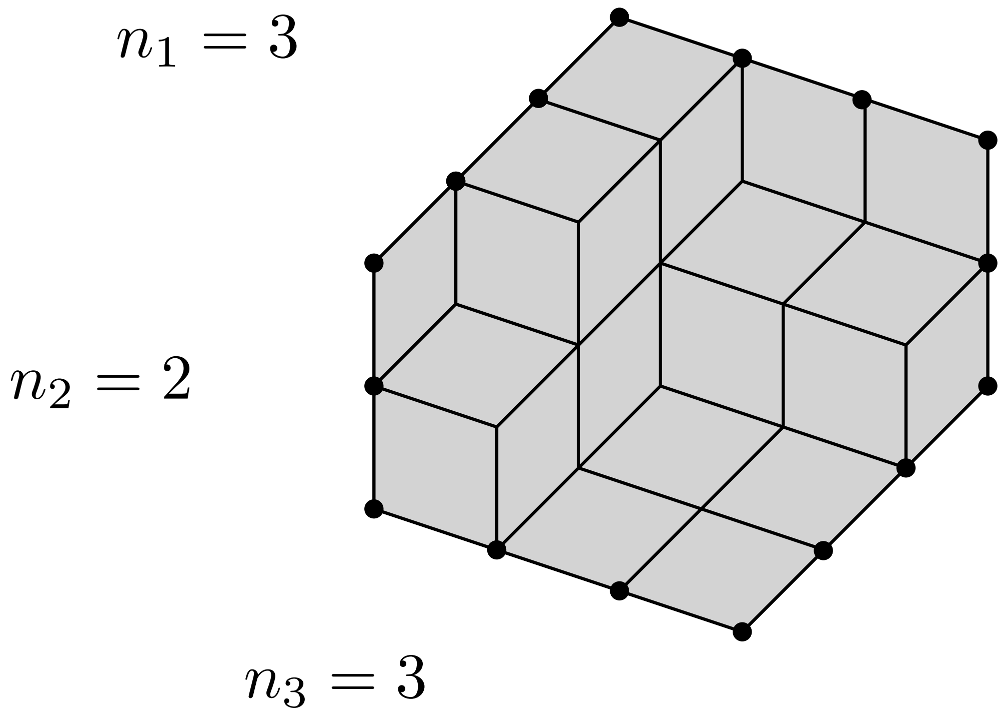

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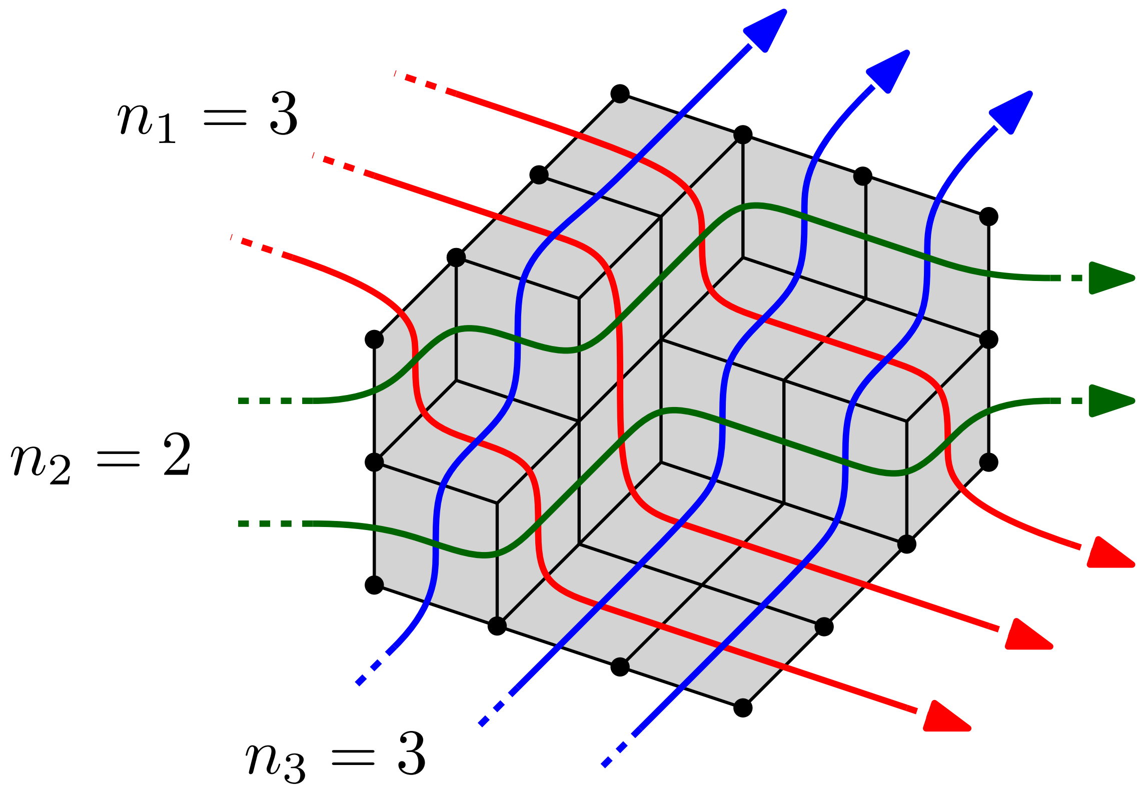

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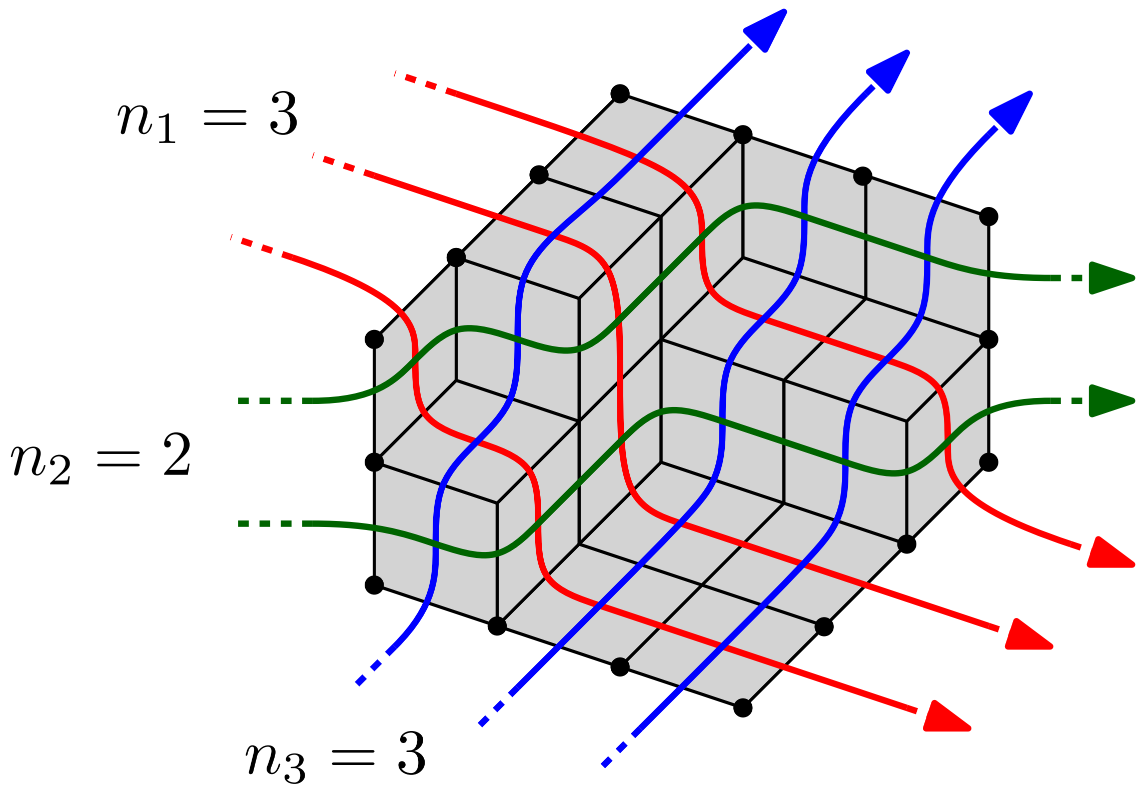
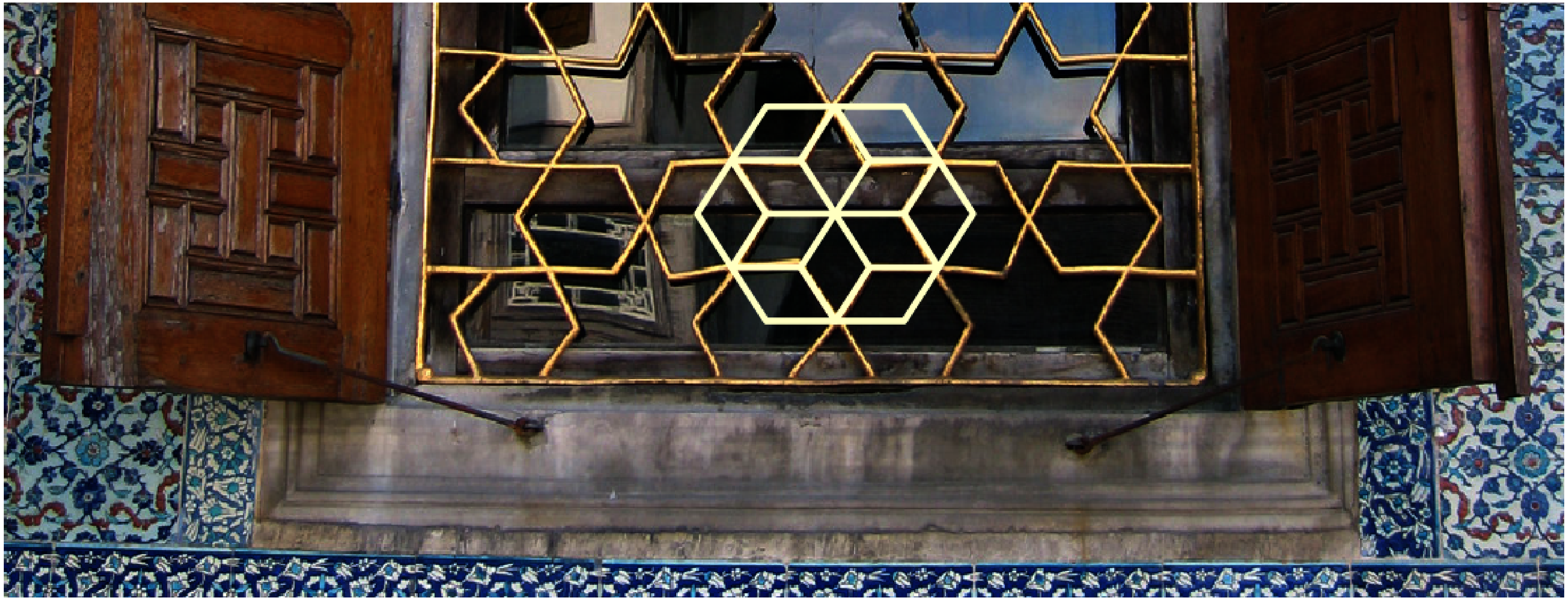


$\Rightarrow$ *Verallgemeinertes Pseudogeradenarrangement*:

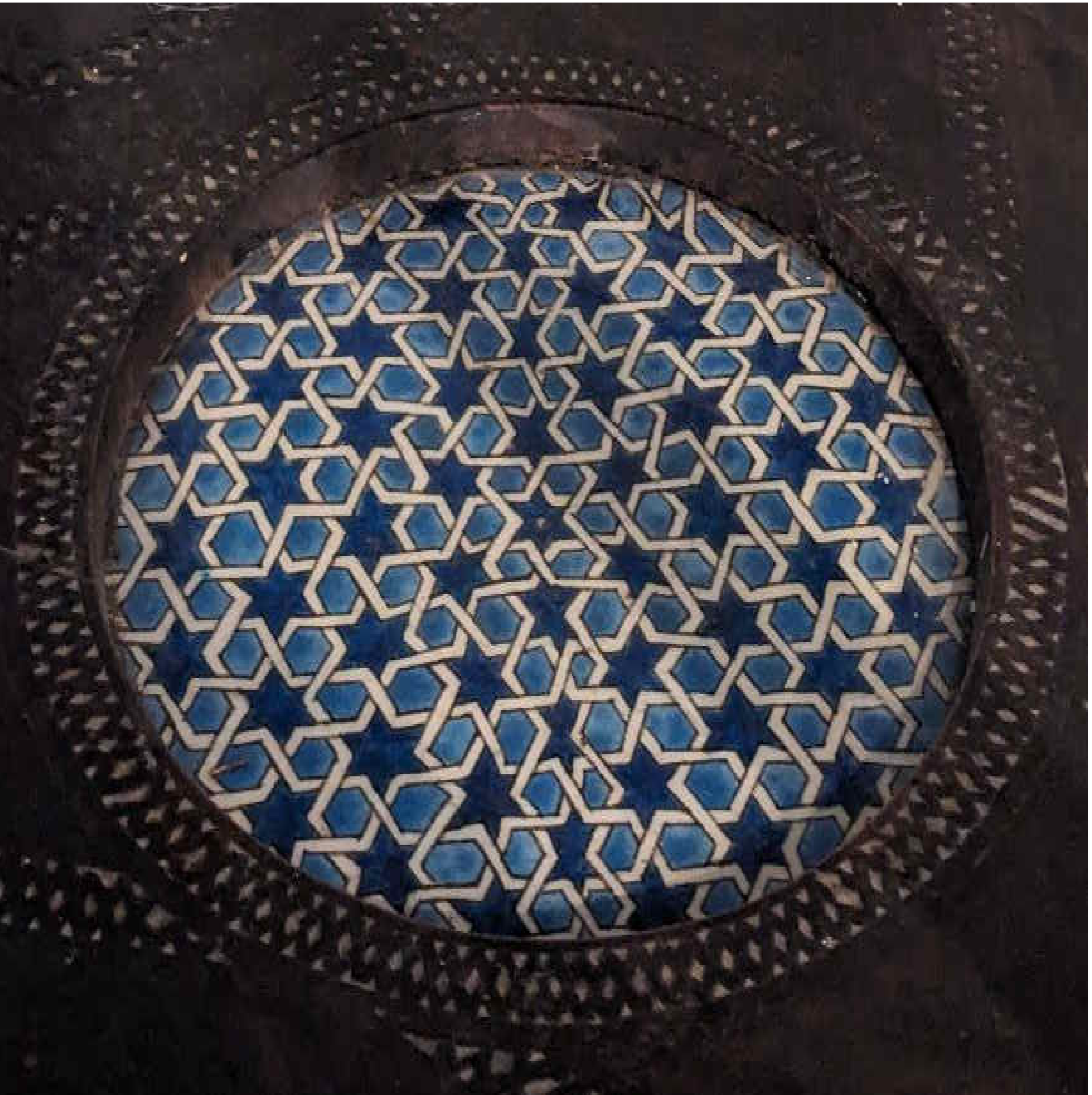
- *Parallelklassen* mit $n_1, ..., n_r$ Pseudogeraden
- (Nur) Pseudogeraden verschiedener Klassen kreuzen sich.

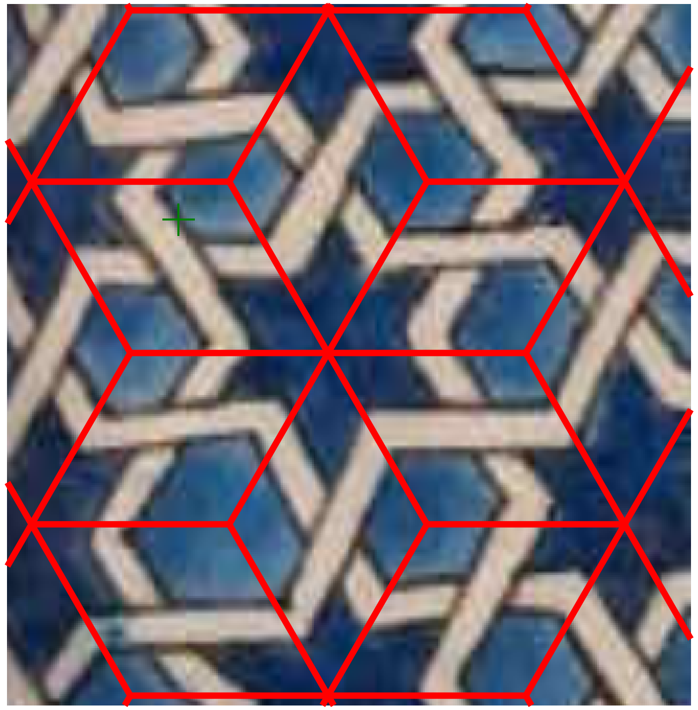
Rhombenpflasterungen



Topkapı Palast, Istanbul, Turkey

Rhombenpflasterungen





Aslan Pasha Moschee
Ioannina, Griechenland

Pseudogeradenarrangements

Drahtdiagramme

Signotope

plane partitions

Permutationen

Rhomben-
pflasterungen

Sortiernetze

Höhere
Bruhat-Ordnung

Standard Young Tableaux

Familien monotoner, nicht
kreuzender Gitterpfade

Orientierte Matroide vom Grad 3

Pseudogeradenarrangements

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Permutationen

Problem:
Wie können PGAs
effizient gleichverteilt
zufällig erzeugt werden?

