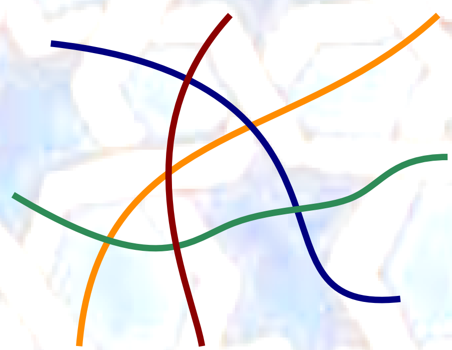
Universidade de Coimbra

World of Pseudoline Arrangements



Sandro M. Roch

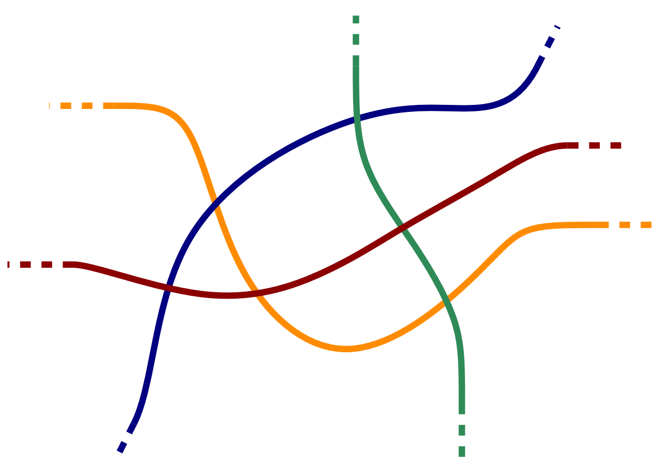
pseudoline arrangements

Def: *pseudoline arrangement*:

- Family of continuous curves $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}^2$ with

$$\lim_{t \to \infty} \|f_i(t)\| = \lim_{t \to -\infty} \|f_i(t)\| = \infty$$

- Each two cross in exactly one point.



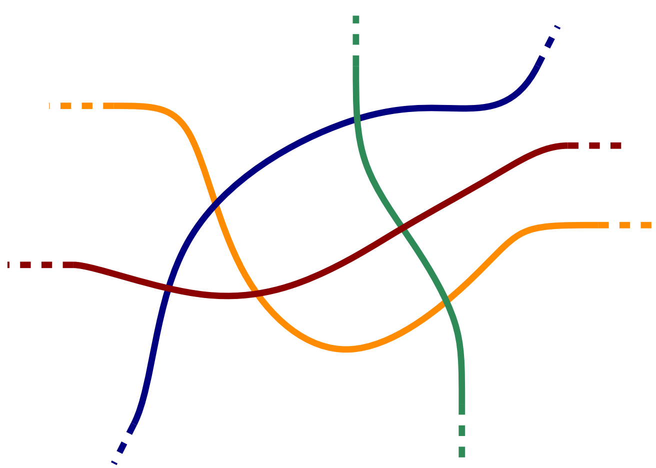
pseudoline arrangements

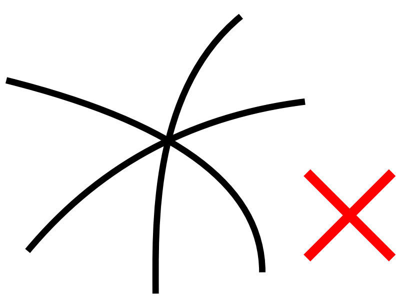
Def: *pseudoline arrangement*:

- Family of continuous curves $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}^2$ with

$$\lim_{t \to \infty} \|f_i(t)\| = \lim_{t \to -\infty} \|f_i(t)\| = \infty$$

- Each two cross in exactly one point.
- No 3 pseudolines cross at a single point.





wiring diagrams

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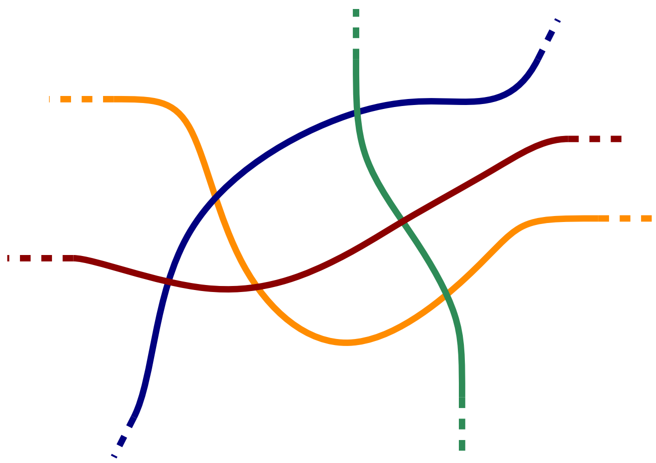

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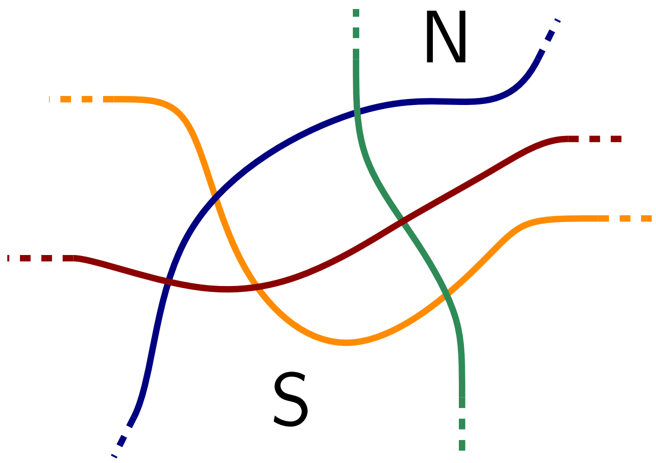

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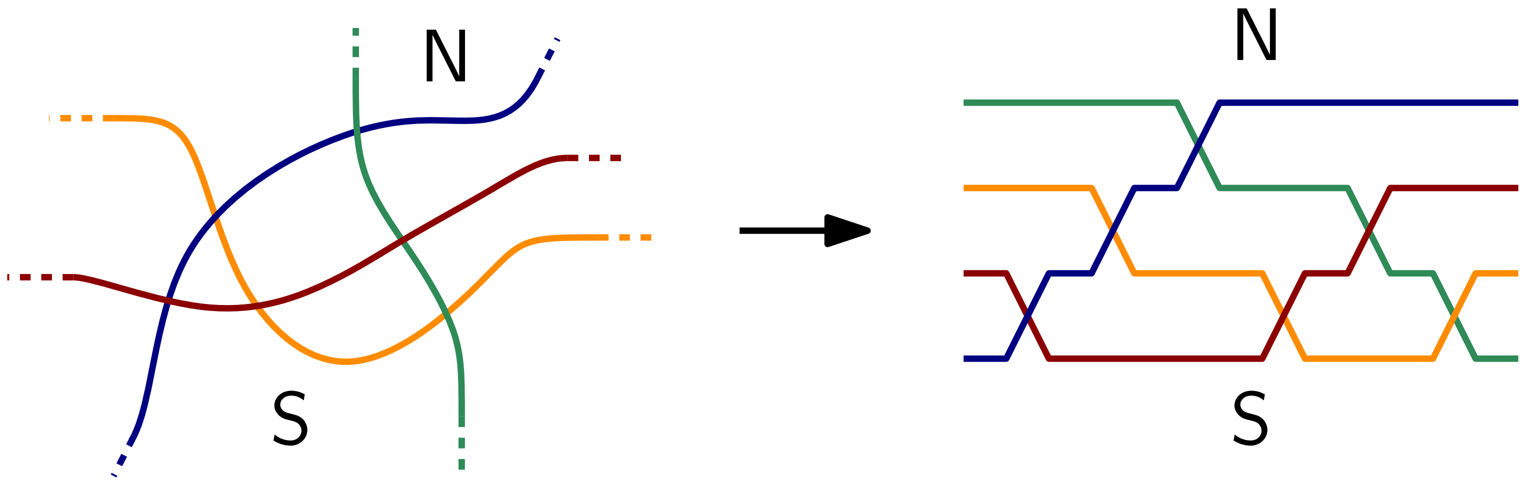

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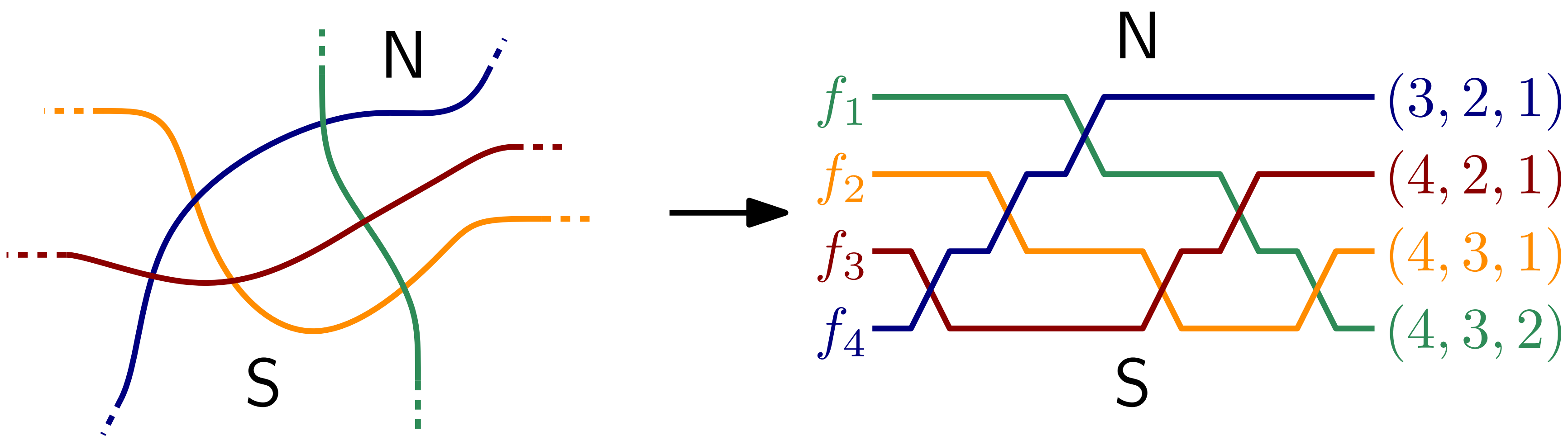
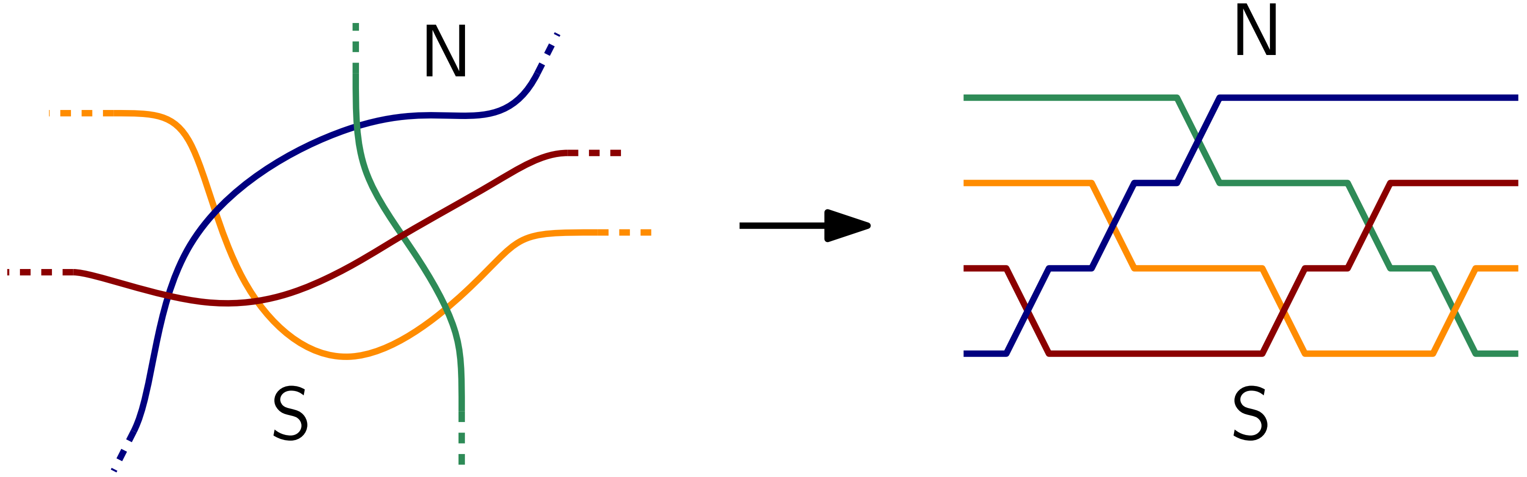
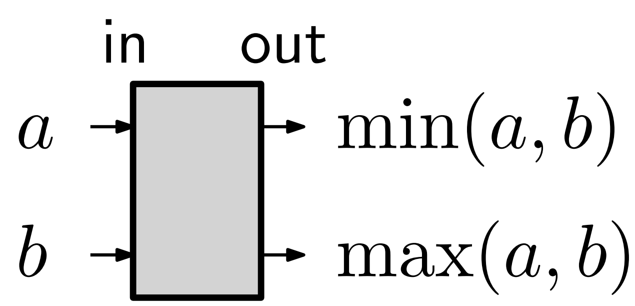


Encoding by permutations:
Permutation $\pi_i \in S_{n-1}$ encodes intersection order of f_i.

wiring diagrams



wiring diagrams as sorting networks:



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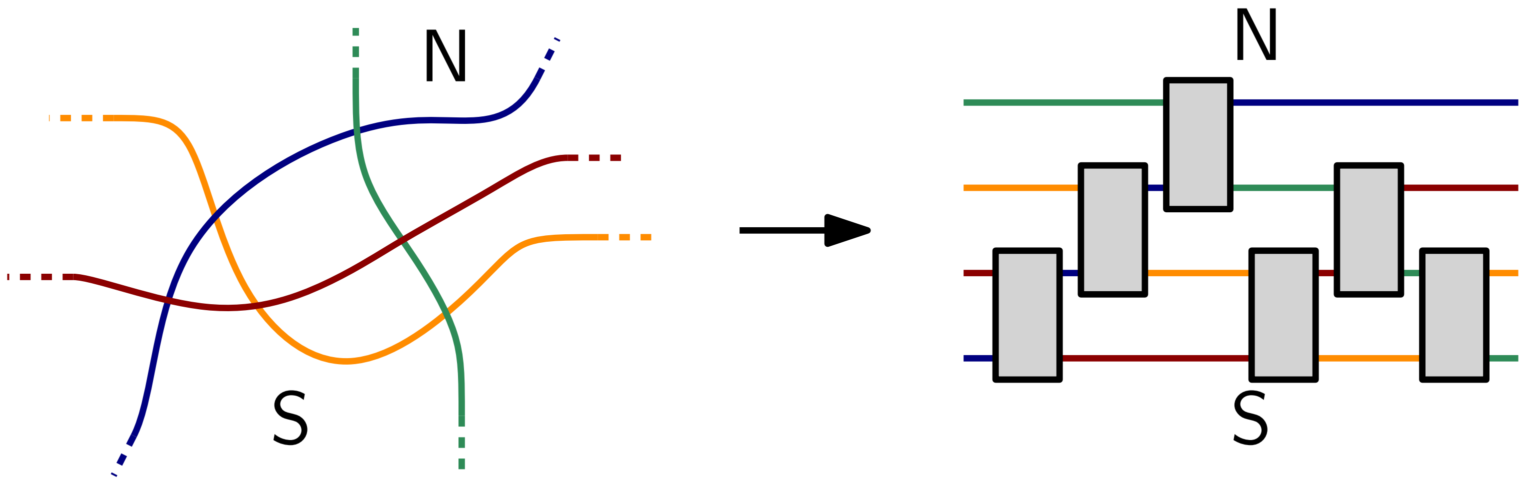

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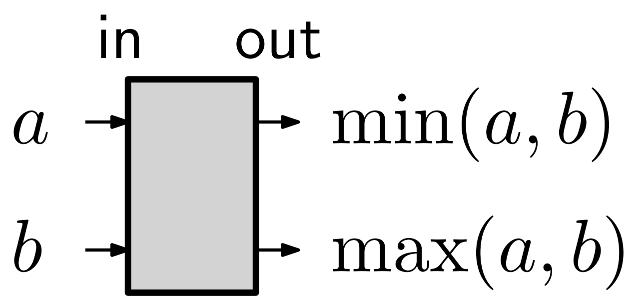

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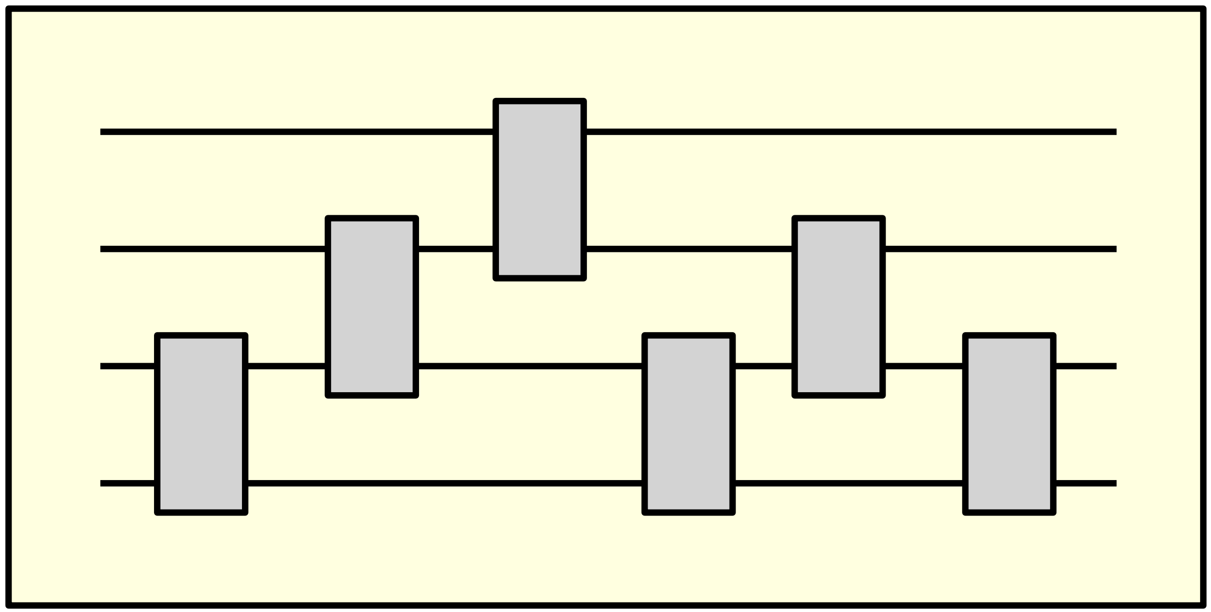

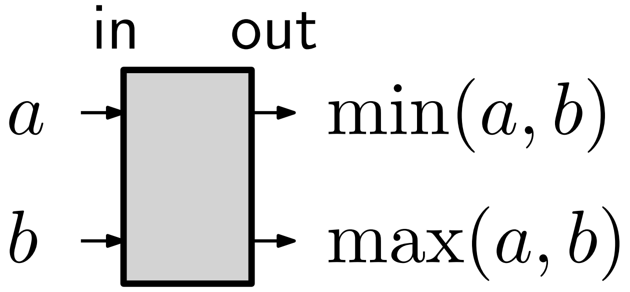

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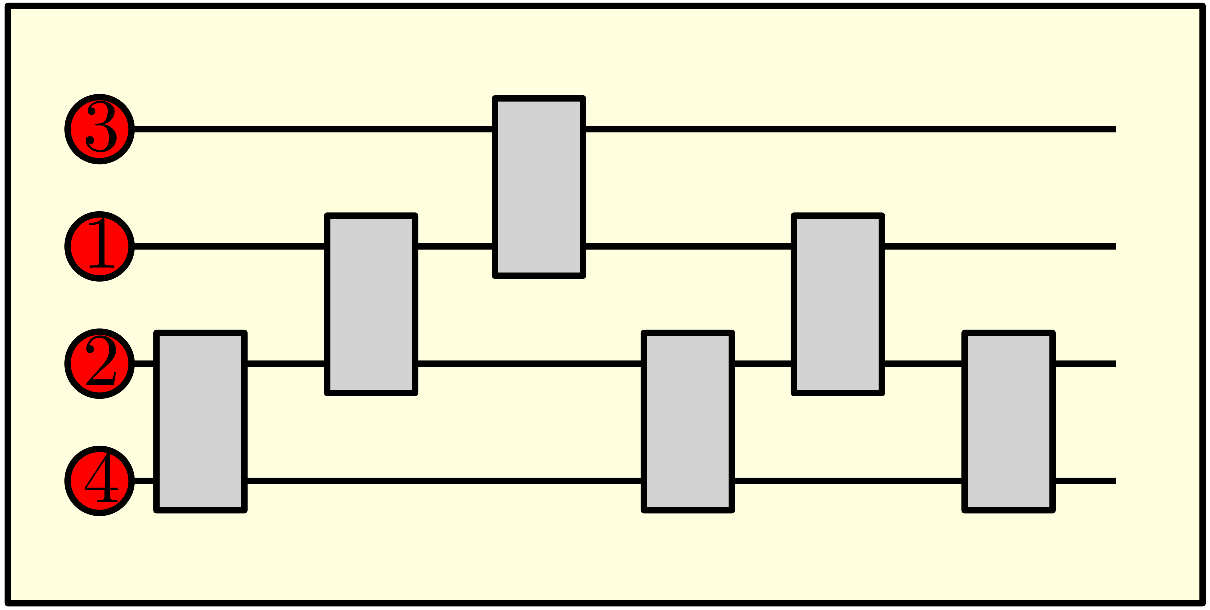

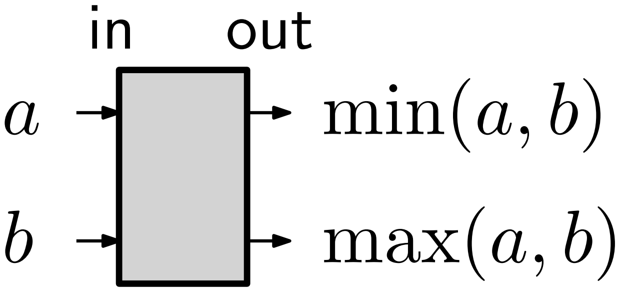

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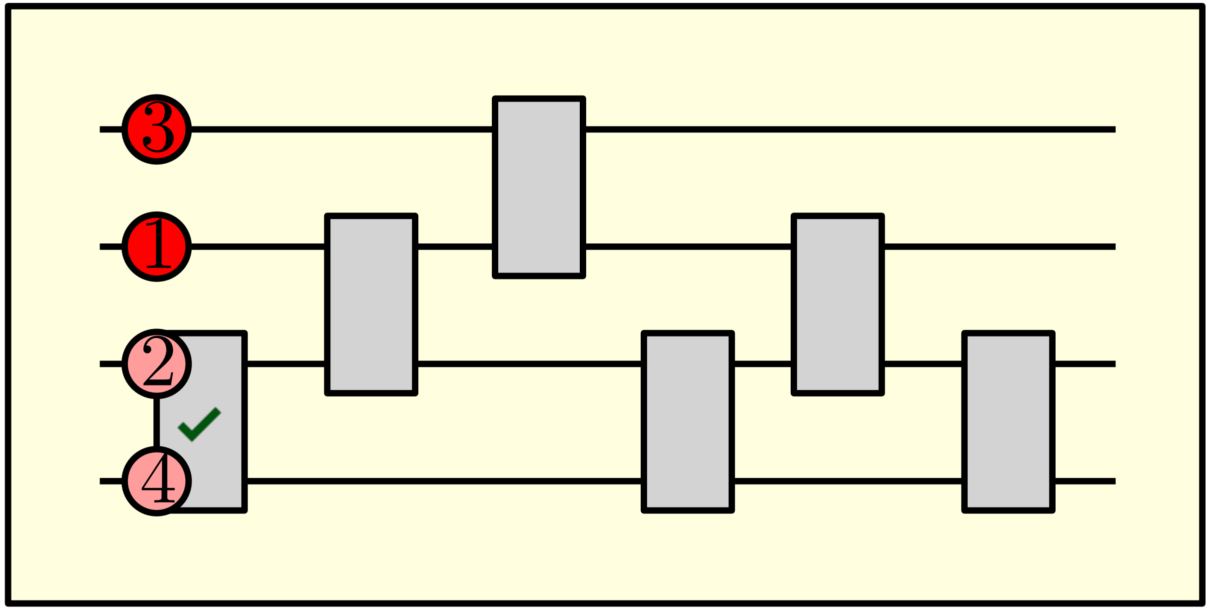

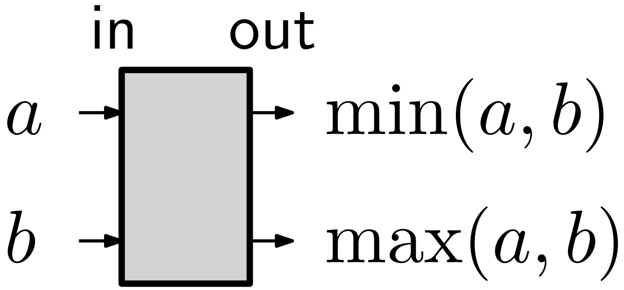

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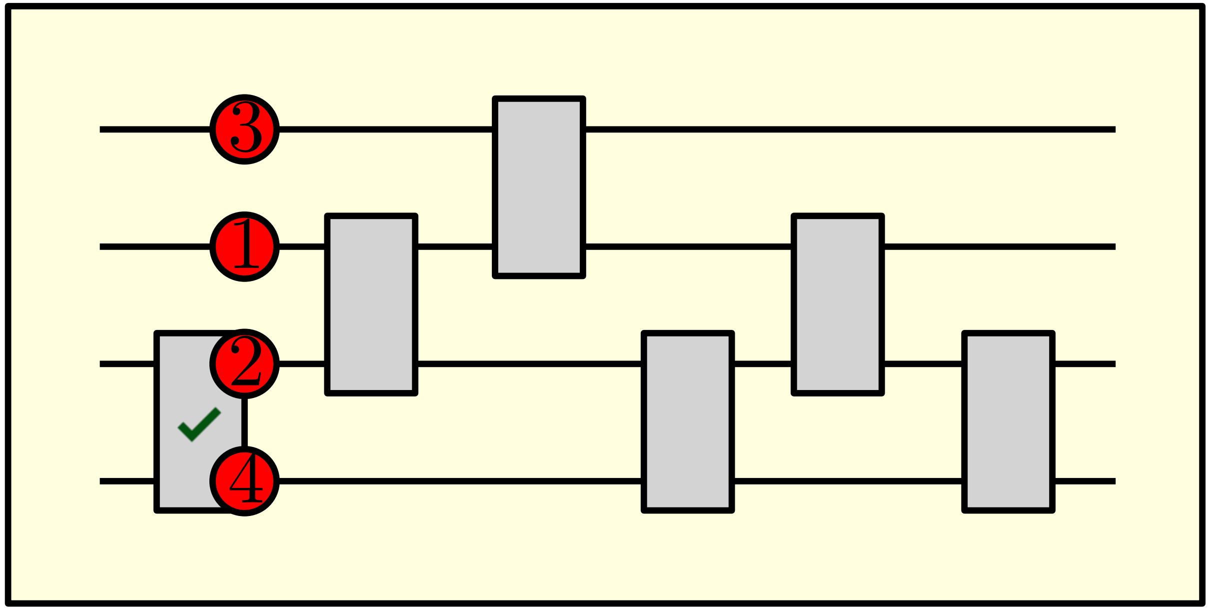

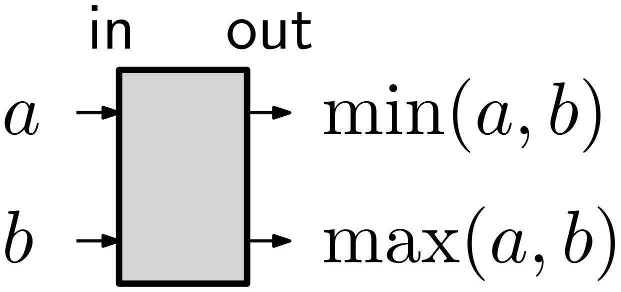

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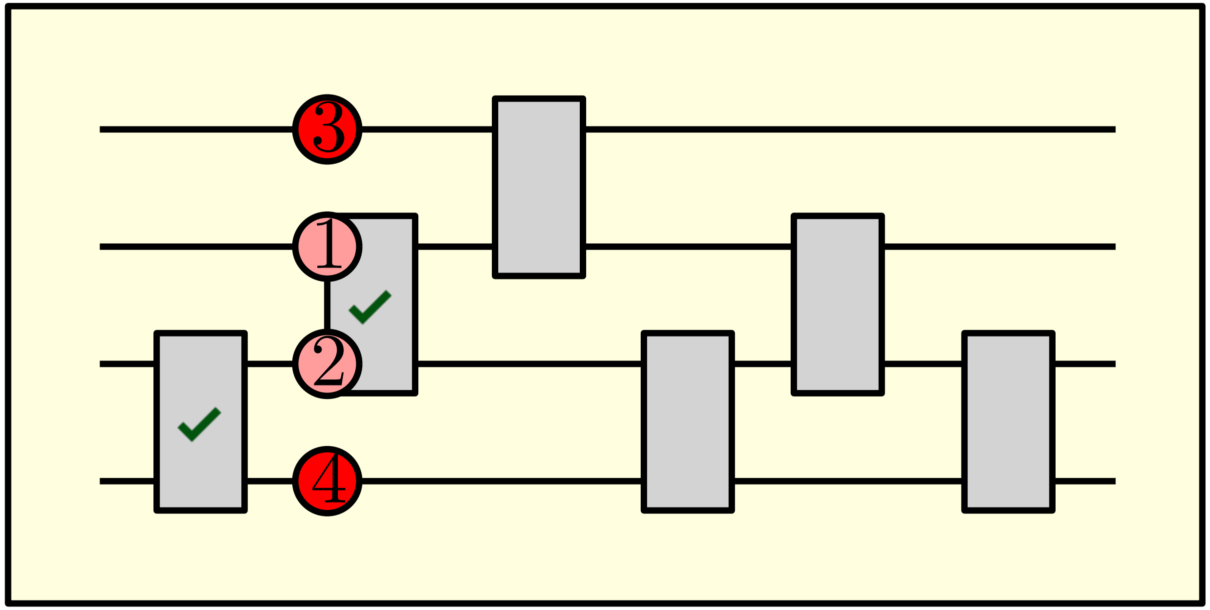

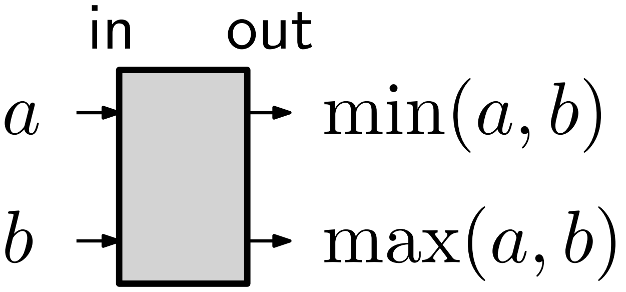

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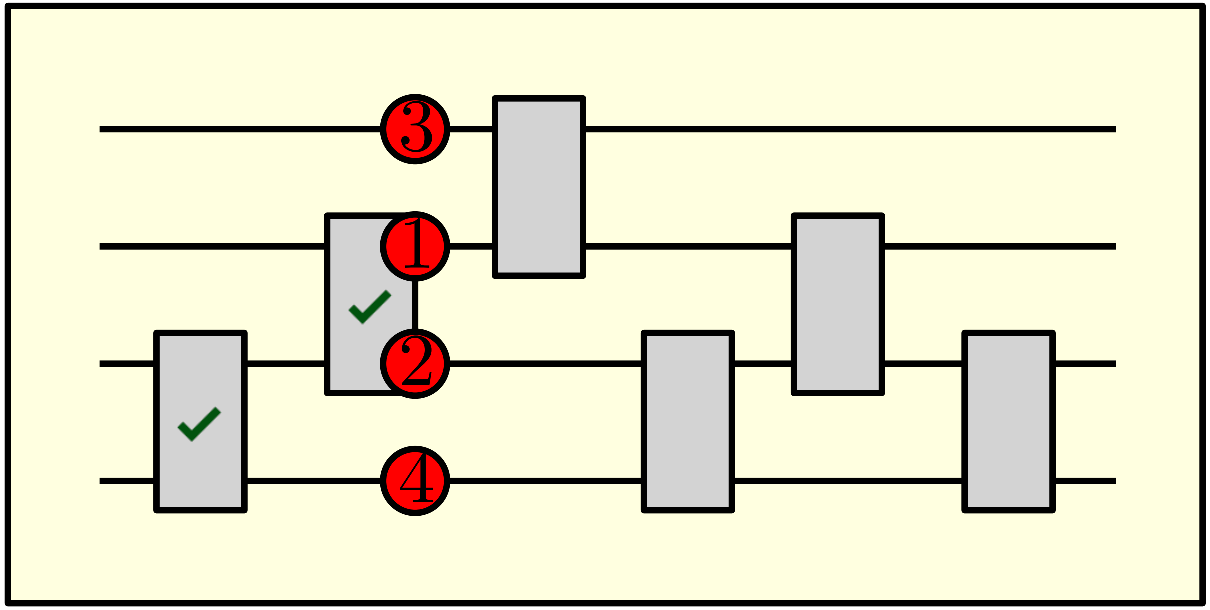

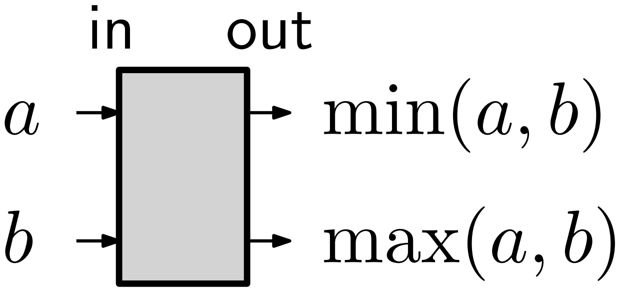

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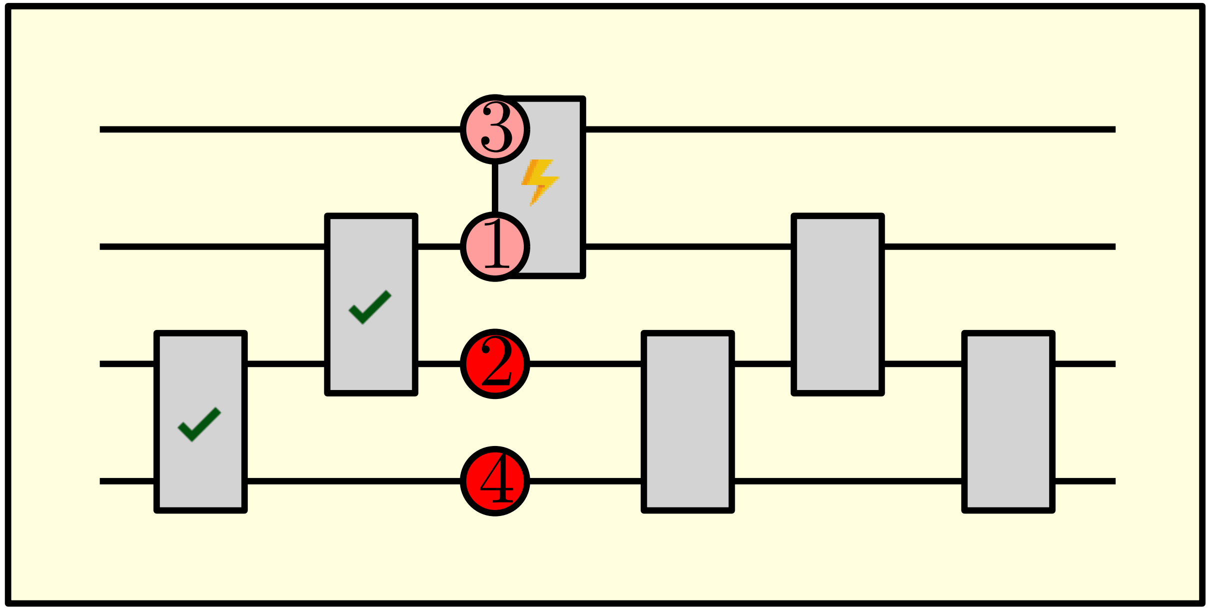