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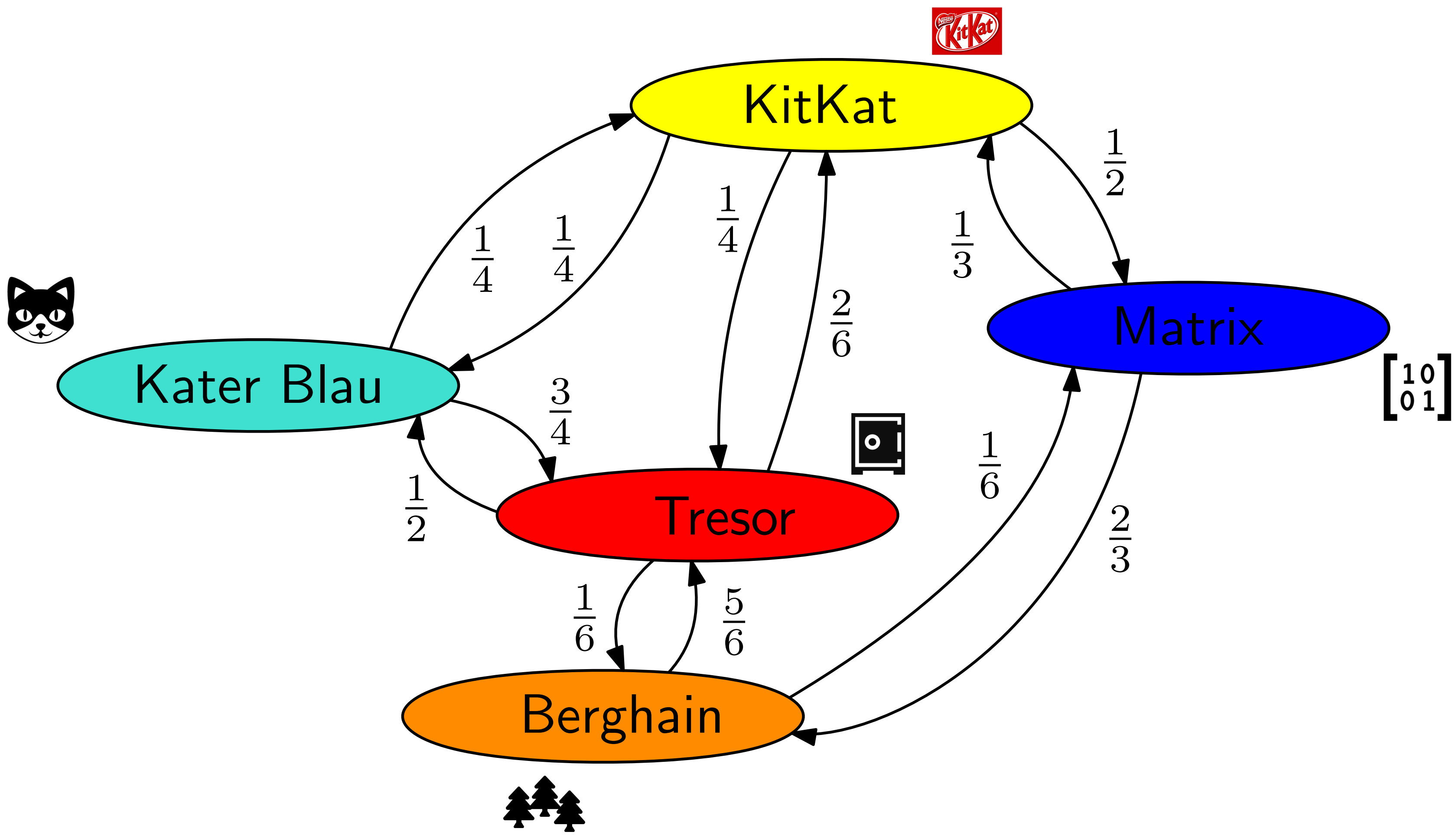
Markov-Ketten

Beispiel einer „nachtaktiven“ Markov-Kette:



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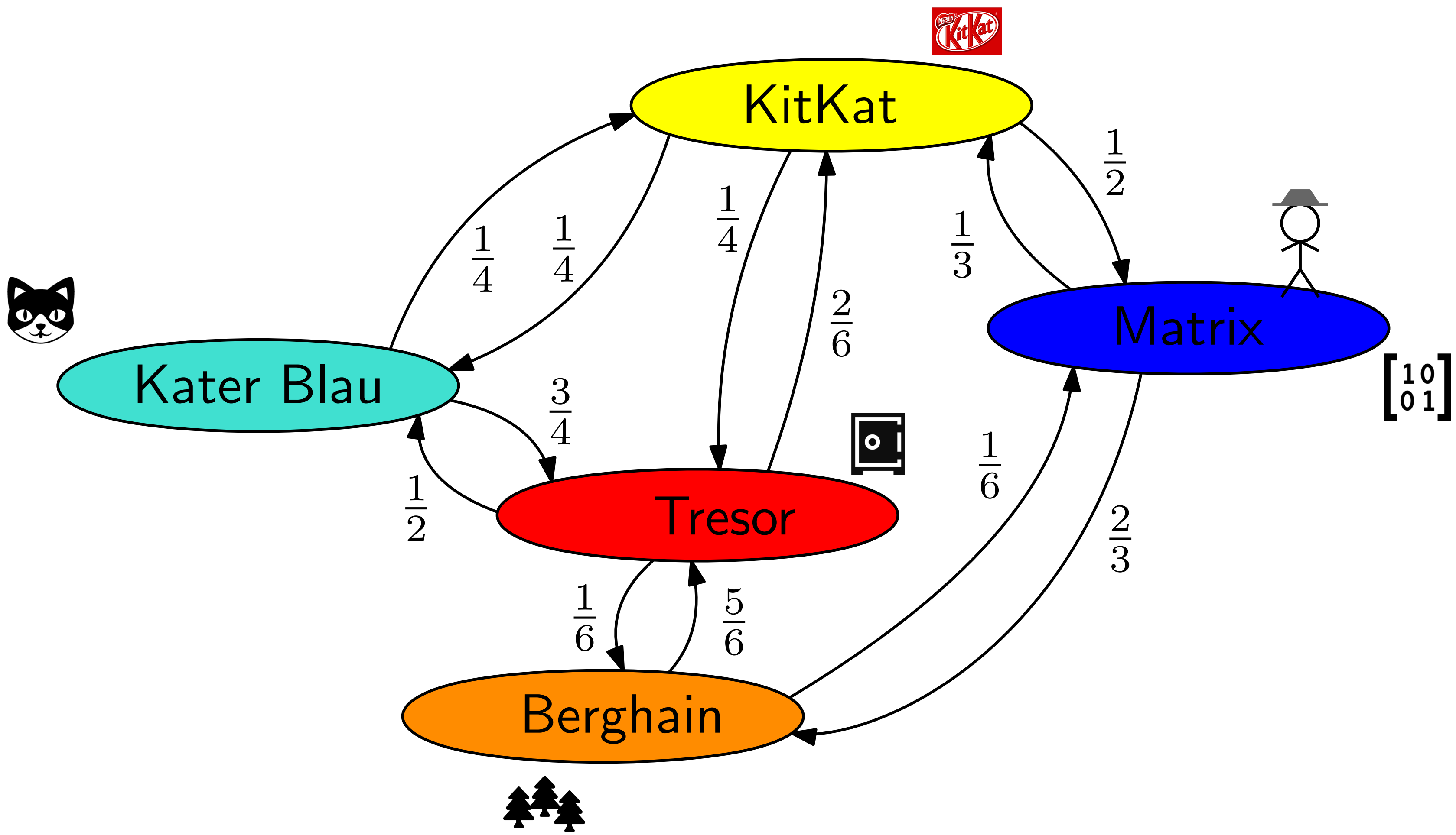

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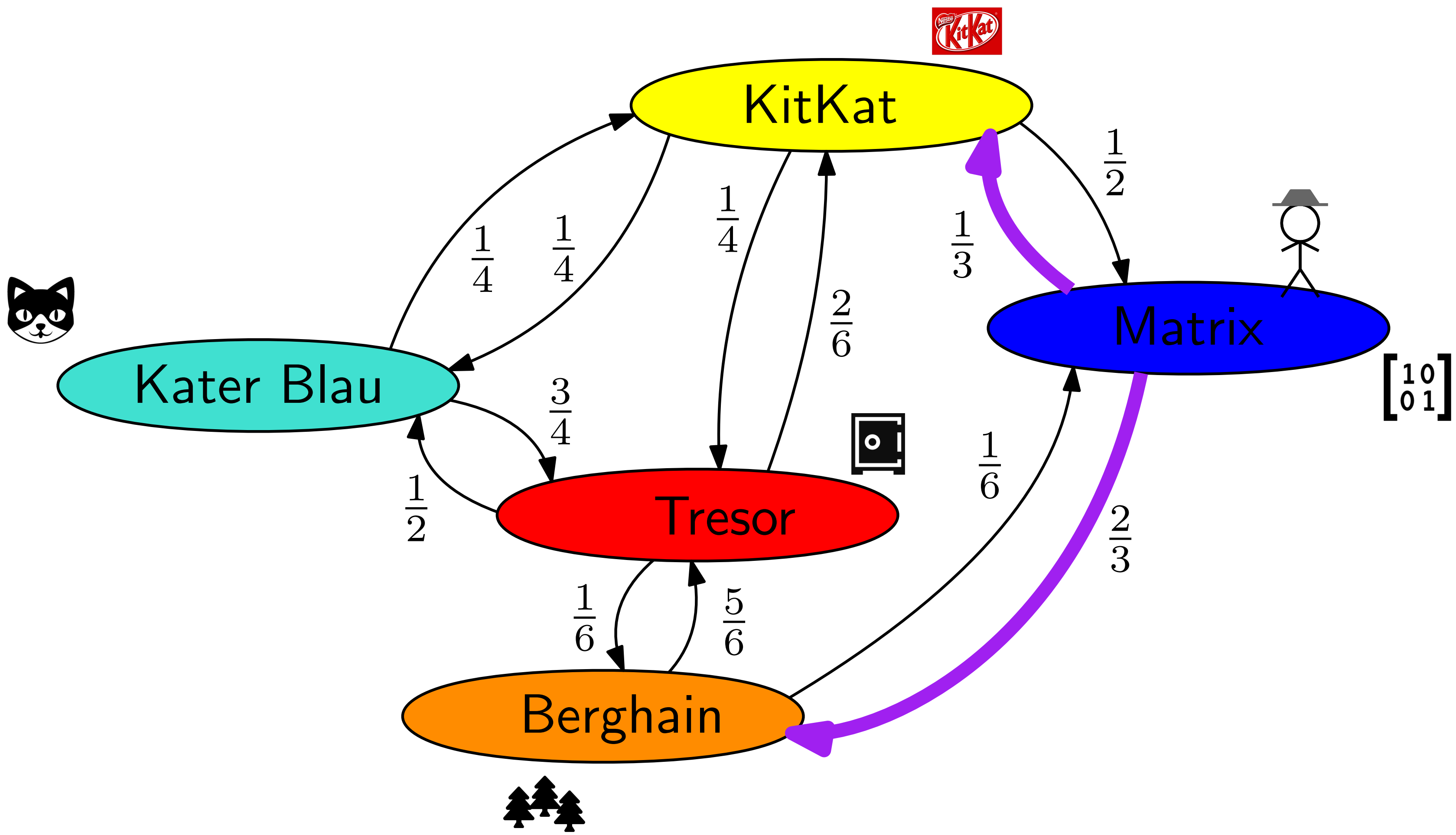

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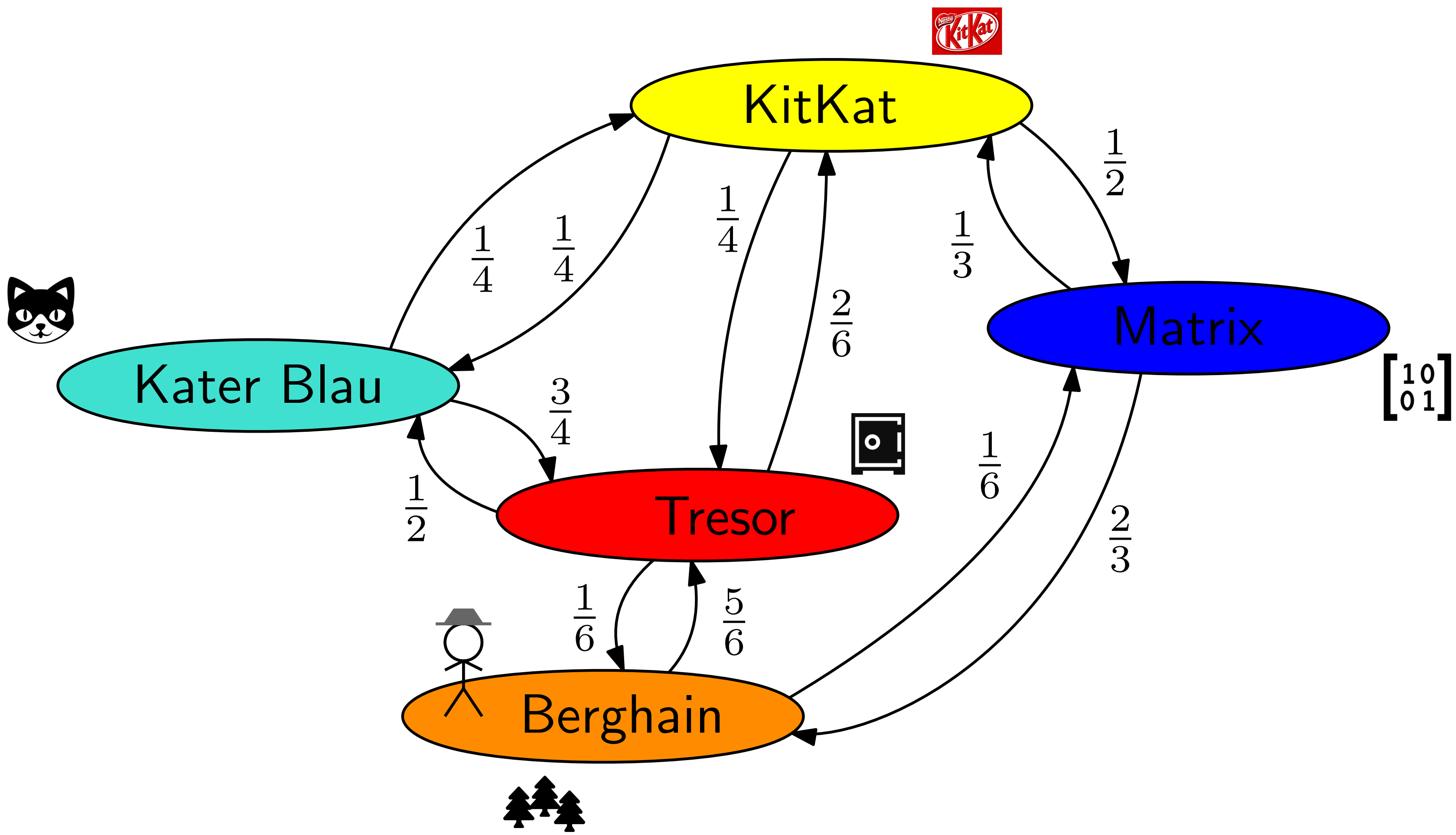

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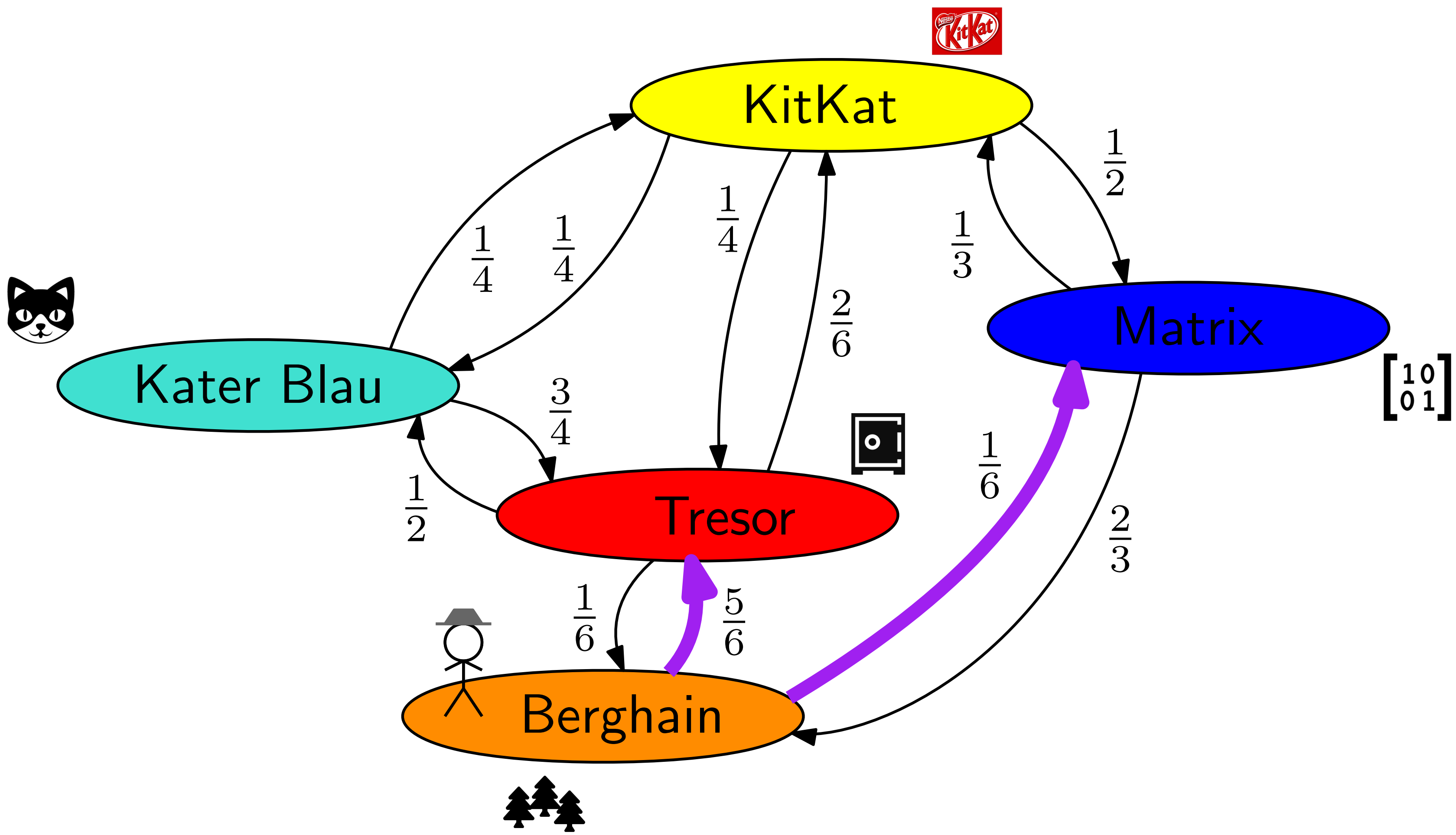