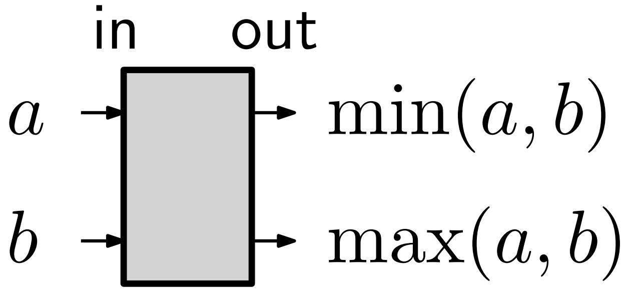

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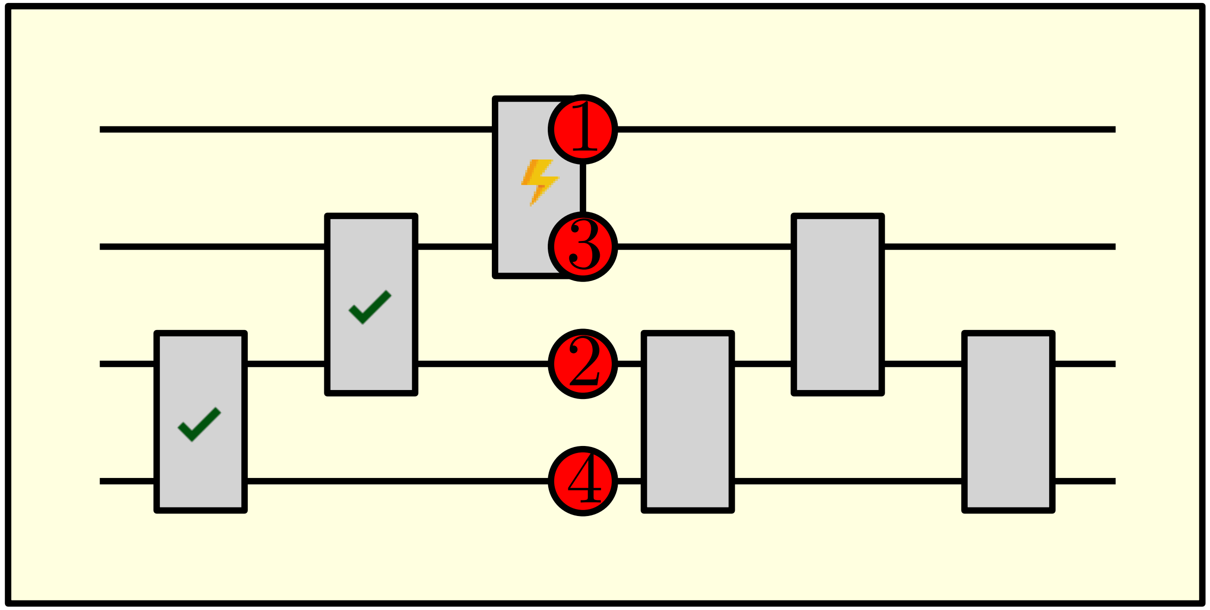

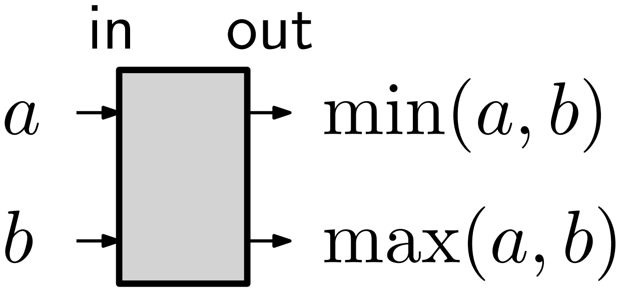

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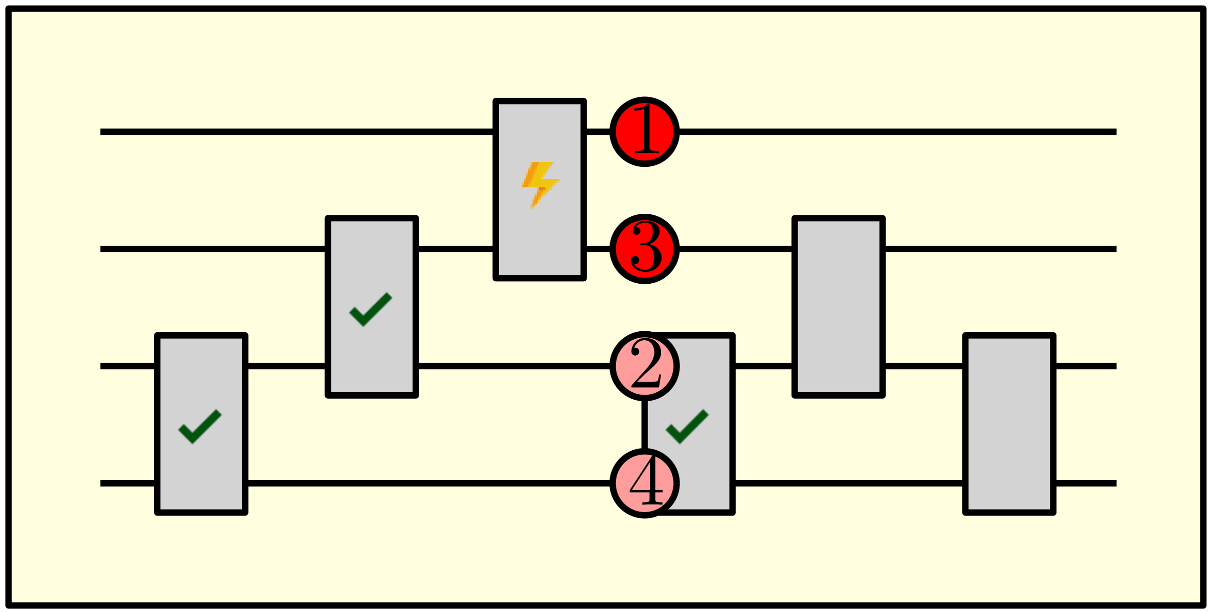

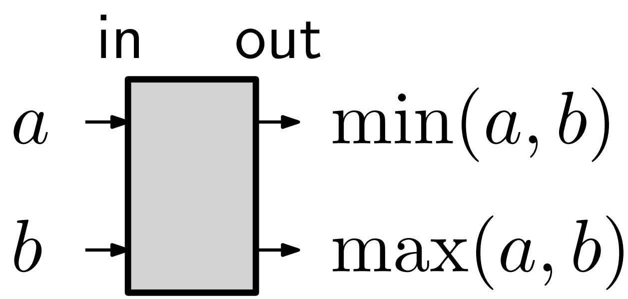

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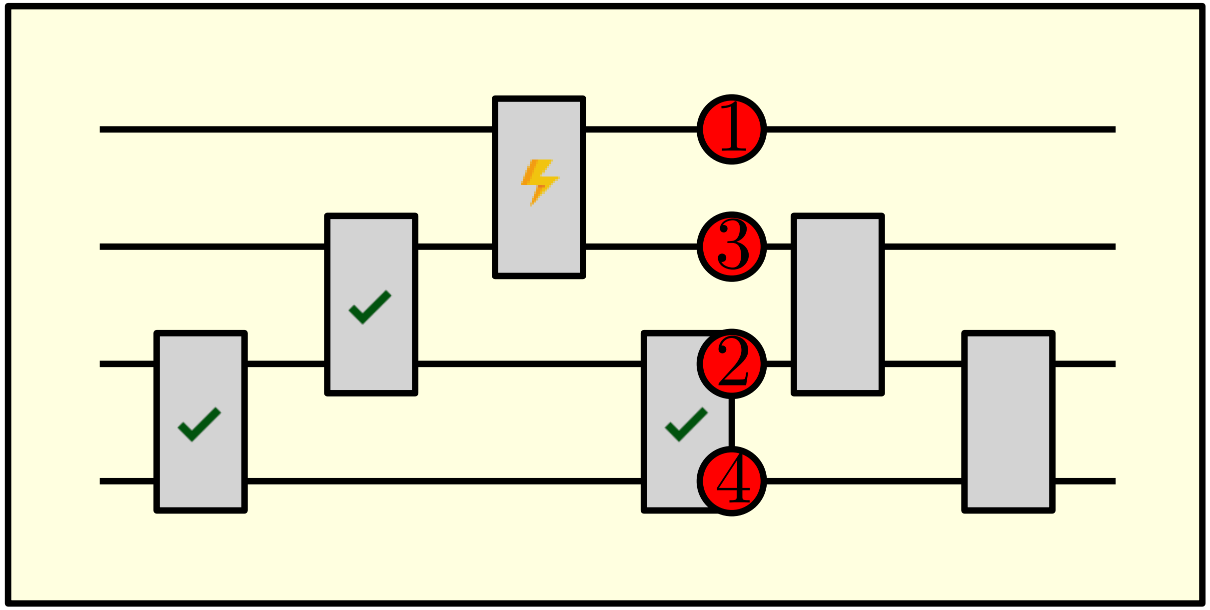

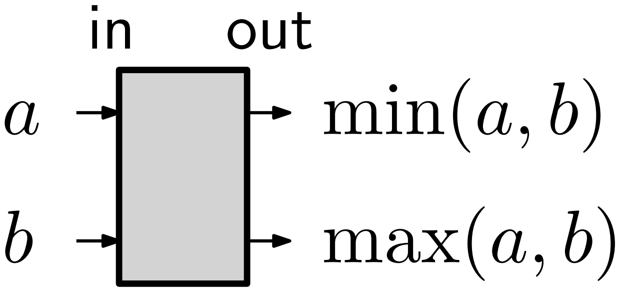

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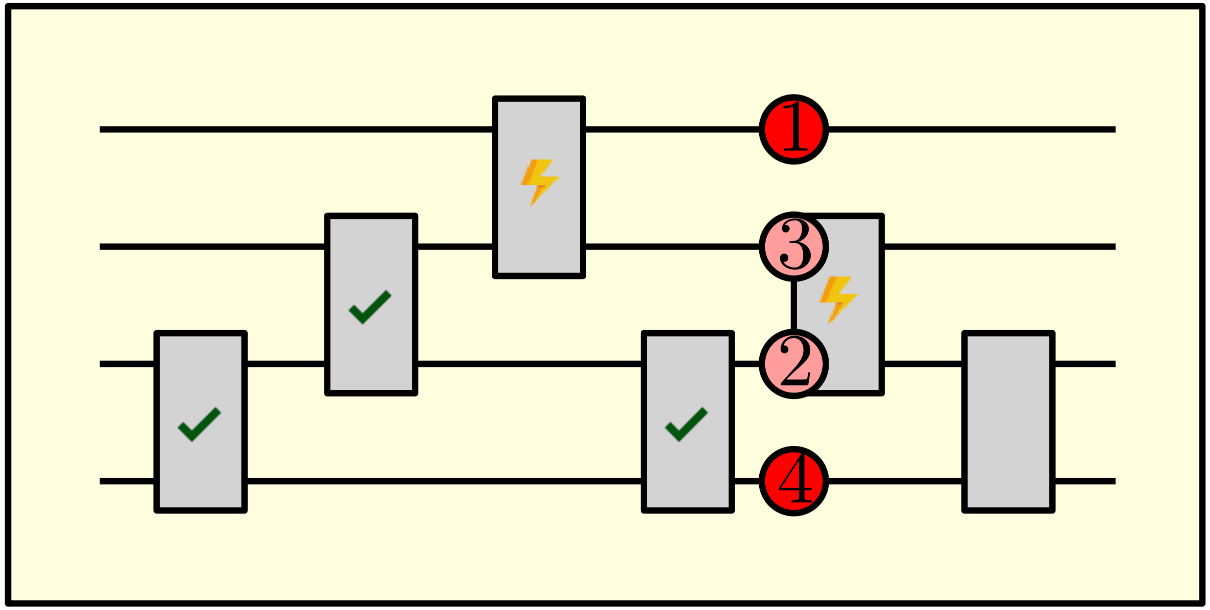

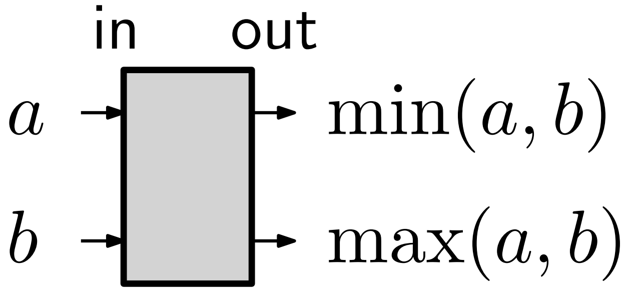

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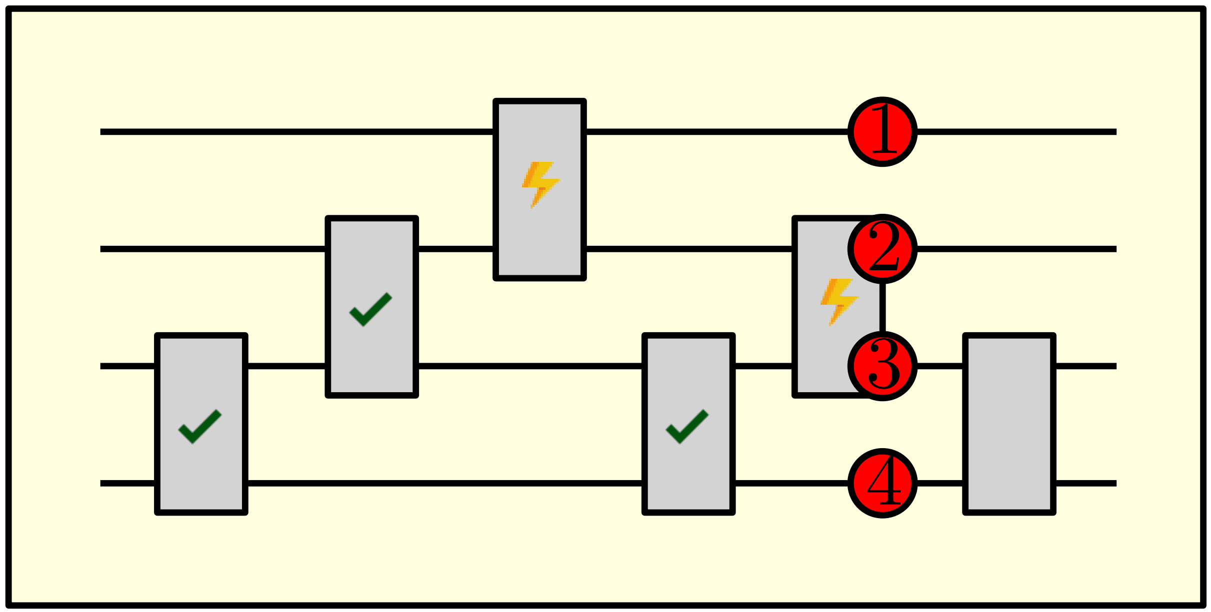

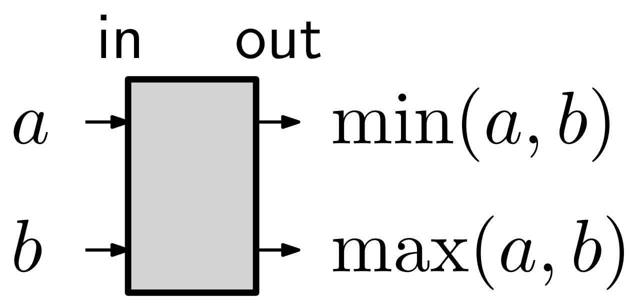

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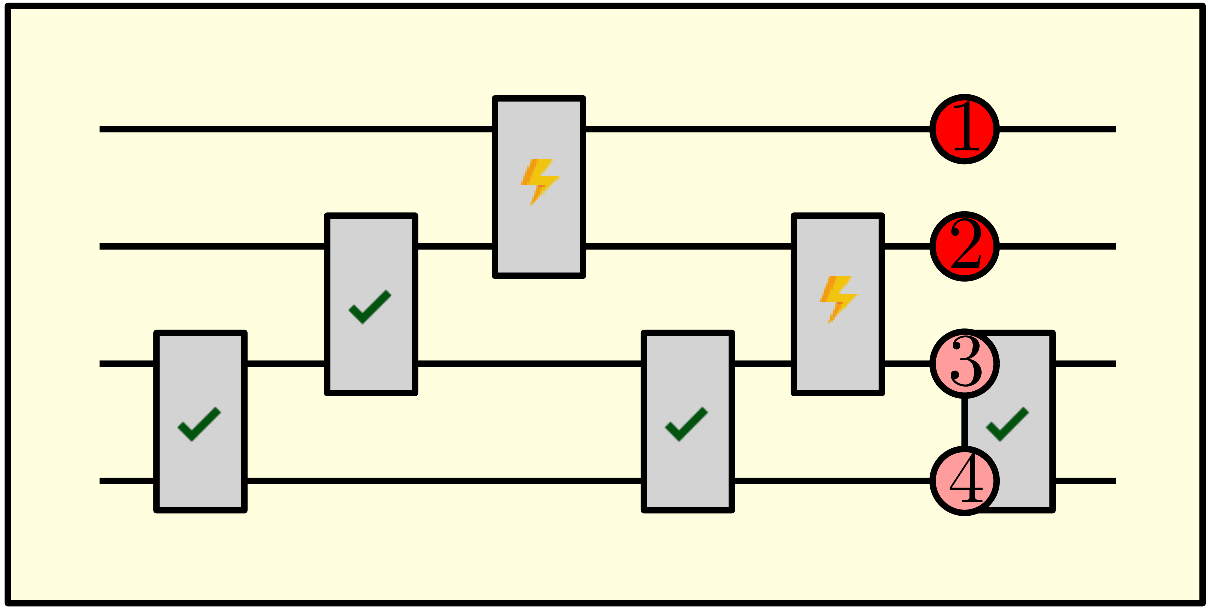

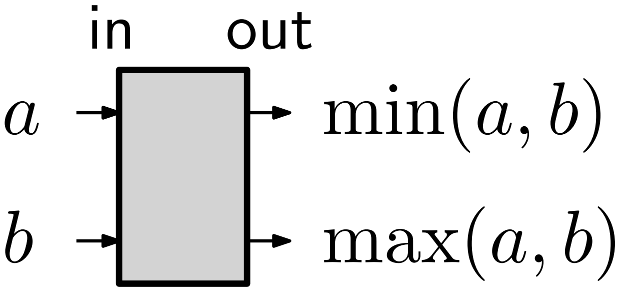

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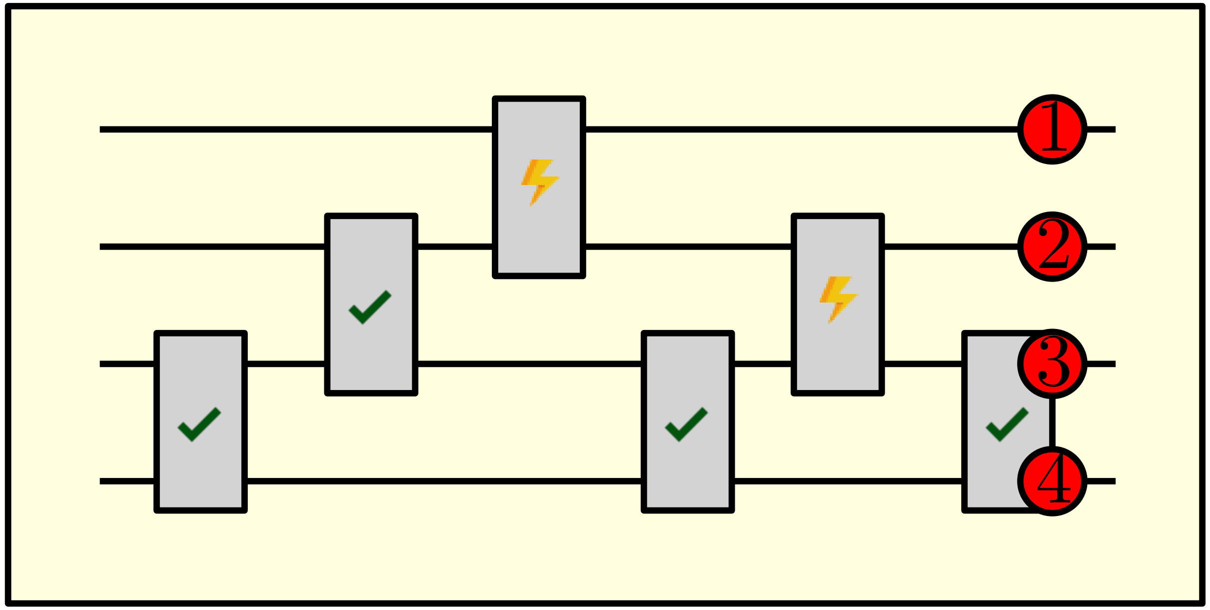

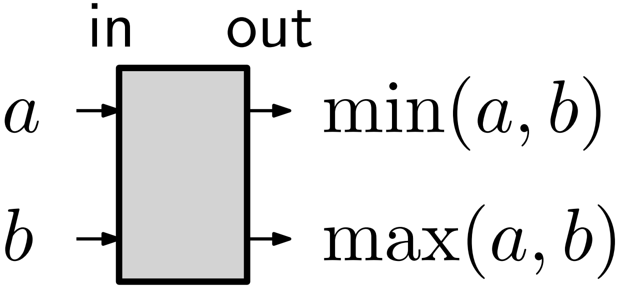

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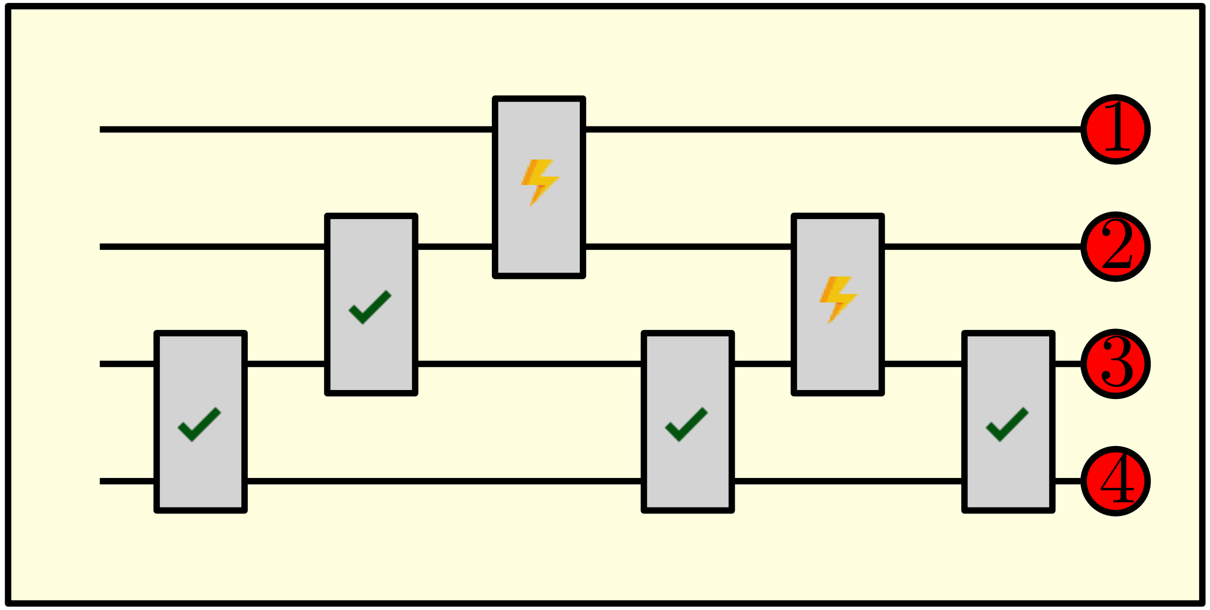

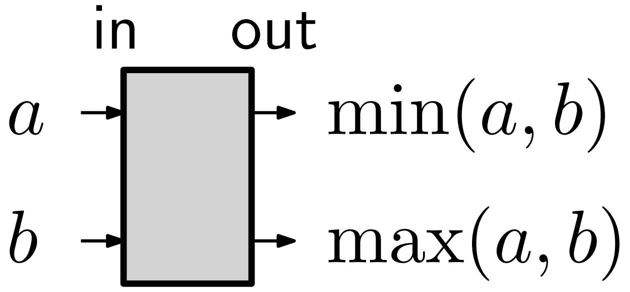

monotonic paths on permutahedron

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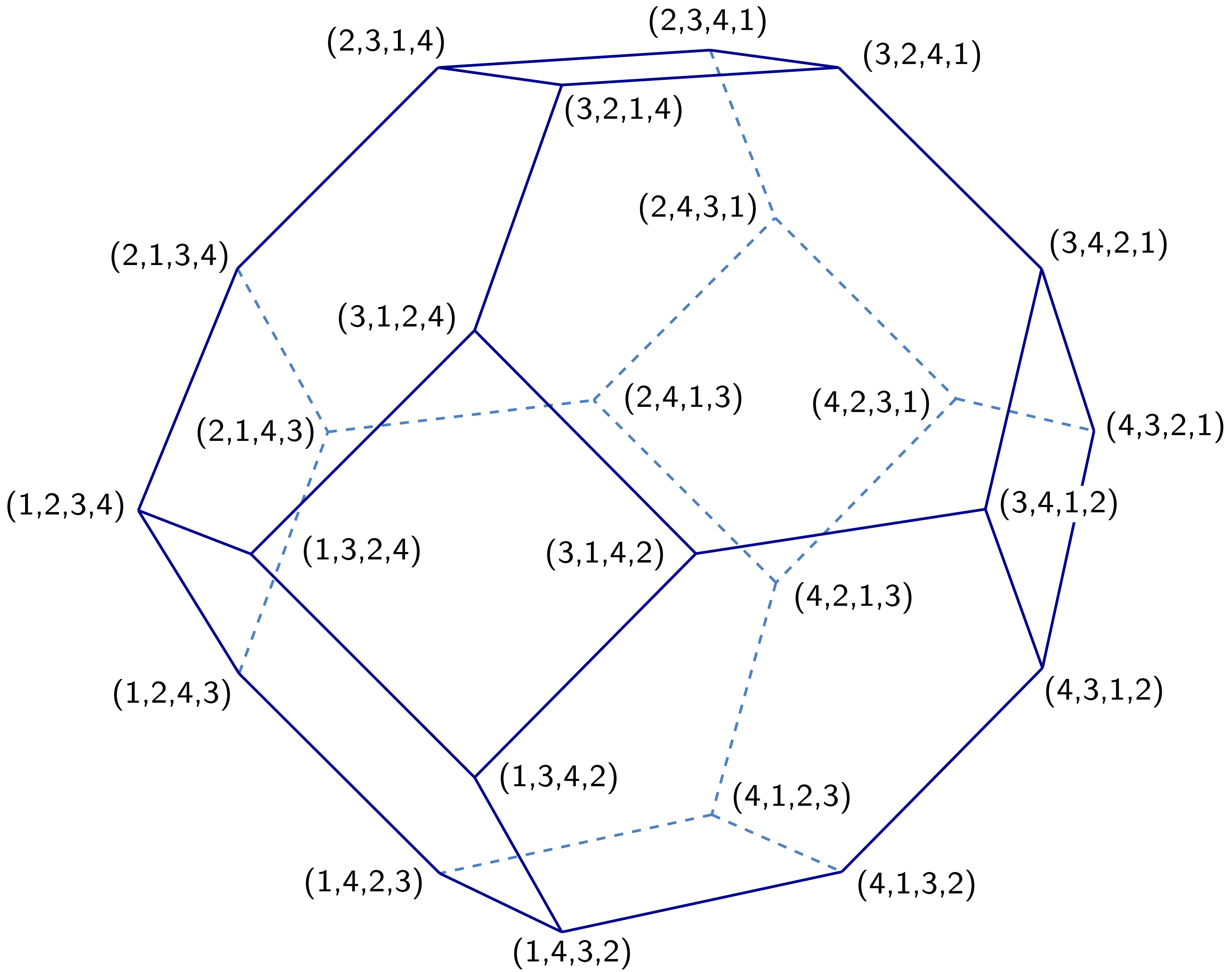

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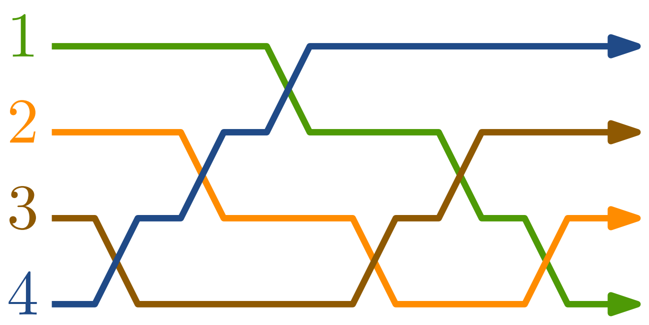

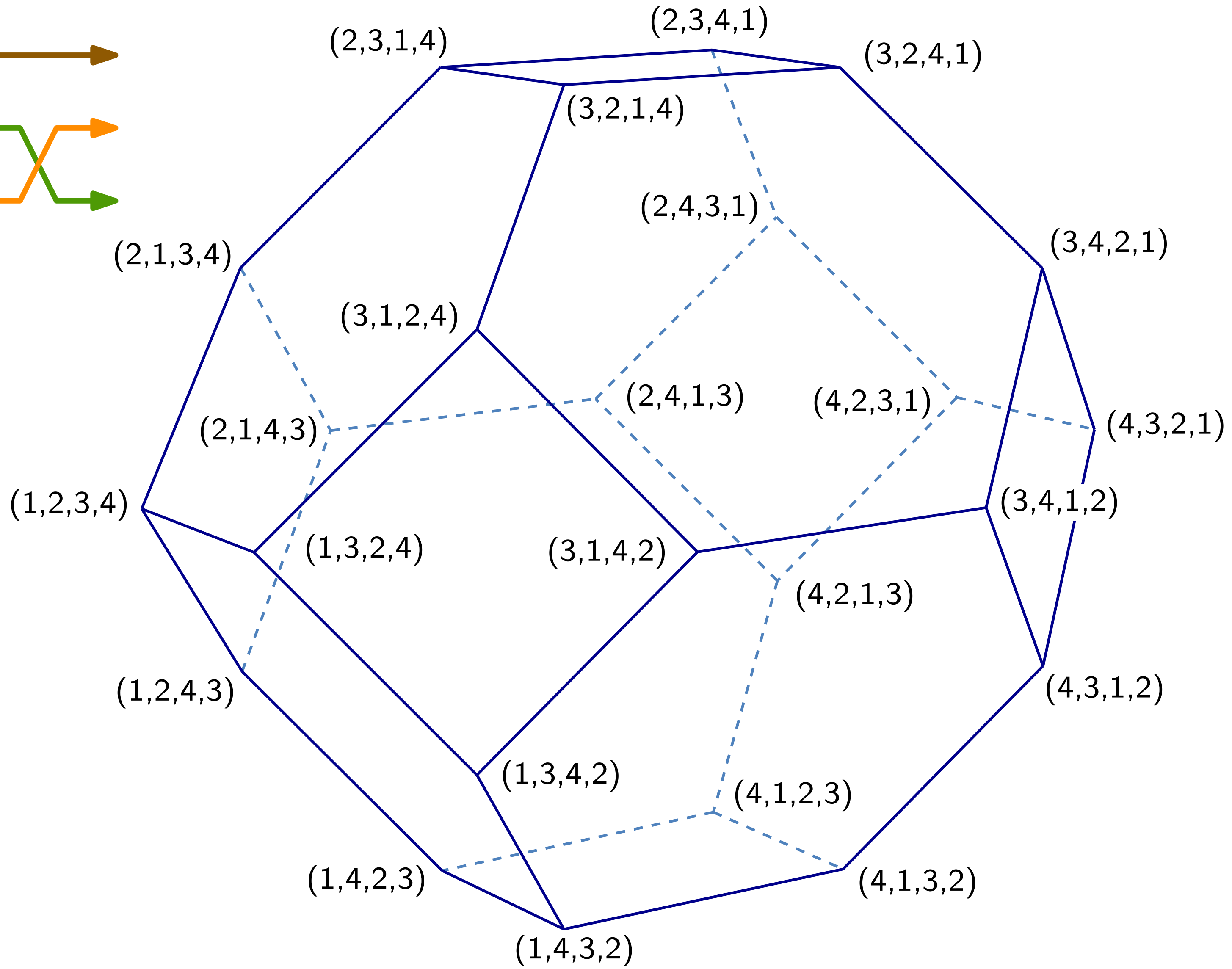

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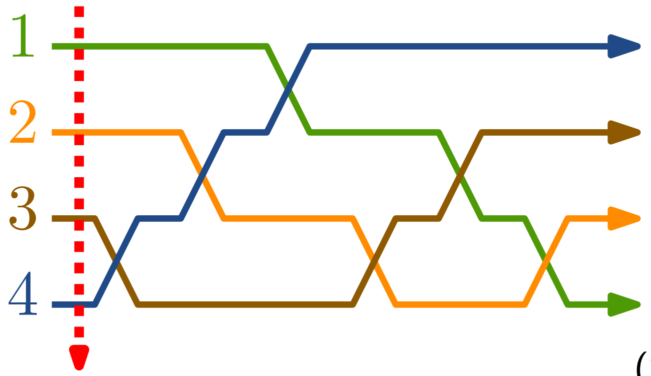

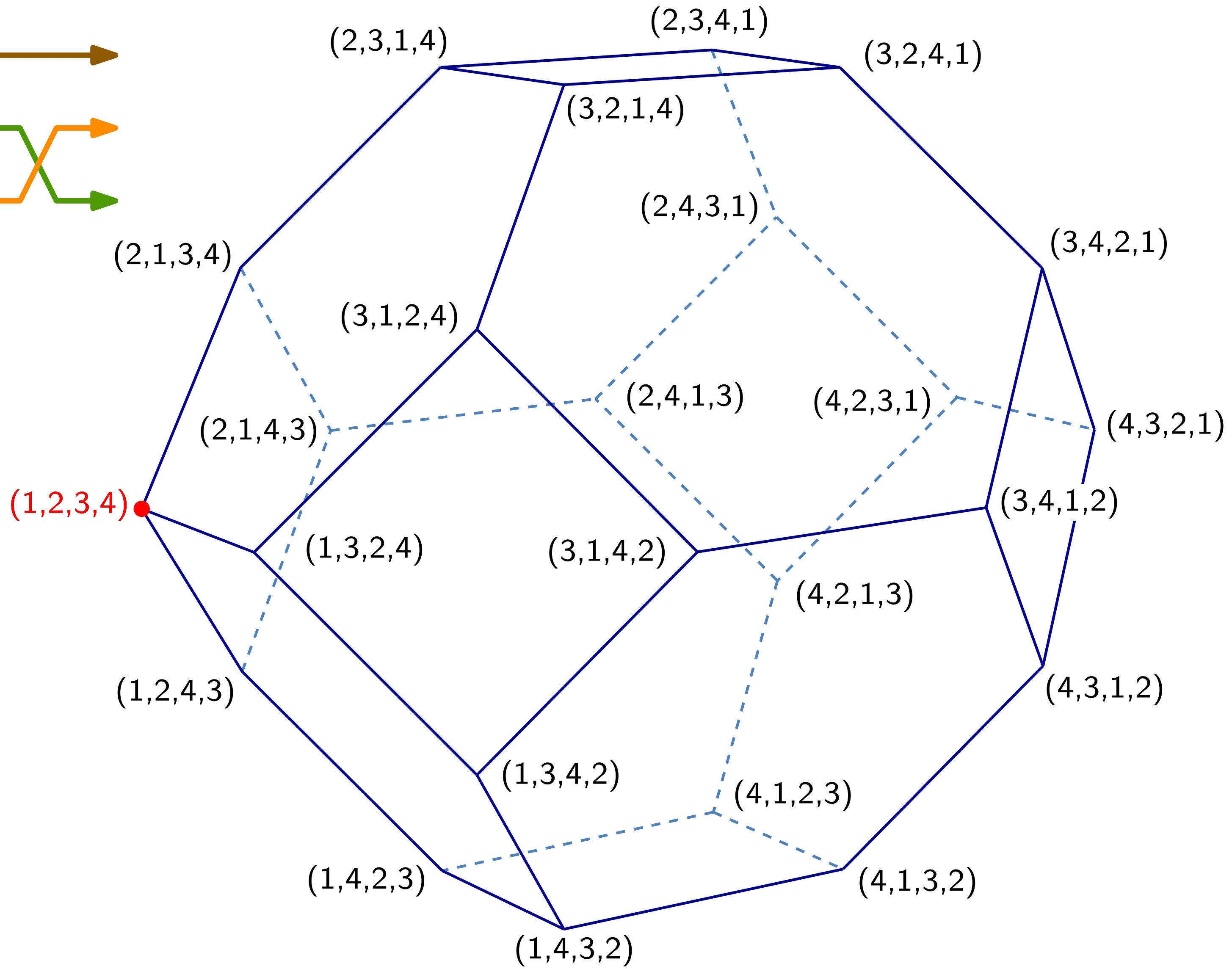

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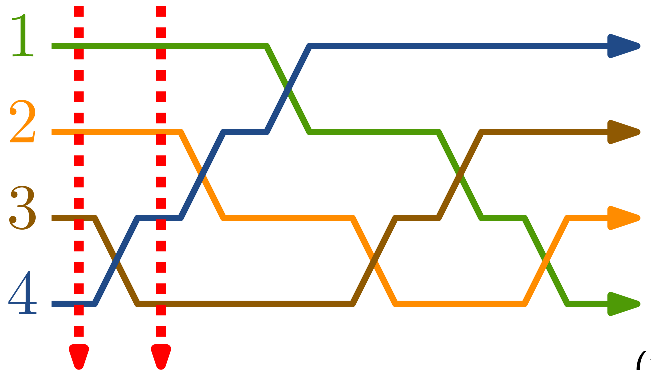

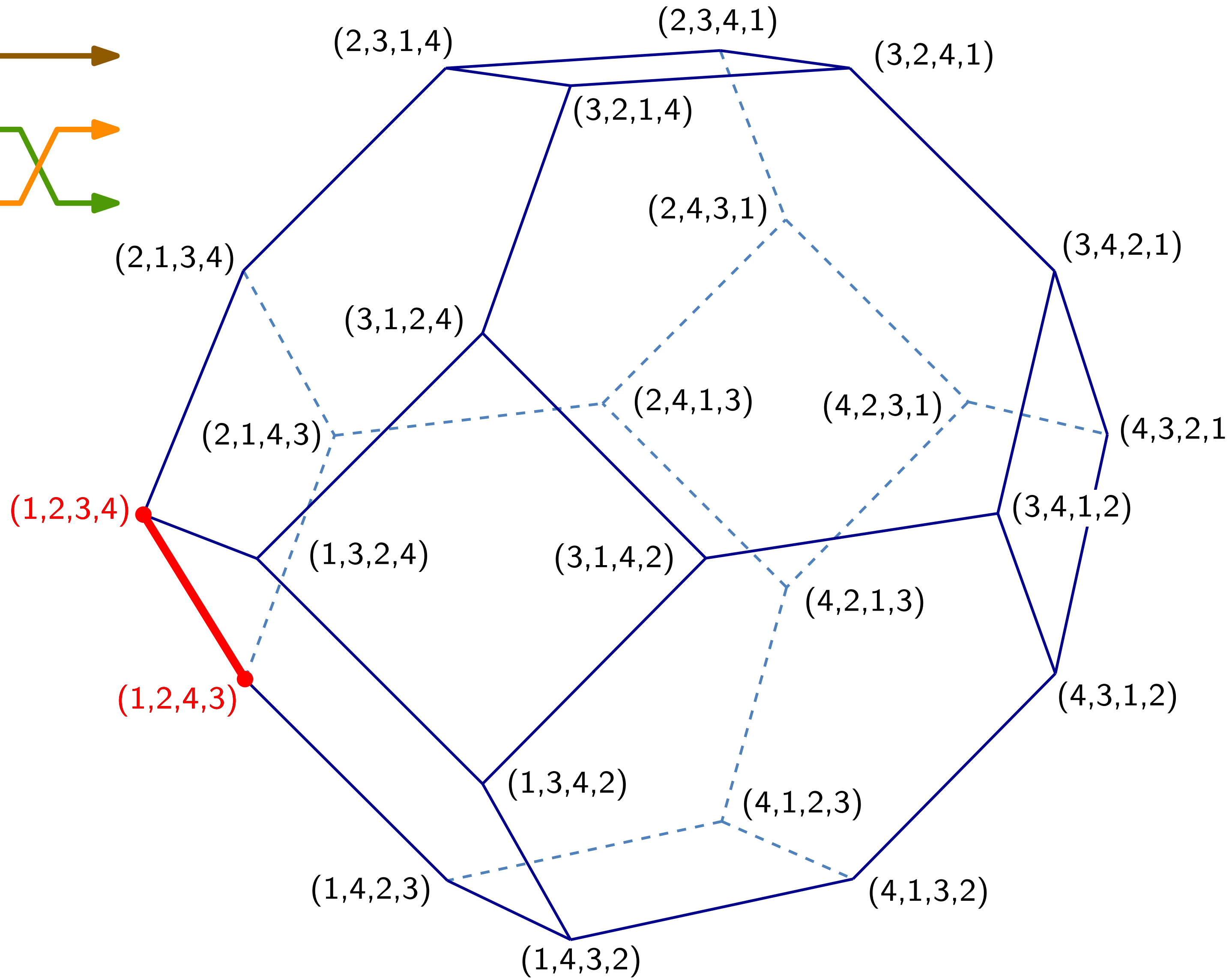

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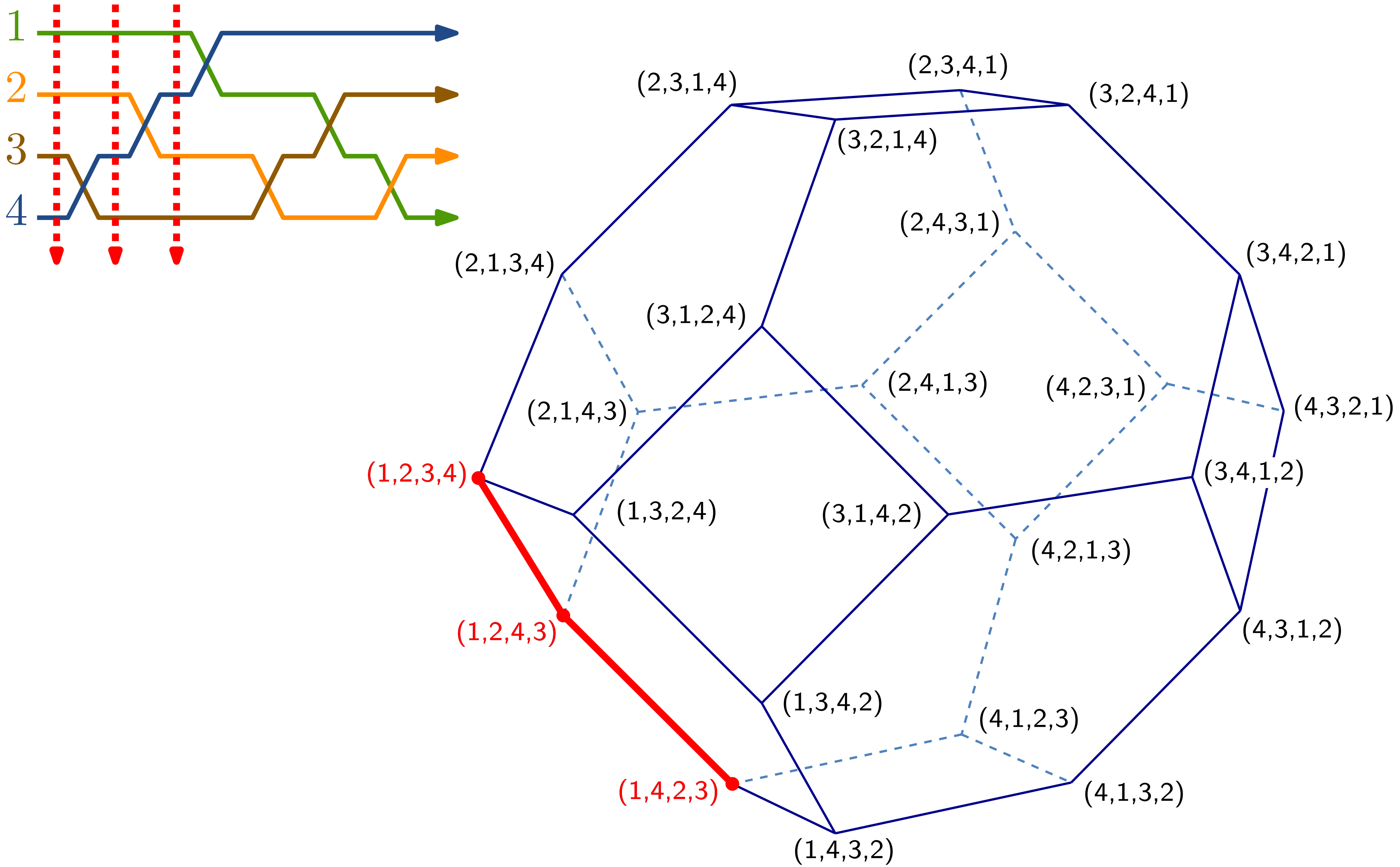

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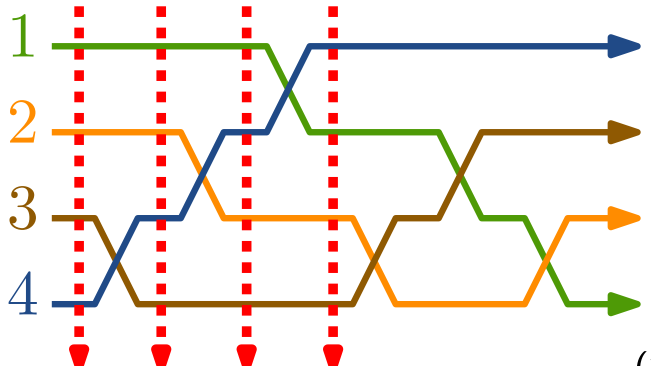

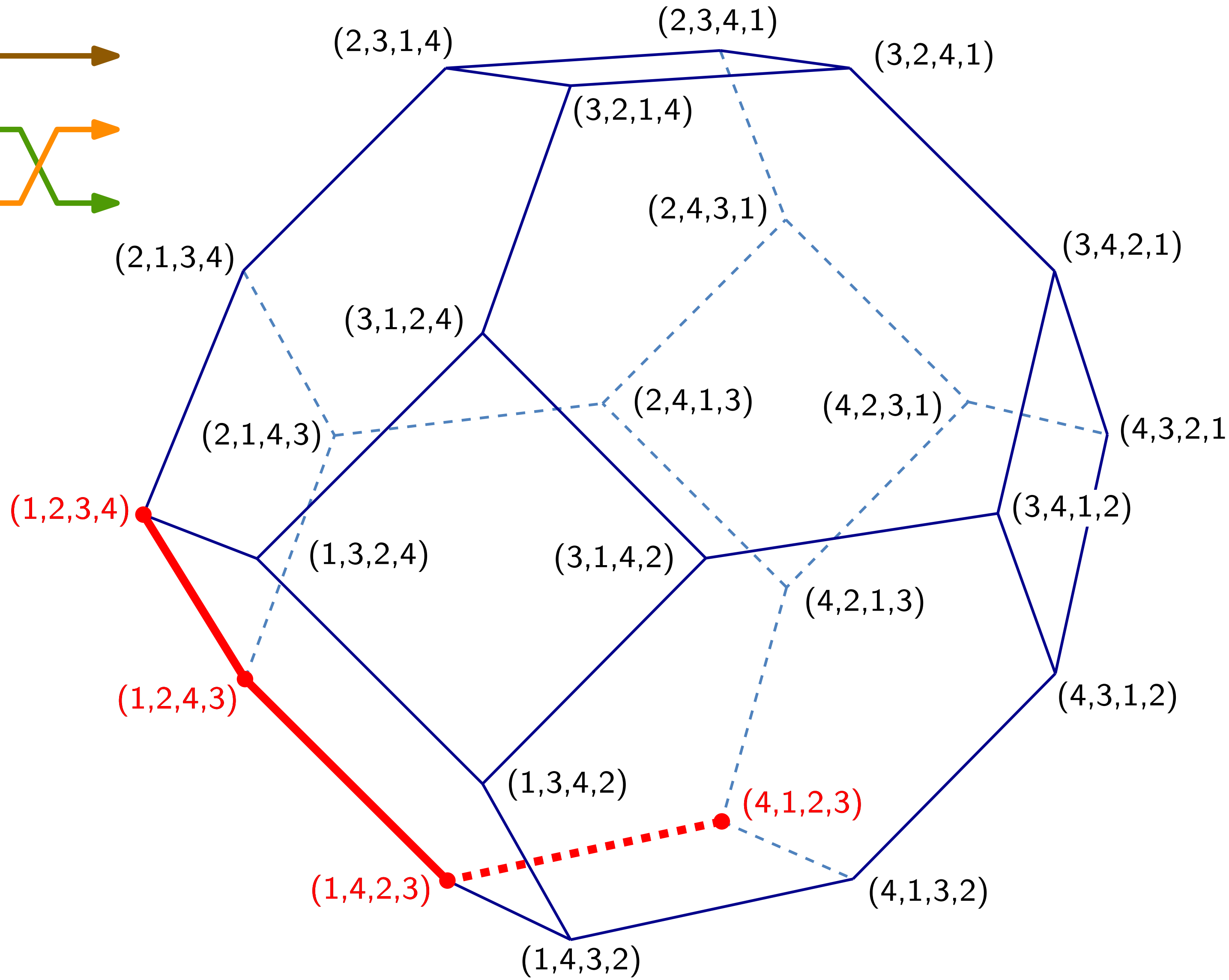

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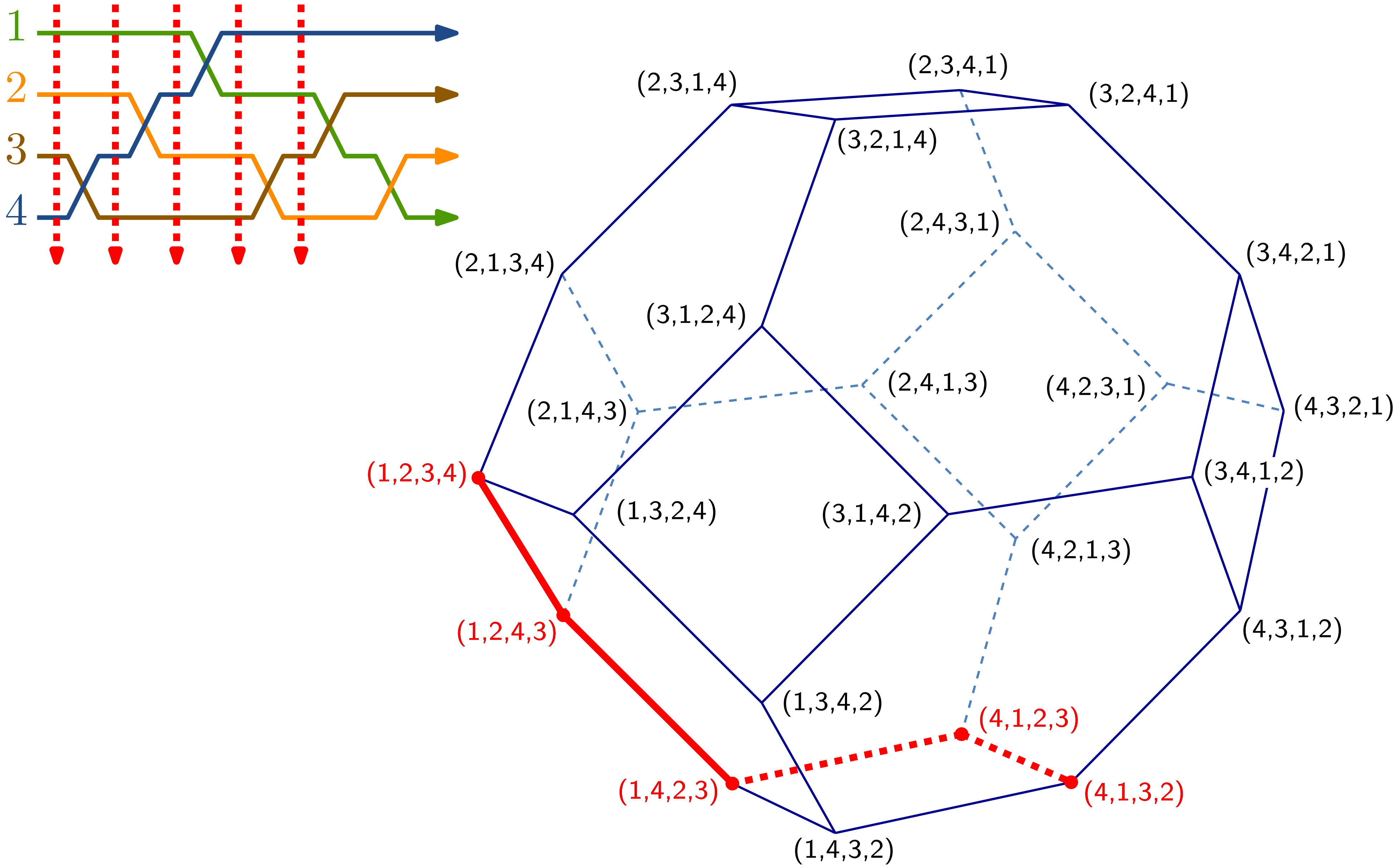

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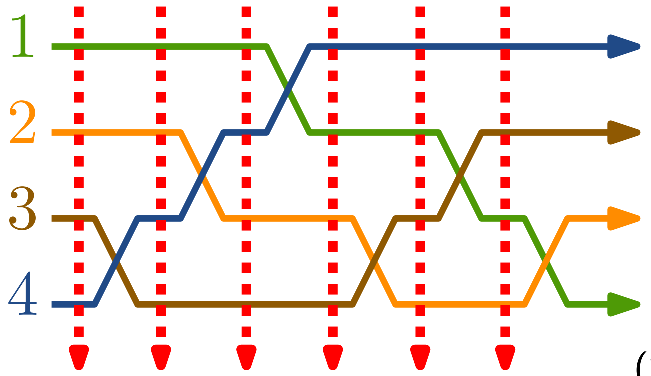

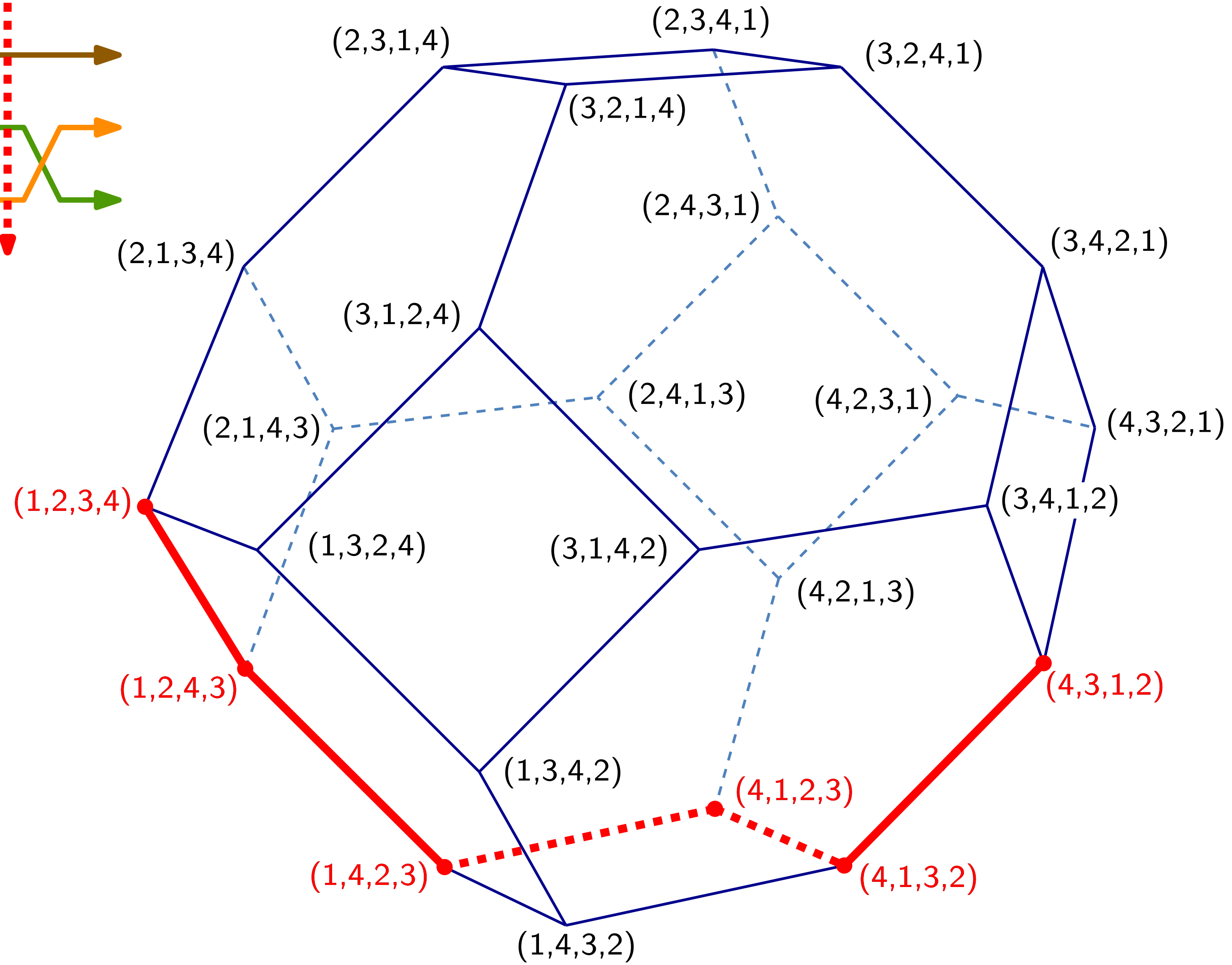

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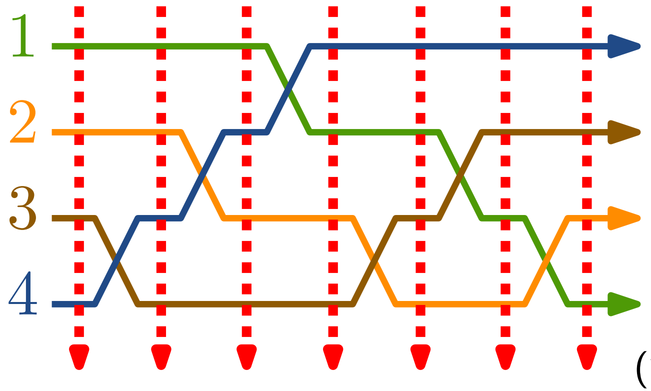

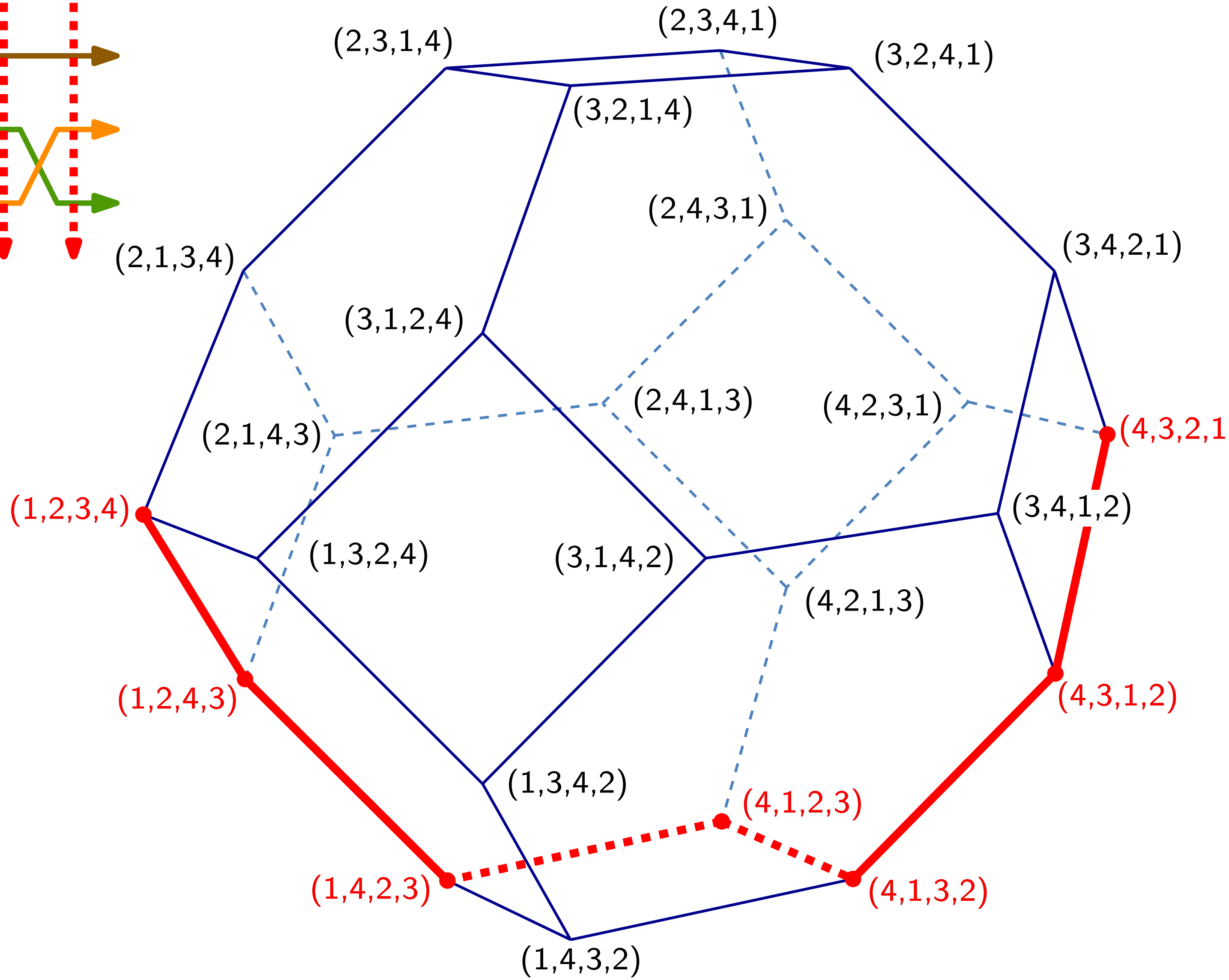

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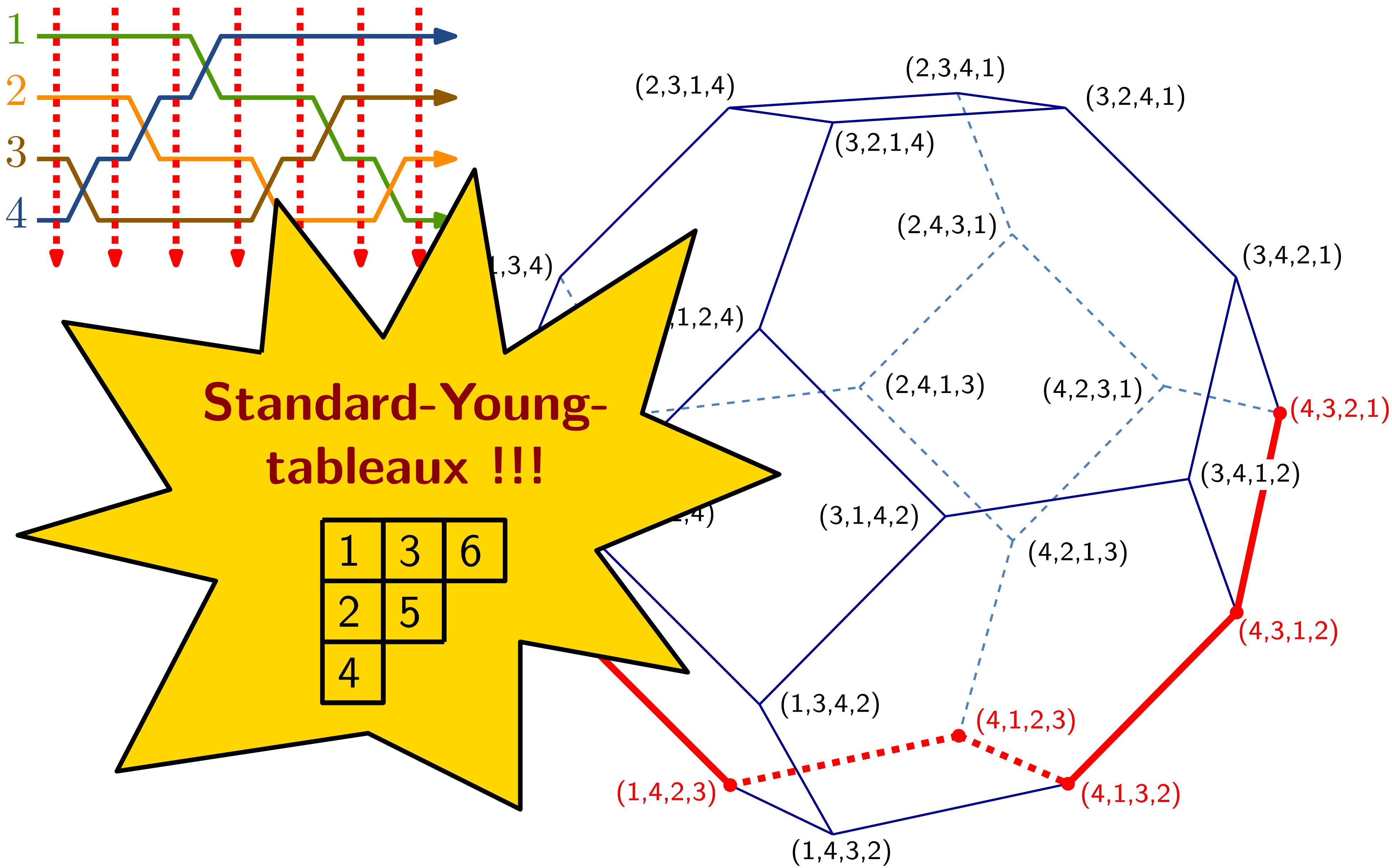
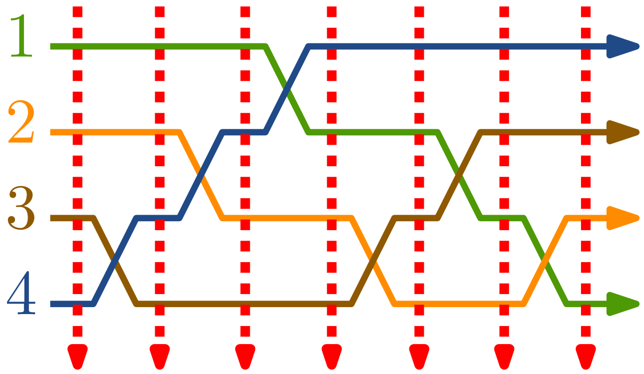


Edelman-Greene bijection



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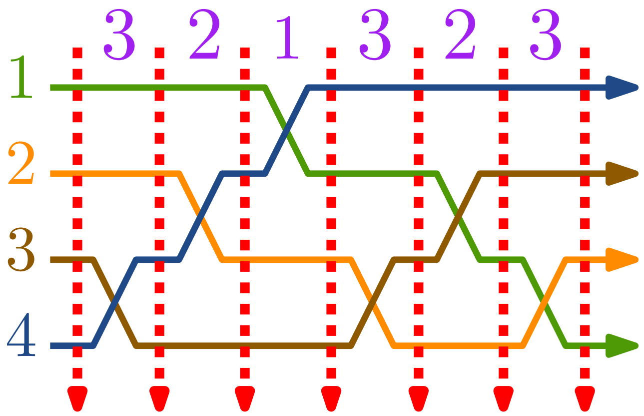

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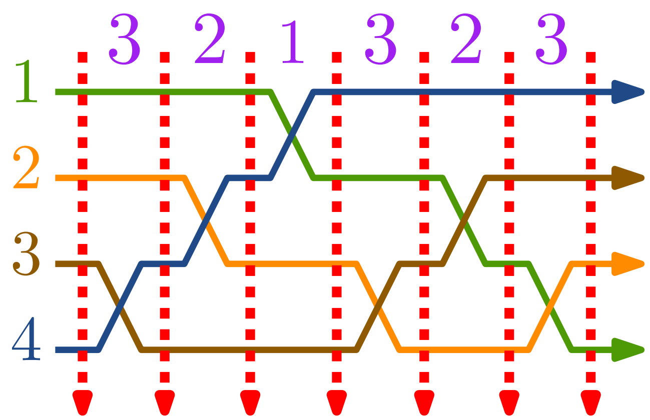

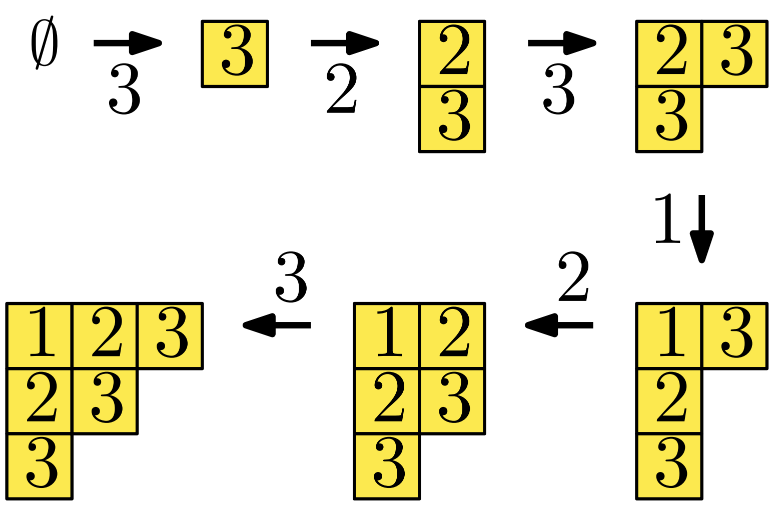

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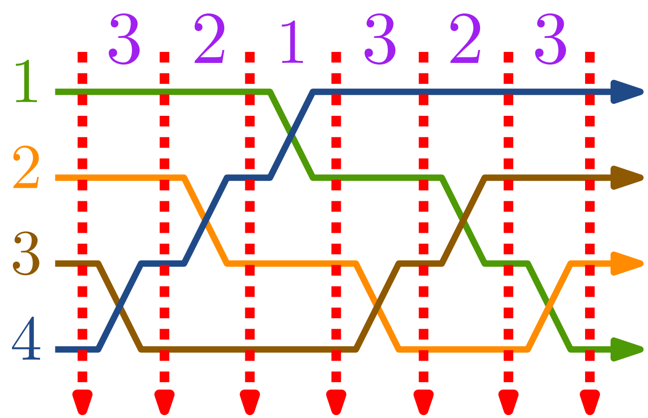
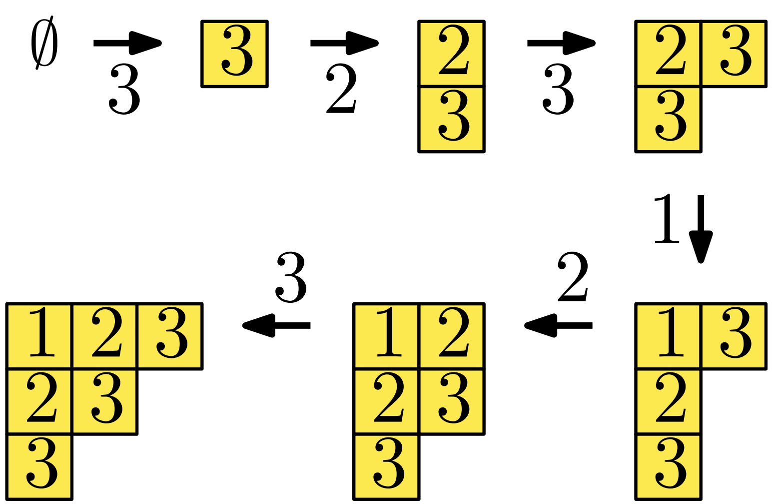


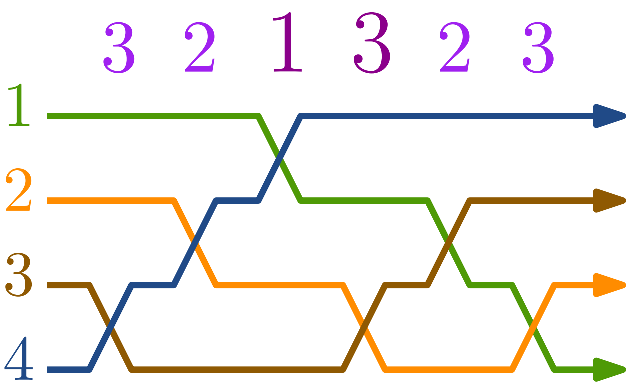


growth:

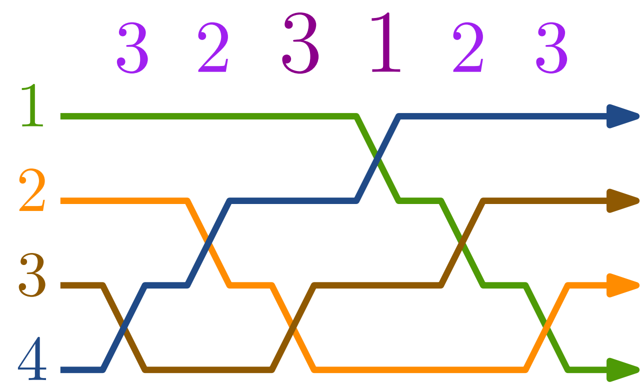
1	3	6
2	5	
4		

classes of wiring diagrams

Same pseudoline arrangement, but different wiring diagrams!



$\cong$



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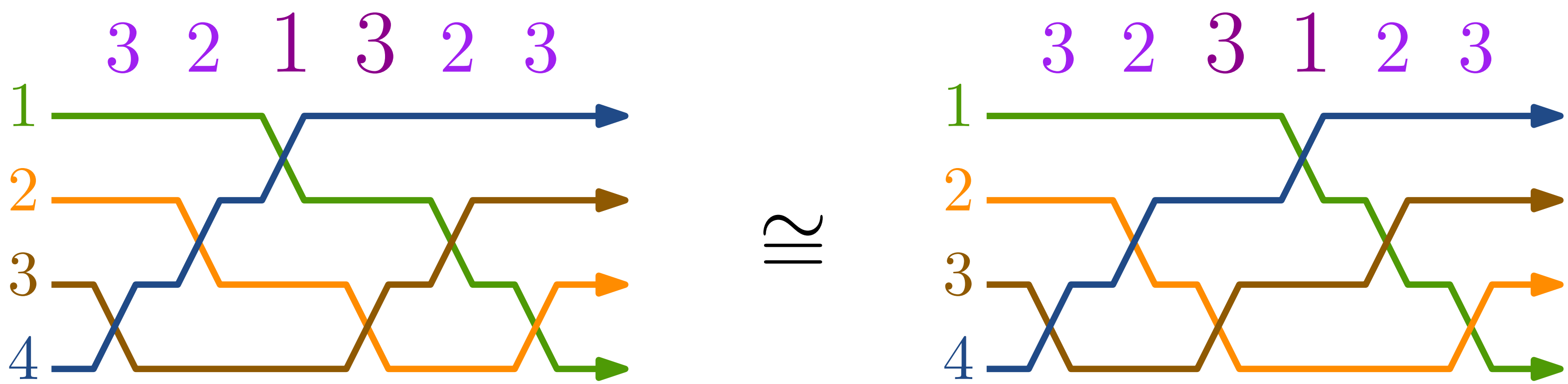

- Two wiring diagrams

$$S = (s_1, \cdots, s_i, s_{i+1}, \cdots, s_{\binom{n}{2}})$$

and $$S' = (s_1, \cdots, s_{i+1}, s_i, \cdots, s_{\binom{n}{2}})$$

are called *directly equivalent*, if $|s_i - s_{i+1}| \geq 2$.