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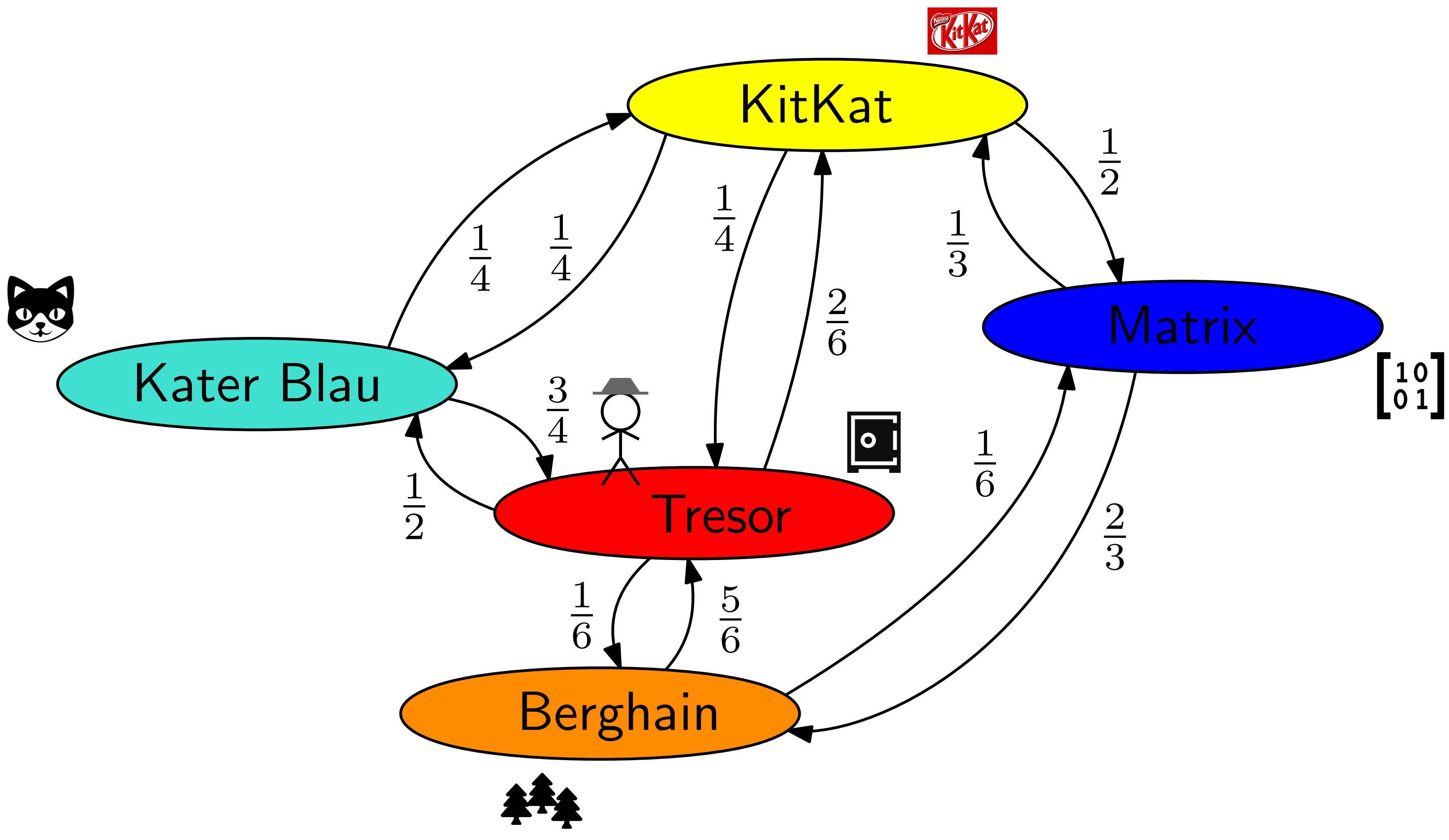

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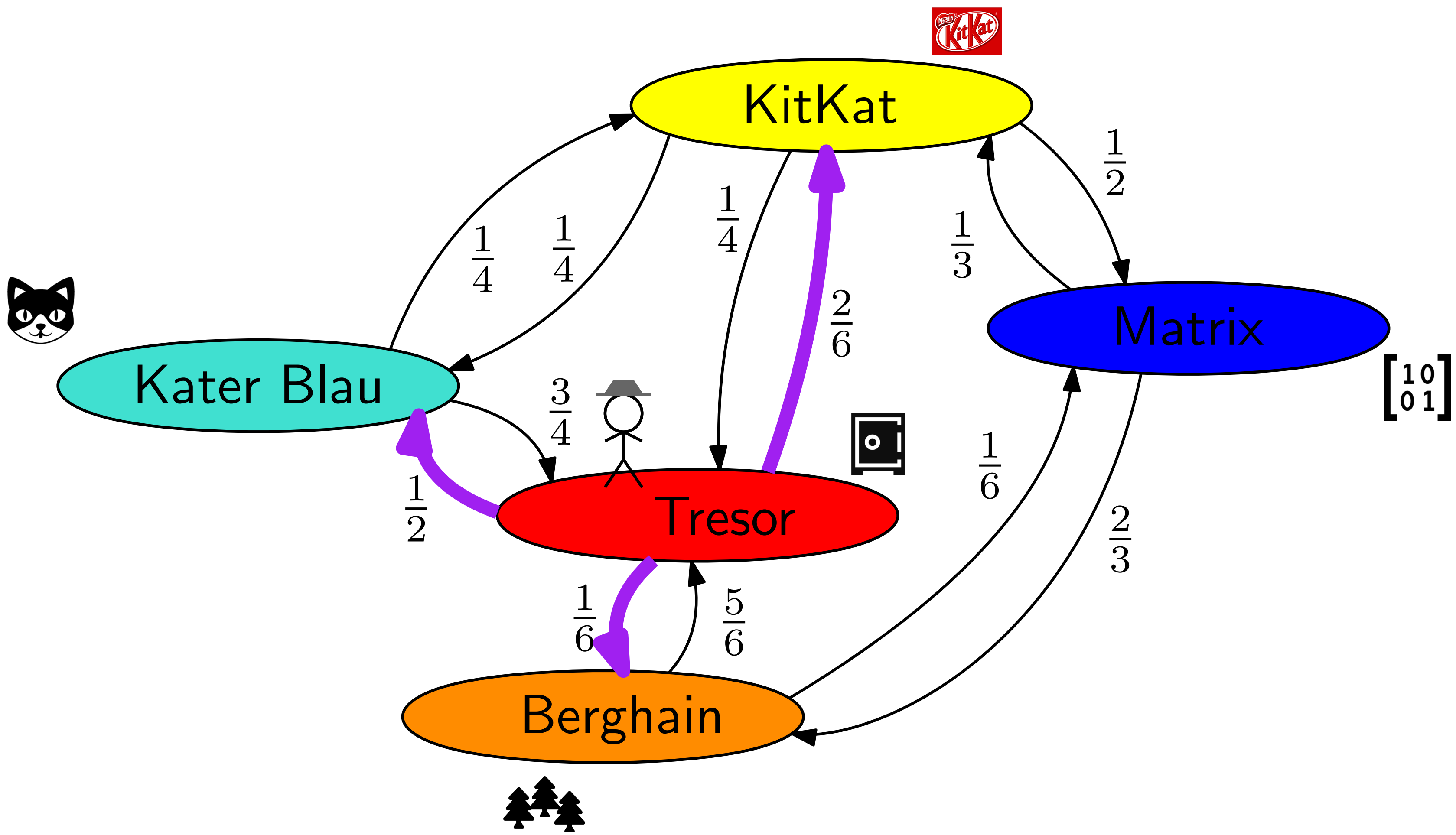

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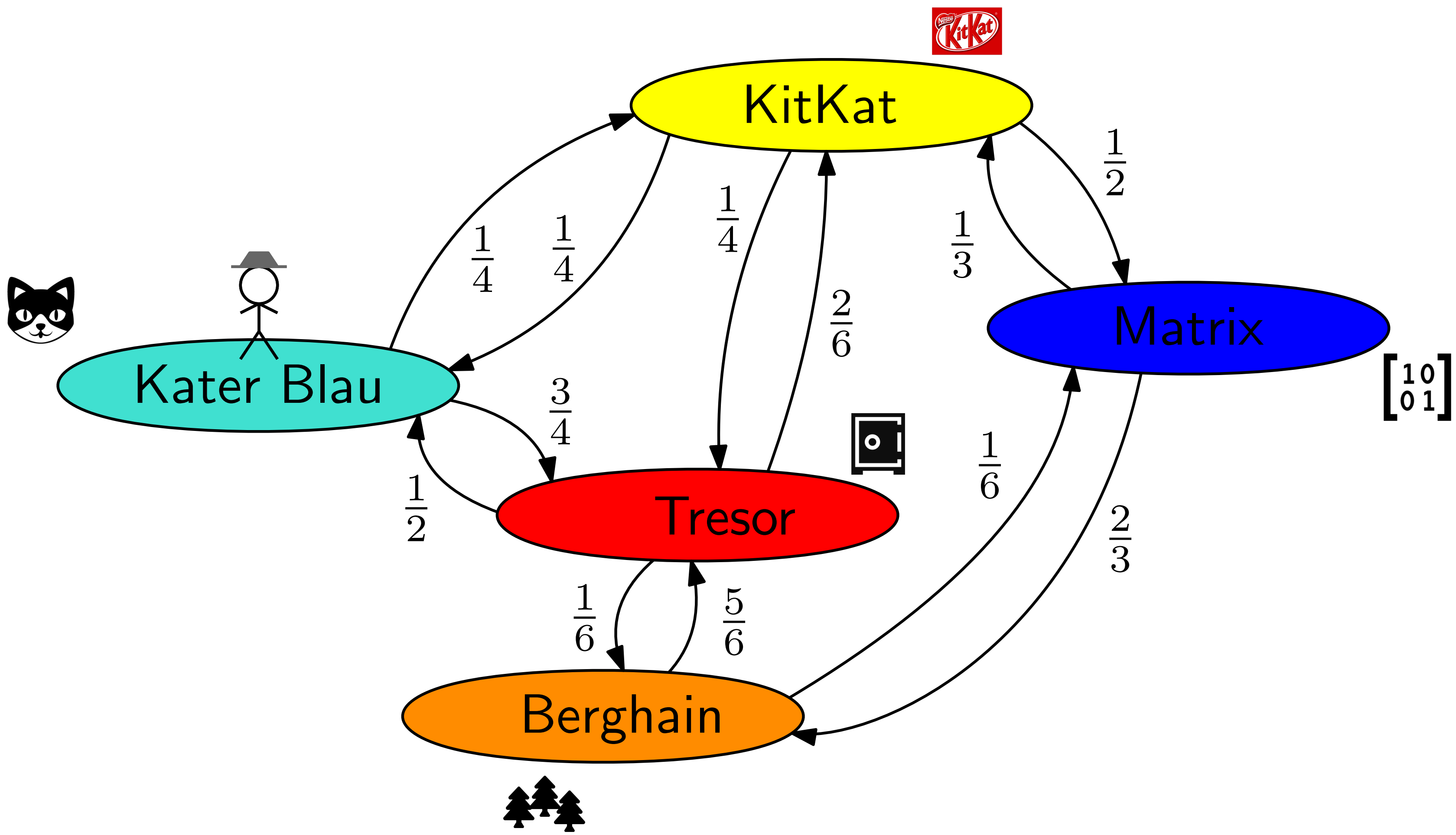

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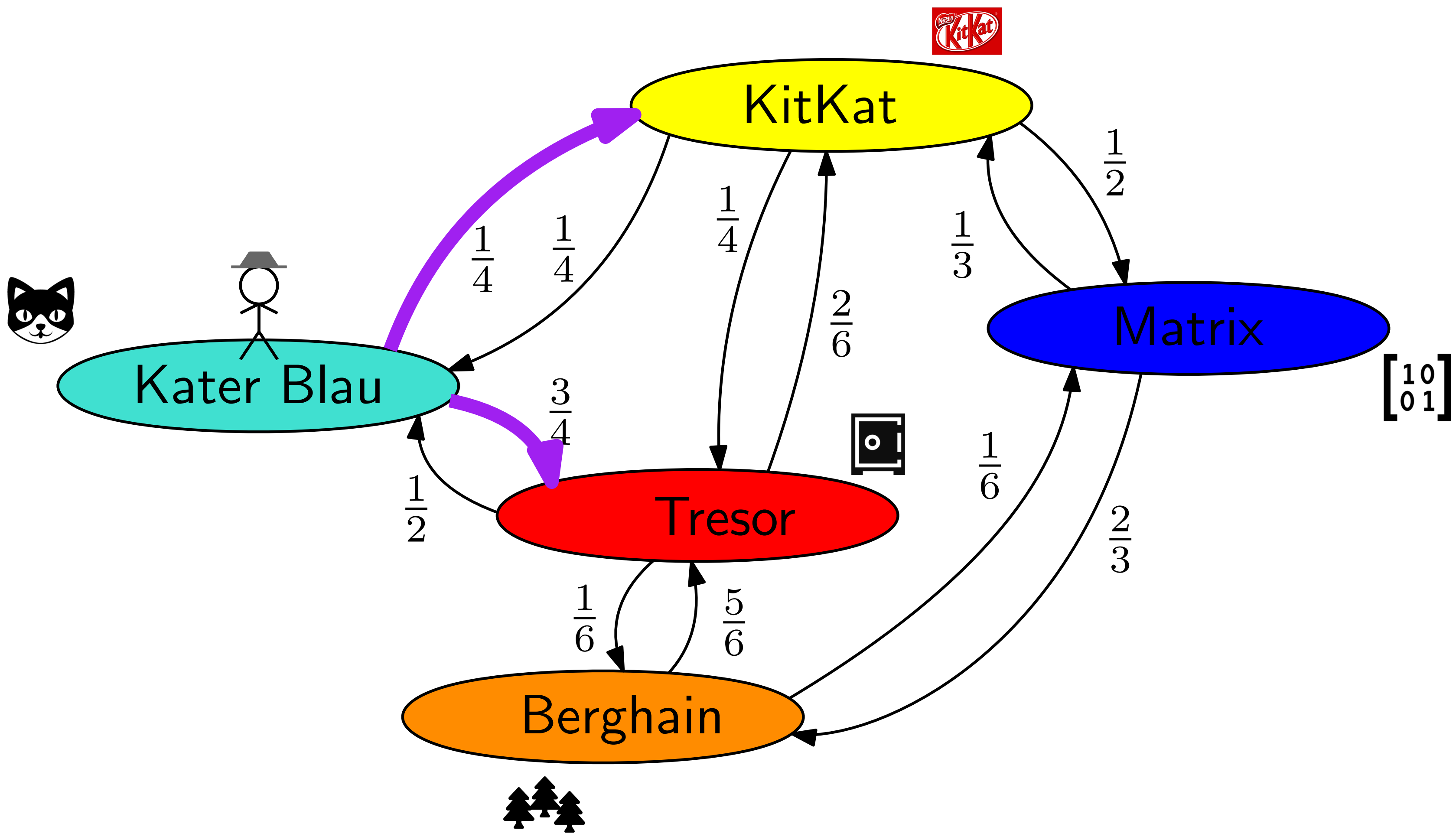

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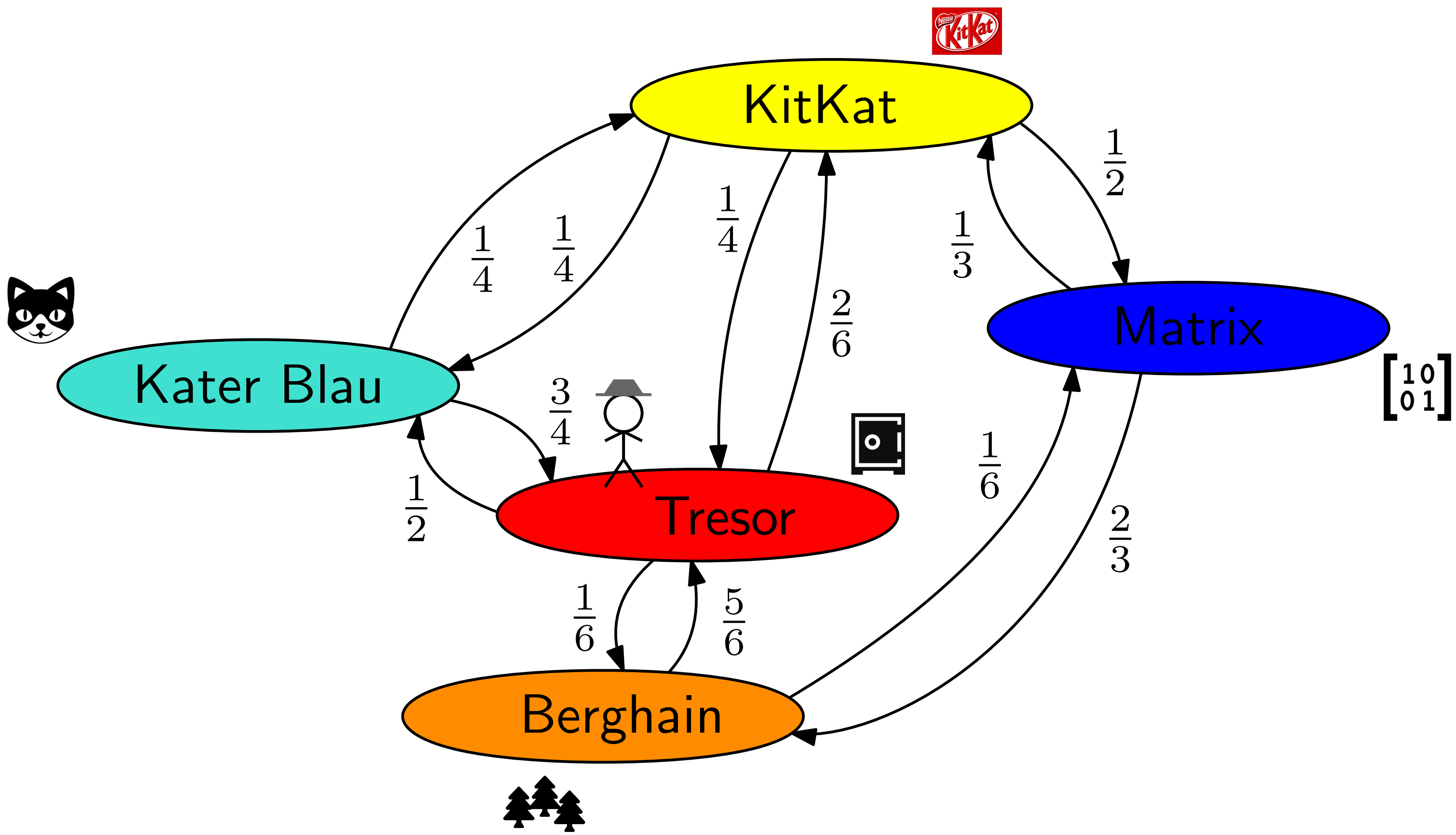

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Markov-Kette $X_0, X_1, X_2, X_3 \cdots$

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- **Fakt:** Eine Markov-Kette, die *symmetrisch*, *irreduzibel* und *aperiodisch* ist, *konvergiert* gegen die *Gleichverteilung*:

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Zufällige Erzeugung mittels Markov-Kette

Idee:

- Zustände $\mathcal{X} = \{\text{Arrangements fester Größe}\}$
- Symmetrische Übergangswahrscheinlichkeiten

$\implies$ Nach vielen Schritten nahe an Gleichverteilung.

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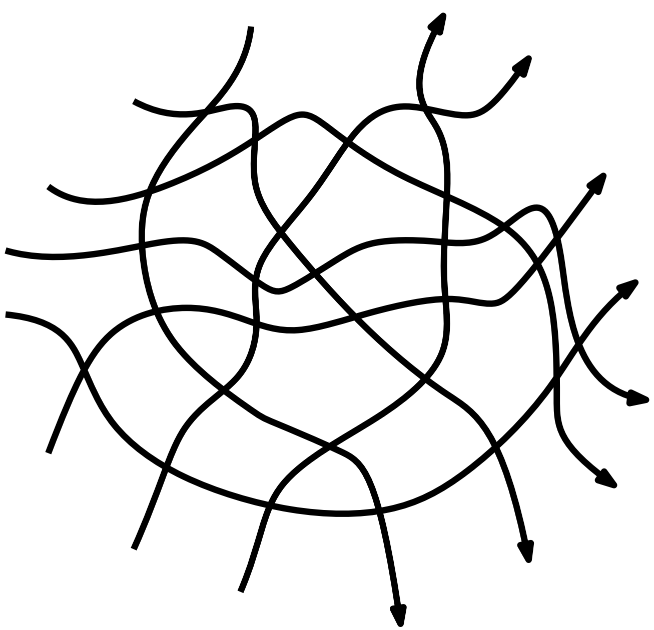

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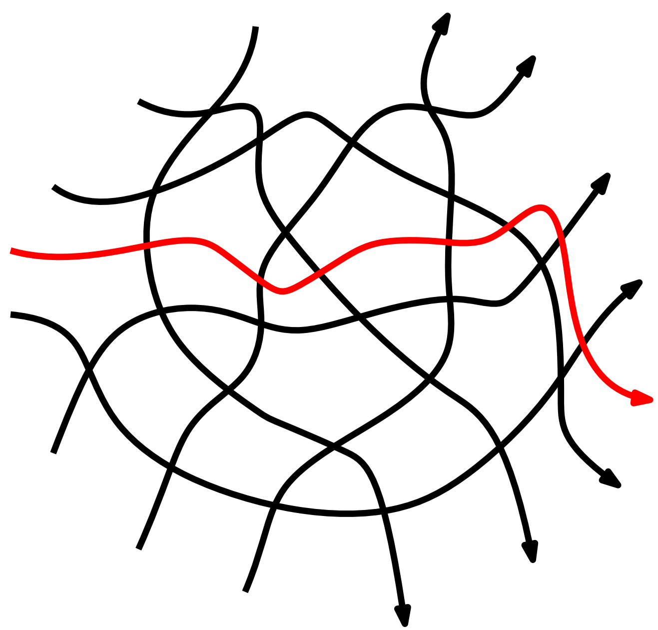

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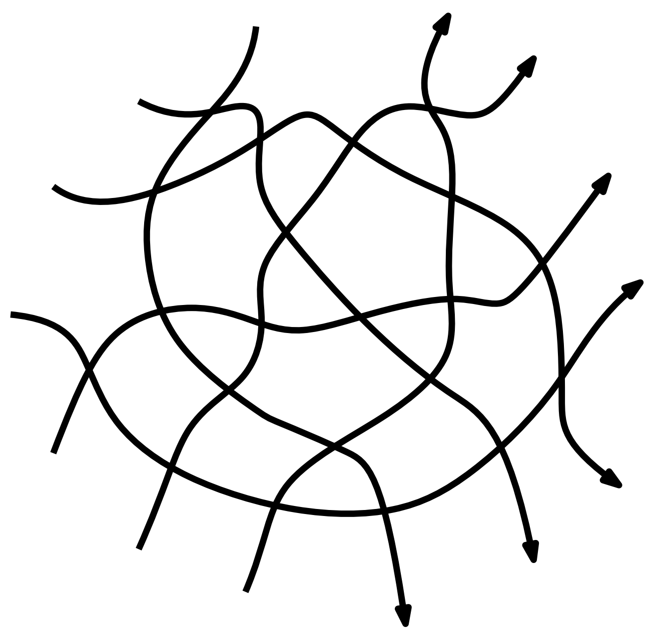

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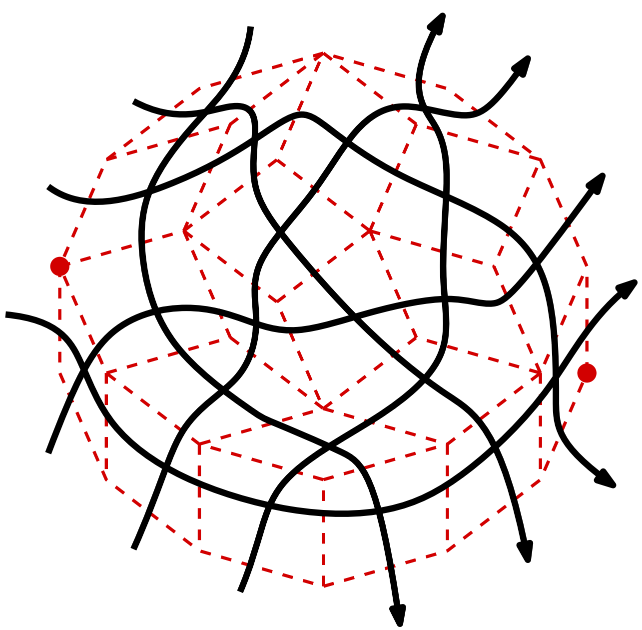

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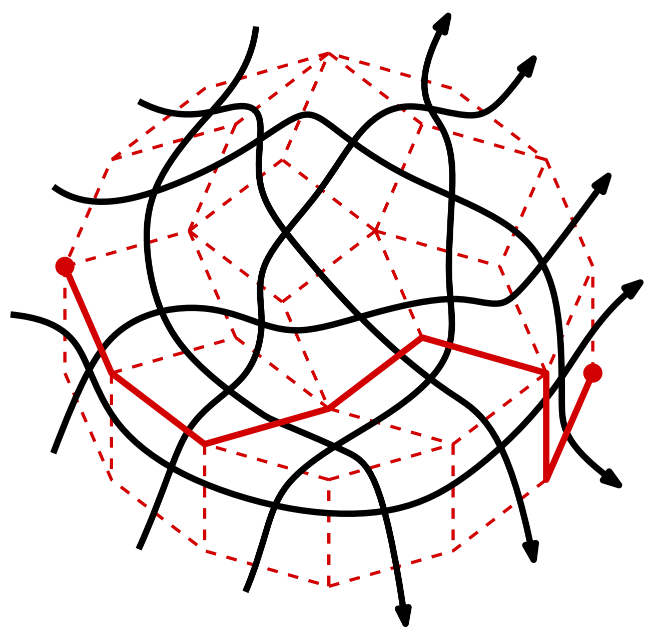

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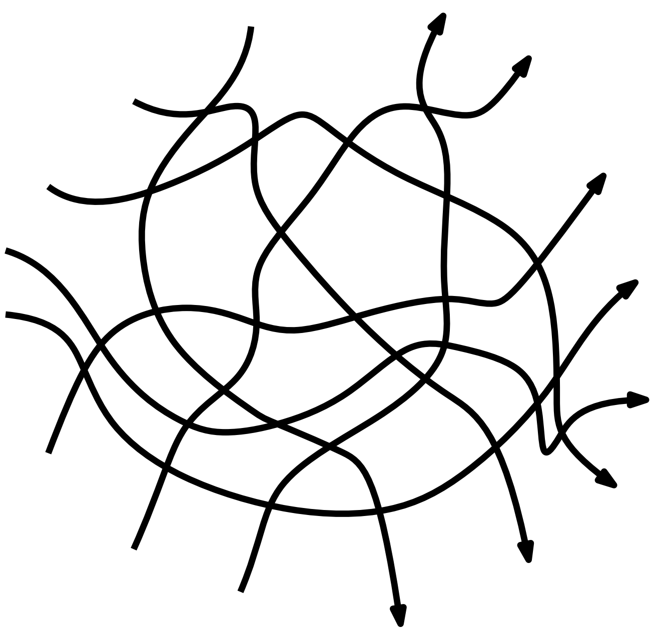

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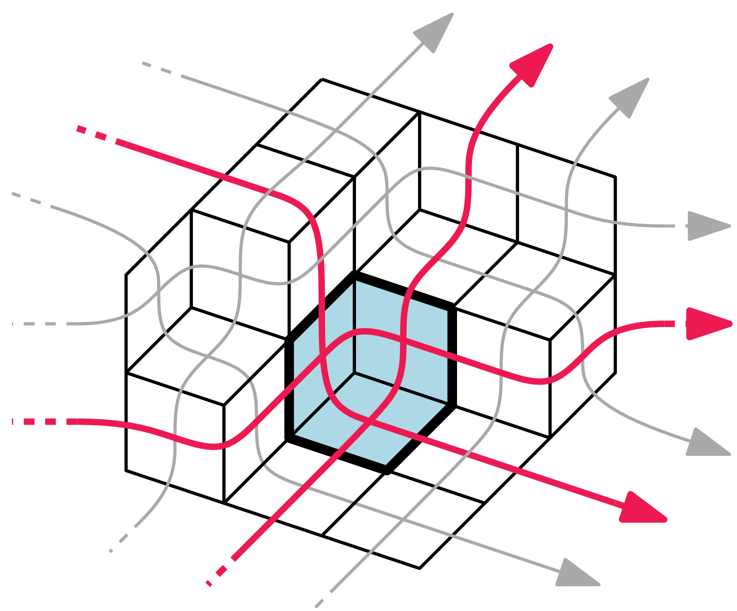

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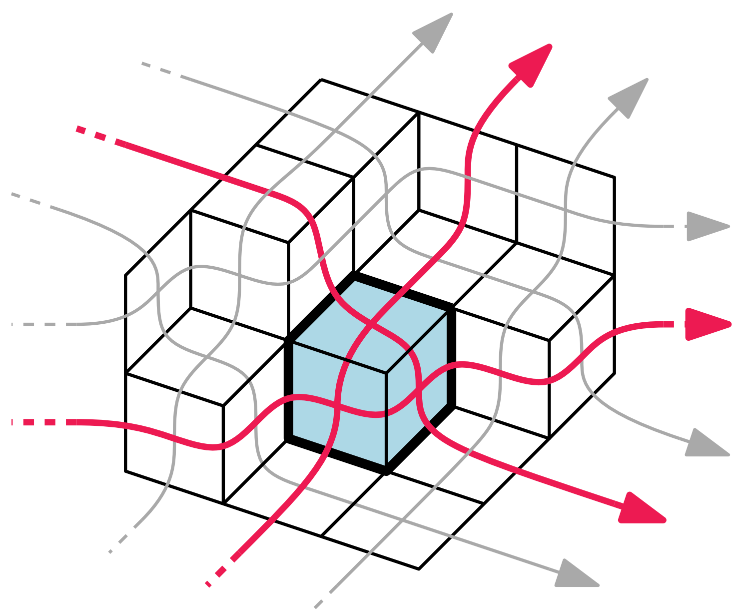
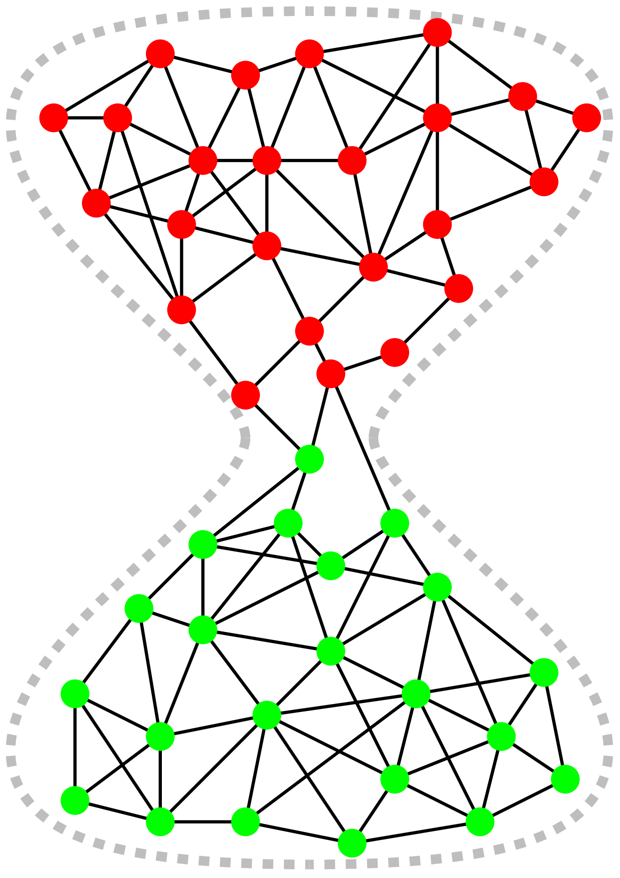
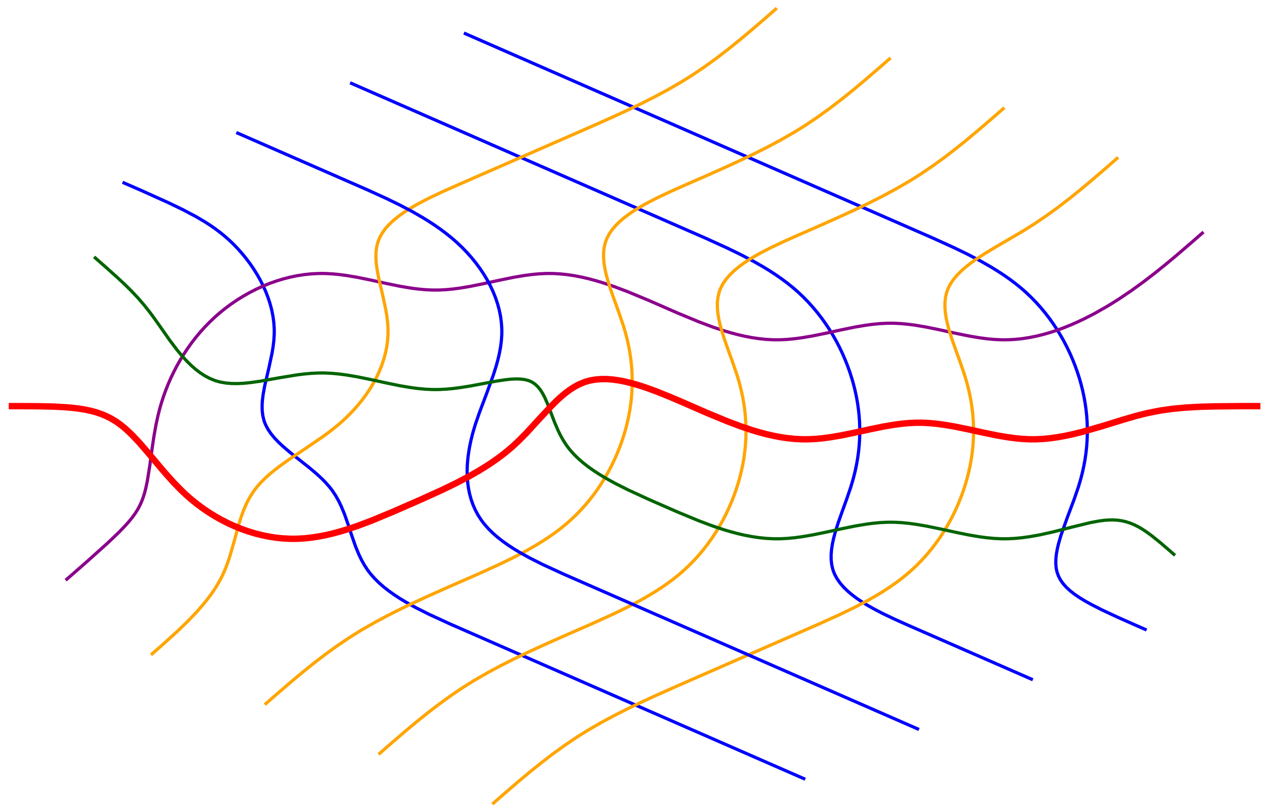


Falschenhals

Markov-Kette mit „Falschenhals“:



Flips an einzelner Pseudogerade



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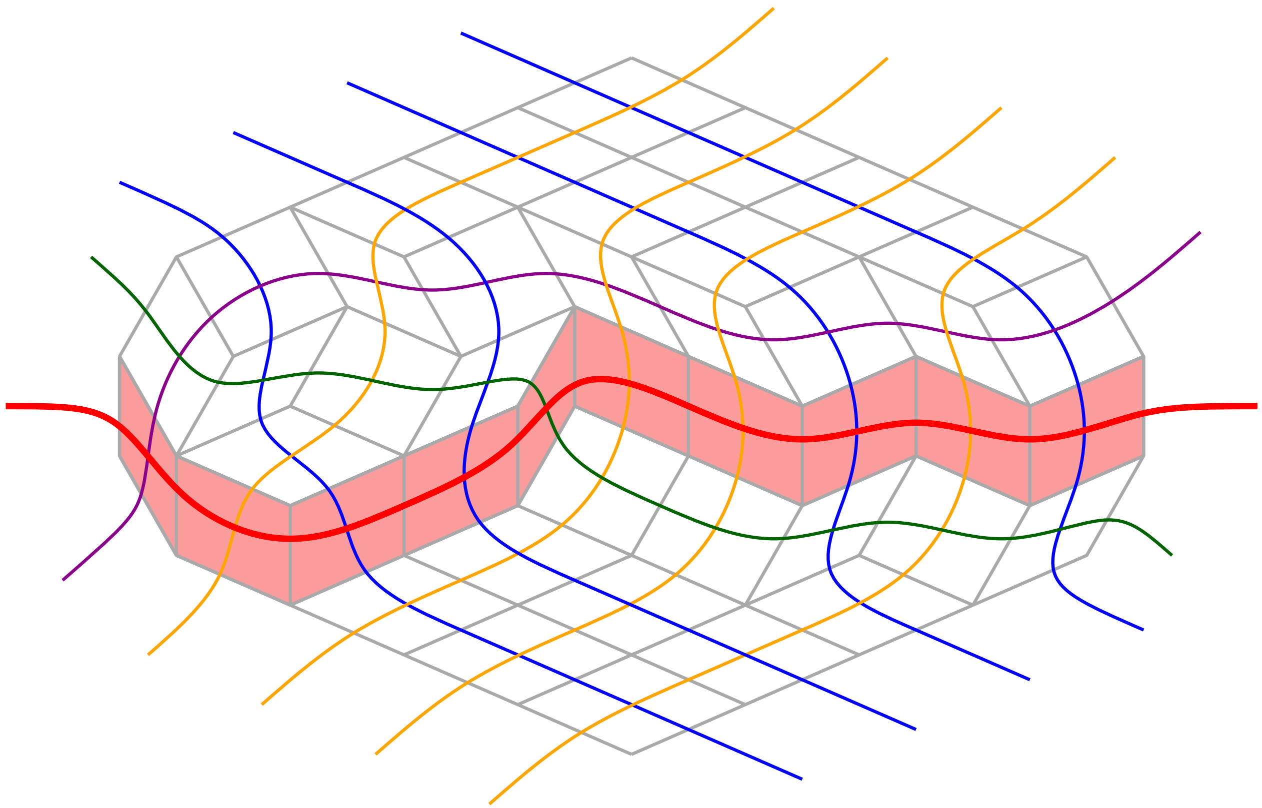

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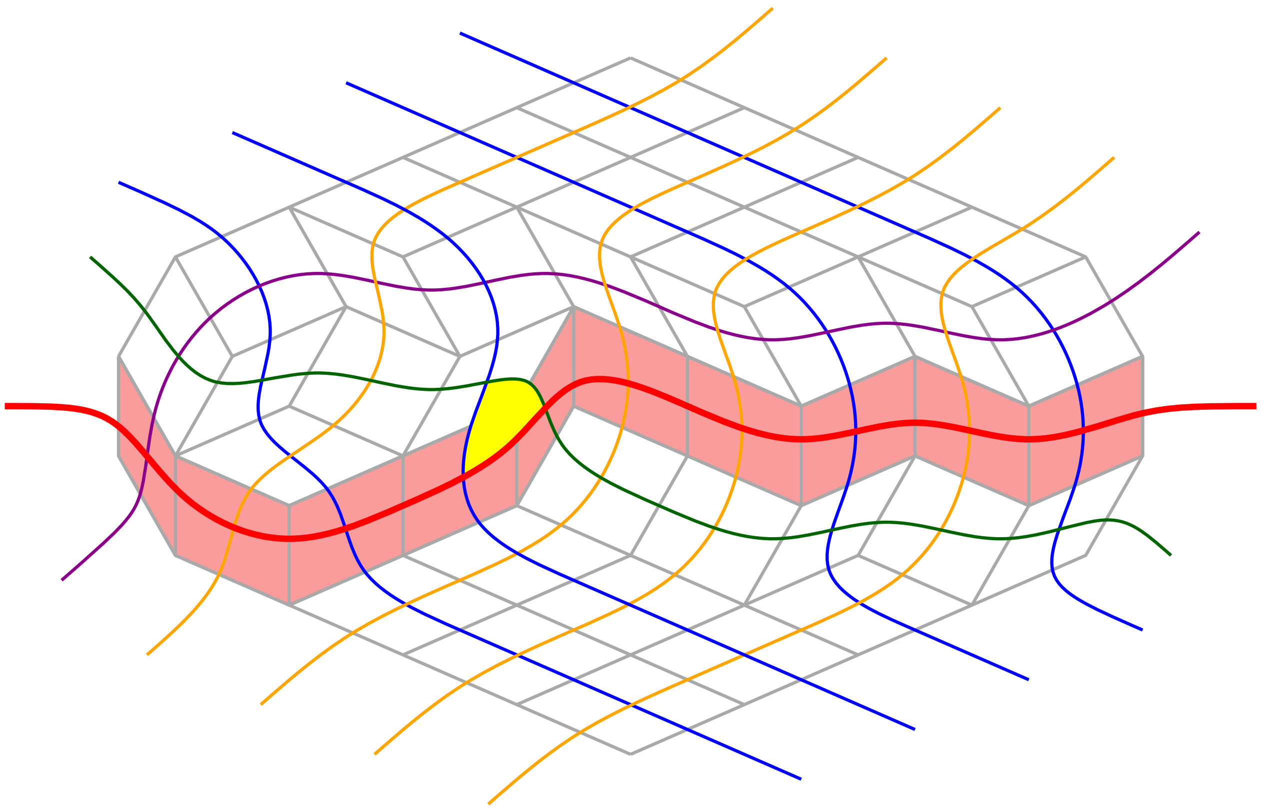

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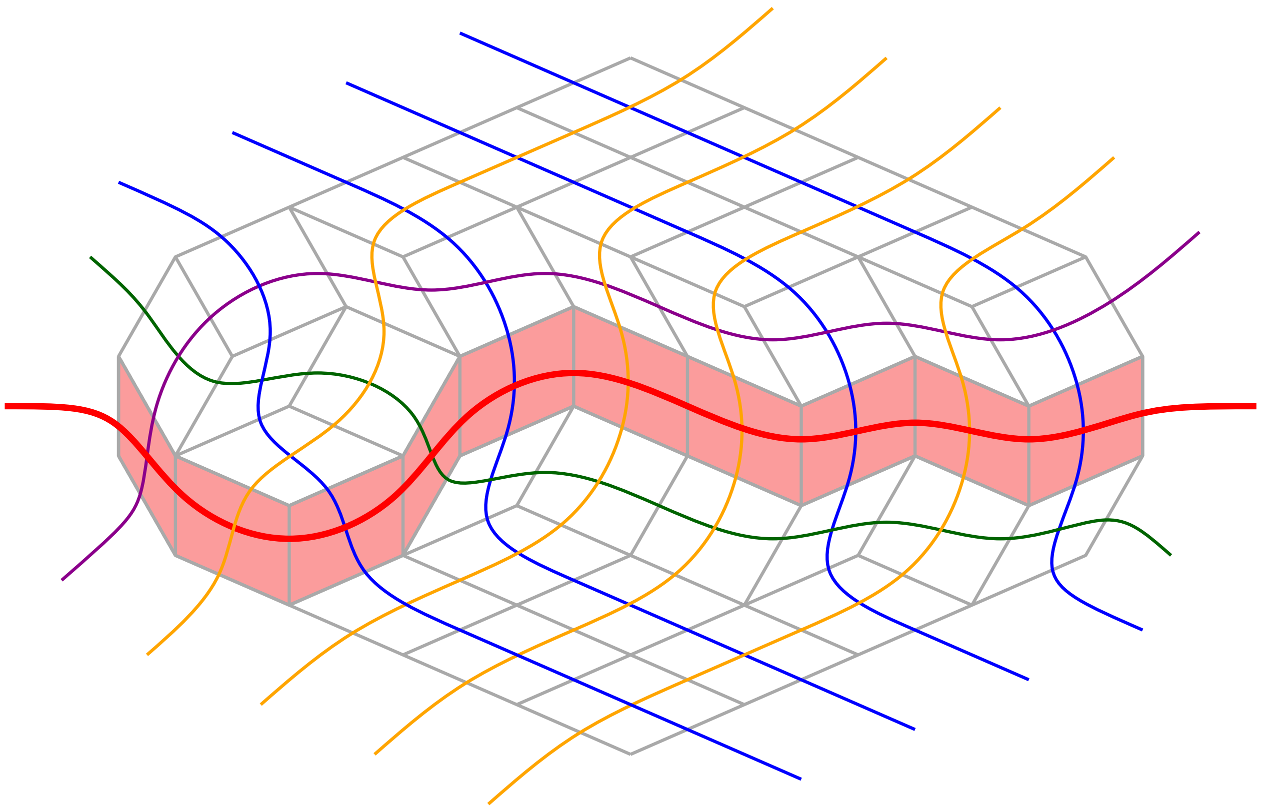

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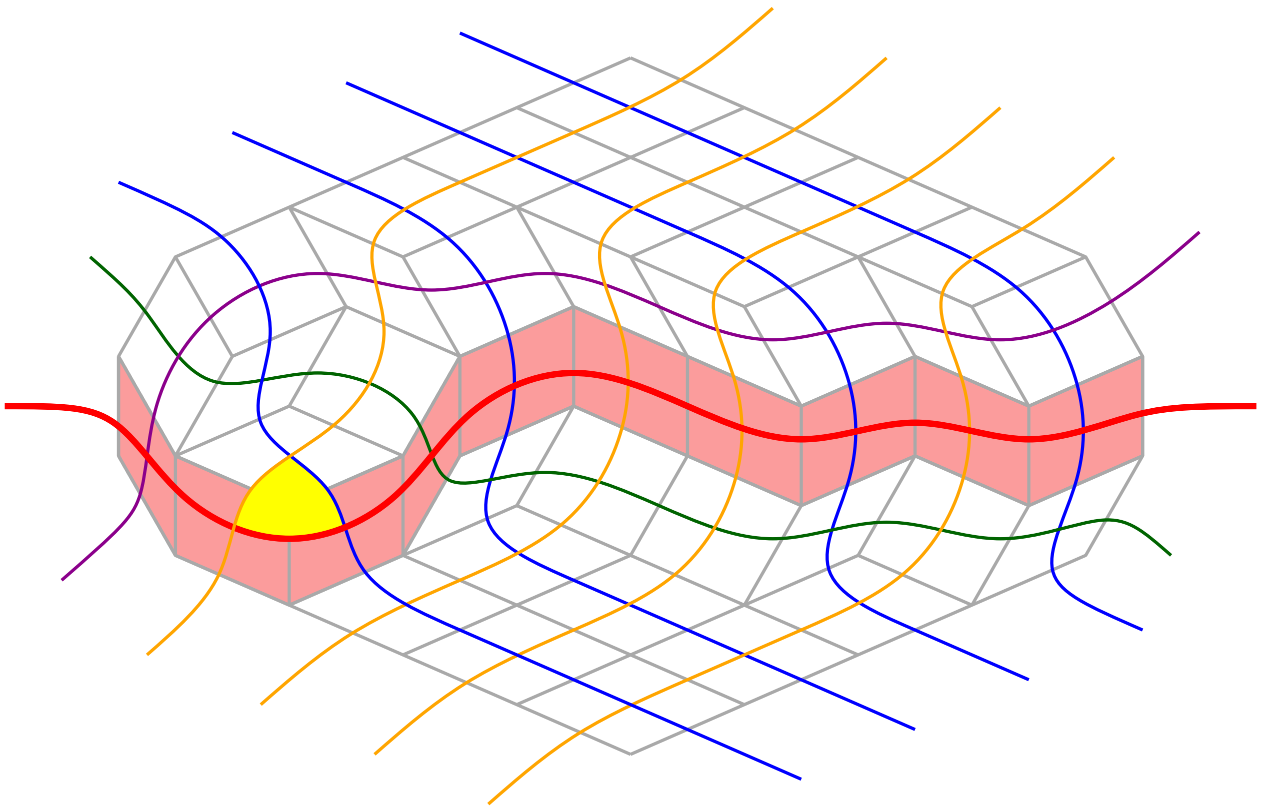

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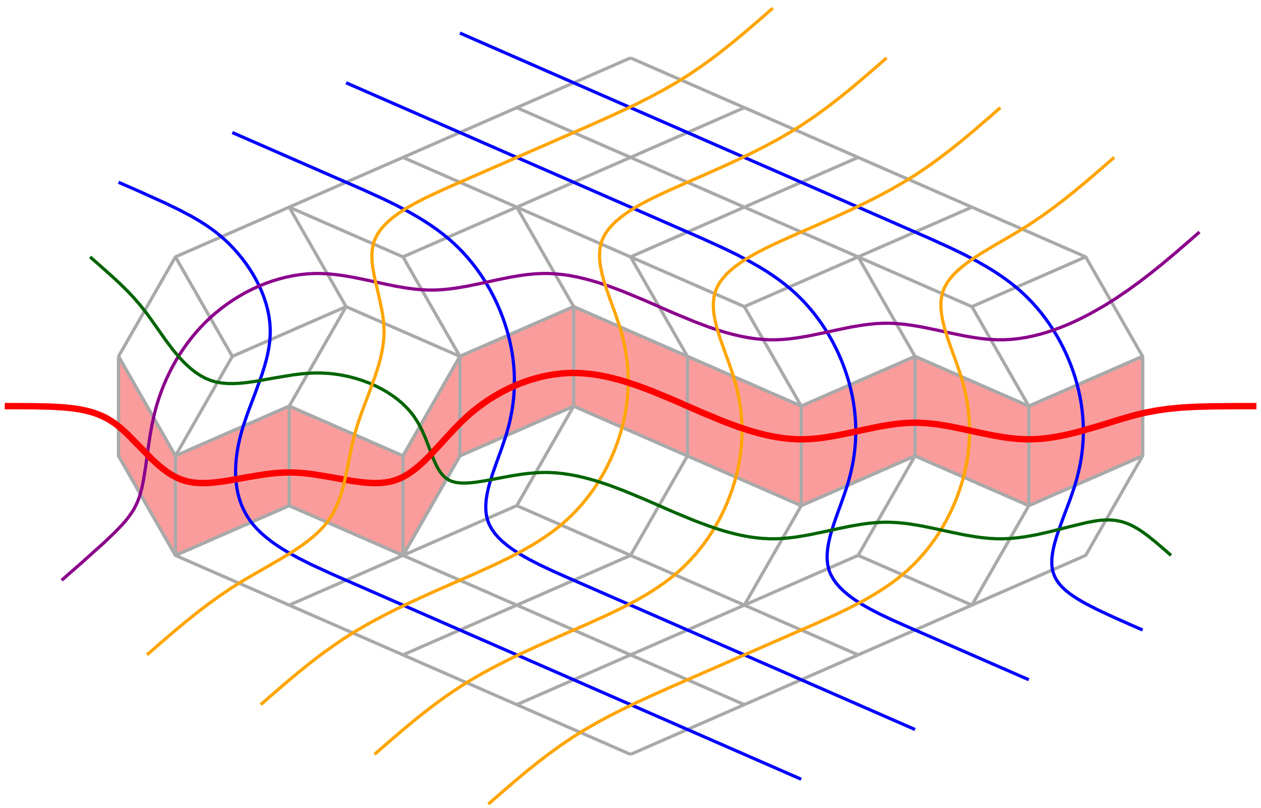

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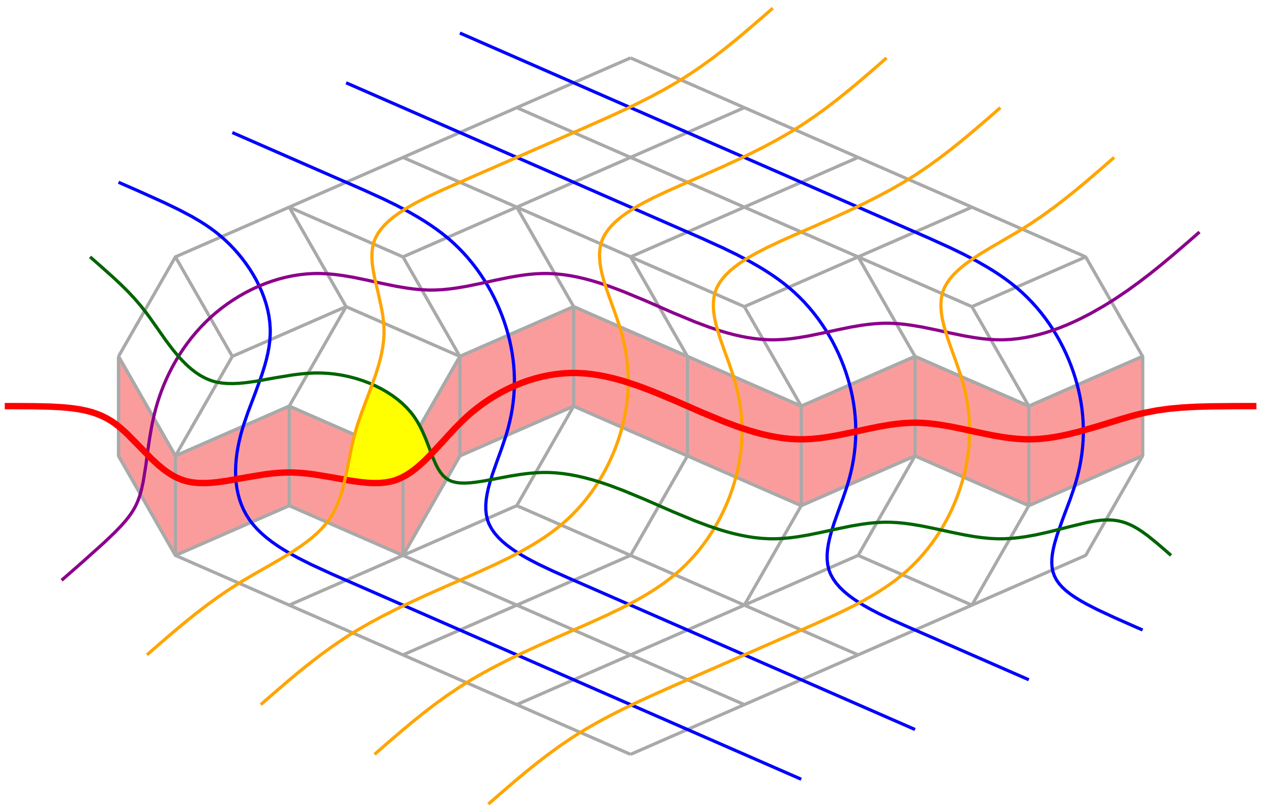