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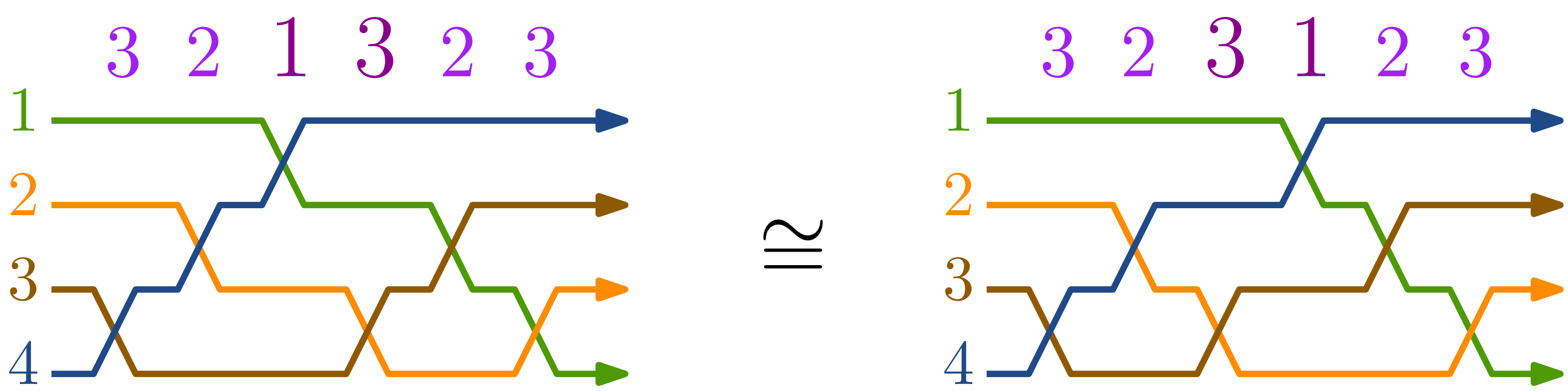

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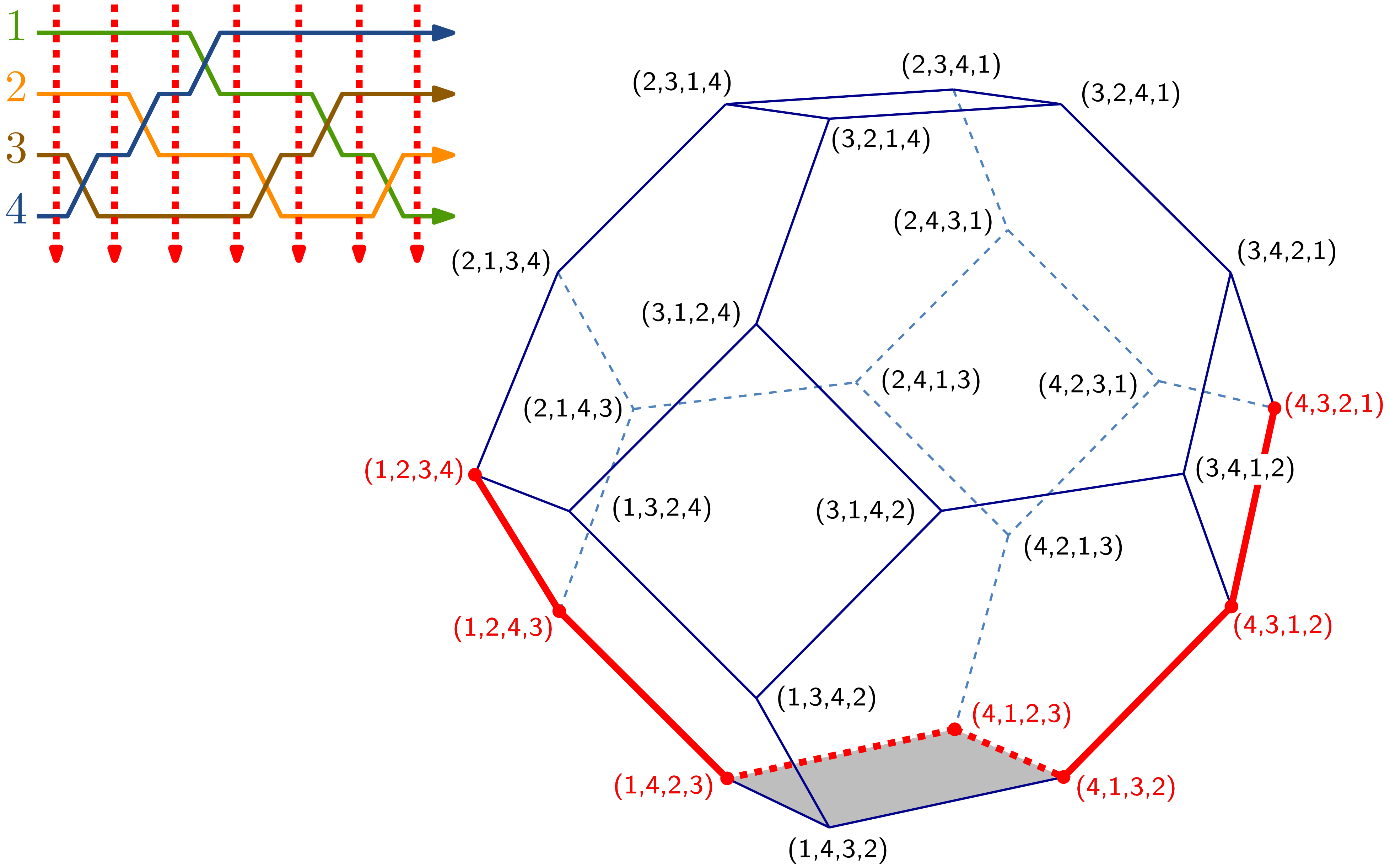
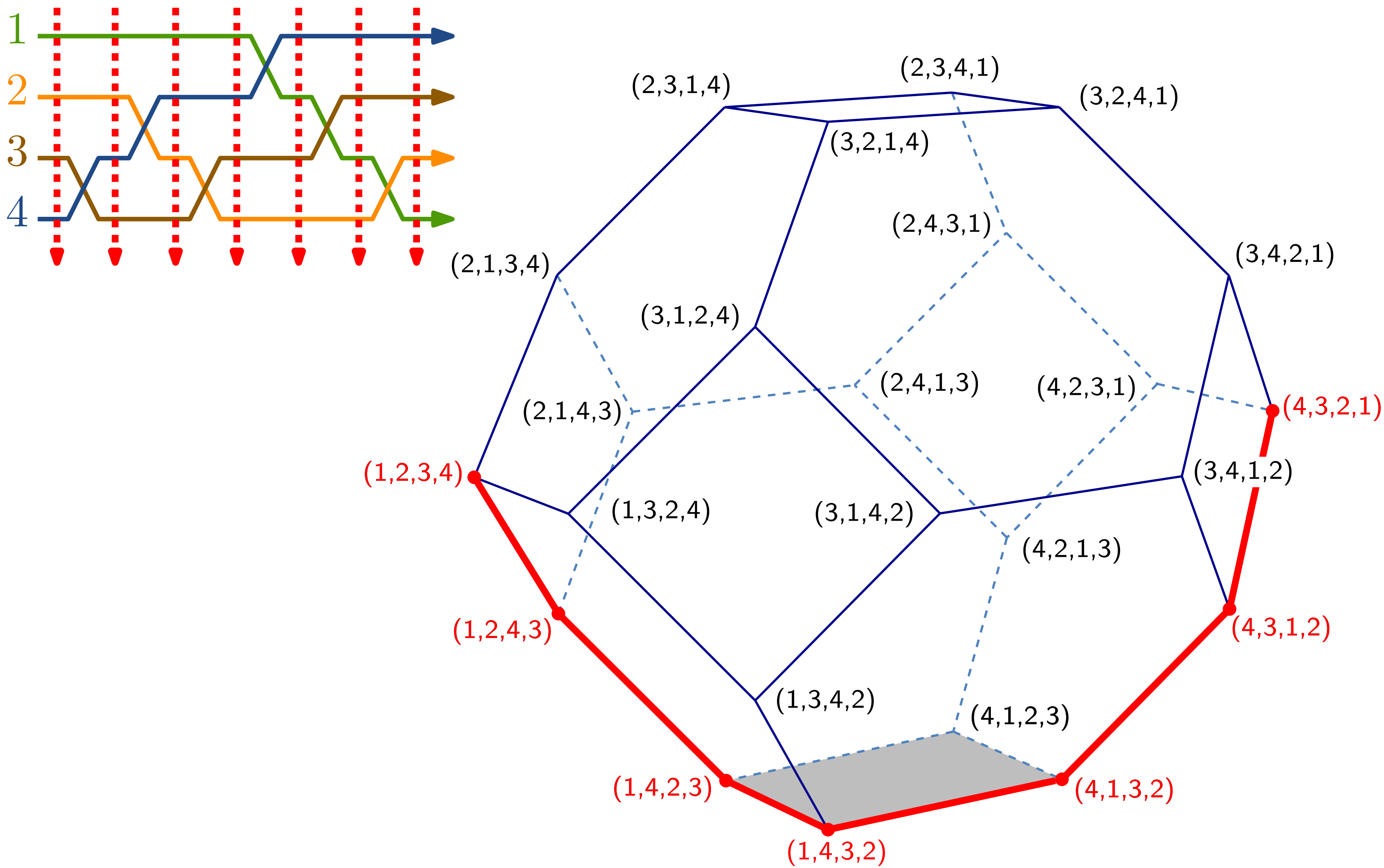
are called *directly equivalent*, if $|s_i - s_{i+1}| \geq 2$.

- Wiring diagrams S and S' are *equivalent* ($S \sim S'$), if there exist $S = S_1, \cdots, S_r = S'$, where S_i and S_{i+1} are directly equivalent.

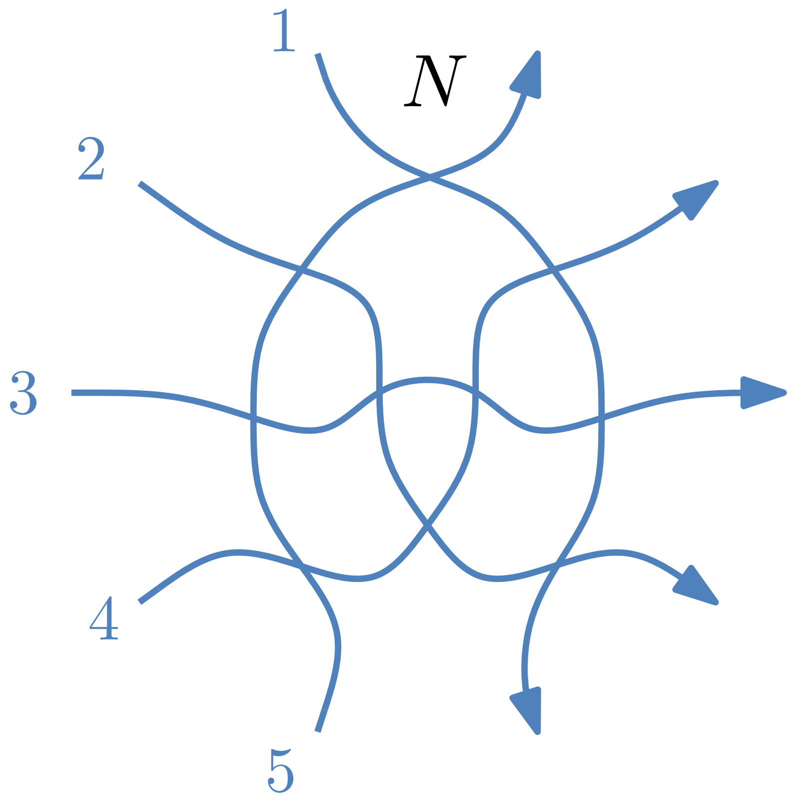
classes of wiring diagrams



classes of wiring diagrams



classes of wiring diagrams



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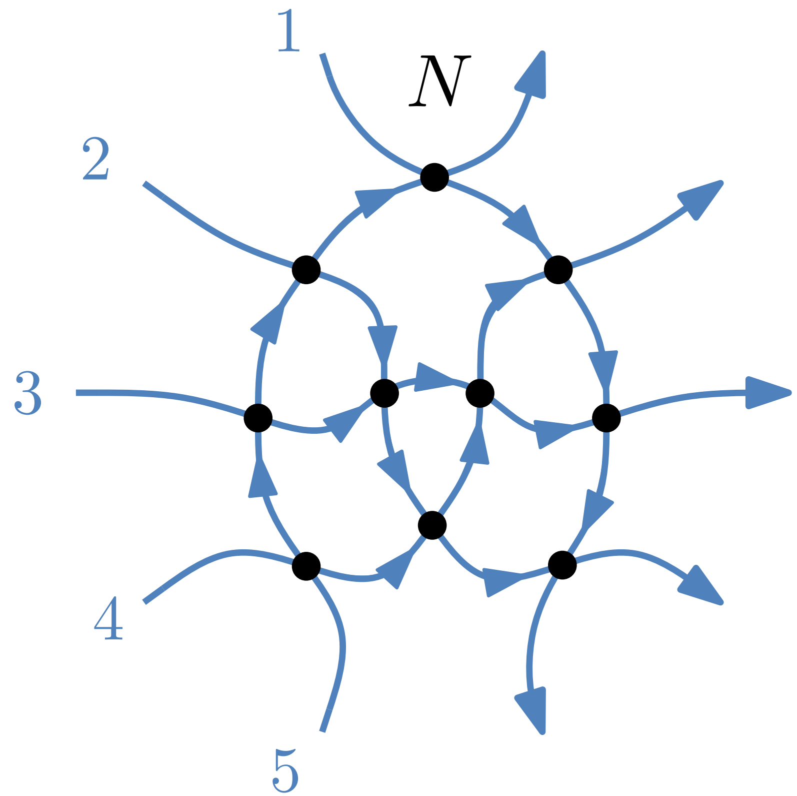

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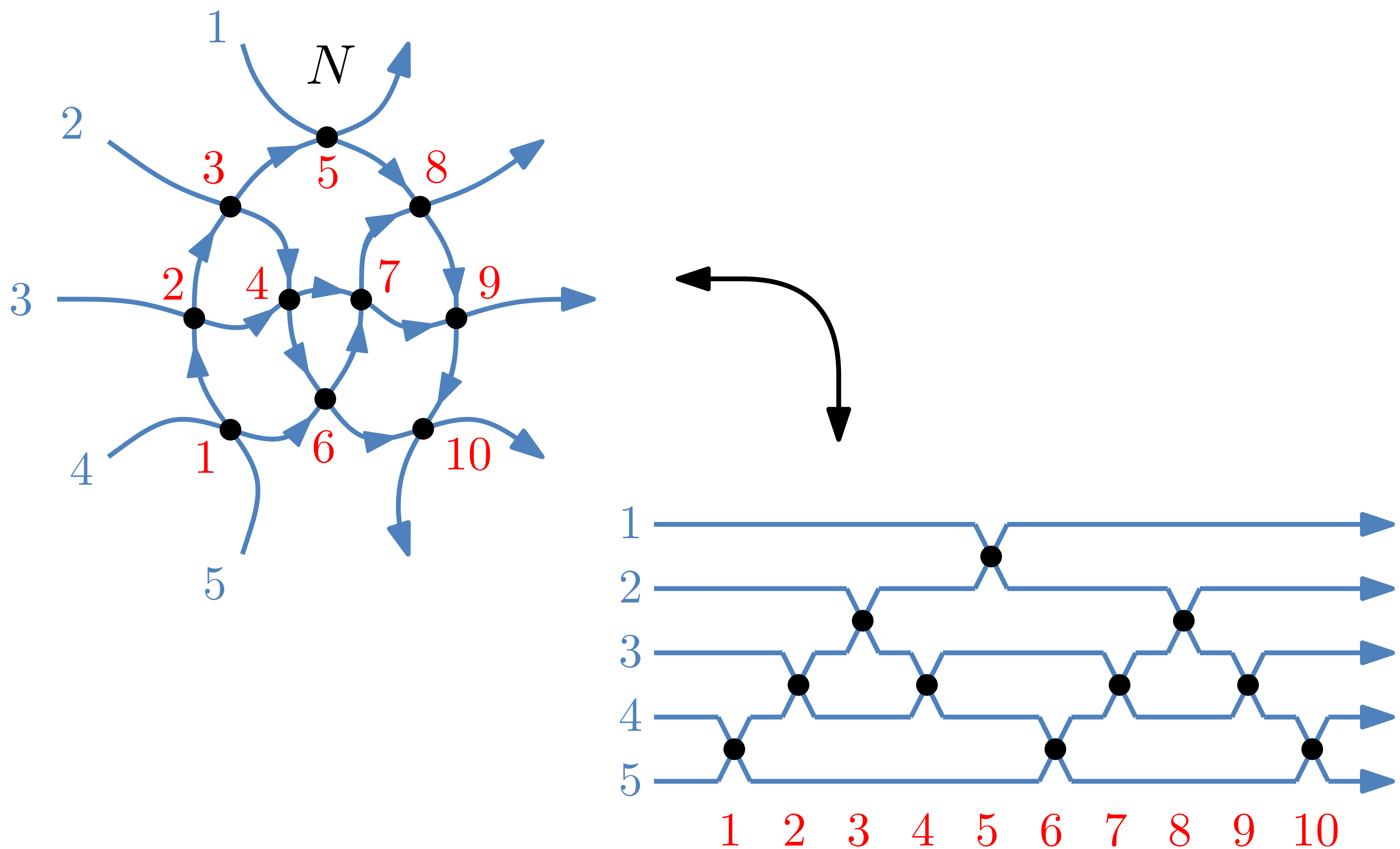
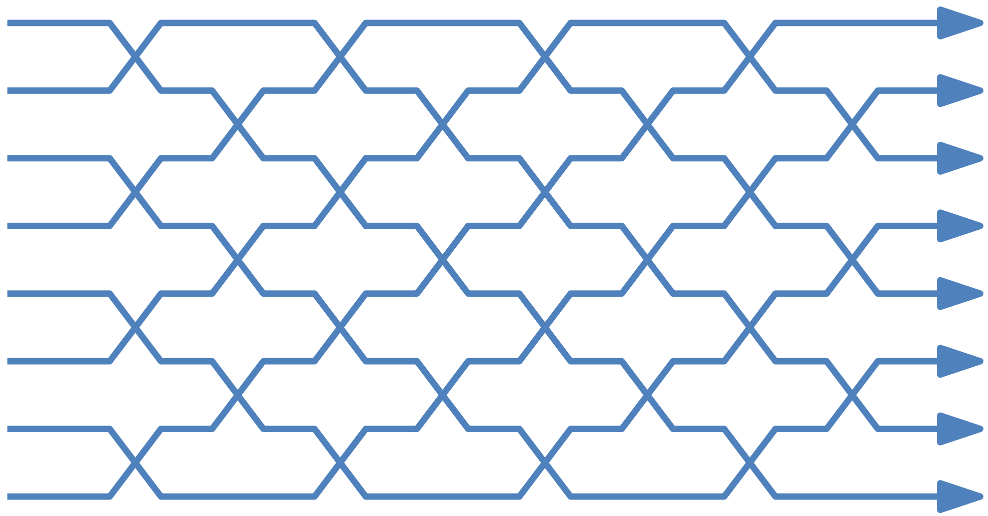


„brick wall conjecture“

Conjecture (Gutierres, Mamede, Santos, 2020)

The *wall arrangements* are the arrangements that maximize the number of corresponding wiring diagrams.



wall arrangement of 8 pseudolines

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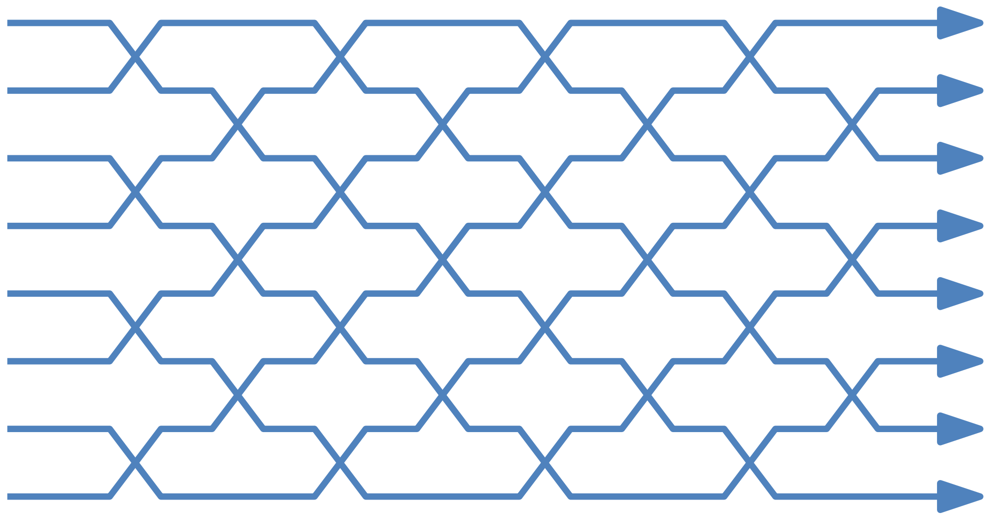

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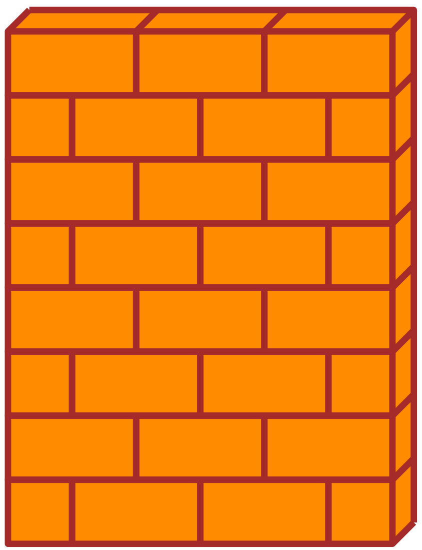

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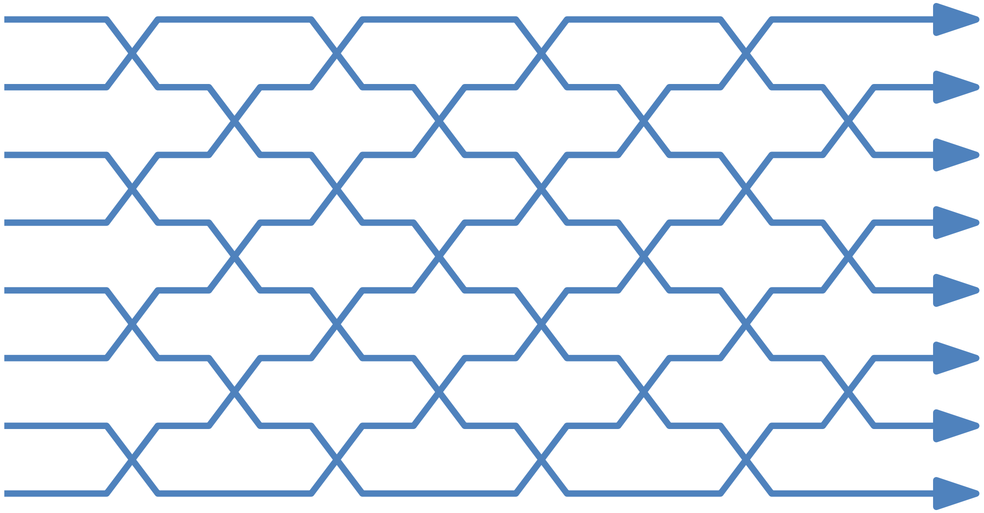

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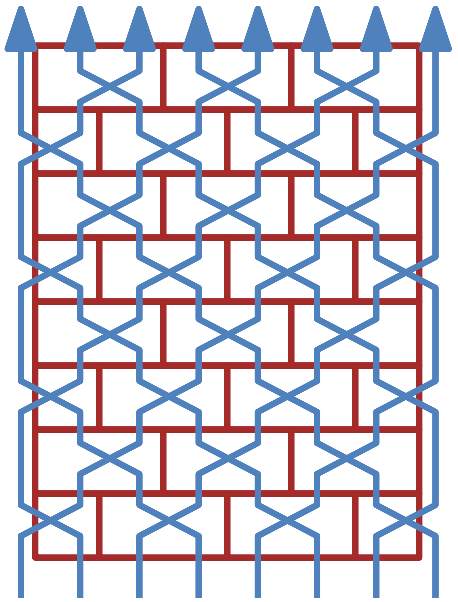
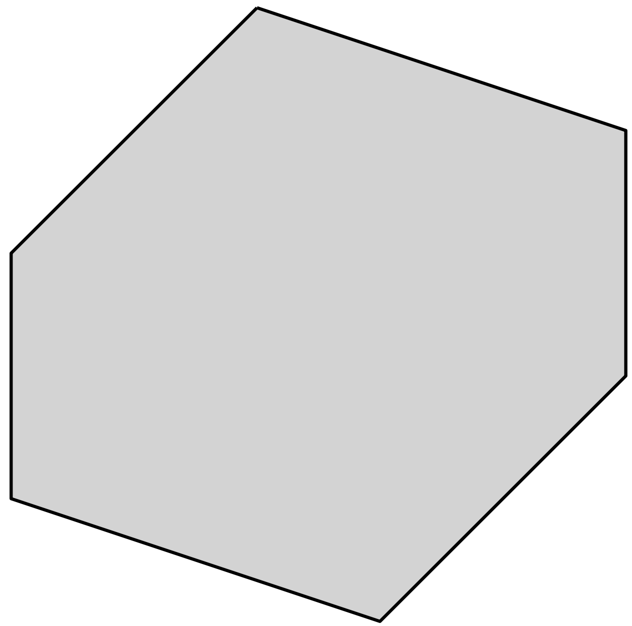
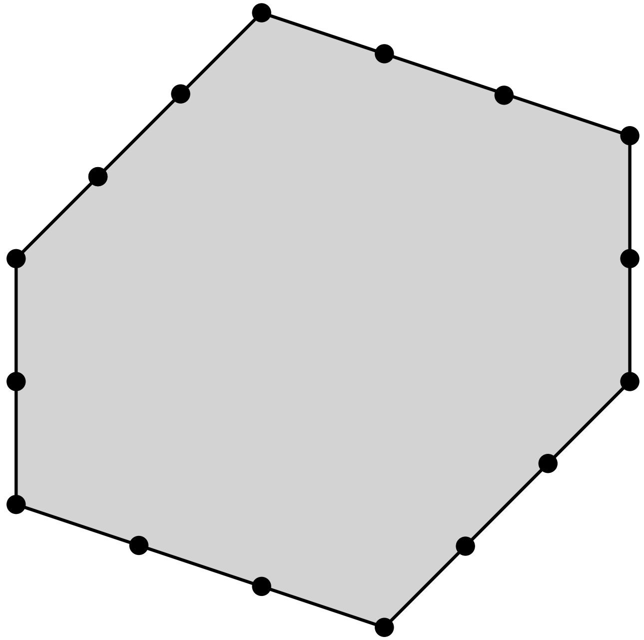


rhombic tilings



rhombic tilings

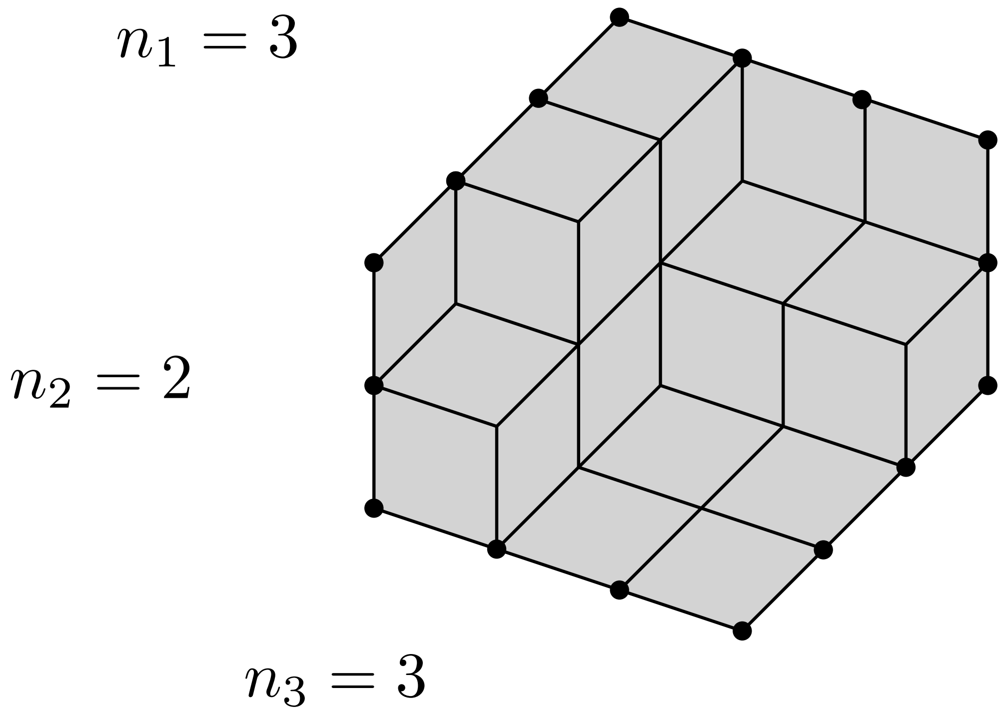
$n_1 = 3$



$n_2 = 2$

$n_3 = 3$

rhombic tilings



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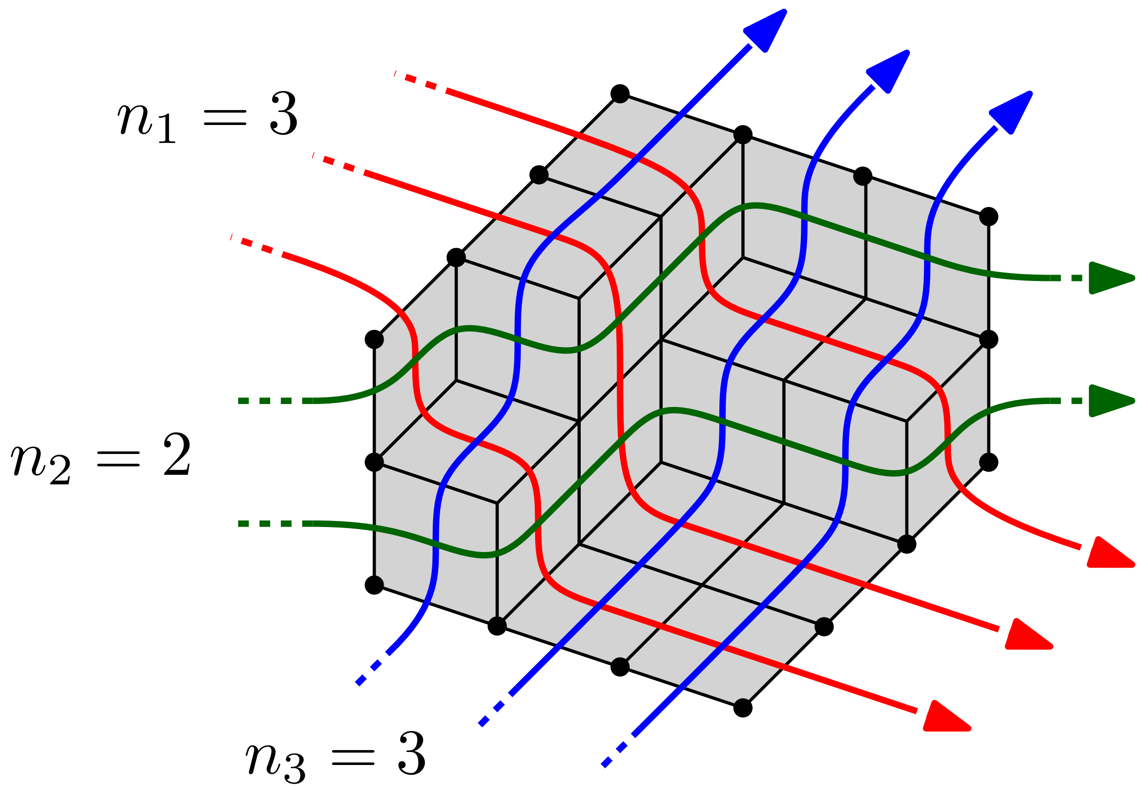

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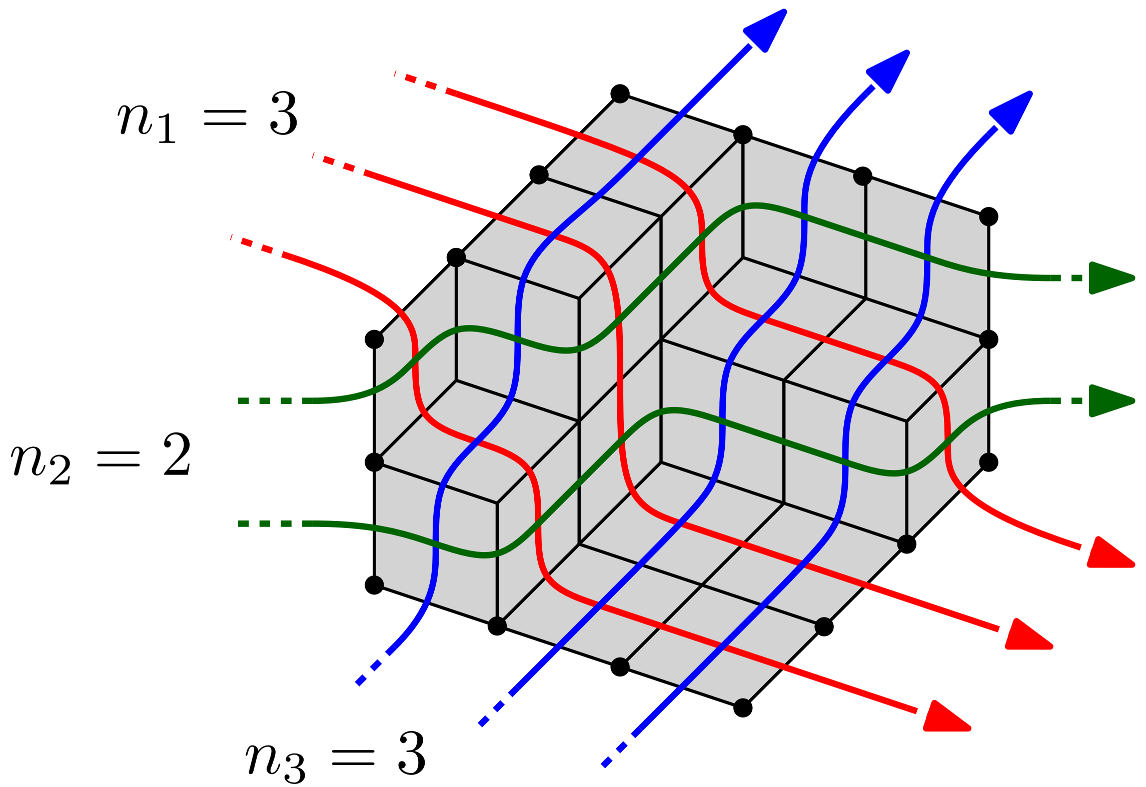
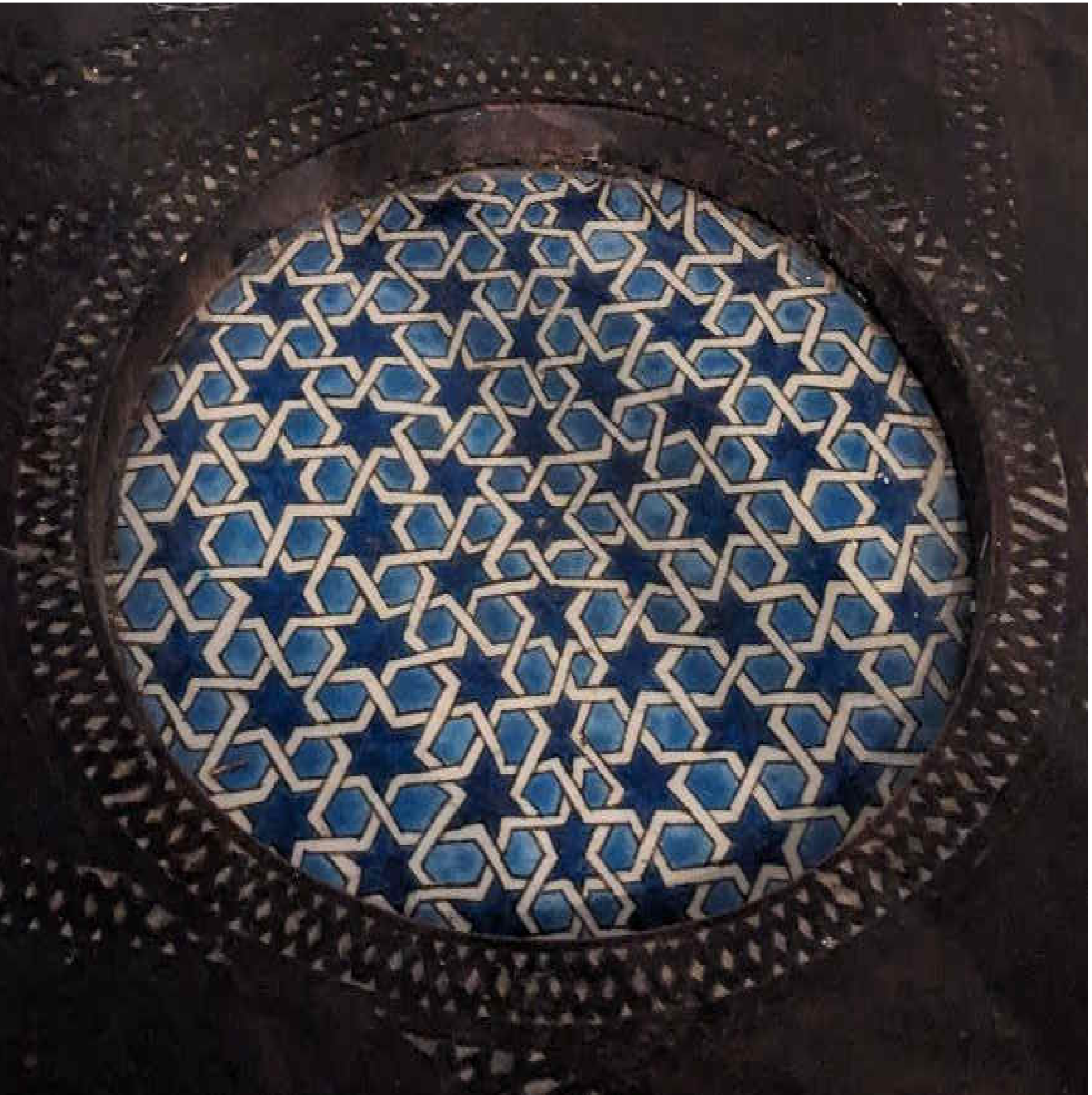
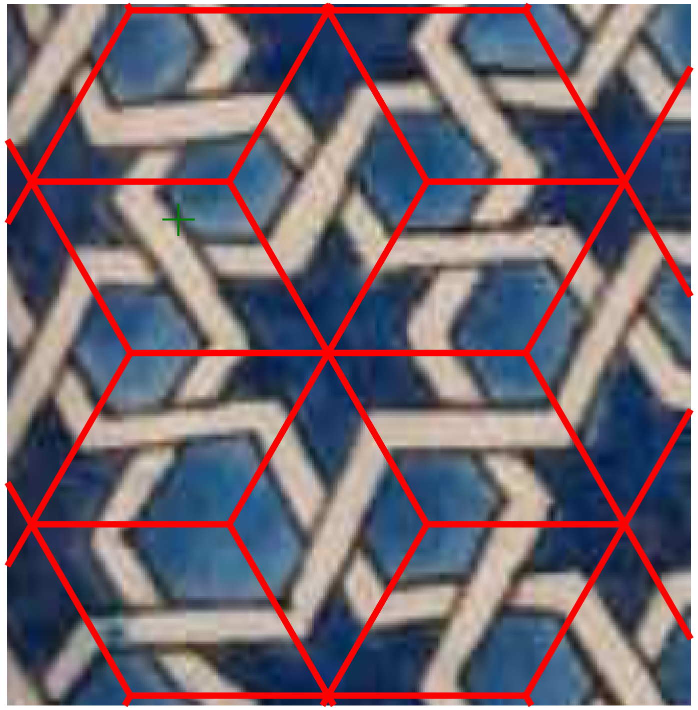


$\Rightarrow$ *generalized pseudoline arrangement*:

- *parallel class* of $n_1, \ldots, n_r$ pseudolines
- (Only) pseudolines of different classes cross

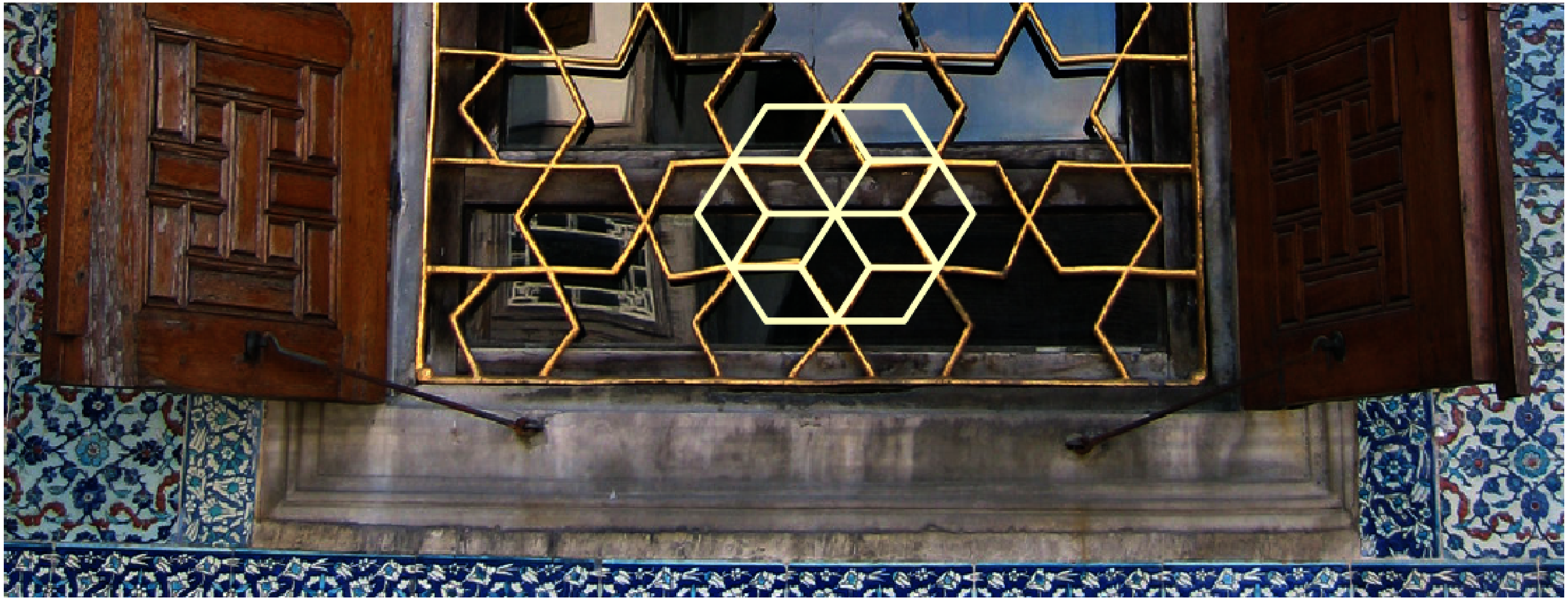
rhombic tilings





Aslan Pasha Mosque
Ioannina, Greece

rhombic tilings



Topkapı Palace, Istanbul, Turkey

plane partitions

plane partitions

Def: matrix $[h_{i,j}] \in {\mathbb{N}_0}^{r \times s}$ is called *plane partition*, if rows and columns are monotonically increasing.

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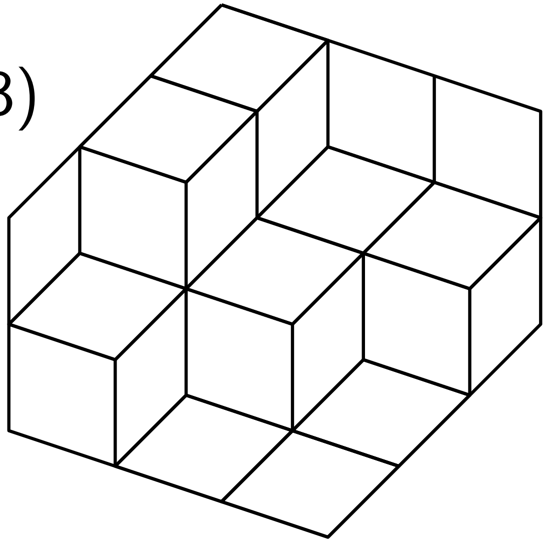
rhombic tiling of size (3, 2, 3)



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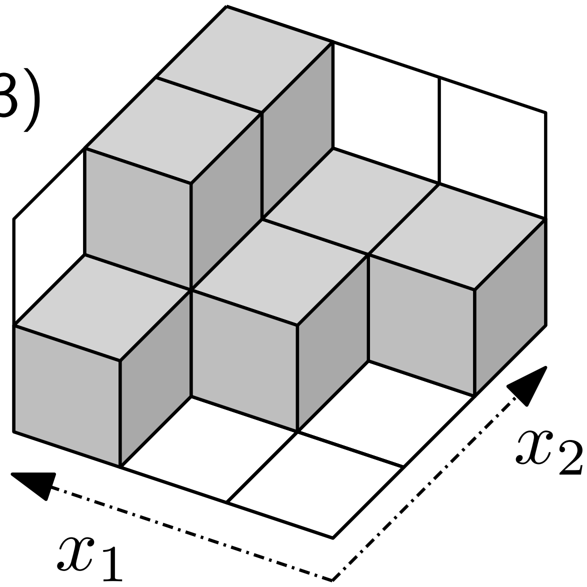

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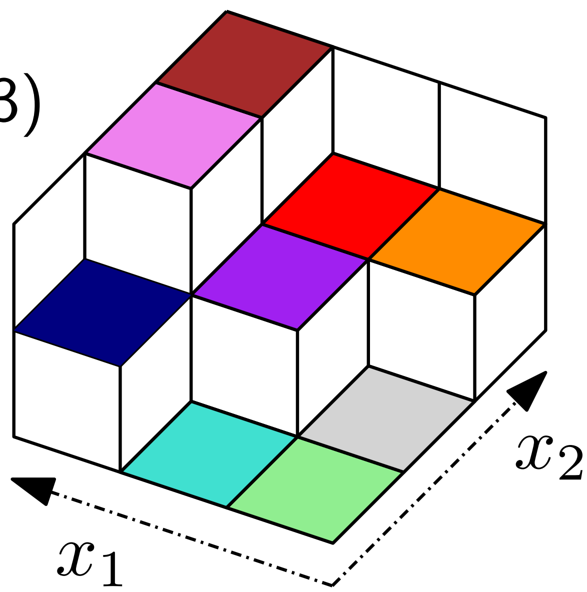

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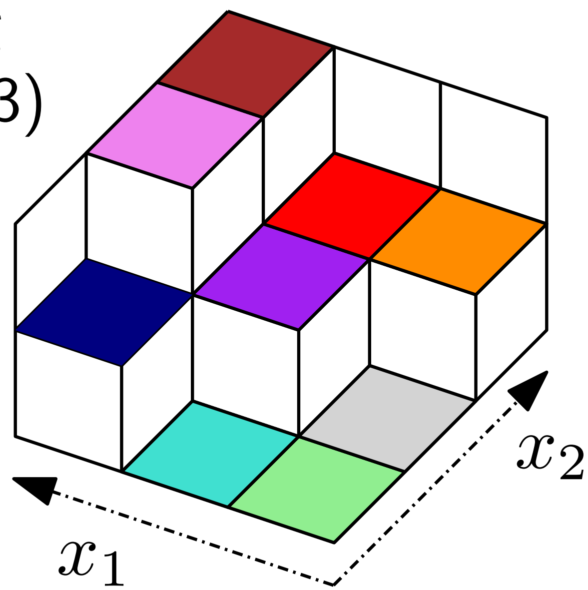

$$\begin{array}{cc} & x_2 \\ x_1 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \end{array}$$

plane partition with entries $h_{i,j} \leq 2$

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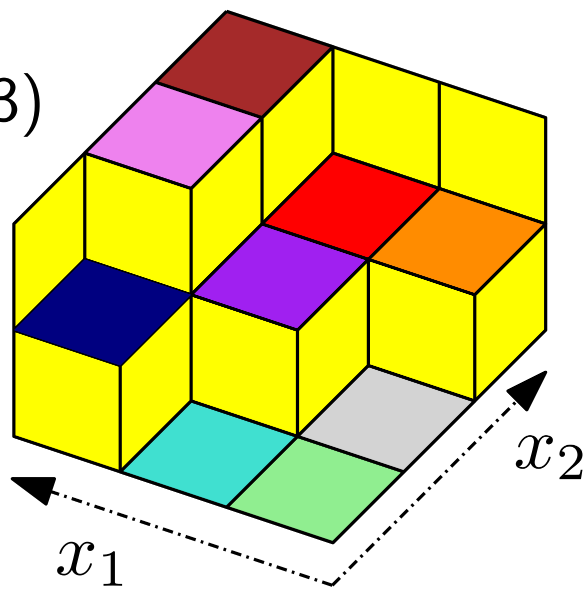

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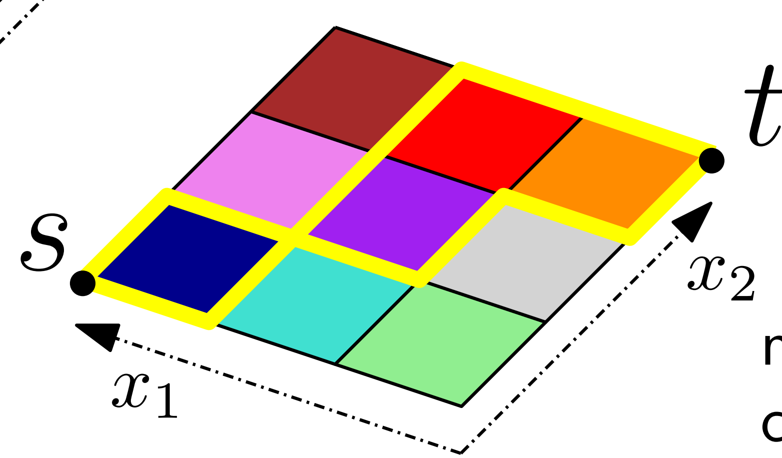

monotonic, non-crossing grid paths

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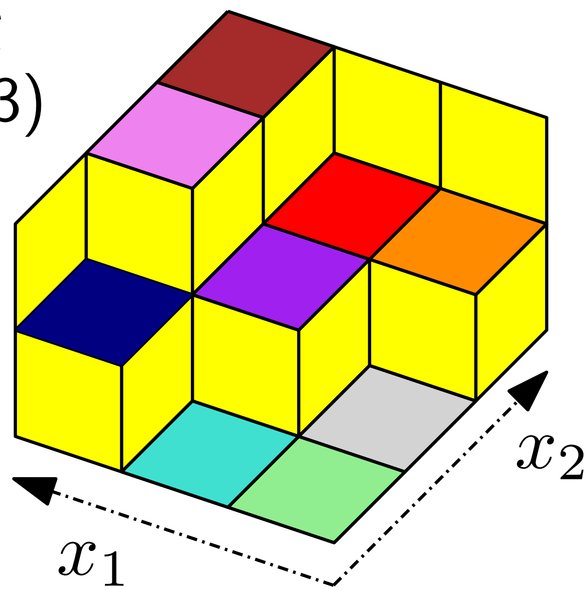
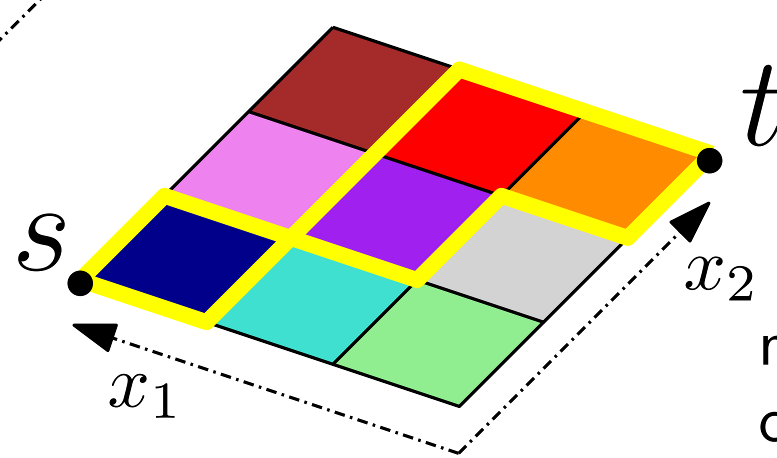


$$\begin{array}{cc} & \xrightarrow{\quad x_2 \quad} \\ x_1 \big\downarrow & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \end{array}$$

plane partition
with entries
$h_{i,j} \leq 2$

min $\sum f_{i,j}(A_{i,j})$
s.t. A p.p., $A_{i,j} \leq h$



monotonic, non-
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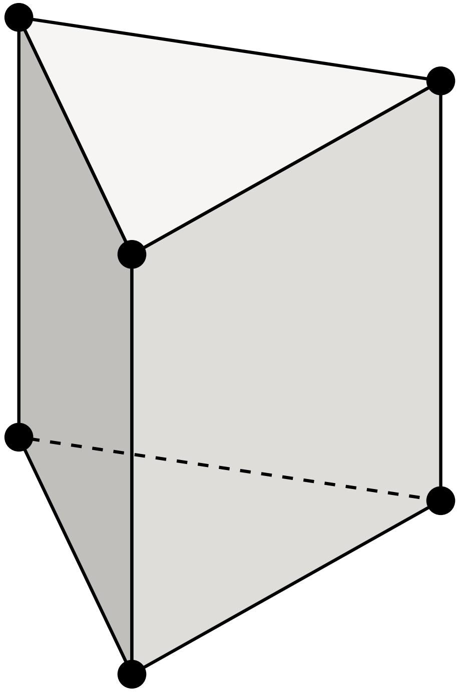
linear orientations



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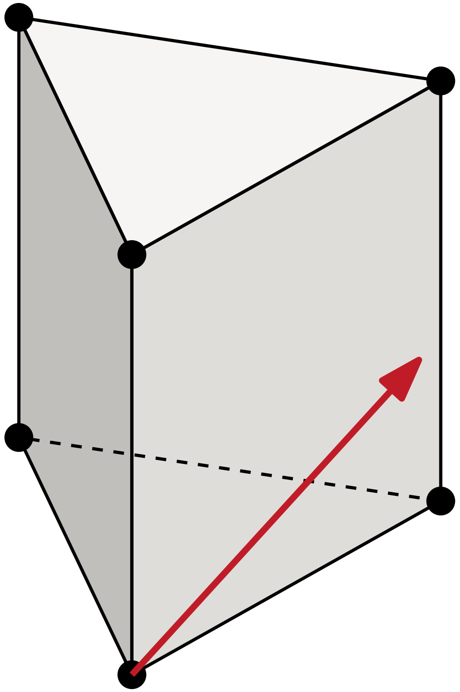
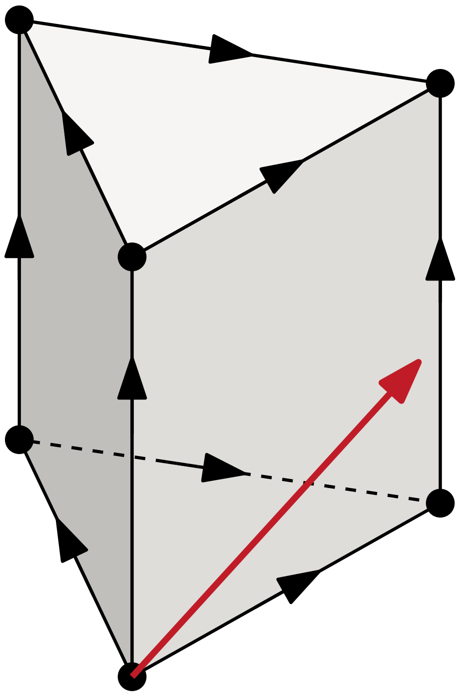


linear orientations



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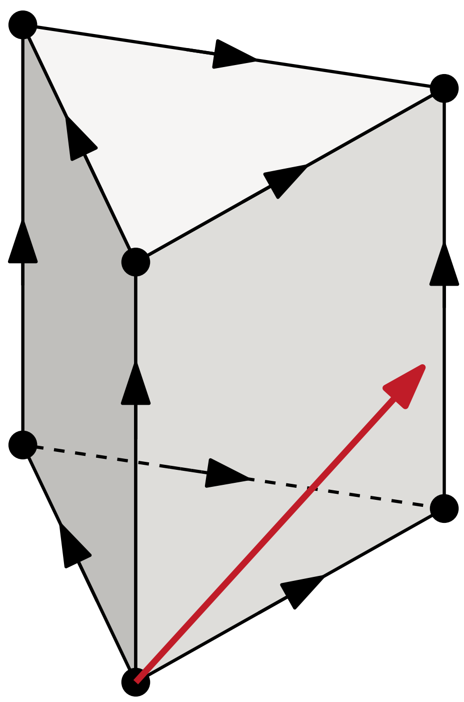
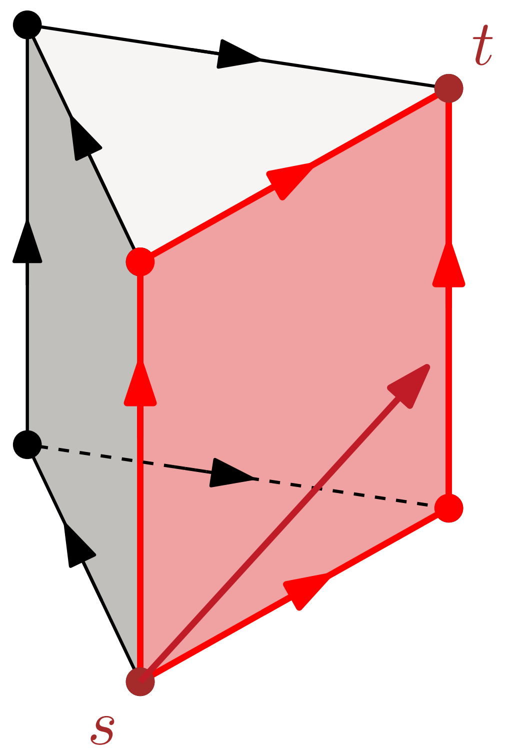


Properties of *linear orientations*:

- acyclic
- Every face contains a unique source and a unique sink (*unique sink orientation USO*).
- *Holt-Klee-property:* On every face F, there exist $\dim F$ many internally disjoint paths from unique source to unique sink.

linear orientations



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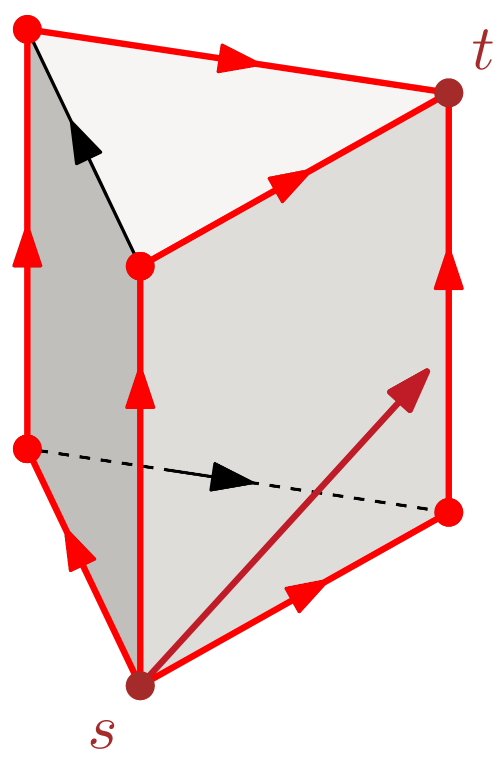

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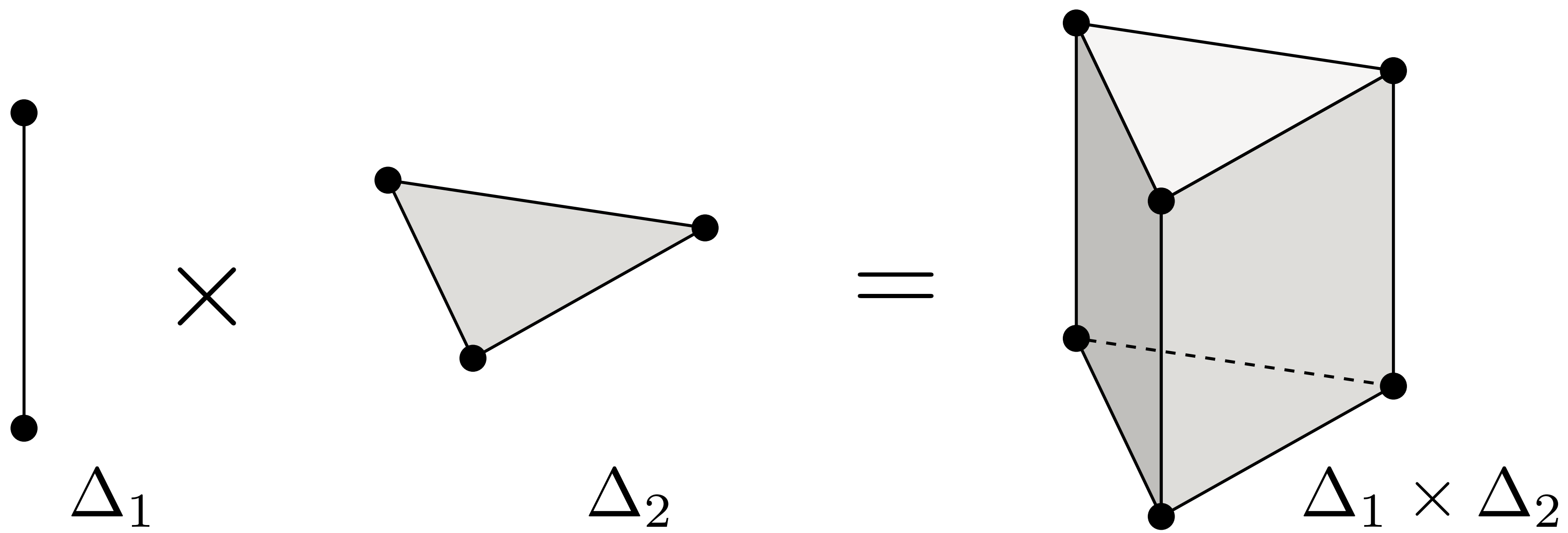
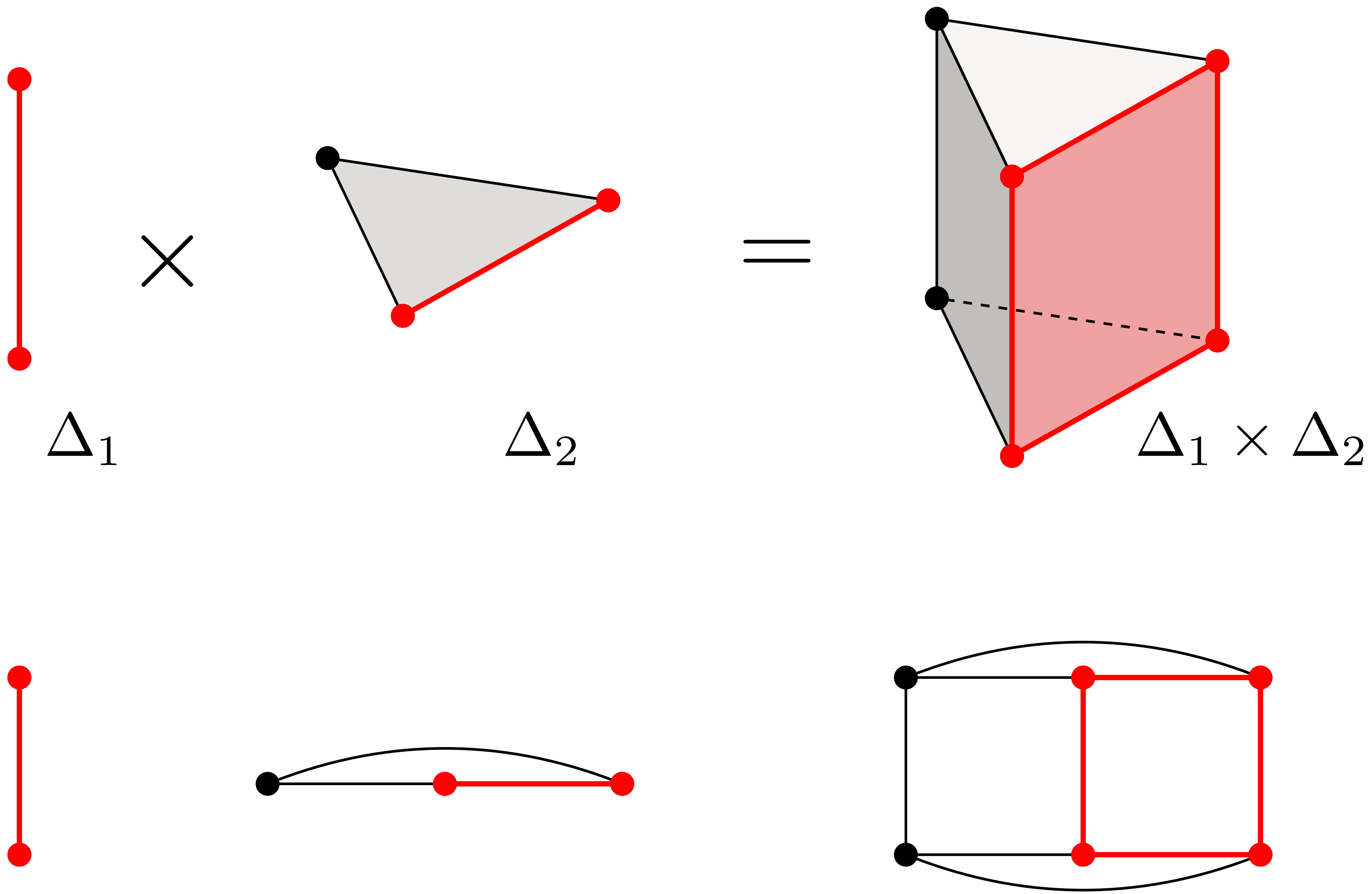
$(d, d+2)$-polytopes

Fact:
(Felsner, Gärtner, Tschirschnitz, 2005)

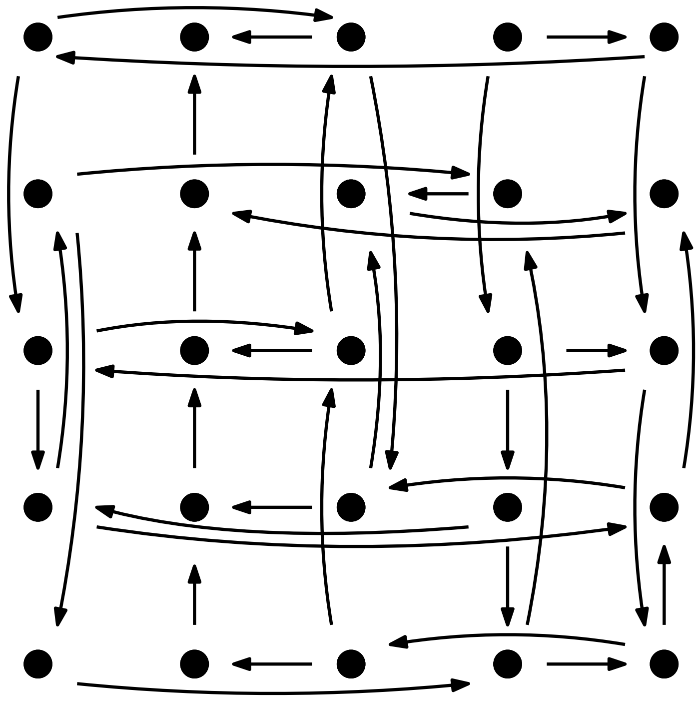
Every simple d-polytope with $d+2$ facets is combinatorially equivalent to a product of two simplices.



$(d, d+2)$-polytopes



unique sink orientations (USOs)



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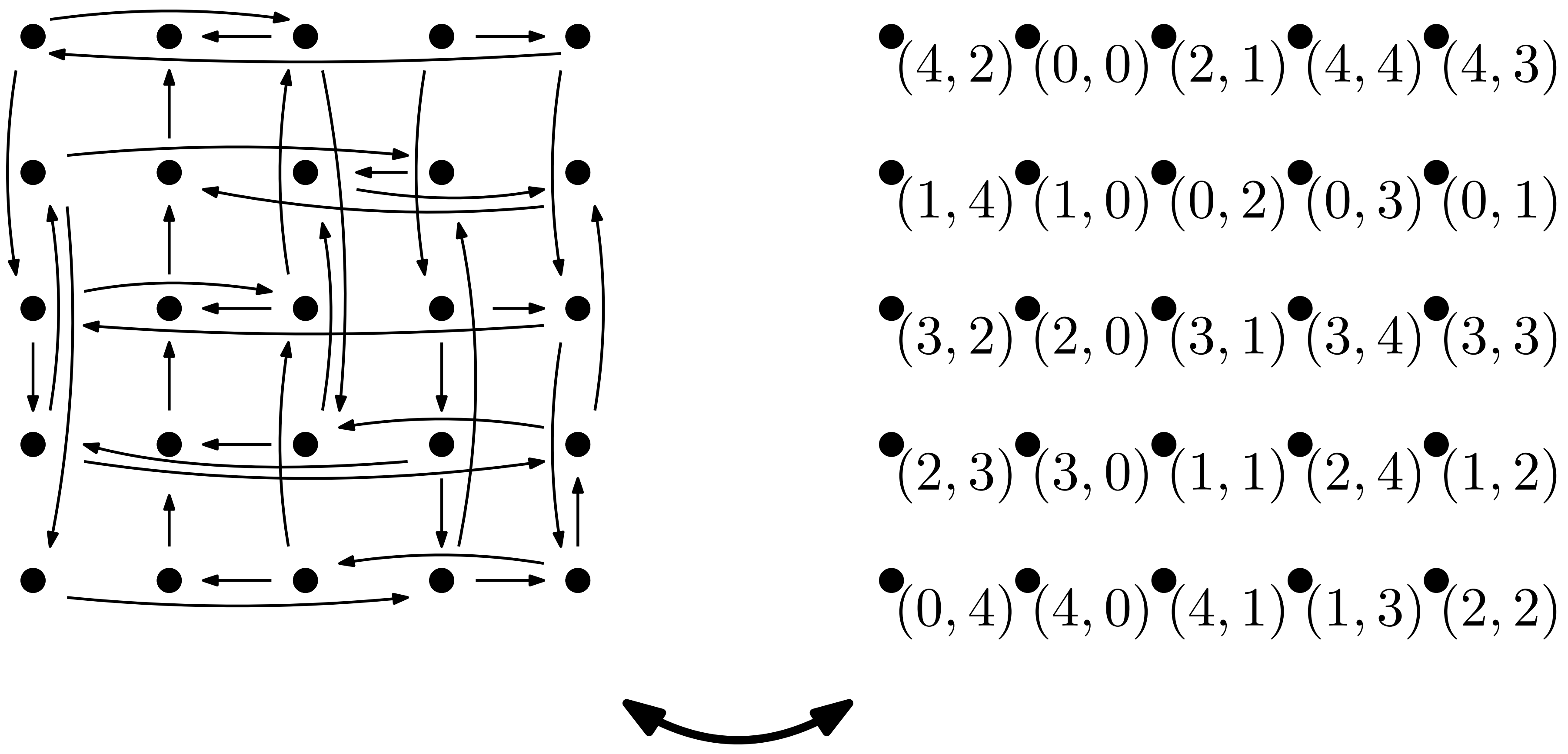

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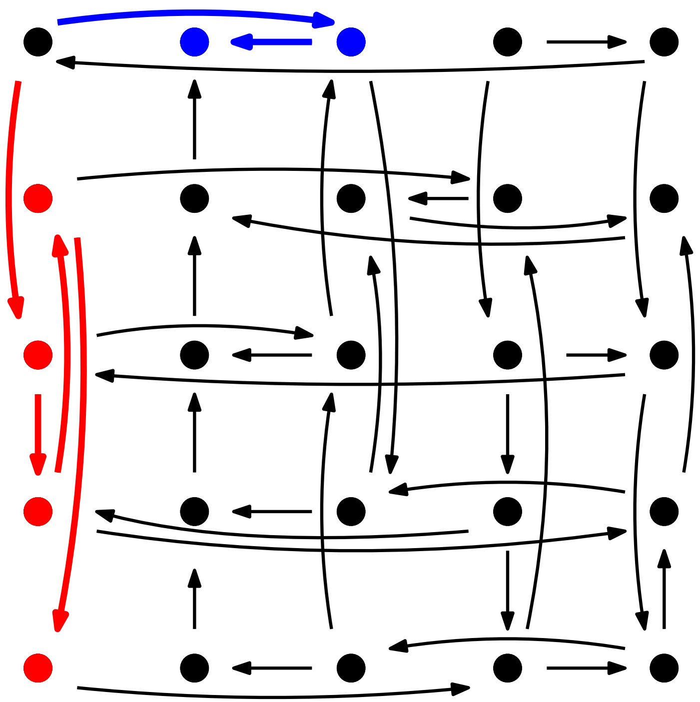
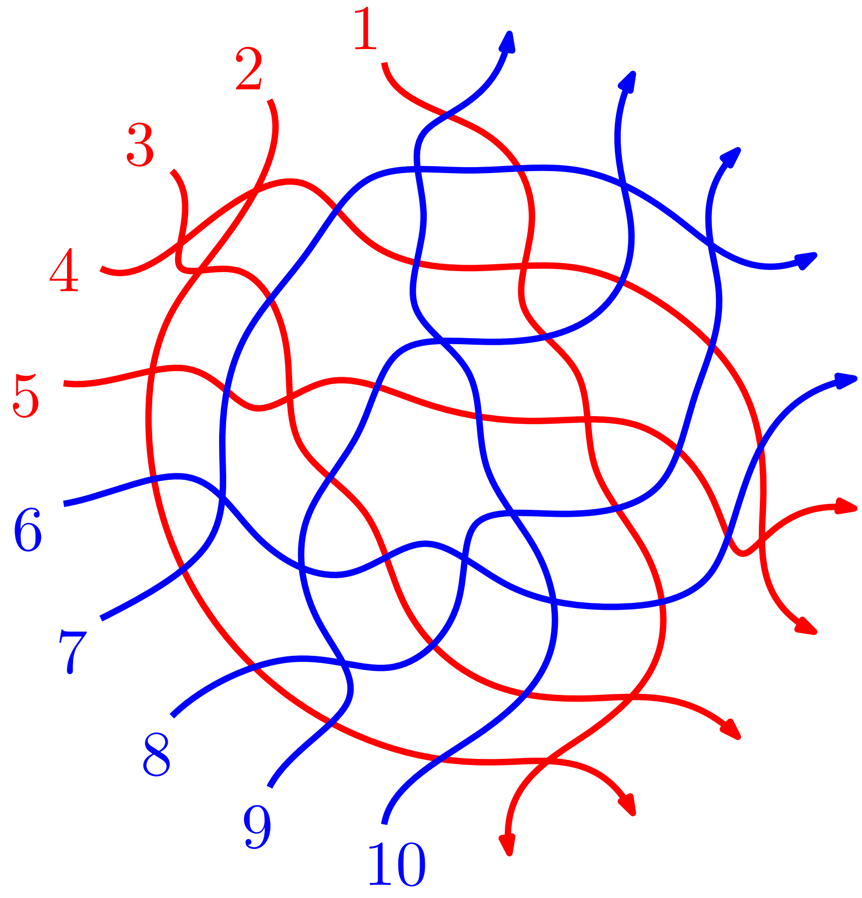


$$\mathrm{rf}(i,j) := (\textcolor{red}{\mathrm{outdeg_v}(i,j)}, \textcolor{blue}{\mathrm{outdeg_h}(i,j)})$$