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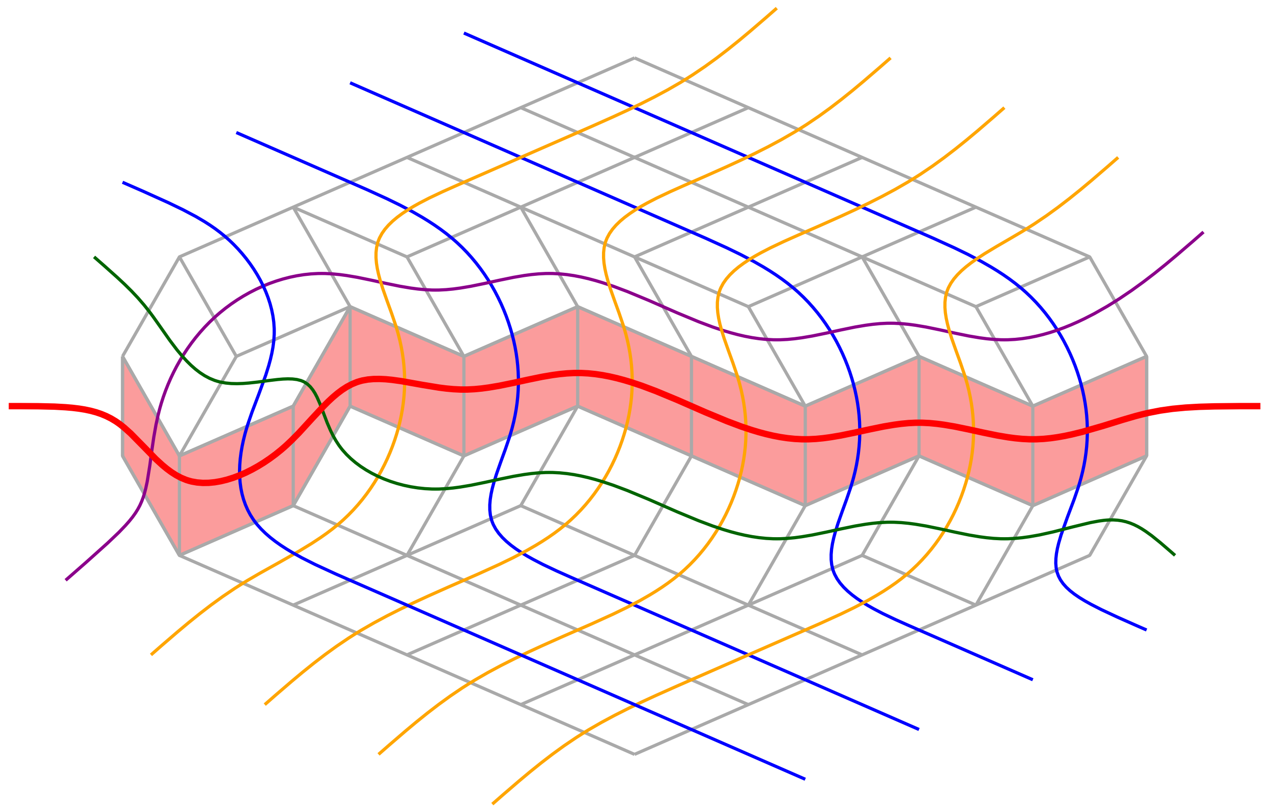

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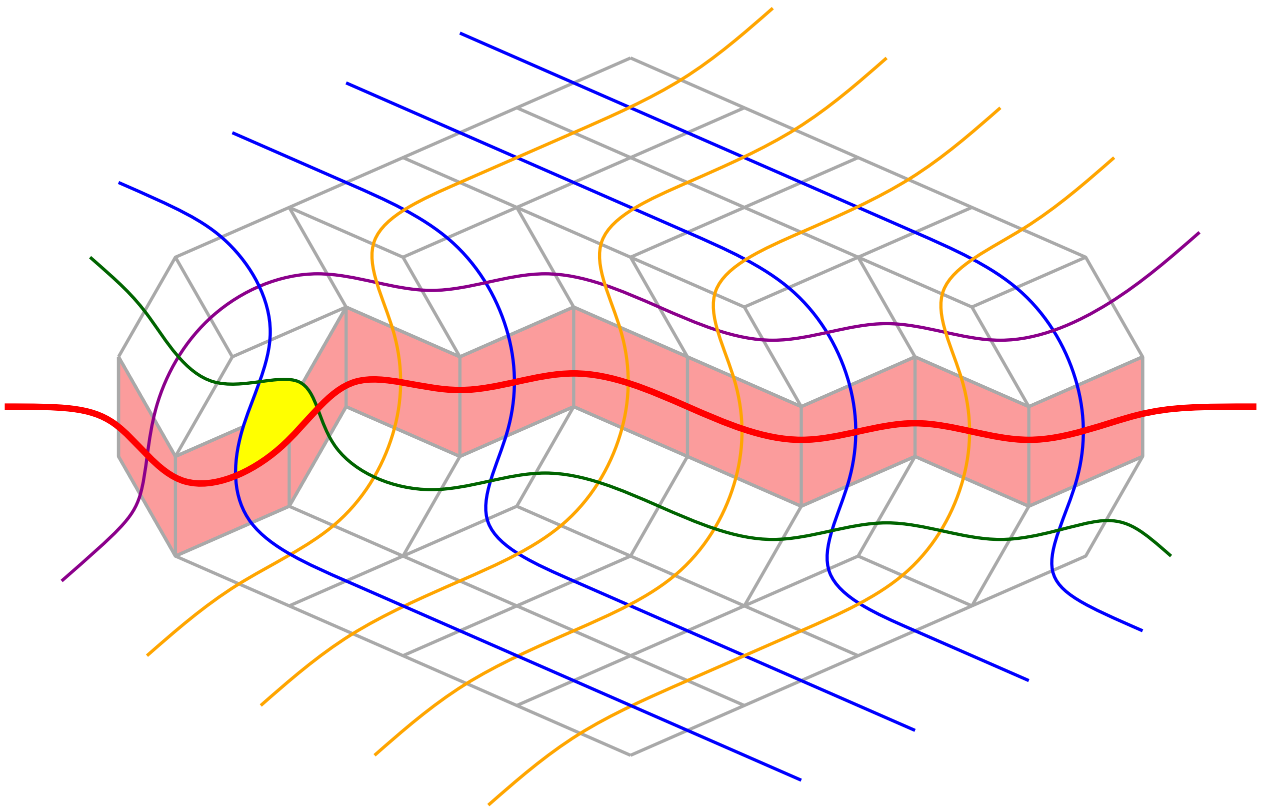

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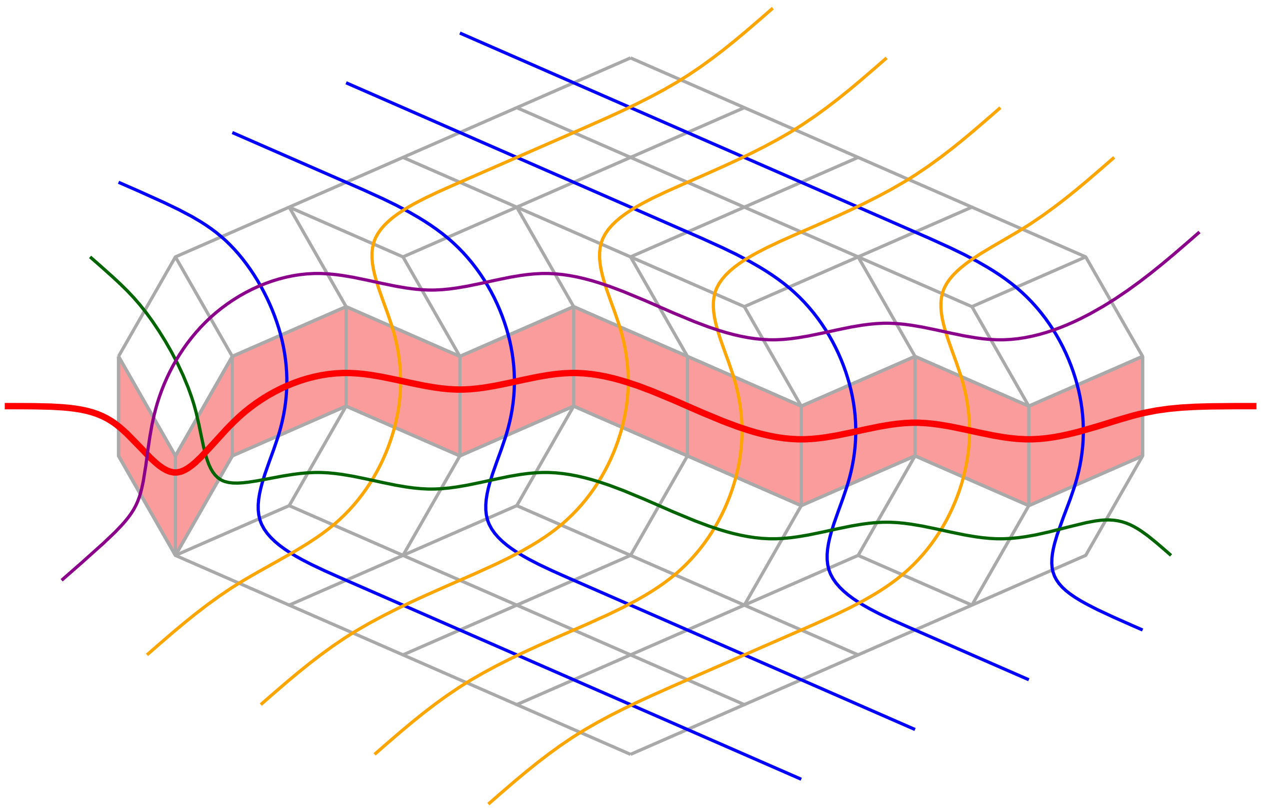

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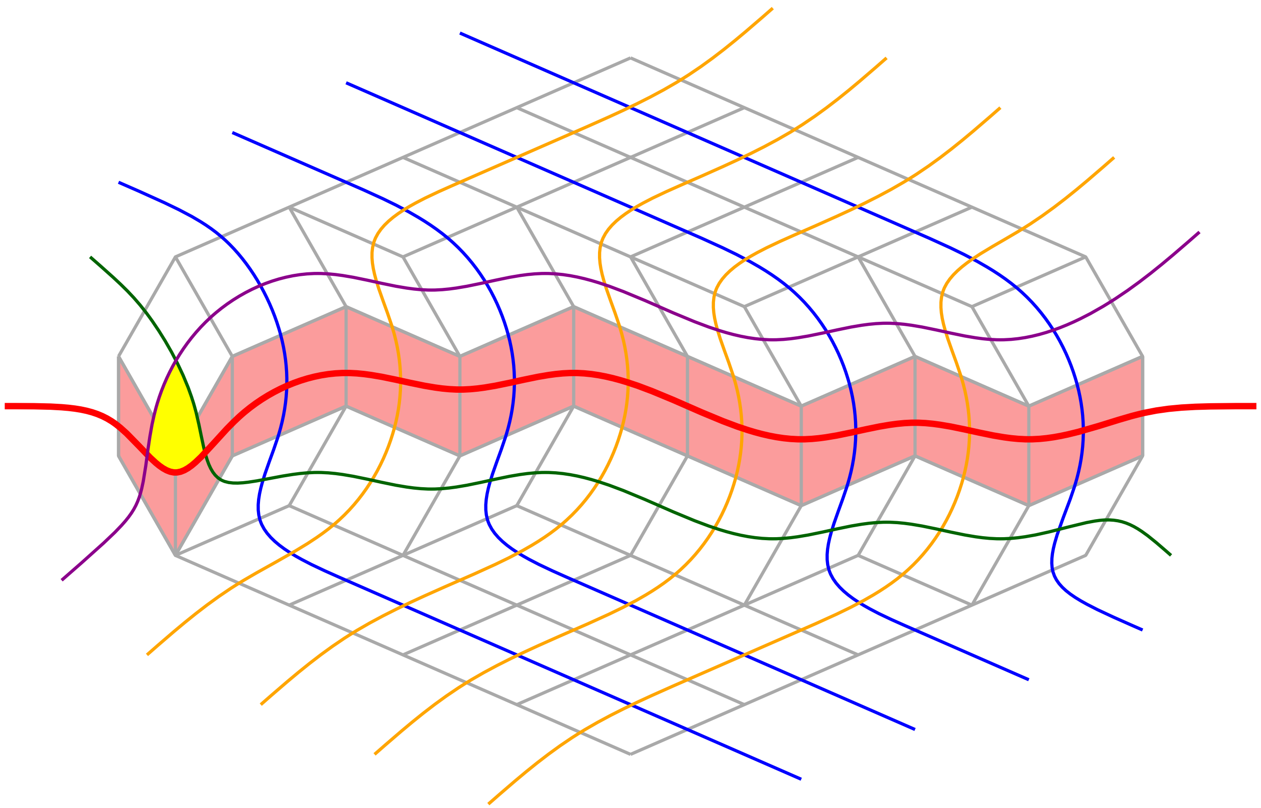

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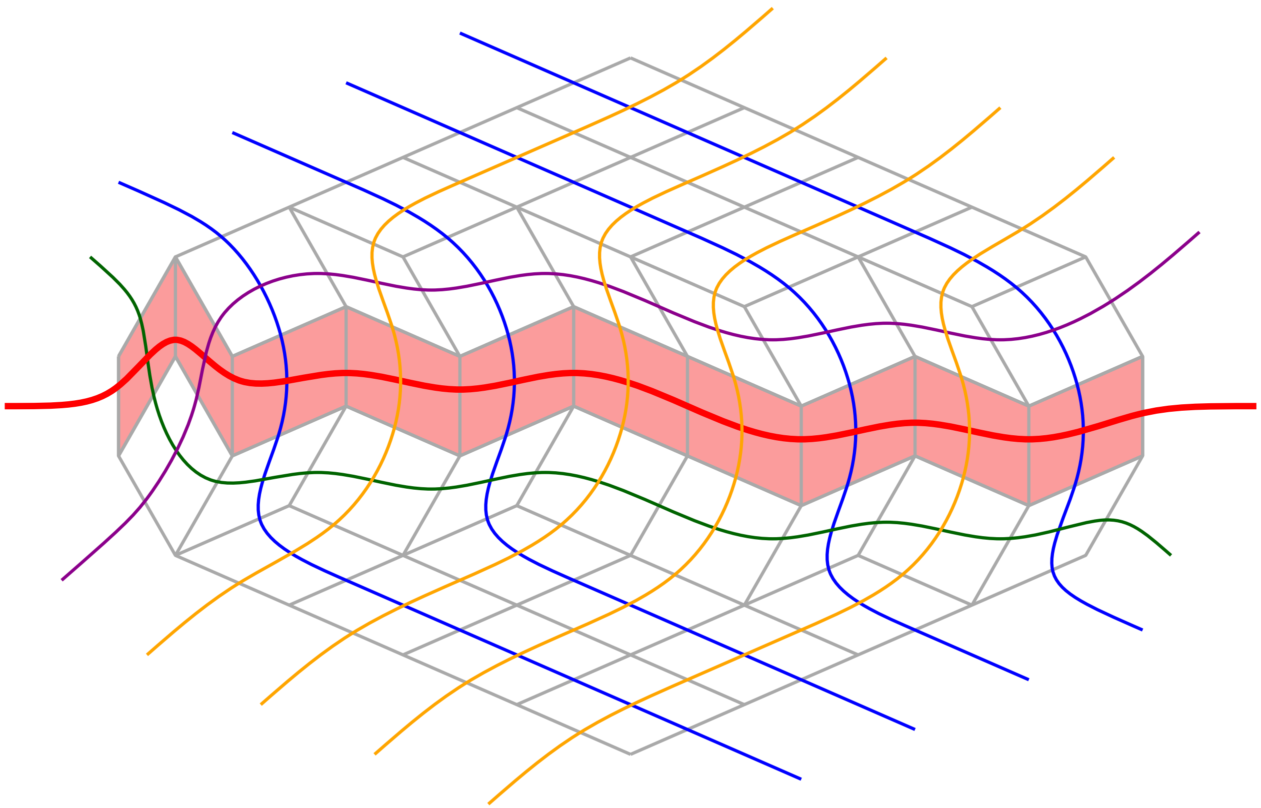

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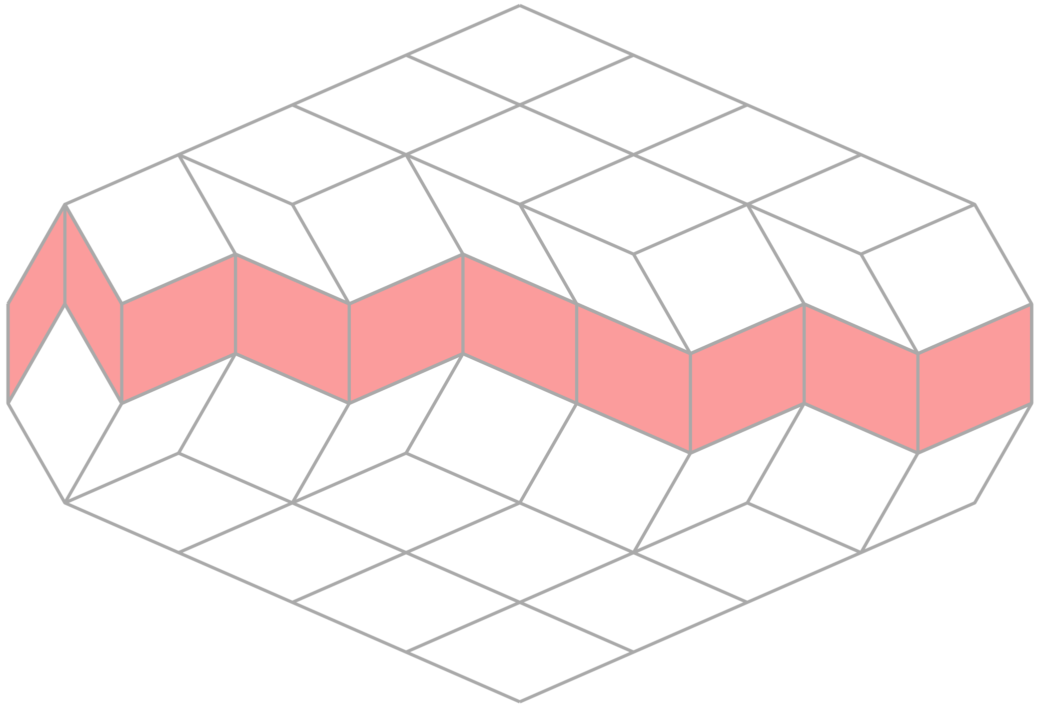