$(\textcolor{red}{4}, \textcolor{blue}{2})$	$(0,0)$	$(2,1)$	$(4,4)$	$(4,3)$
$(1,4)$	$(1,0)$	$(0,2)$	$(0,3)$	$(0,1)$
$(3,2)$	$(2,0)$	$(3,1)$	$(3,4)$	$(3,3)$
$(2,3)$	$(3,0)$	$(1,1)$	$(2,4)$	$(1,2)$
$(0,4)$	$(4,0)$	$(4,1)$	$(1,3)$	$(2,2)$

red-blue-arrangement $\to$ USO

red-blue-arrangement → USO



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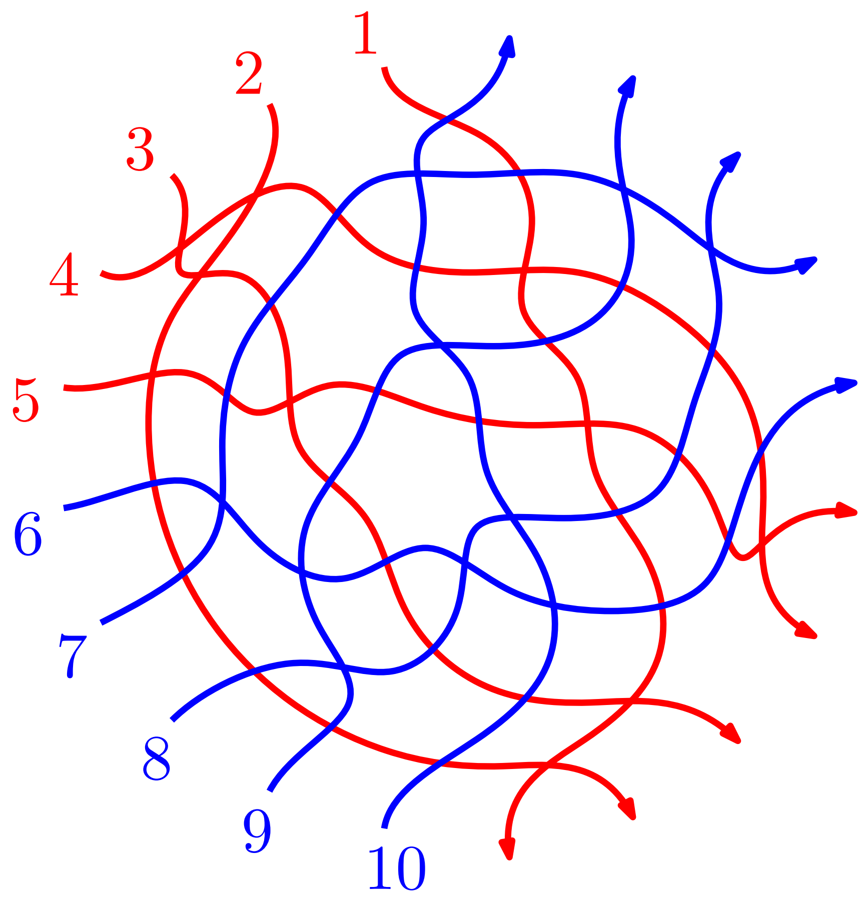

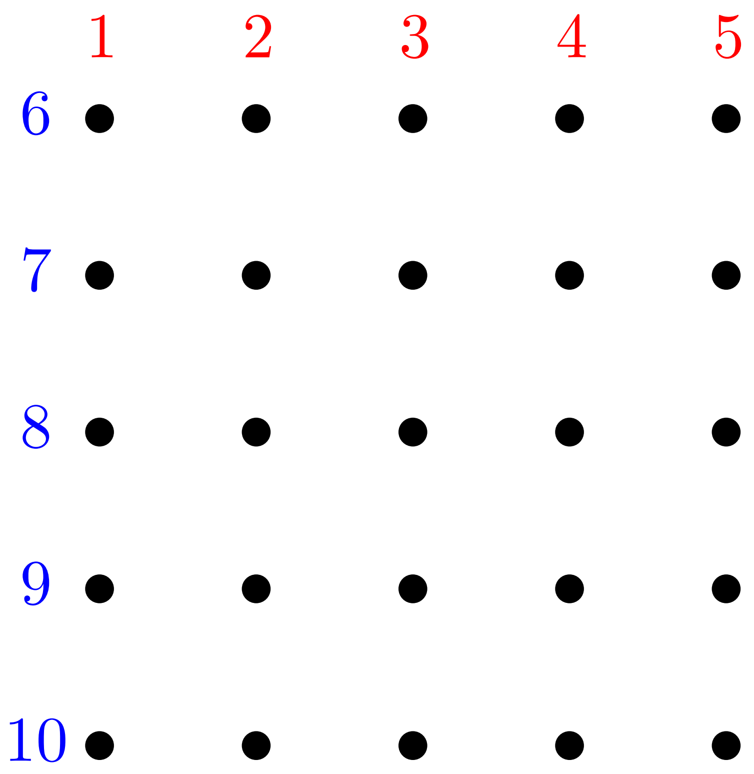

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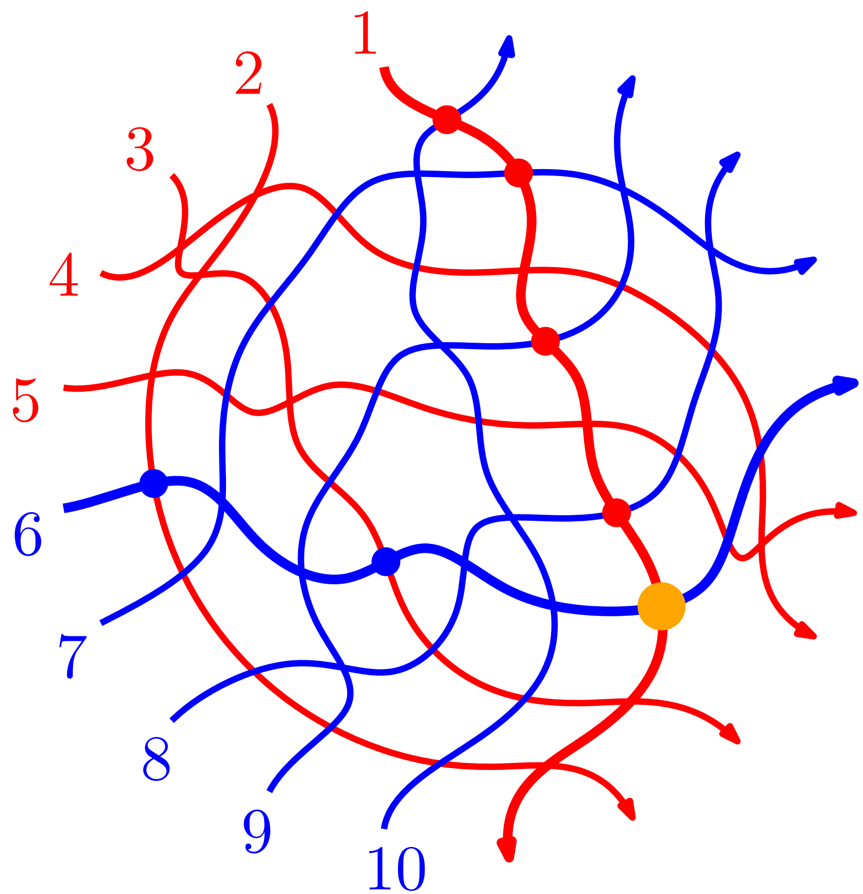

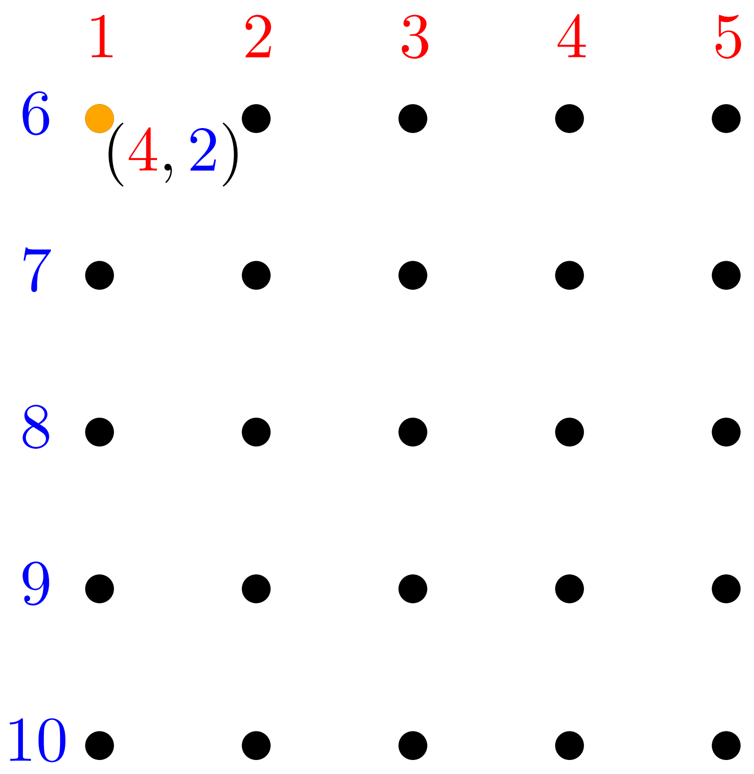

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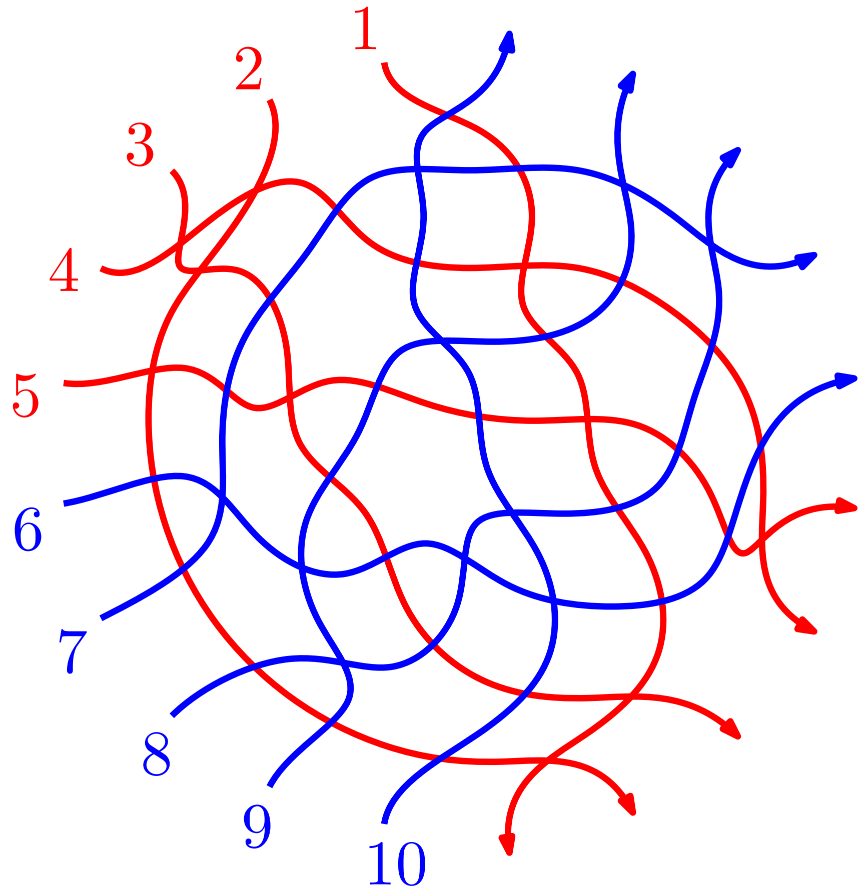
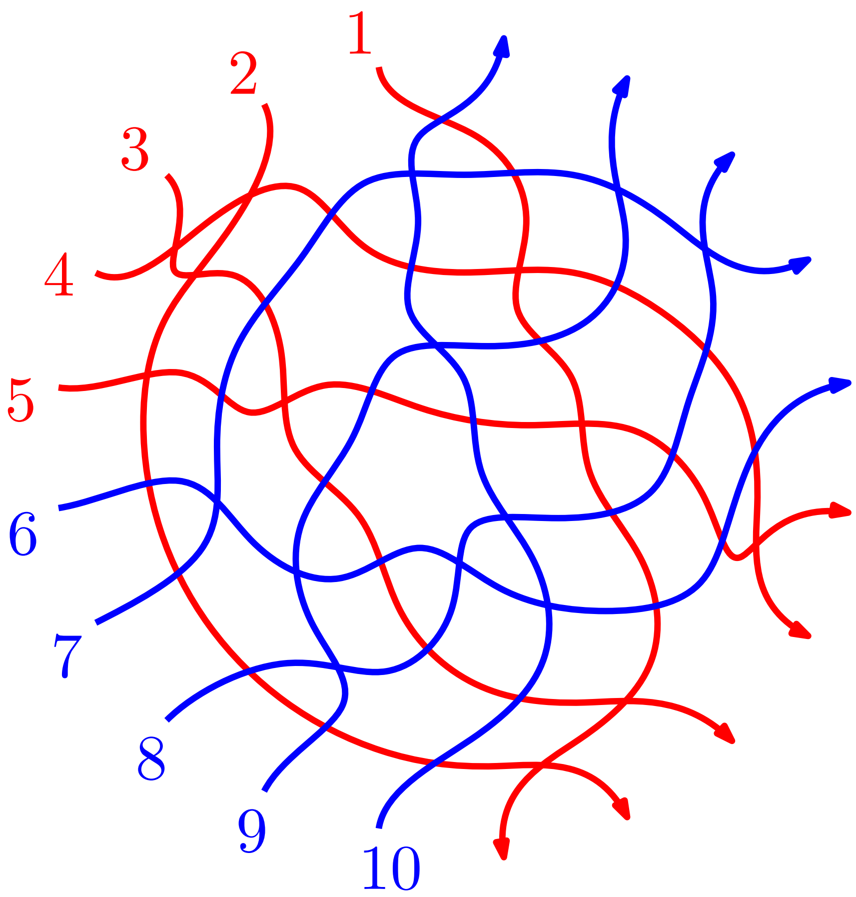
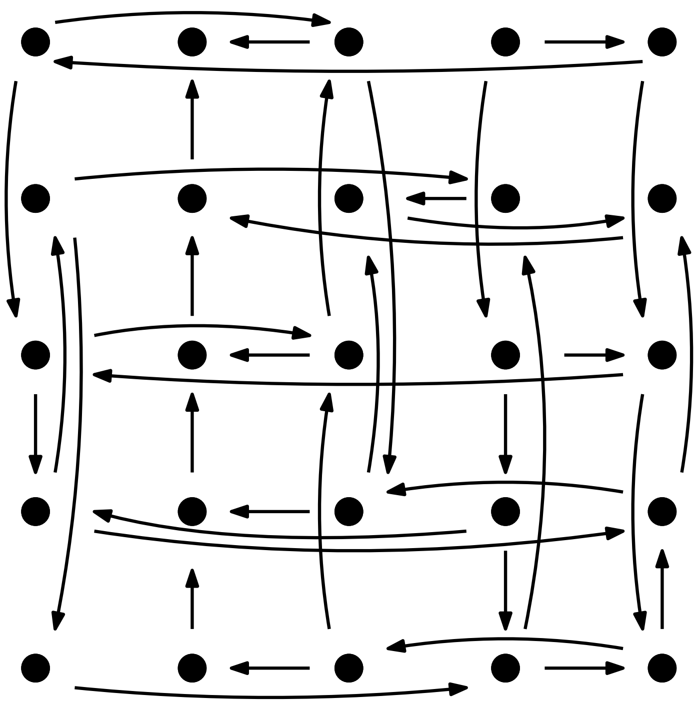


	1	2	3	4	5
6	$(4,2)$	$(0,0)$	$(2,1)$	$(4,4)$	$(4,3)$
7	$(1,4)$	$(1,0)$	$(0,2)$	$(0,3)$	$(0,1)$
8	$(3,2)$	$(2,0)$	$(3,1)$	$(3,4)$	$(3,3)$
9	$(2,3)$	$(3,0)$	$(1,1)$	$(2,4)$	$(1,2)$
10	$(0,4)$	$(4,0)$	$(4,1)$	$(1,3)$	$(2,2)$

red-blue-arrangement $\rightarrow$ USO





USO $\rightarrow$ red-blue-arrangement

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$(4,2)$ $(0,0)$ $(2,1)$ $(4,4)$ $(4,3)$

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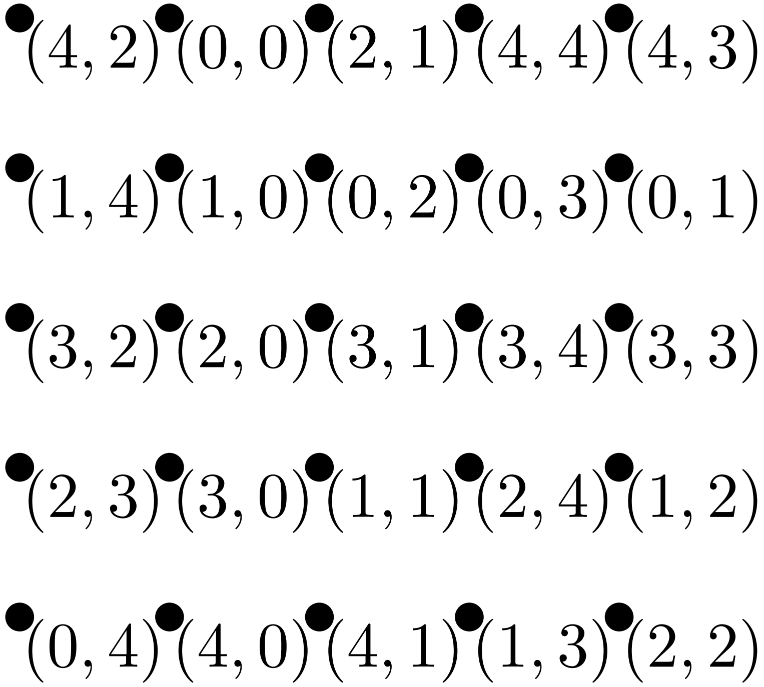

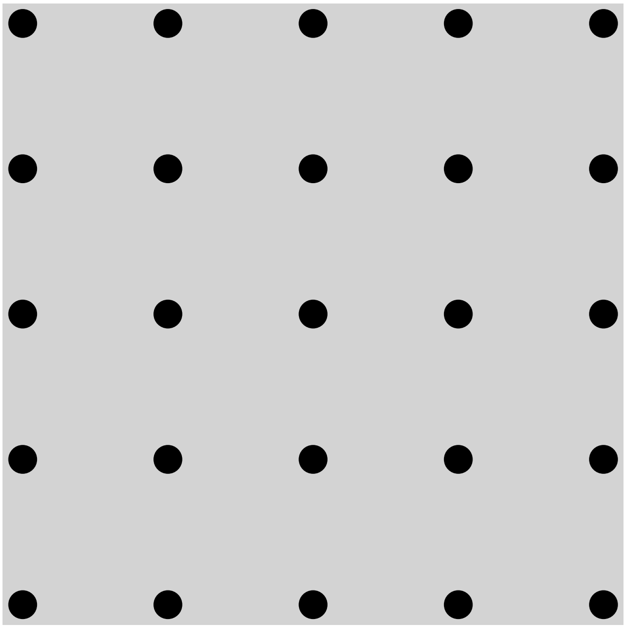

USO $\to$ red-blue-arrangement

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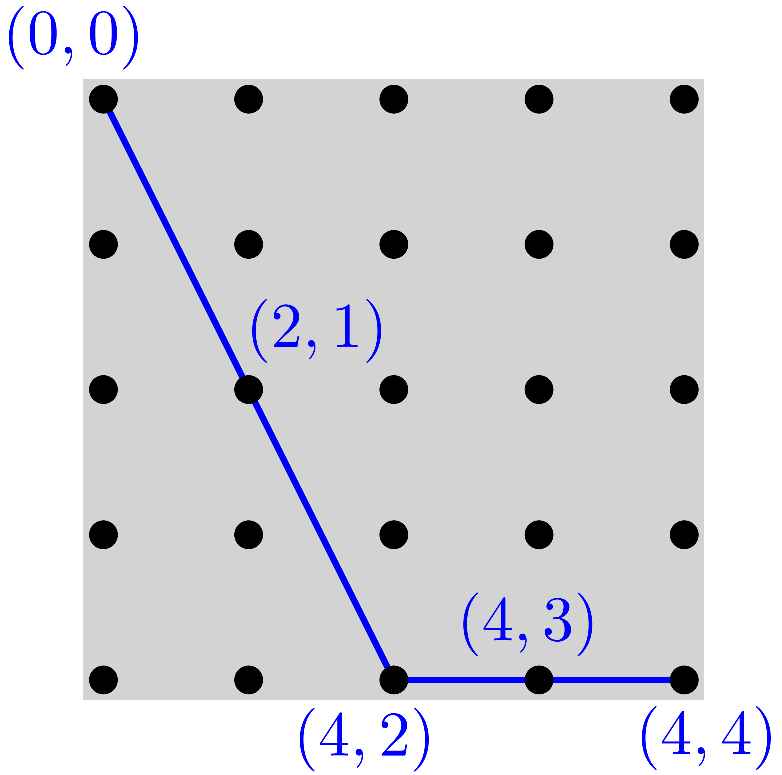

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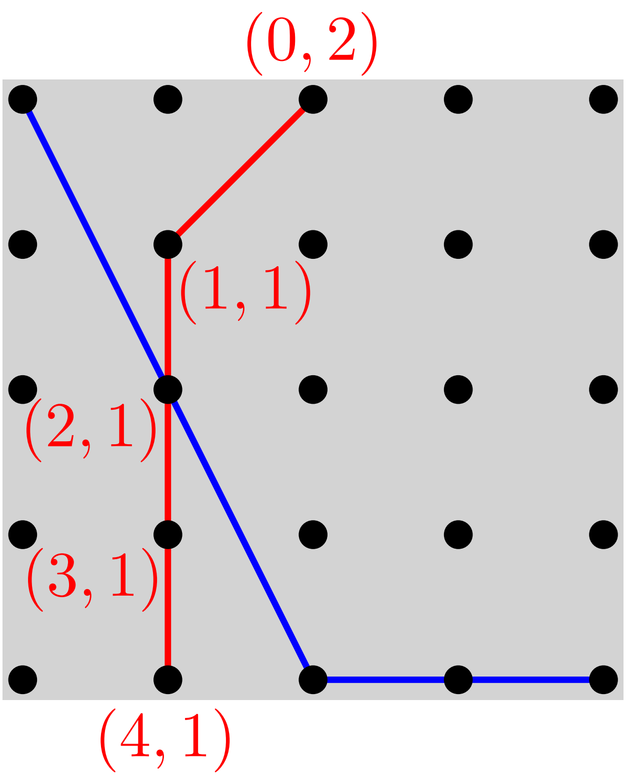

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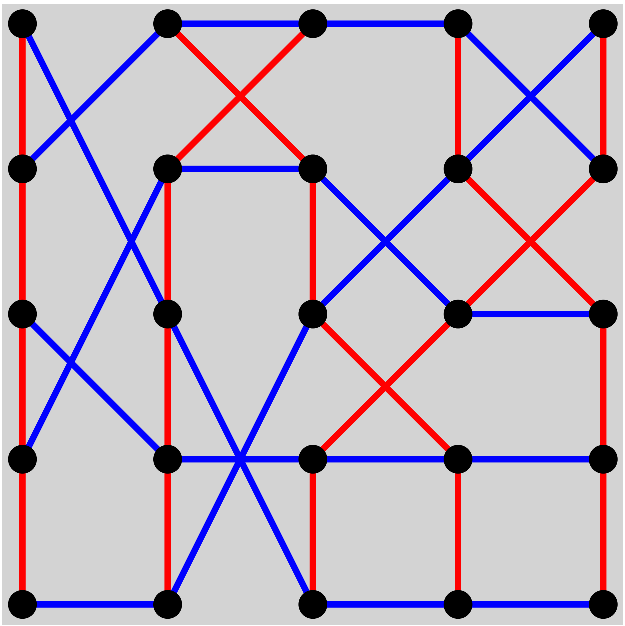

USO $\rightarrow$ red-blue-arrangement

$(4,2)\ (0,0)\ (2,1)\ (4,4)\ (4,3)$

$(1,4)\ (1,0)\ (0,2)\ (0,3)\ (0,1)$

$(3,2)\ (2,0)\ (3,1)\ (3,4)\ (3,3)$

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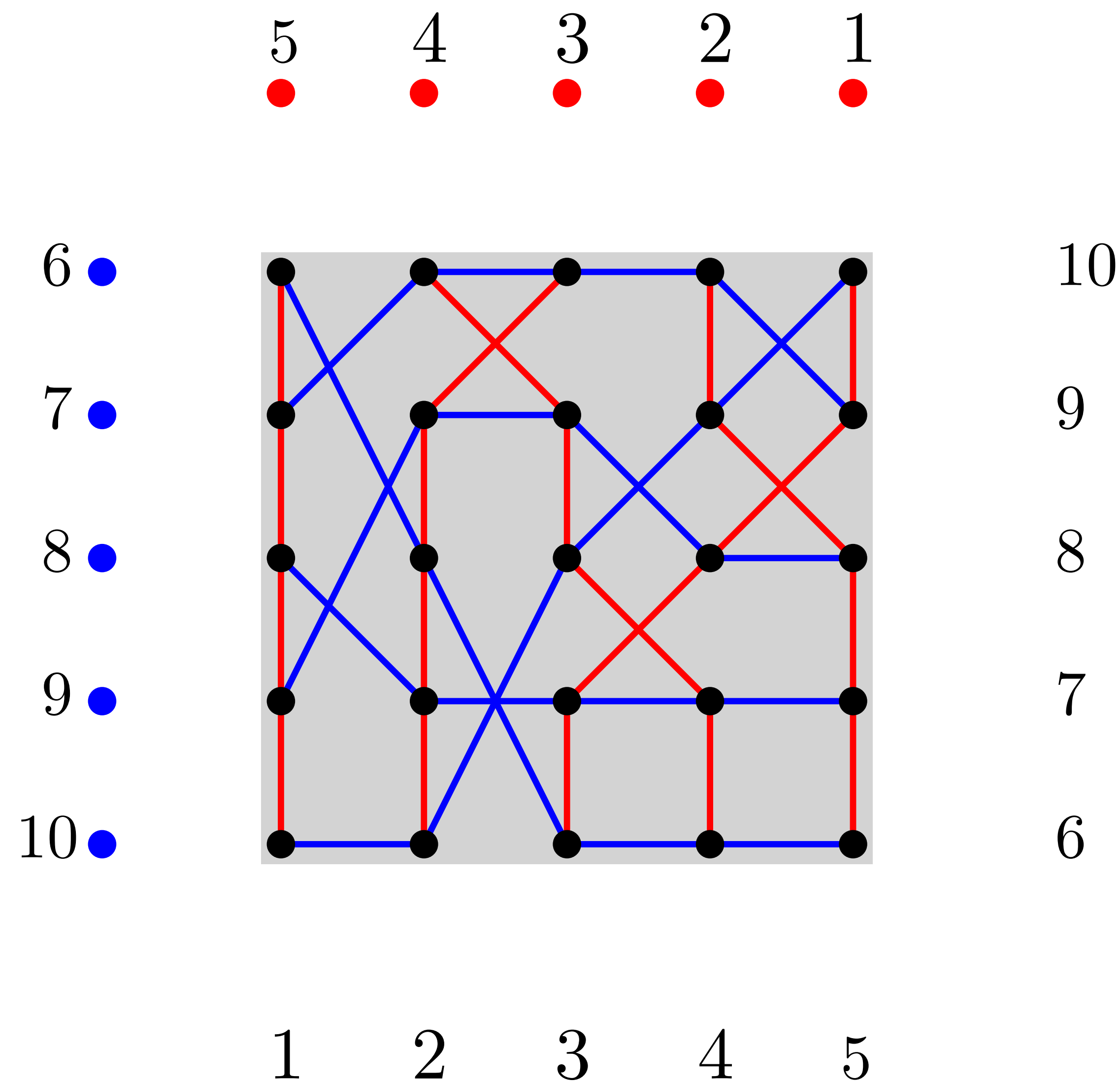

USO $\rightarrow$ red-blue-arrangement

$(4,2)$	$(0,0)$	$(2,1)$	$(4,4)$	$(4,3)$
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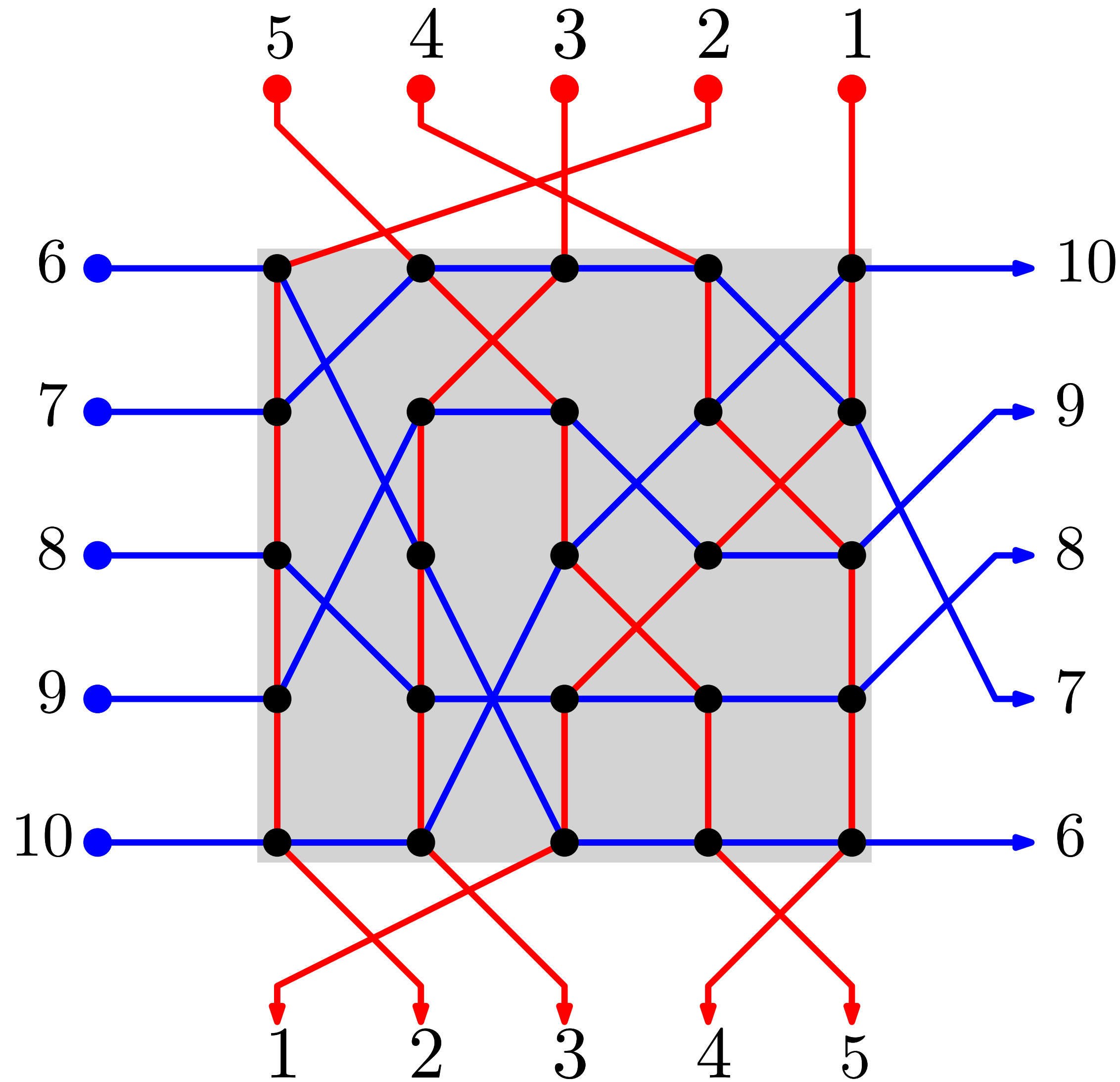

USOs with Holt-Klee $\leftrightarrow$ red-blue-arrangements

Theorem
(Felsner, Gärtner, Tschirschnitz, 2005)

1) USOs with Holt-Klee-property are exactly the orientations induced by red-blue-arrangements.
2) They are linear orientations if and only if the arrangement is *stretchable*.

USOs with Holt-Klee $\leftrightarrow$ red-blue-arrangements

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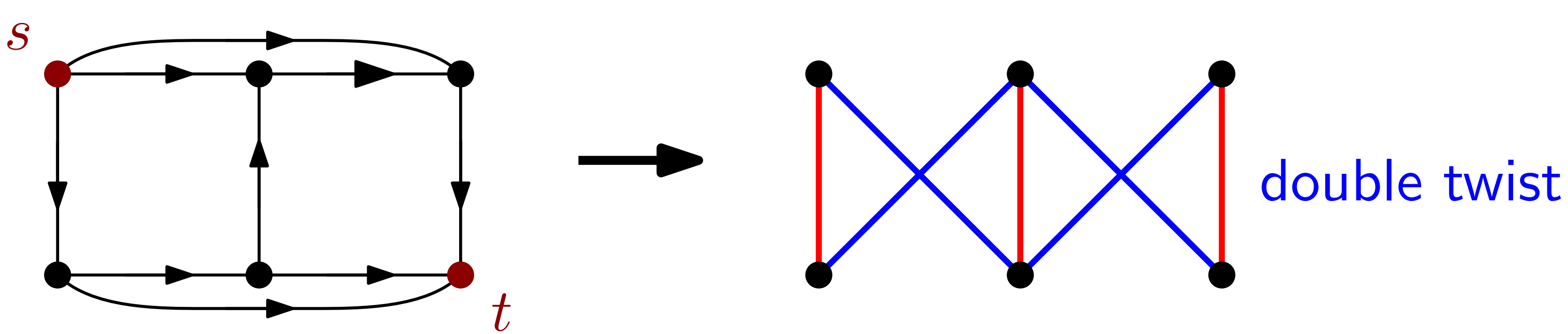
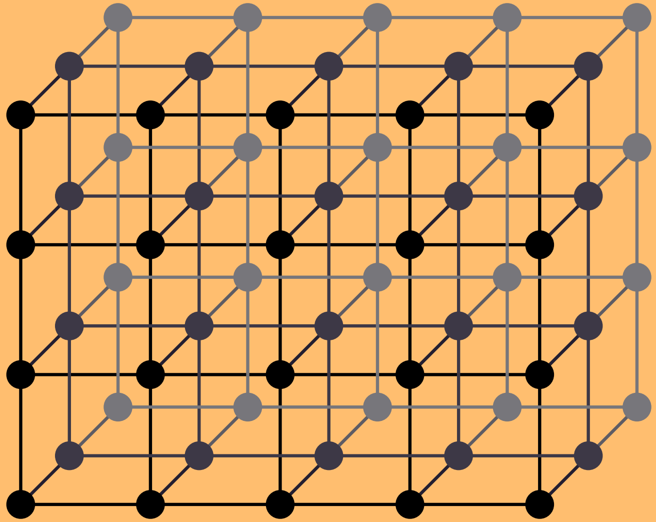
Acyclic USO which violates Holt-Klee-property:



USOs with Holt-Klee $\leftrightarrow$ red-blue-arrangements

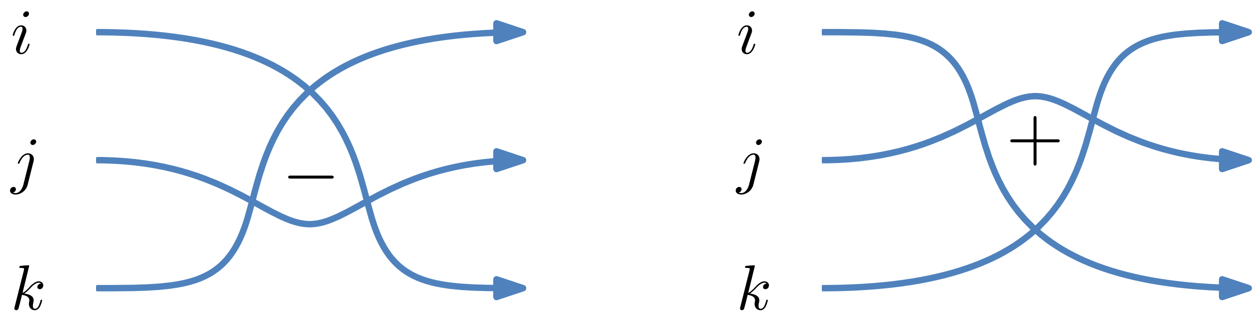
Generalize this to
higher dimensional grids ???



$$\cong \quad \Delta_2 \times \Delta_4 \times \Delta_2$$

arrangements $\leftrightarrow$ 3-signotopes

Two cases for pseudolines $i < j < k$:



Pseudoline arrangement $\mathscr{A}$ defines map: $\chi_{\mathscr{A}} : \binom{[n]}{3} \to \{-, +\}$

arrangements $\leftrightarrow$ 3-signotopes

Definition:
A map $\chi : \binom{[n]}{3} \to \{-,+\}$ is called 3-*signotope* if for all 4-tuples $1 \leq i < j < k < l \leq n$ we have:

$$(\chi(jkl), \chi(ikl), \chi(ijl), \chi(jkl)) \in \Big\{(+ + ++), (+ + +-), \\ (+ + --), (+ - --), \\ (- - --), (- - -+), \\ (- - ++), (- + ++)\Big\}$$

arrangements $\leftrightarrow$ 3-signotopes

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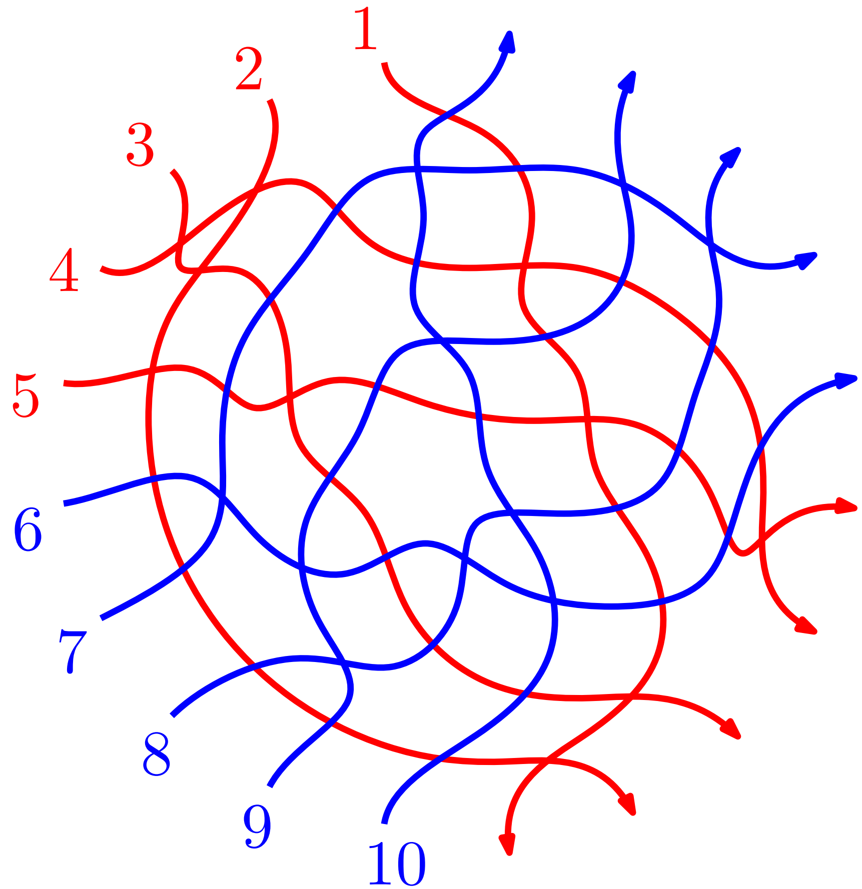
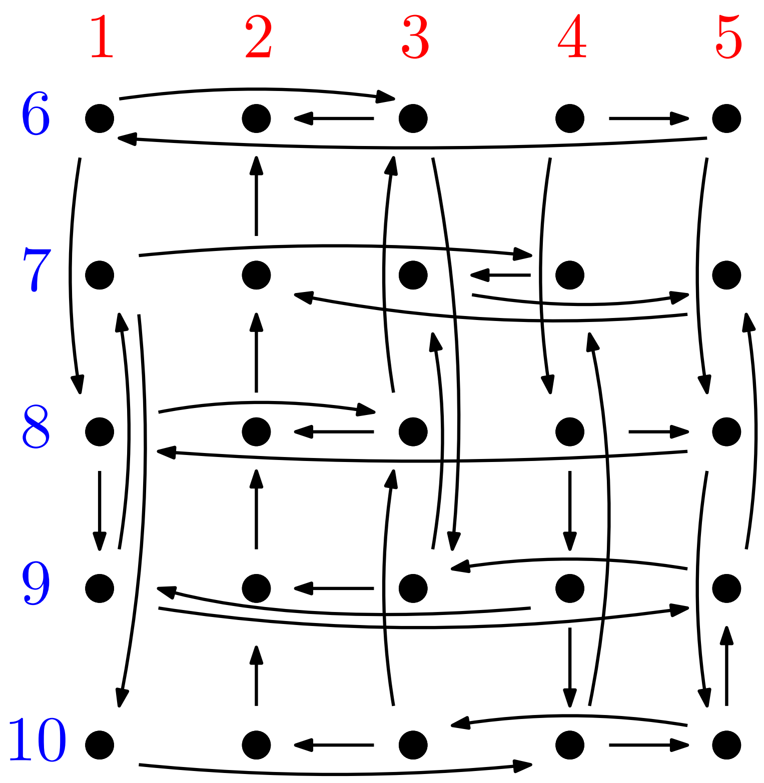
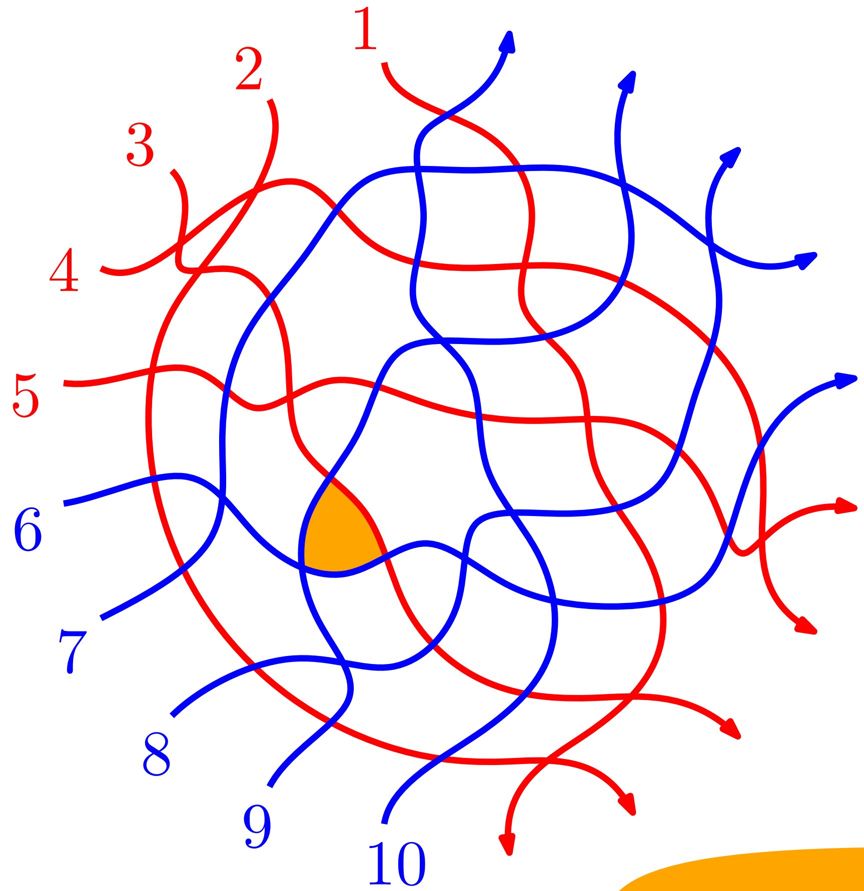
- Bijection: pseudoline arrangements $\leftrightarrow$ 3-signotopes
- Define more generally r-*signotopes* $\chi : \binom{[n]}{r} \to \{-,+\}$

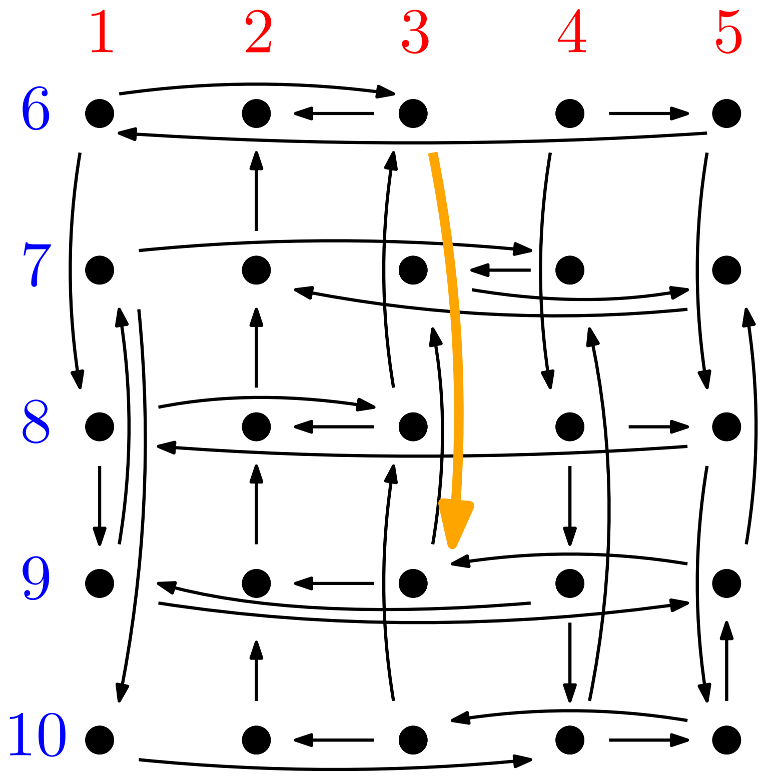
3-signotopes $\leftrightarrow$ USOs





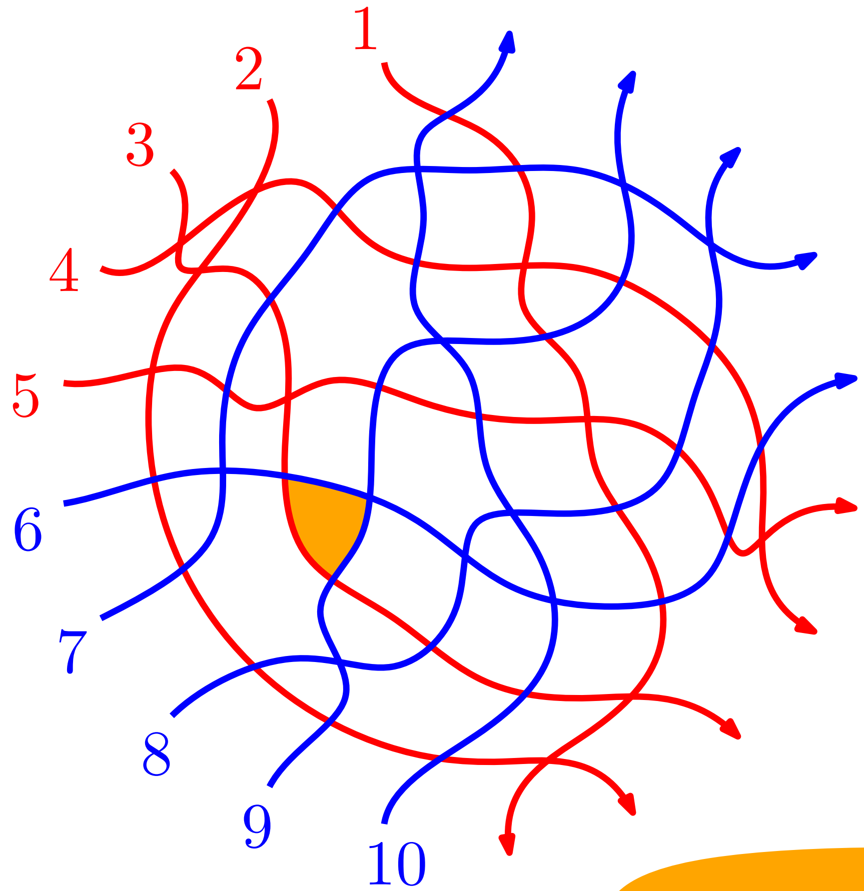
3-signotopes $\leftrightarrow$ USOs

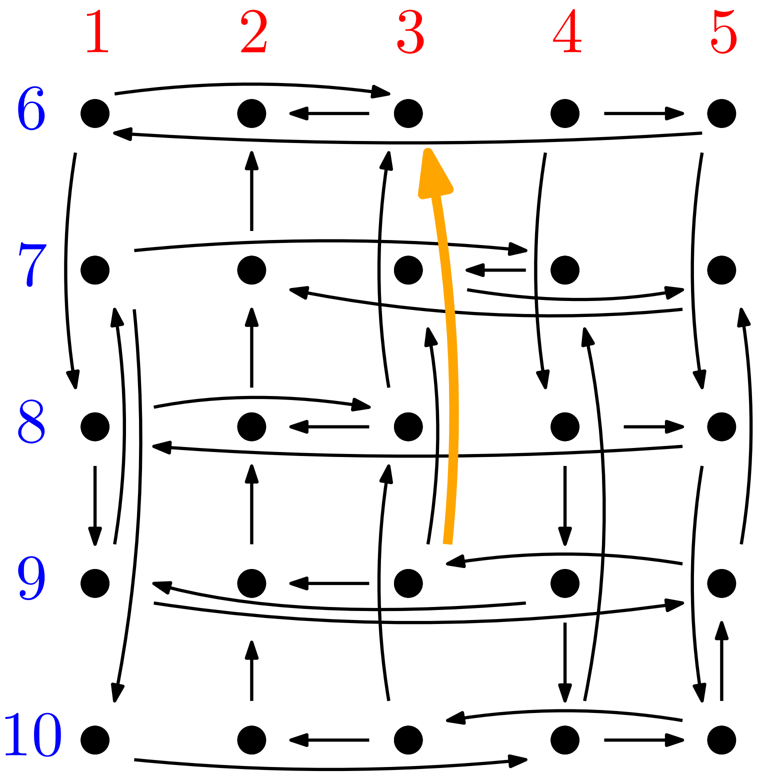




$\chi(3, 6, 9) = -$

3-signotopes $\leftrightarrow$ USOs





$$\chi(3, 6, 9) = +$$

4-signotopes $\leftrightarrow$ 3-dimensional USOs

Assume 4 signotope $\chi : \binom{[n]}{4} \to \{-,+\}$
and *block partition* $[n] = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

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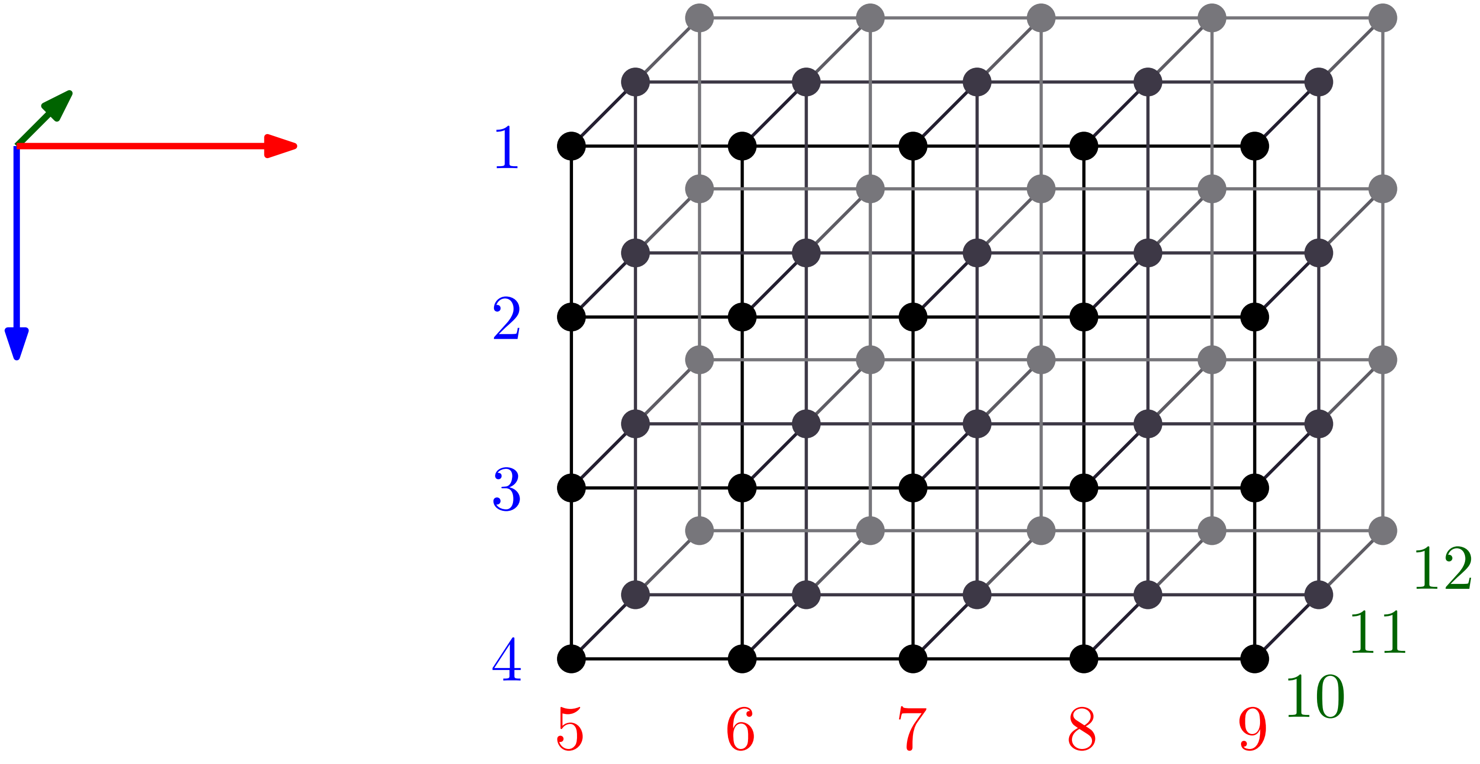

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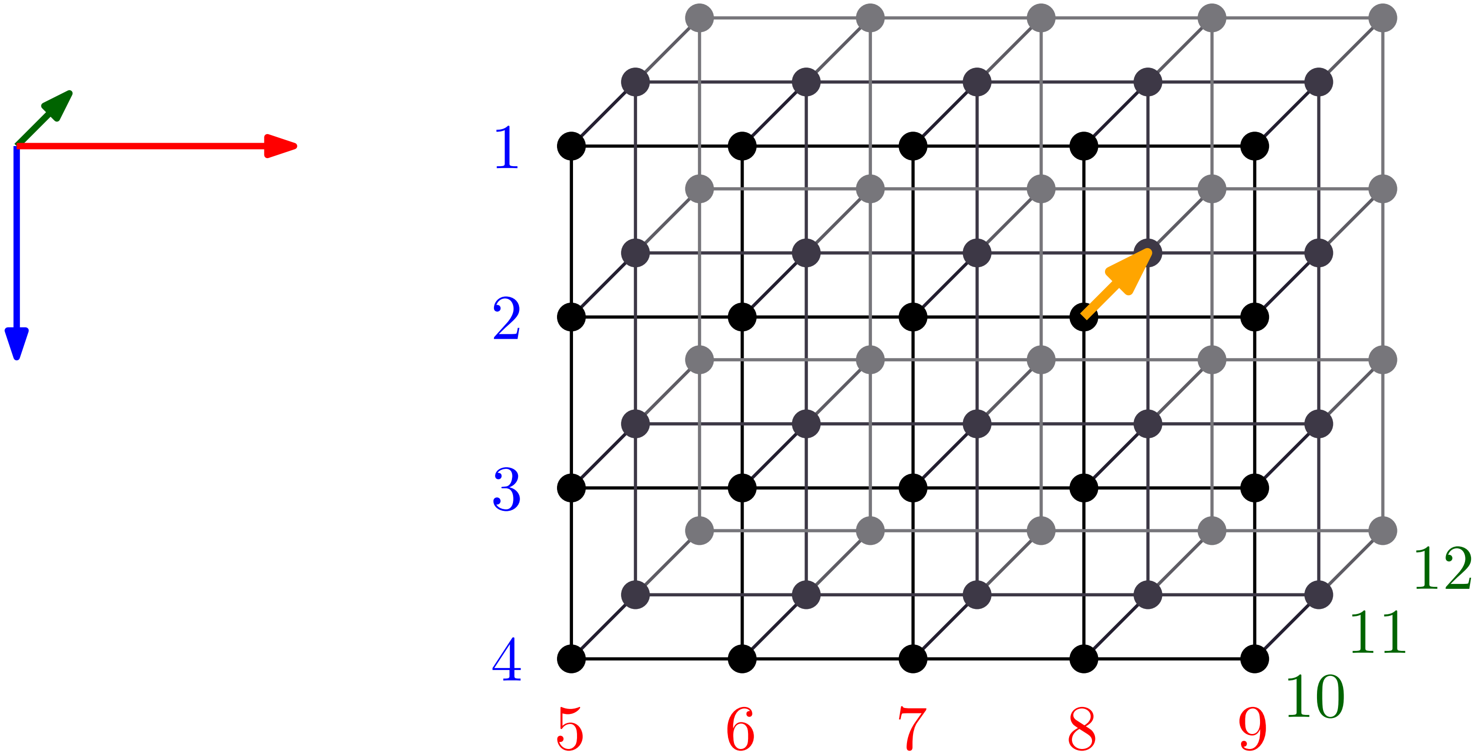

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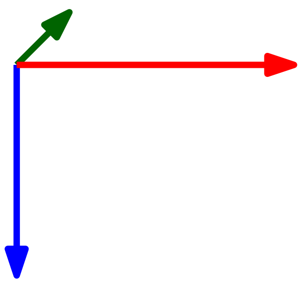

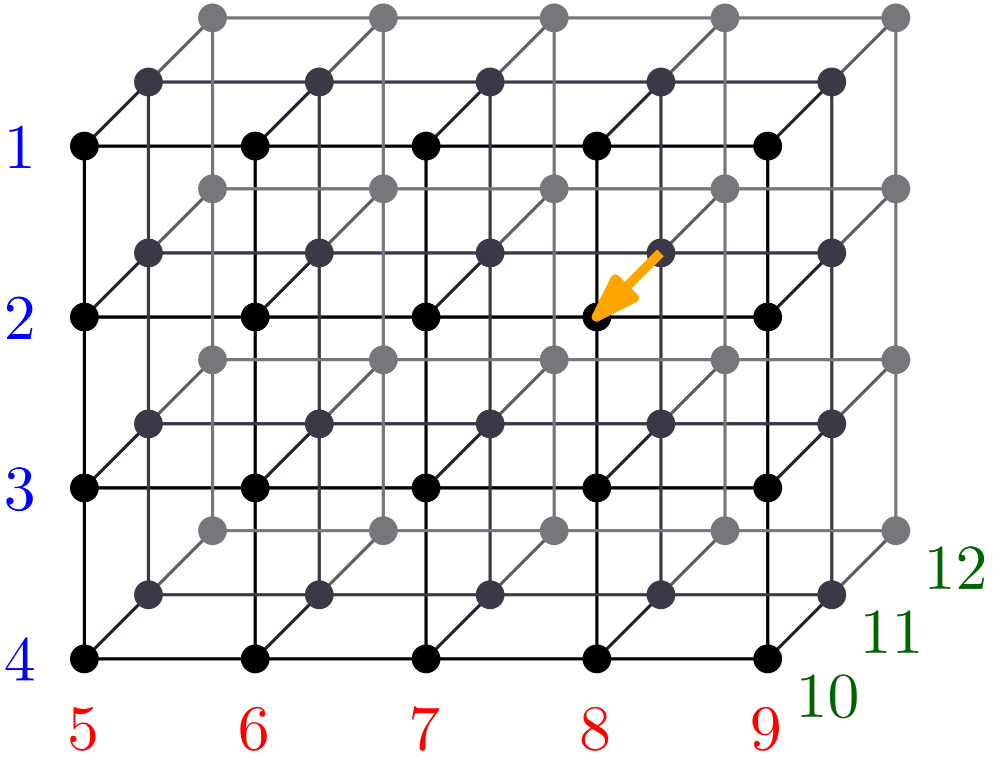

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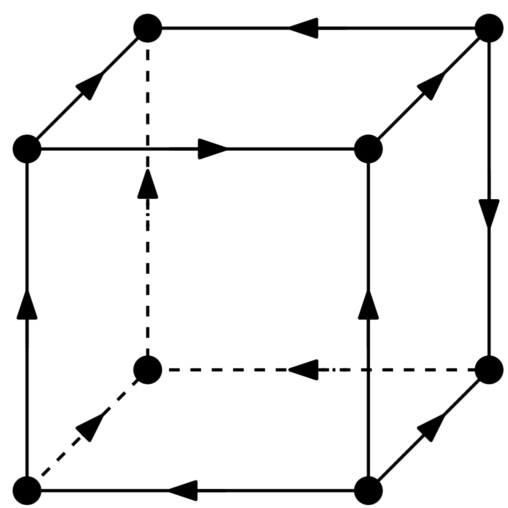
4-signotopes $\leftrightarrow$ 3-dimensional USOs

Theorem (R. 2025)

1) USOs induced by 4-signotopes are acyclic USOs with Holt-Klee-property.
2) There exist 3-dimensional acyclic USOs with Holt-Klee-property that are **not** induced by 4-signotopes.



acyclic USO with Holt-Klee. But not induced by signotope):

4-signotopes $\leftrightarrow$ 3-dimensional USOs

Theorem (R. 2025)

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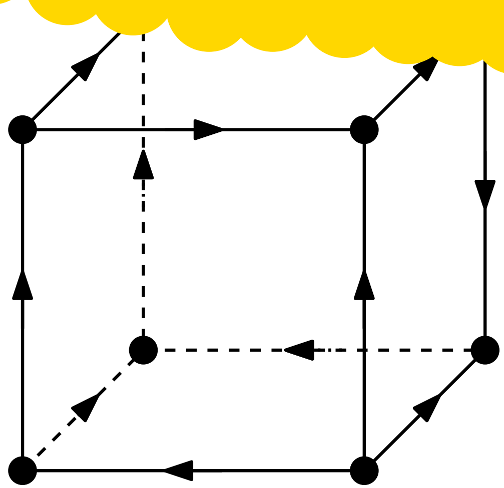
gnotopes.

Open problem:

$(r+1)$-signotopes / r-dimensional grids ??



acyclic USO with
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pseudoline arrangements

wiring diagrams

signotopes

plane partitions

permutations

rhombic
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sorting networks

higher Bruhat
orders

Standard Young tableaux

families of monotonic
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oriented matroid of rank 3

pseudoline arrangements

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Problem:
How can pseudoline arrangements be efficiently generated uniformly at random?

rhombic tilings

sorting networks

higher Bruhat orders

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rapidly mixing Markov chains

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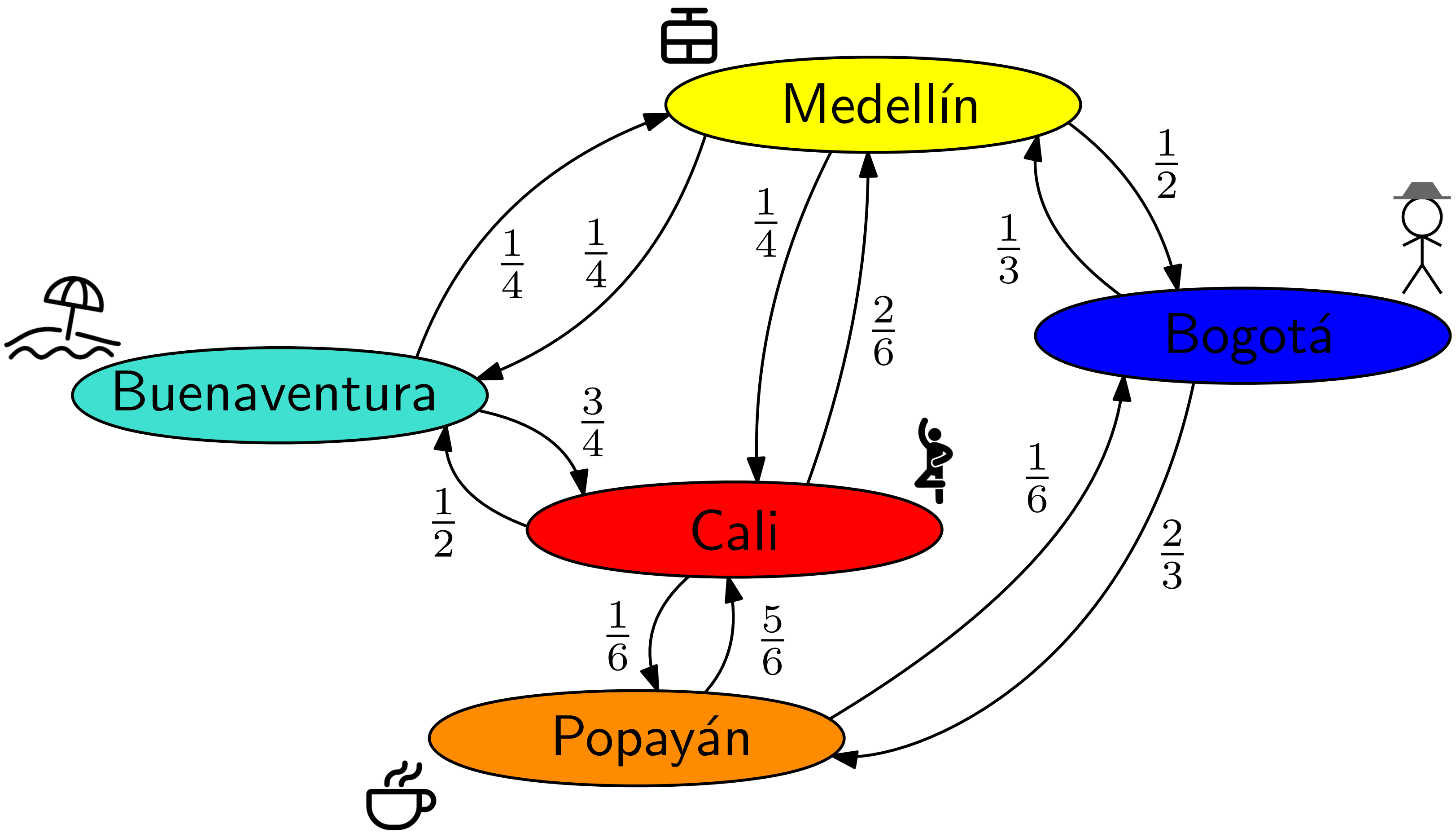
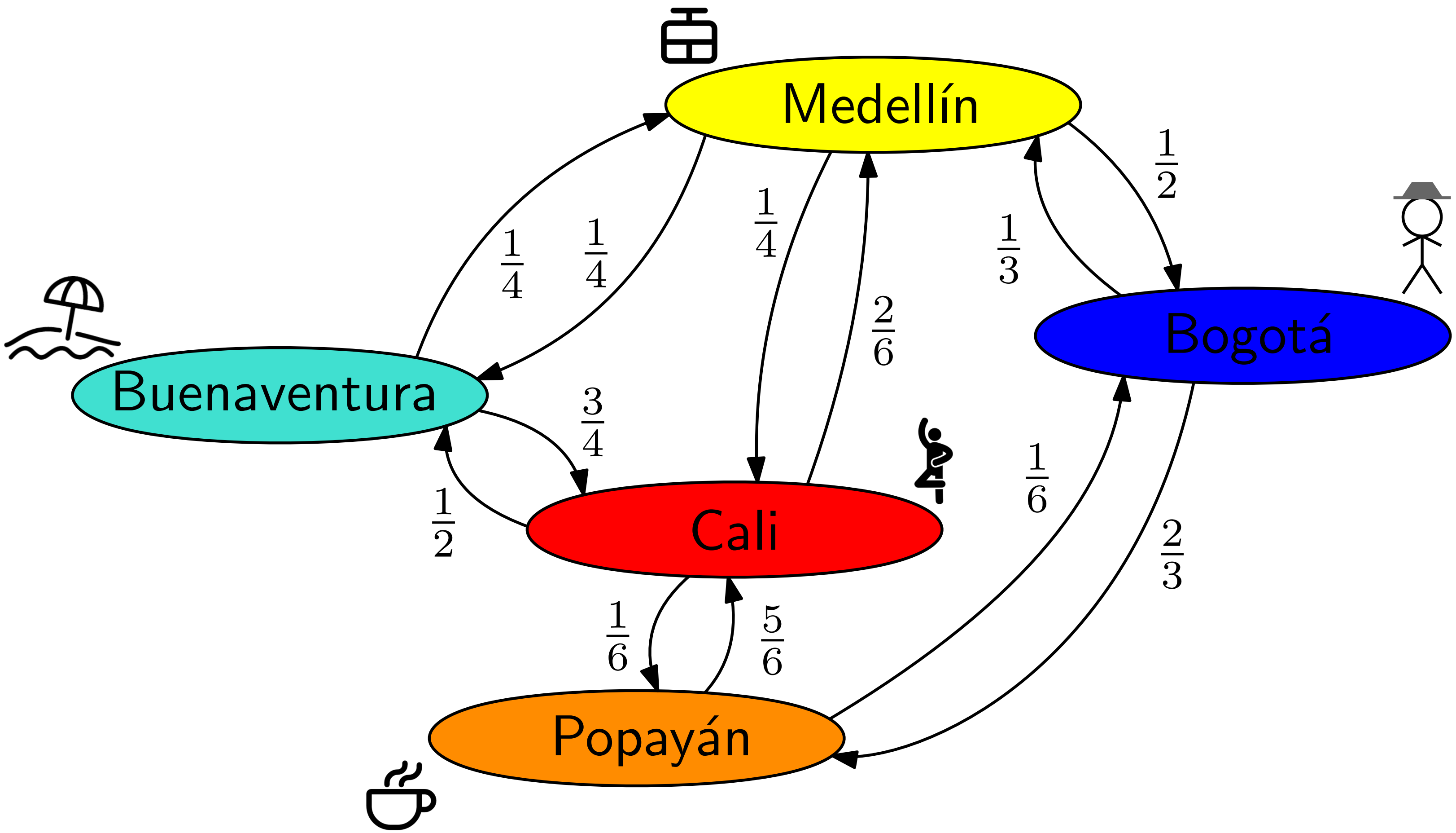


rapidly mixing Markov chains

Markov chain (X_t), state space $\mathscr{X}$, transition prob. $P : \mathscr{X} \times \mathscr{X} \to [0, 1]$



random generation using Markov chains

Idea:

- States $\mathscr{X} = \{\text{arrangements of fixed size}\}$
- Symmetric transition probabilities

$\implies$ After many steps get almost uniform arrangement

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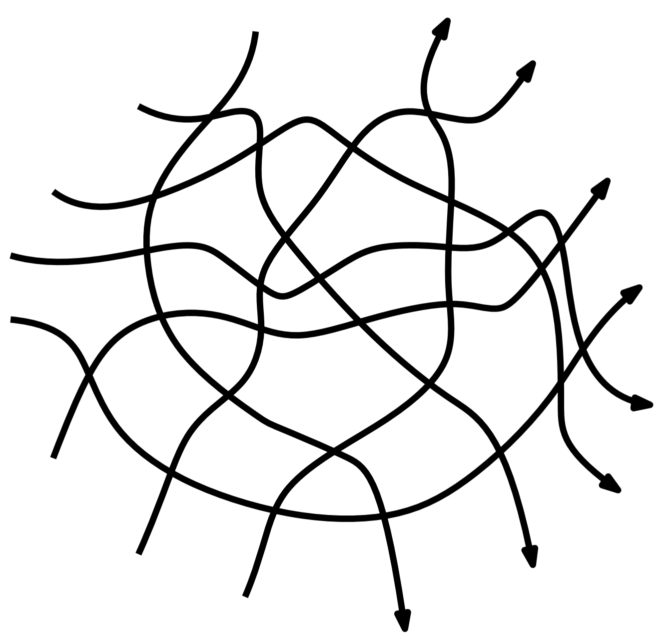

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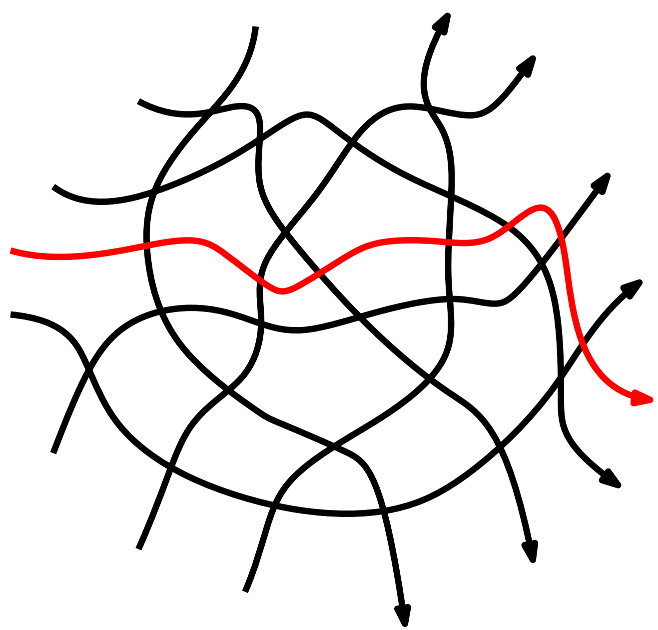

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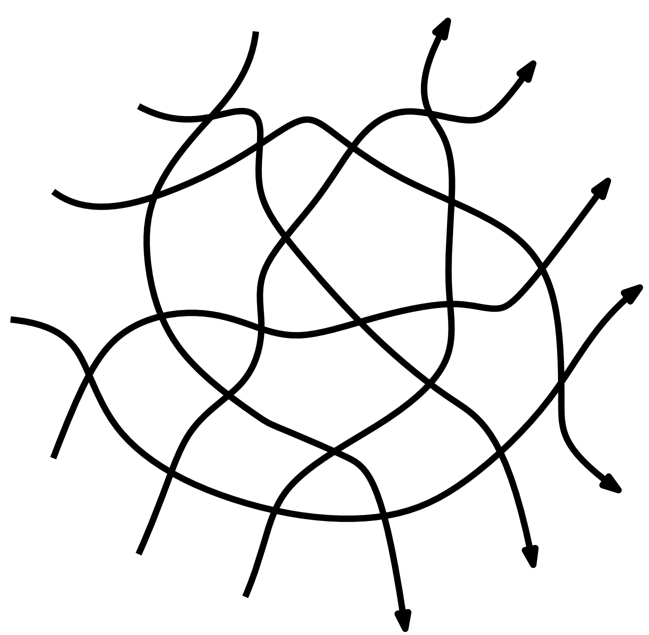