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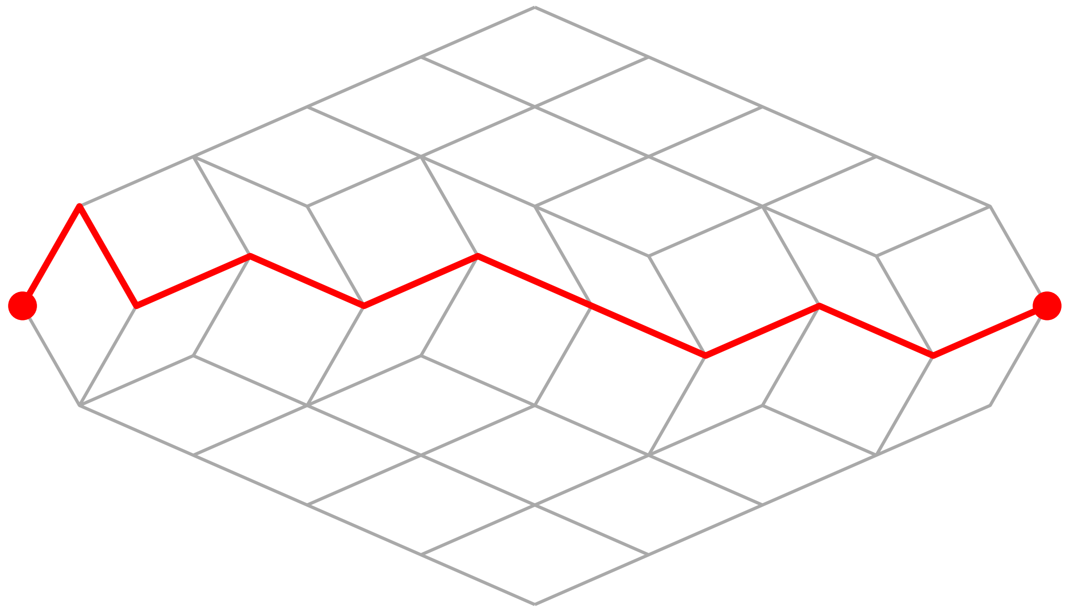

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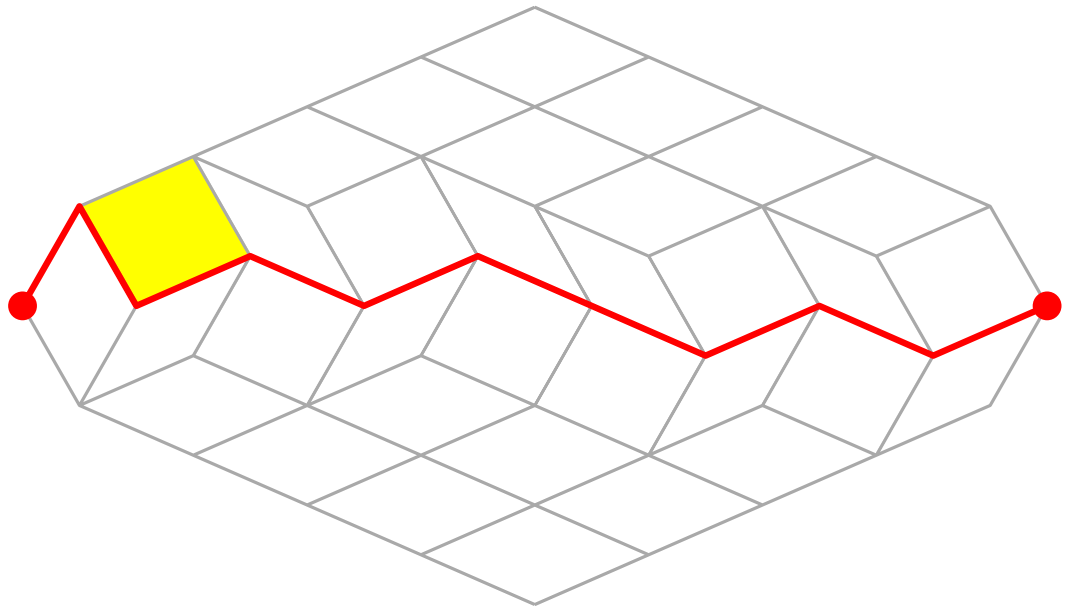

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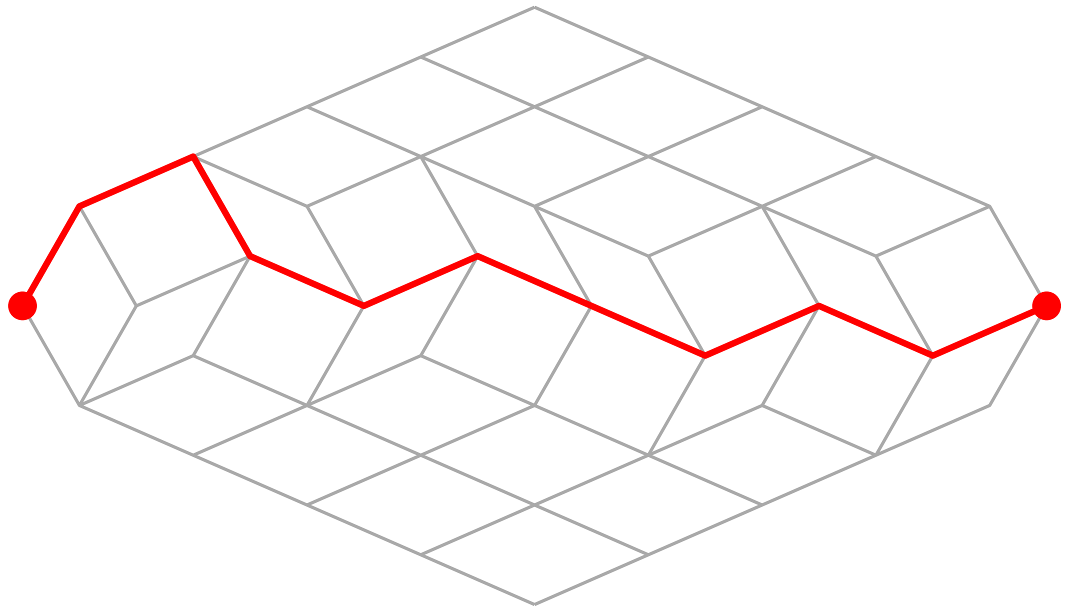

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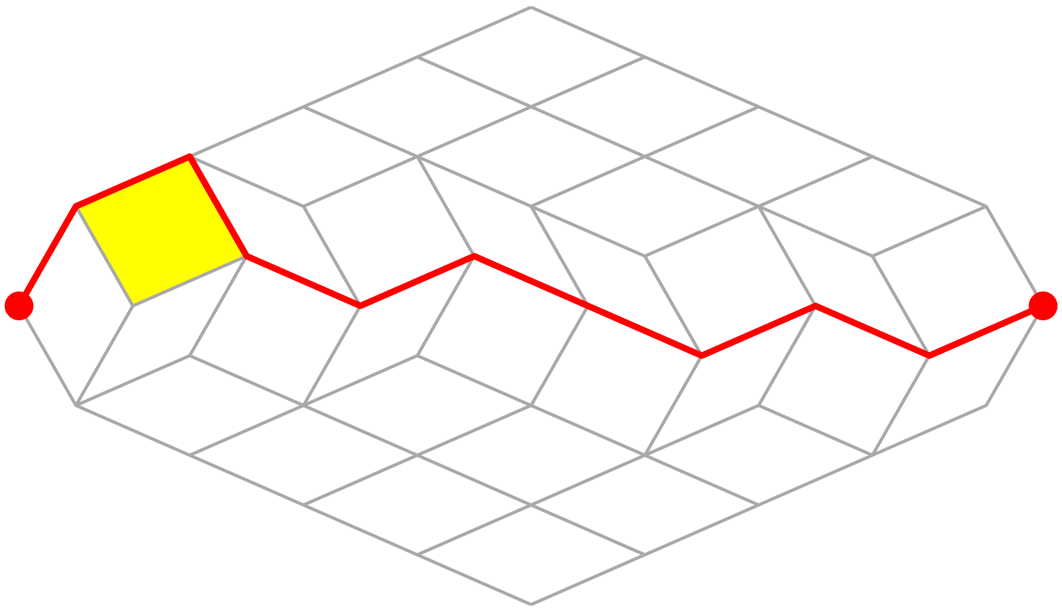

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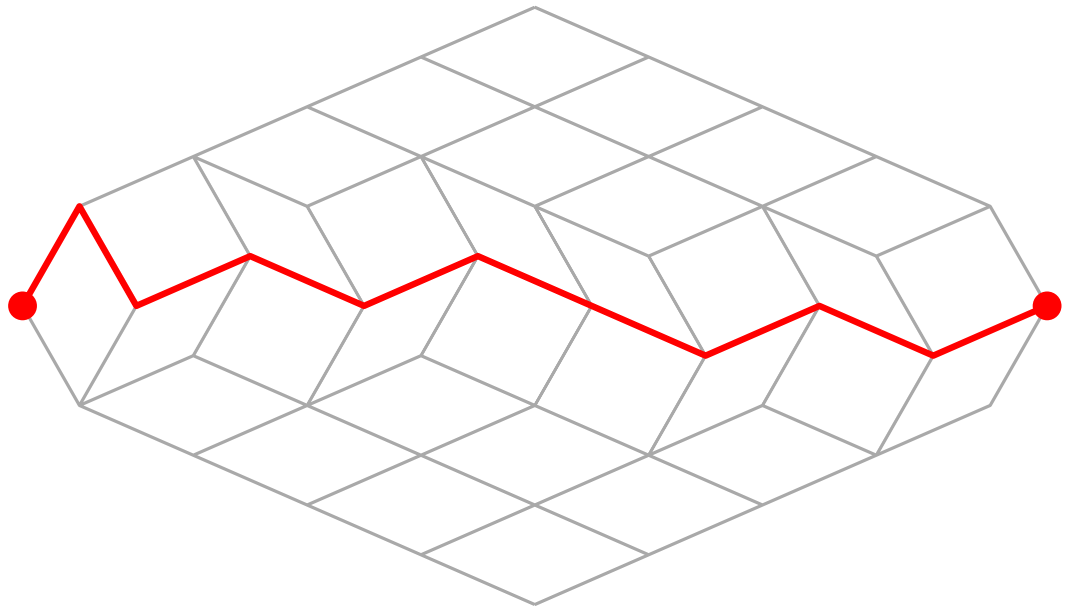

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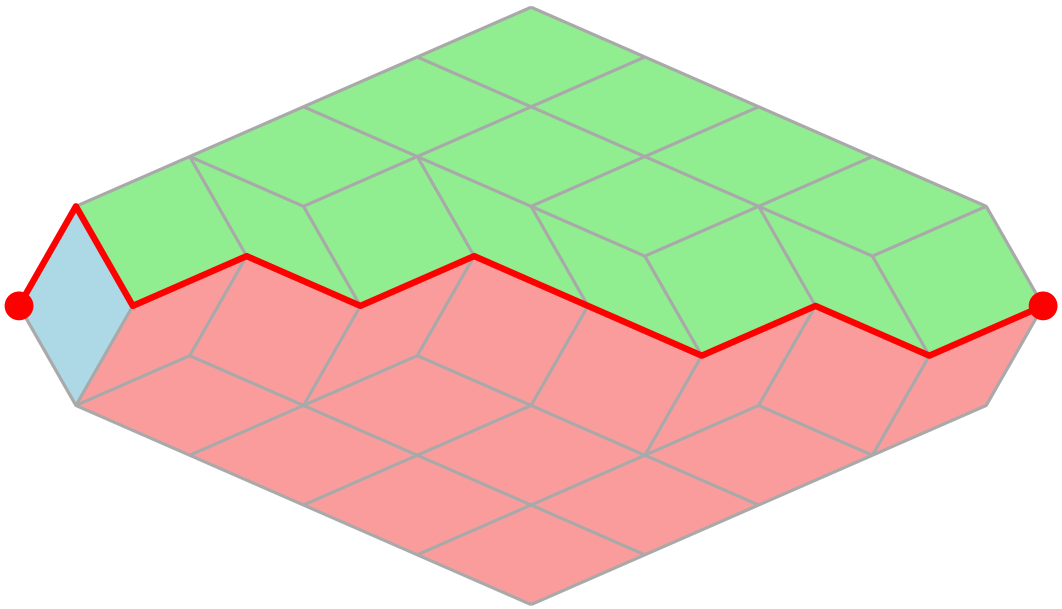
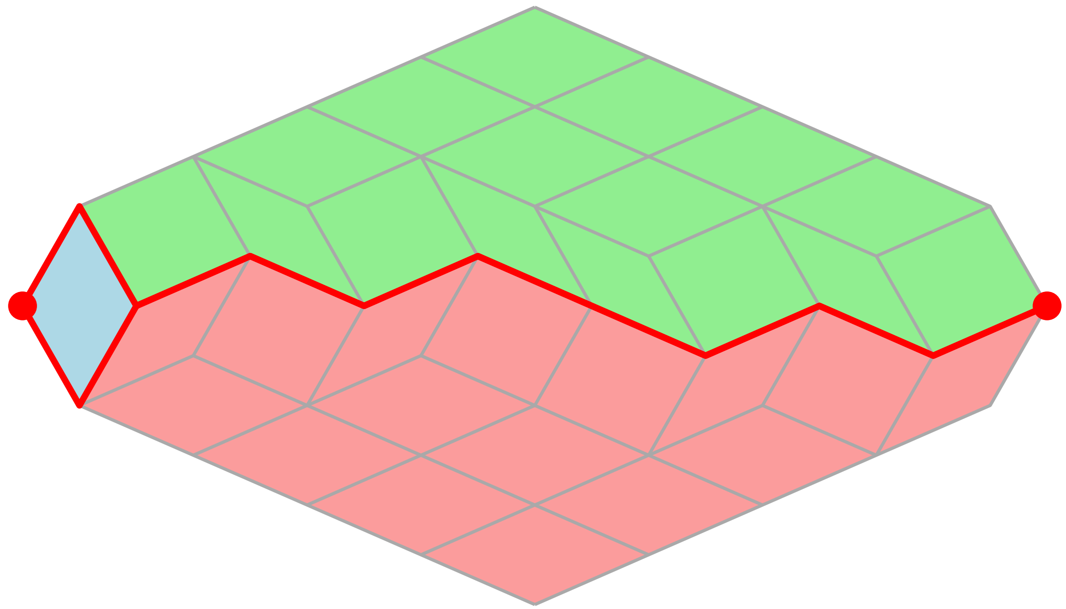


- Partition der Zustände in zwei Klassen:
  - Pfade **über blauem Rhombus**
  - Pfade **unter blauem Rhombus**

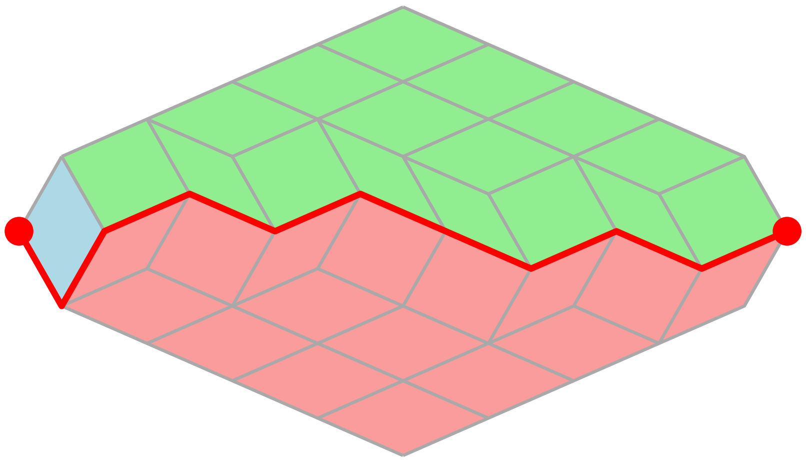
Flips an einzelner Pseudogerade



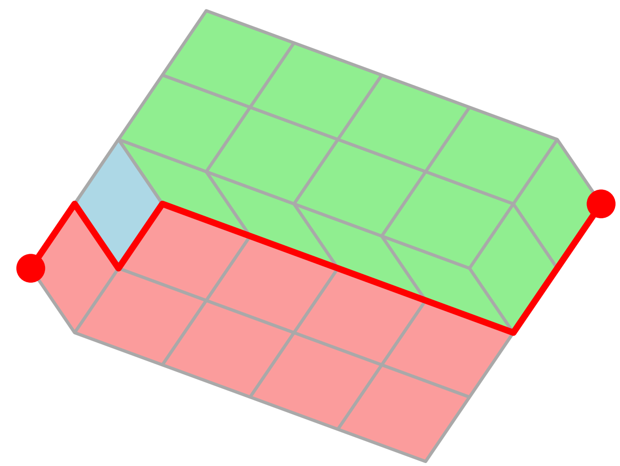
- Partition der Zustände in zwei Klassen:
  - Pfade **über blauem Rhombus**
  - Pfade **unter blauem Rhombus**
- Nur Flip an blauem Rhombus verbindet beide Klassen!

Flips an einzelner Pseudogerade

$r = 5$ Parallelklassen:
(verallgemeinerbar zu $r > 5$)



$r = 4$ Parallelklassen:



- Partition der Zustände in zwei Klassen:
  - Pfade **über blauem Rhombus**
  - Pfade **unter blauem Rhombus**
- Nur Flip an blauem Rhombus verbindet beide Klassen!

Flips an einzelner Pseudogerade

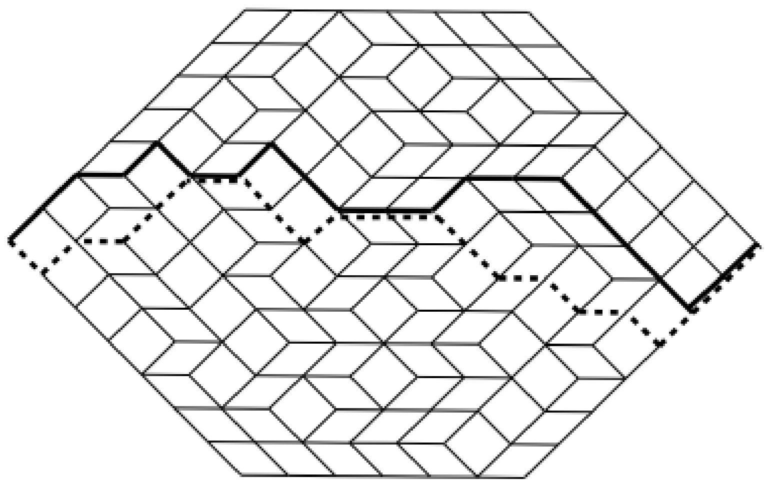
Theorem (R., 2021):
Die Markov-Kette, welche auf verallgemeinerten Pseudogeradenarrangements operiert und Dreiecke mit Beteiligung einer ausgezeichneten Parallelklasse flipt, ist

- ... **schnell mischend** bei 3 Parallelklassen, und...
- ... i.A. **nicht schnell mischend** bei 4 oder mehr Klassen.

Aussage für 3 Klassen bekannt aus
(Luby, Randall & Sinclair, 1995)

Flips an einzelner Pseudogerade

Destainville, 2001: *Mixing times of plane rhombus tilings*



„Nevertheless, the above arguments do not exclude definitively the existence of rare slow fibers, [...]“

Wissen jetzt: „slow fibers“ gibt es tatsächlich!

Drei Parallelklassen

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Def: Matrix $[h_{i,j}] \in {\mathbb{N}_0}^{r \times s}$ heißt *Plane partition*, falls Zeilen und Spalten monoton wachsen.

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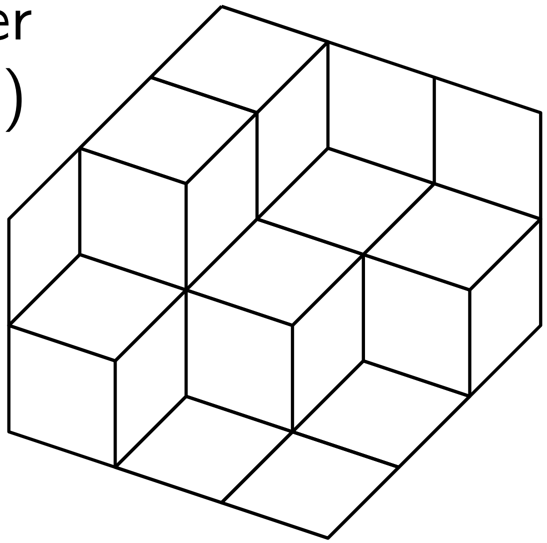
Pflasterung der Größe (3, 2, 3)



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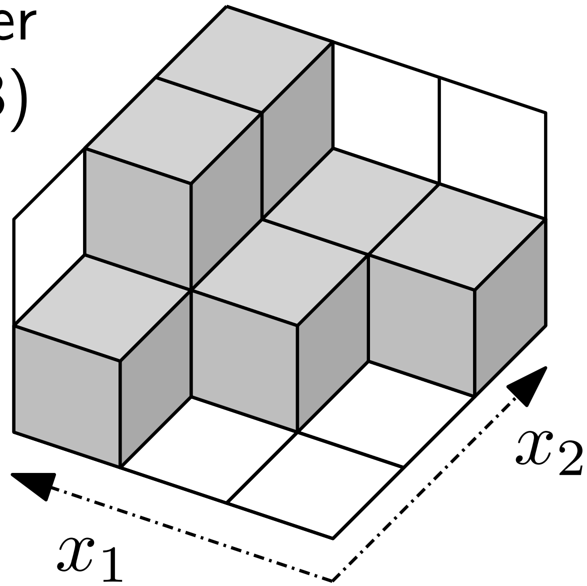

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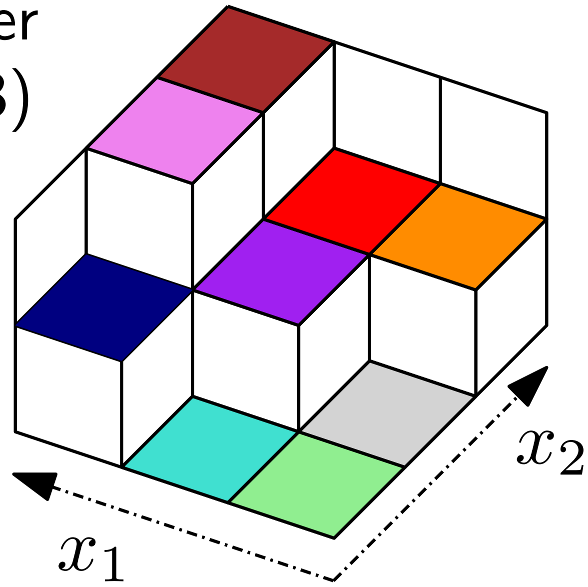

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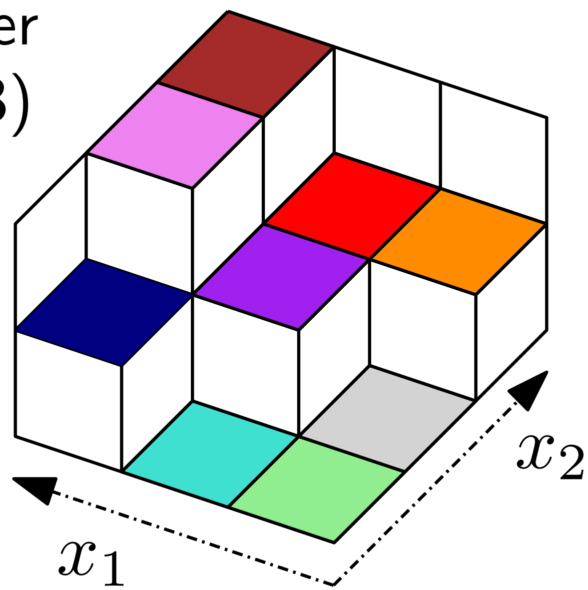

$$\begin{array}{cc} & x_2 \\ x_1 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \end{array}$$

Plane partition mit Einträgen $h_{i,j} \leq 2$

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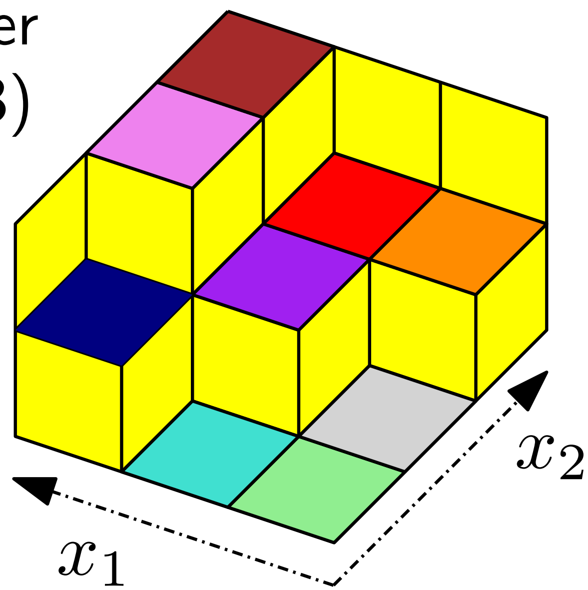

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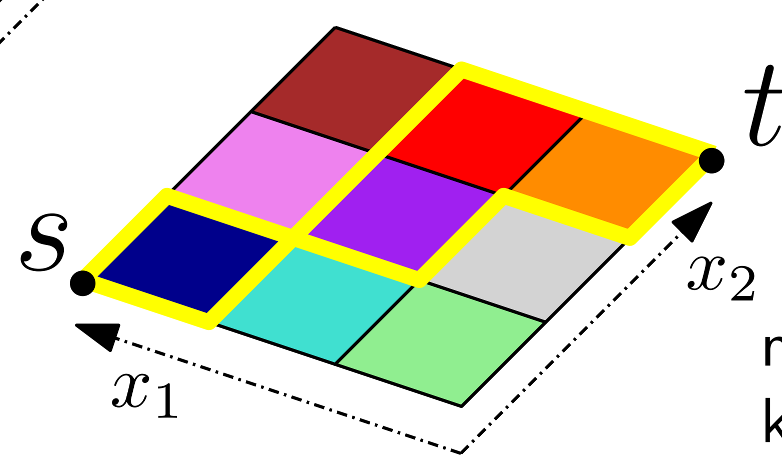

monotone, nicht-kreuzende Pfade

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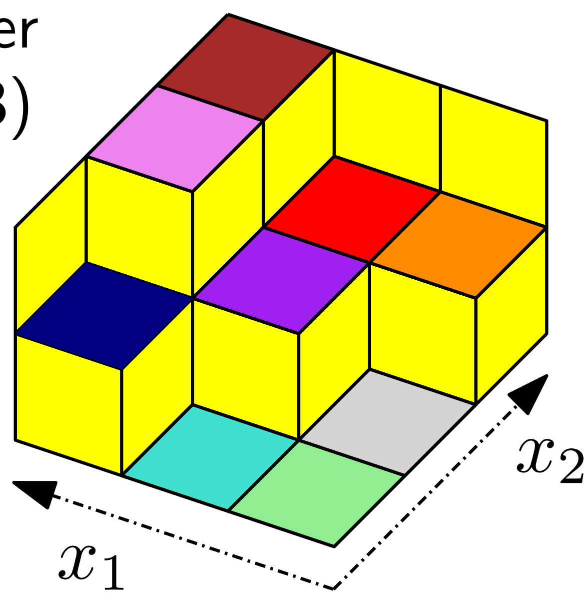

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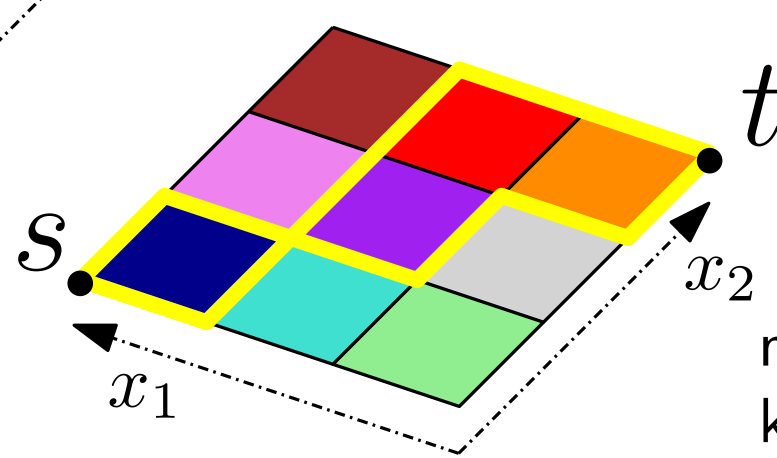
Plane partition mit Einträgen $h_{i,j} \leq 2$

min $\sum f_{i,j}(A_{i,j})$

s.d. A p.p., $A_{i,j} \leq h$



monotone, nicht-kreuzende Pfade

Bichromatische Dreiecksvermutung

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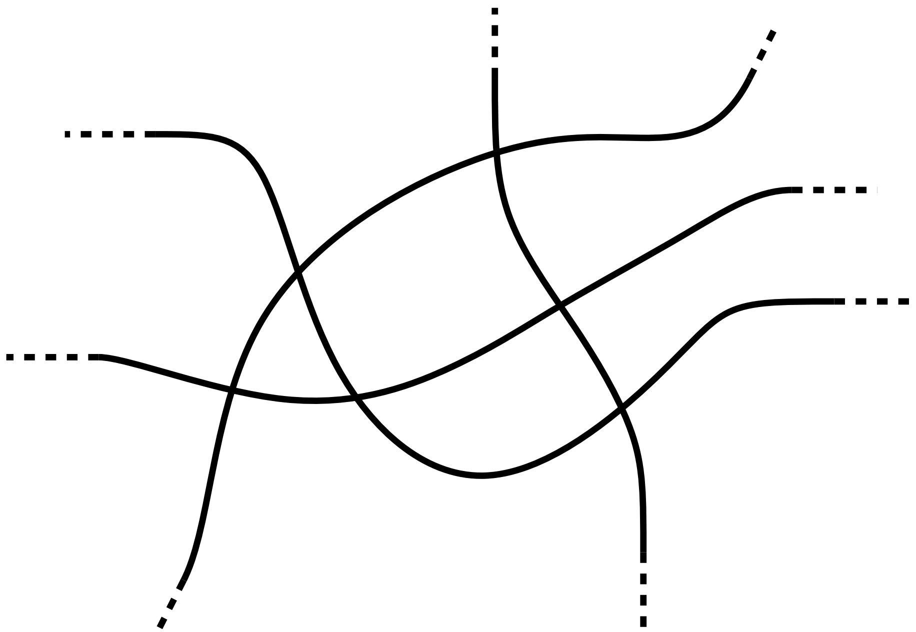

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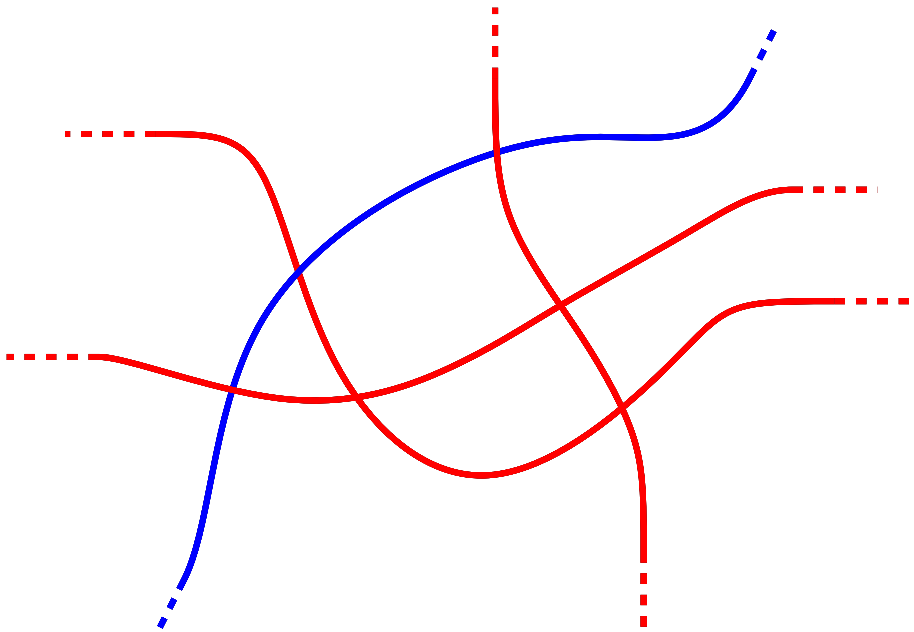

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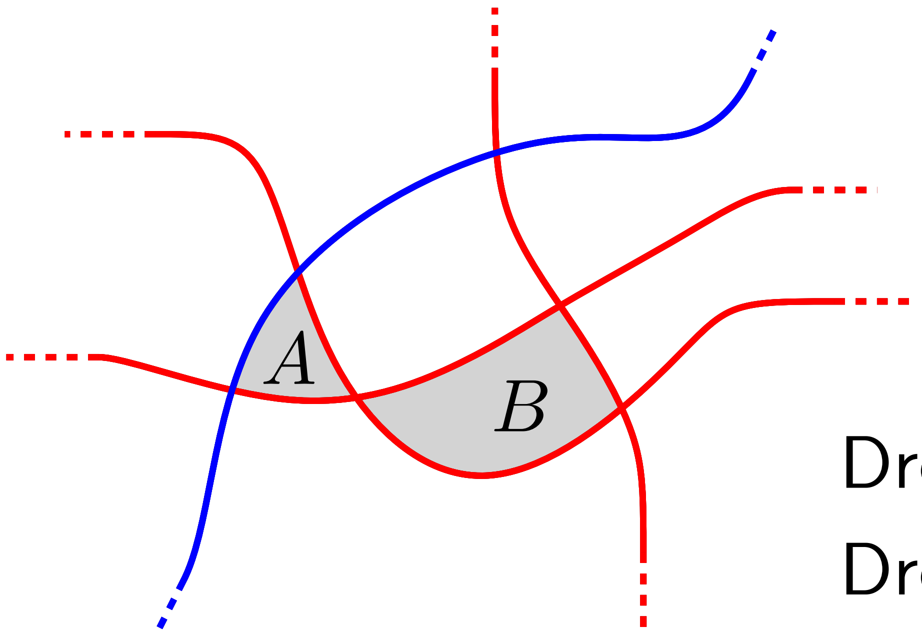
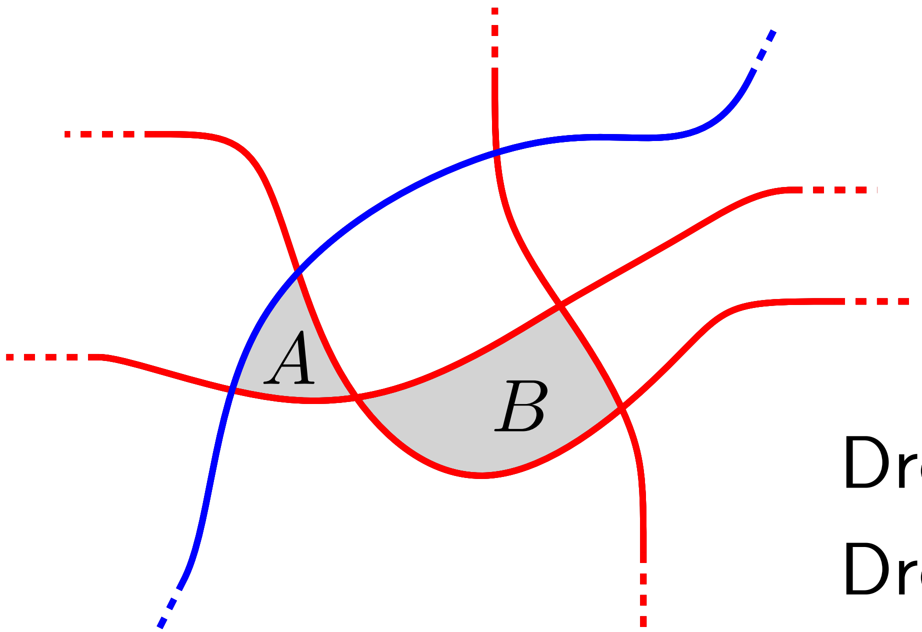


Dreieck A zweifarbig

Dreieck B einfarbig

Bichromatische Dreiecksvermutung



Dreieck A zweifarbig

Dreieck B einfarbig

Vermutung:
(Björner, Las Vergnas, Sturmfels, White, Ziegler, 1999)

Jedes echt zweigefärbte Arrangement mit mindestens drei Pseudogeraden enthält ein zweifarbiges Dreieck.

Fragen?