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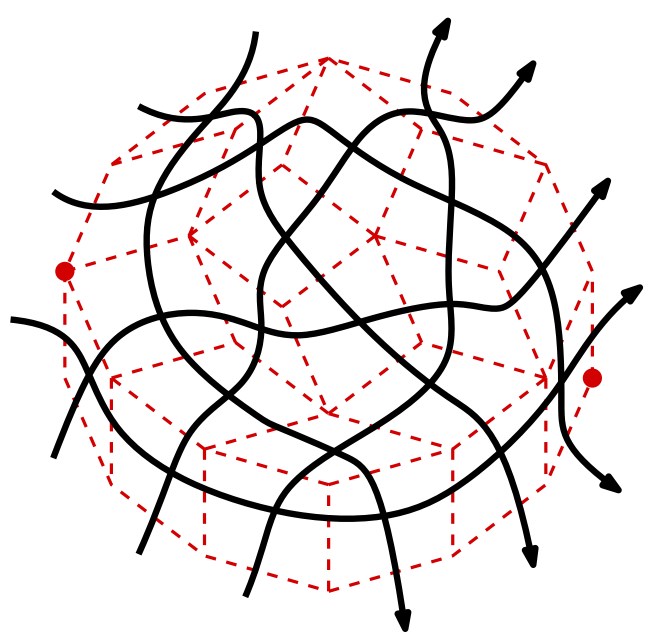

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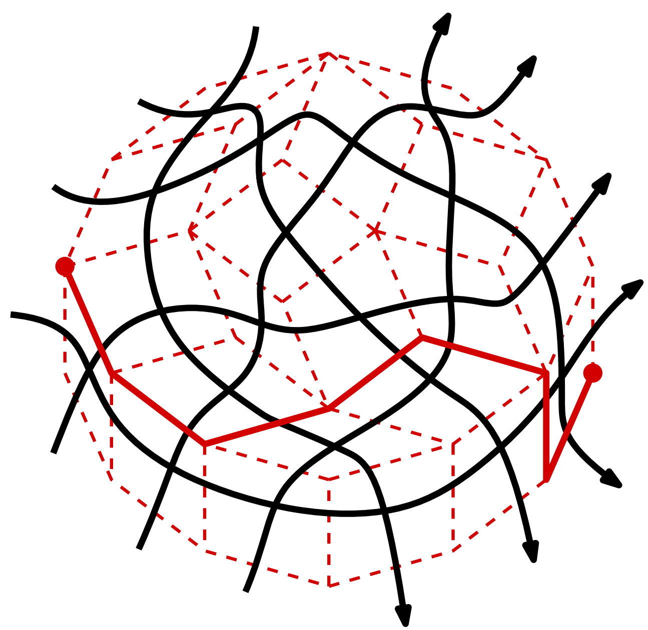

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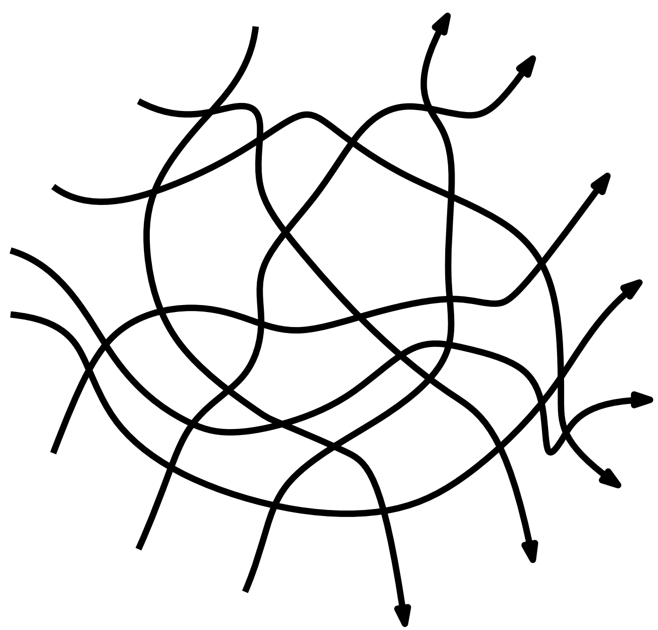

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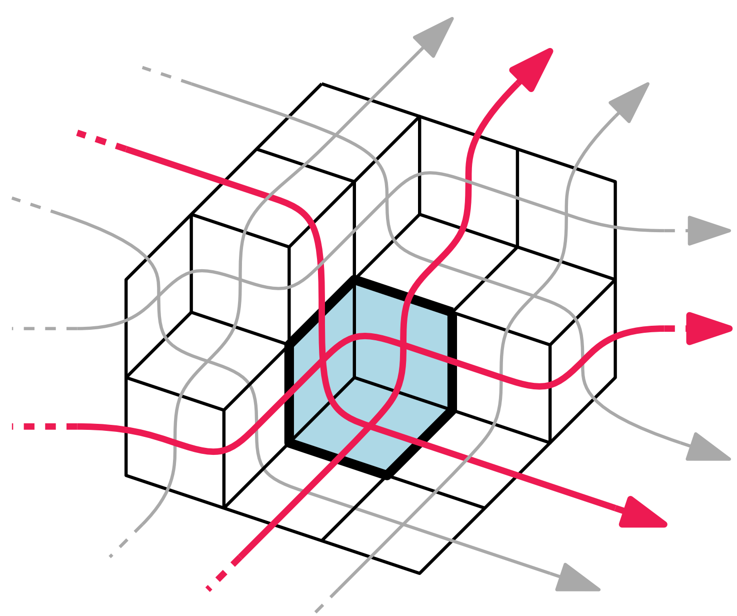
Markov chain II: random triangle flip



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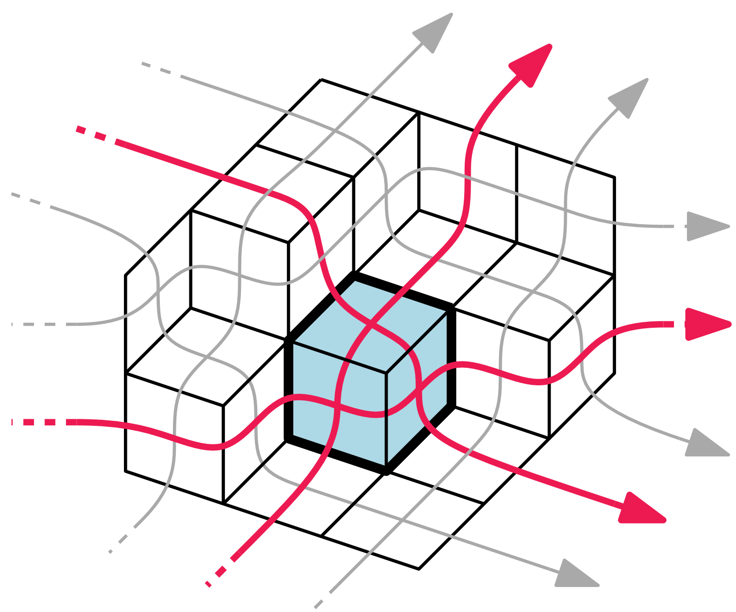
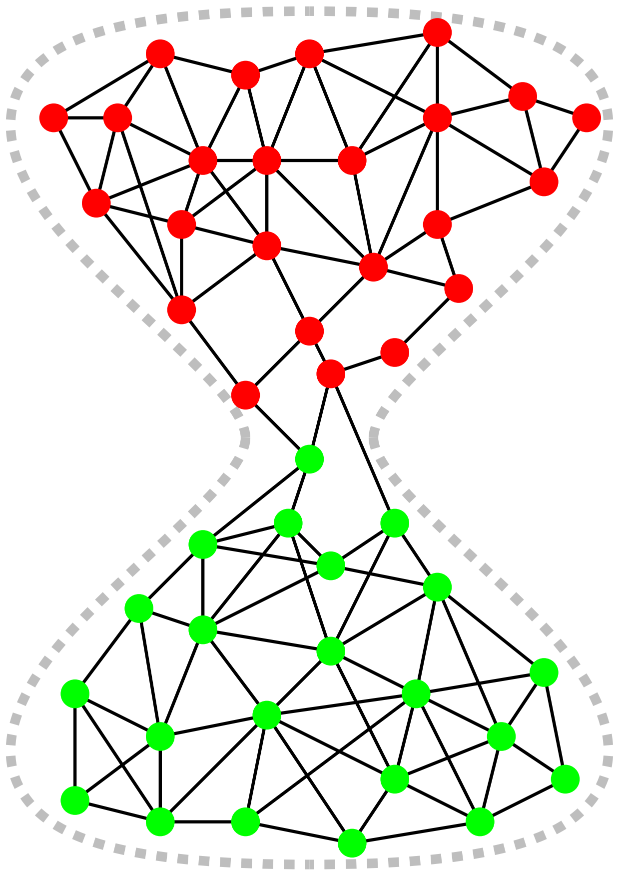
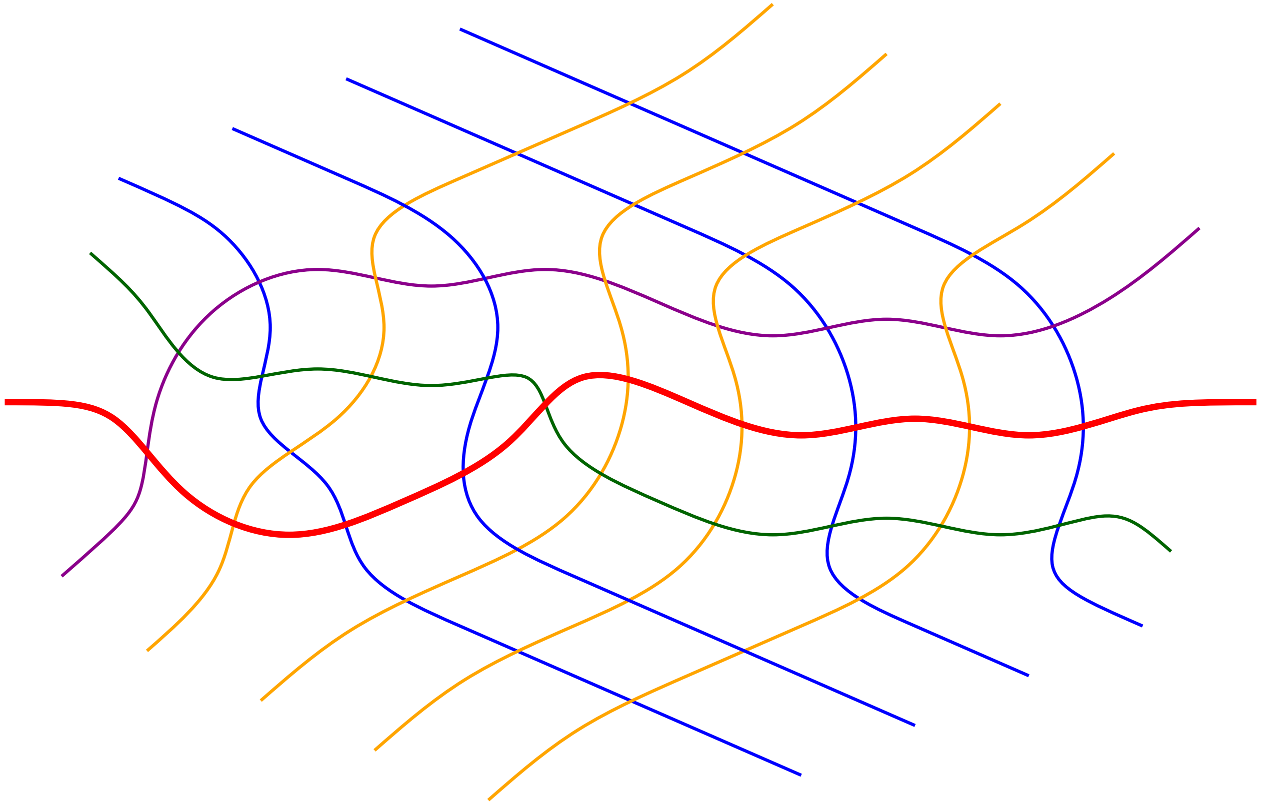


bottleneck

Markov chain having a „bottleneck“:



flips on single pseudoline



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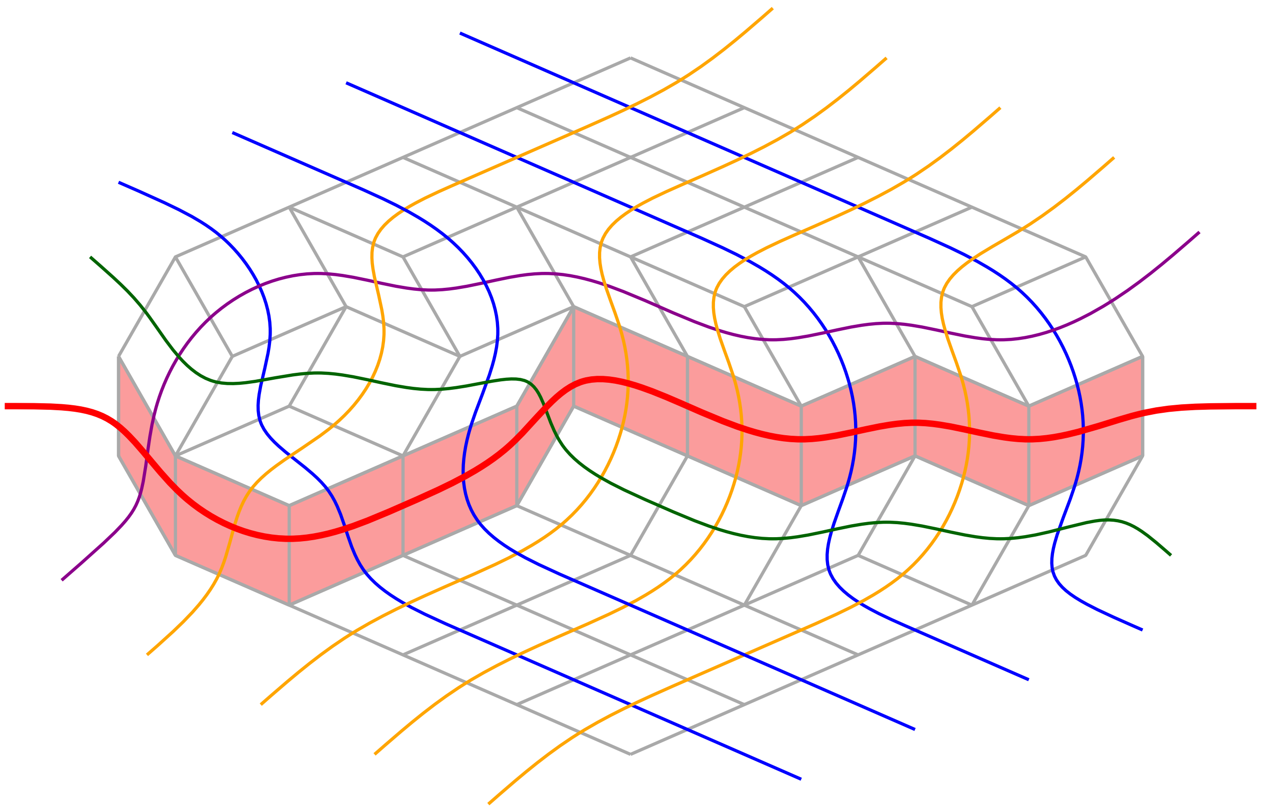

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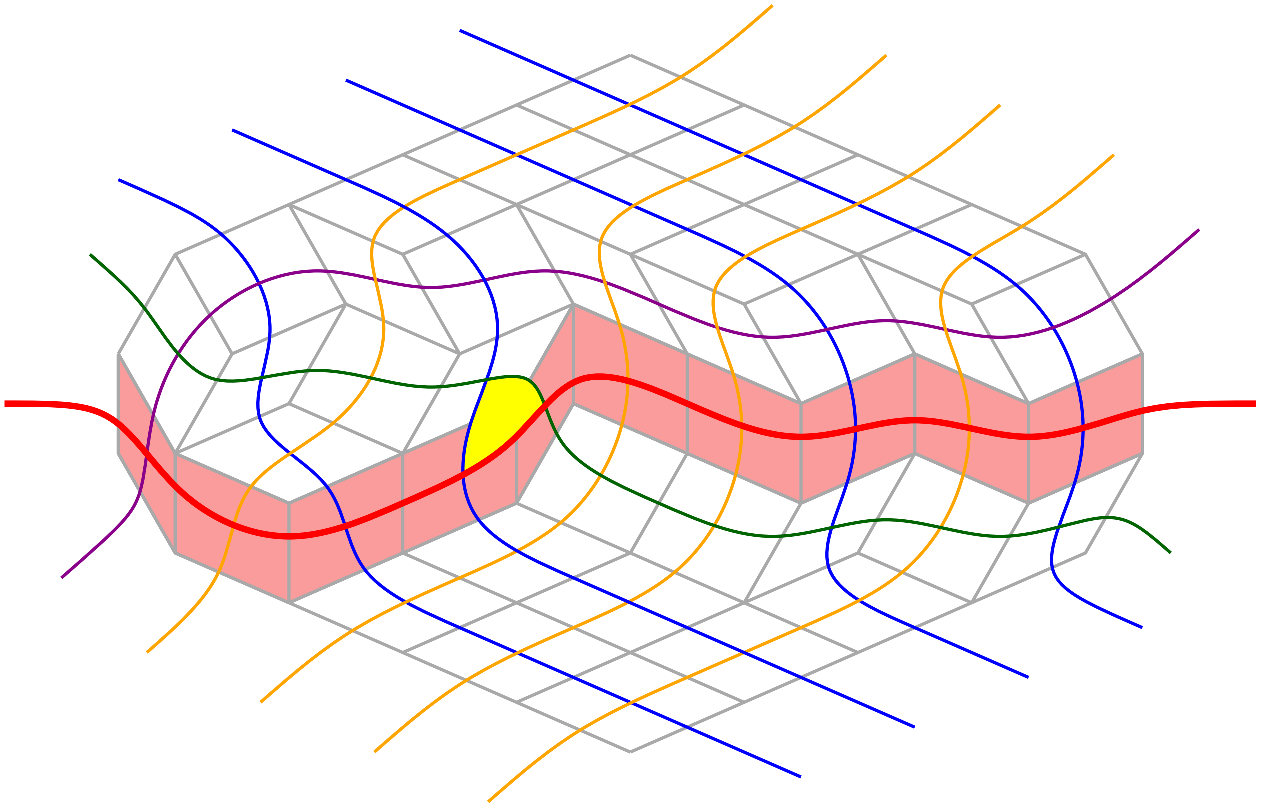

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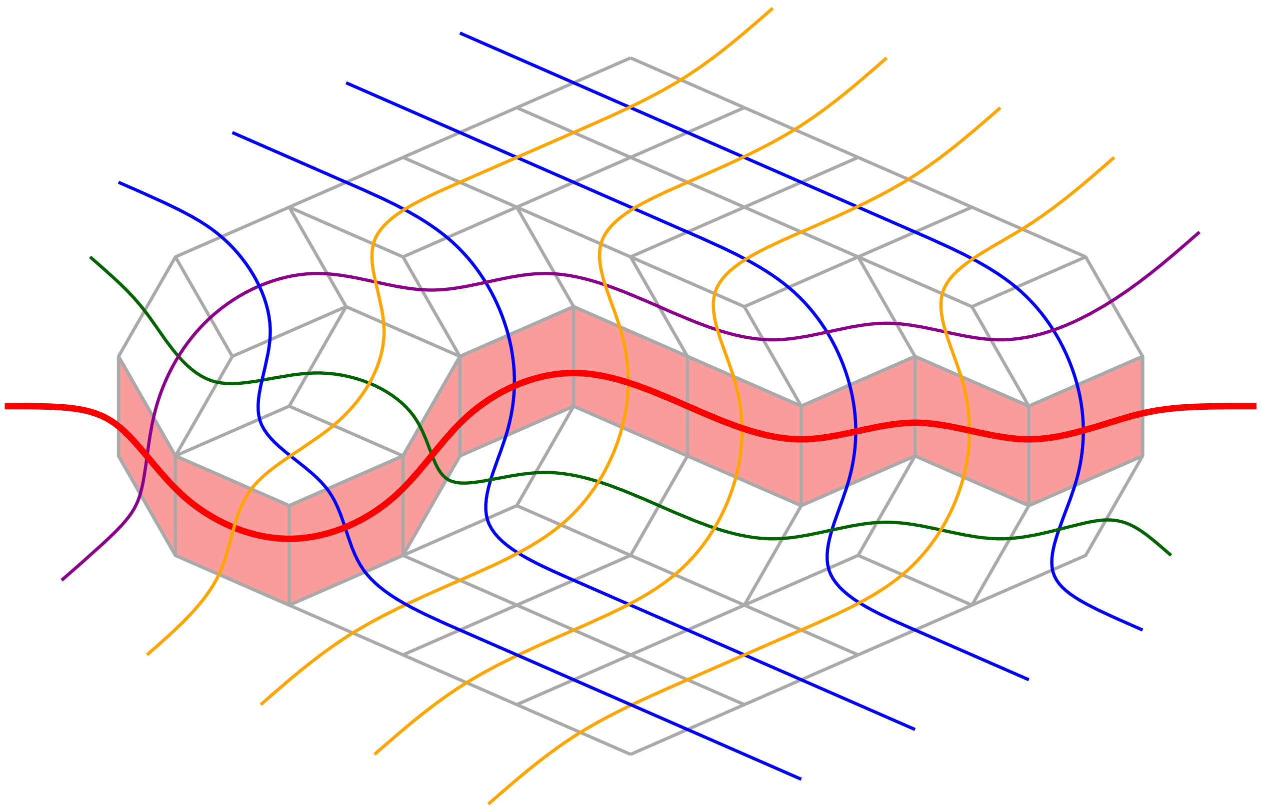

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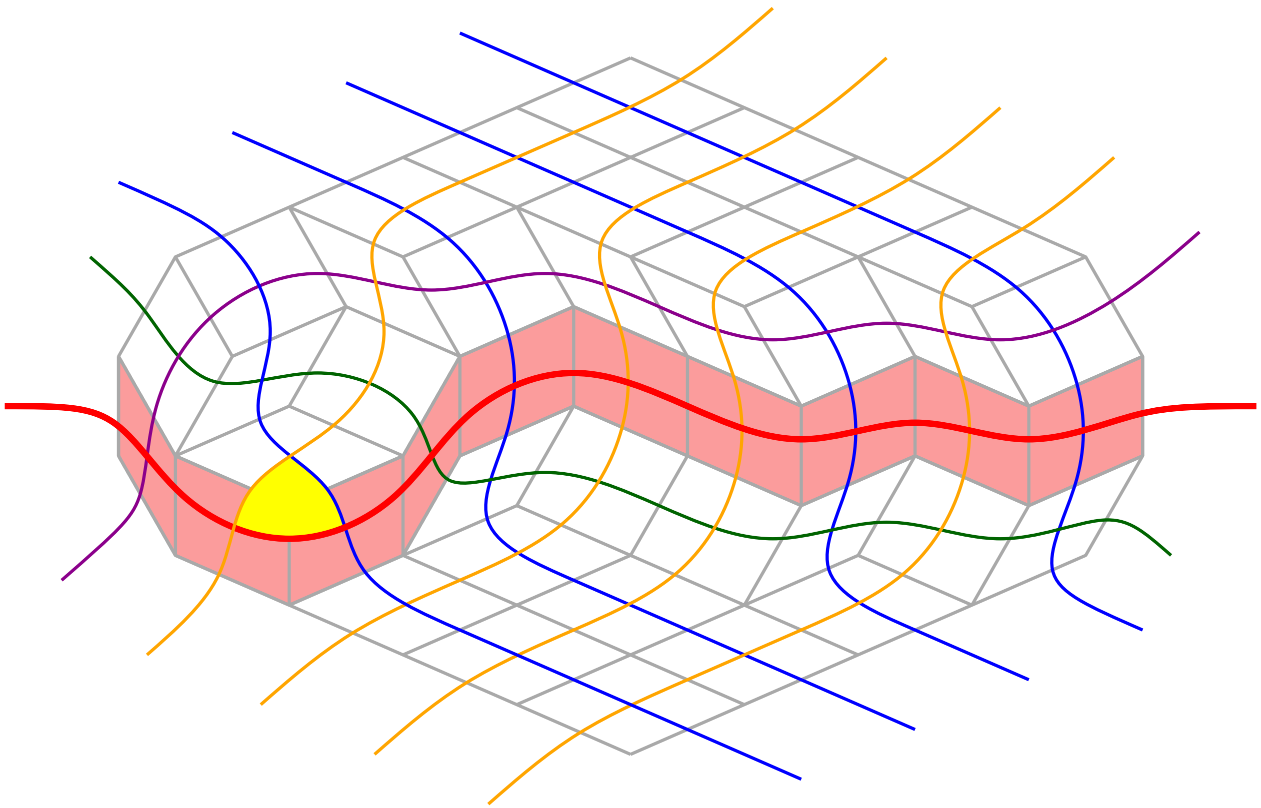

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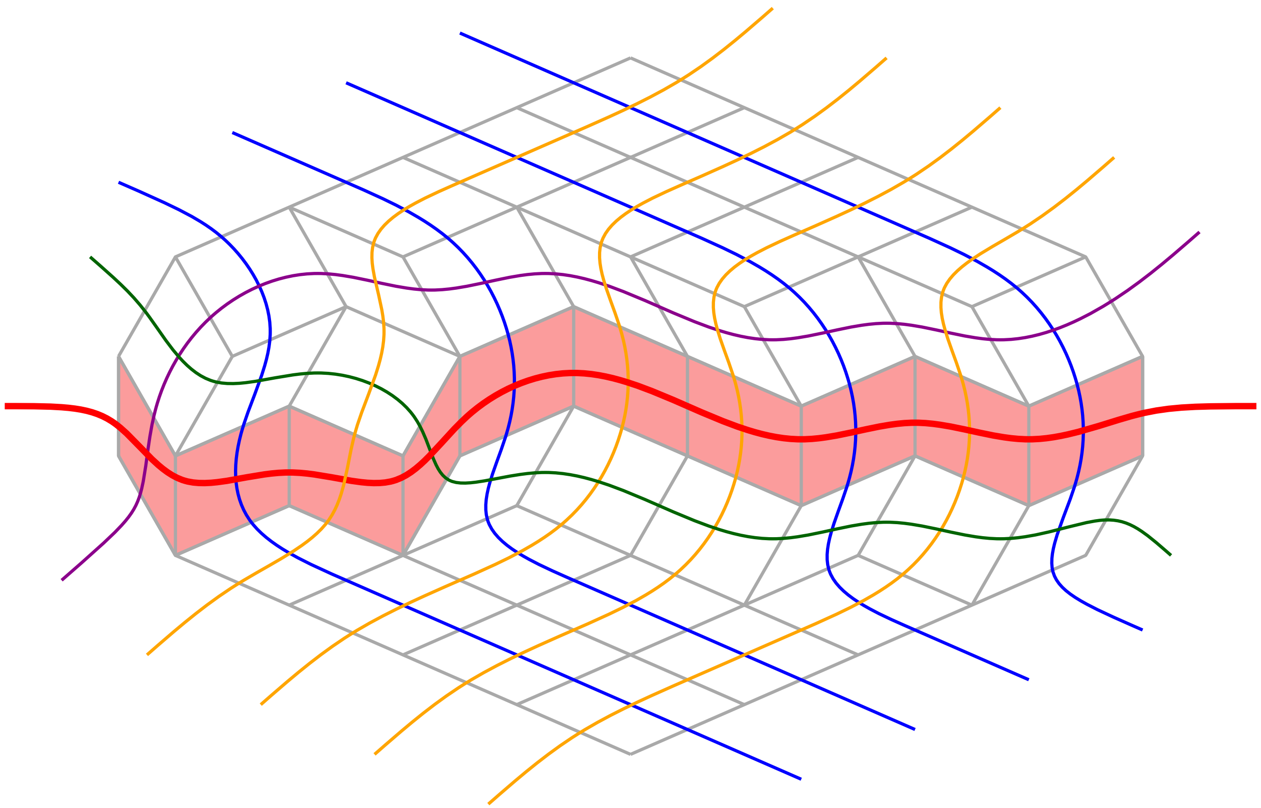

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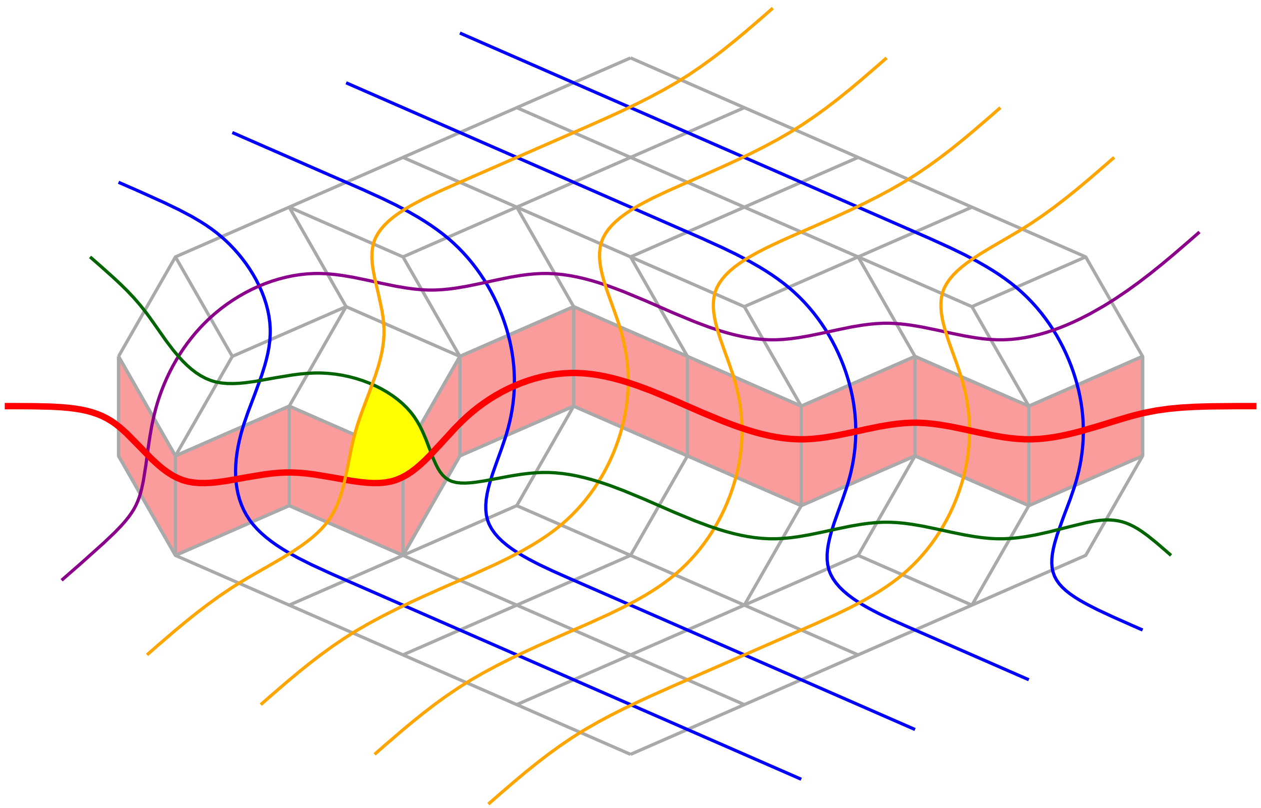

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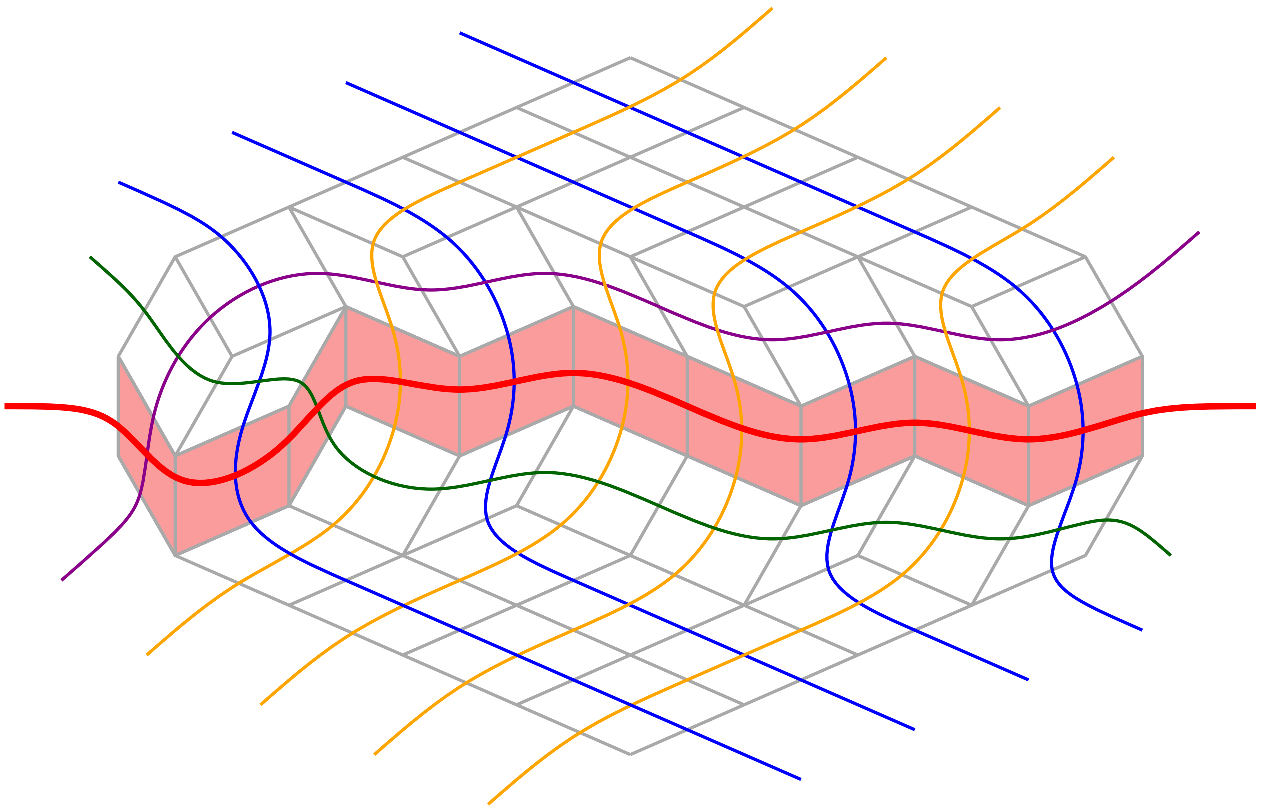

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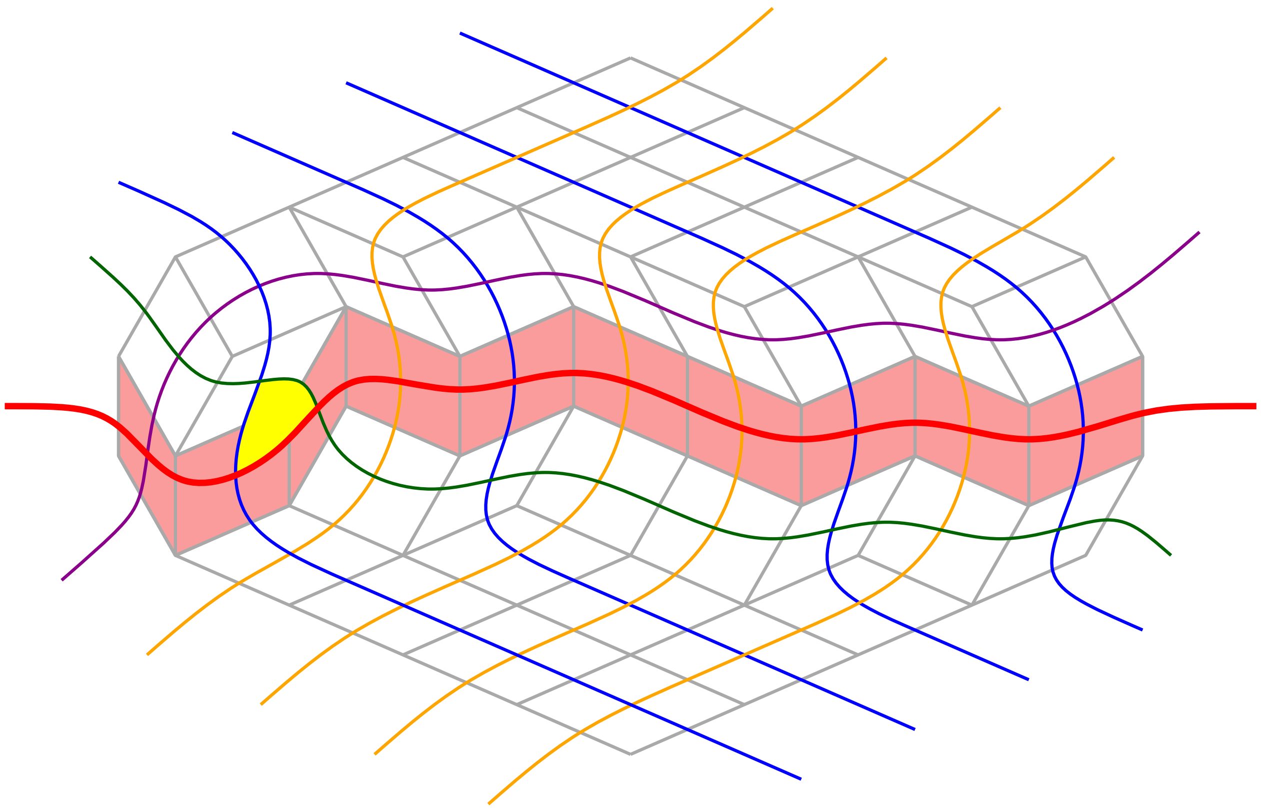

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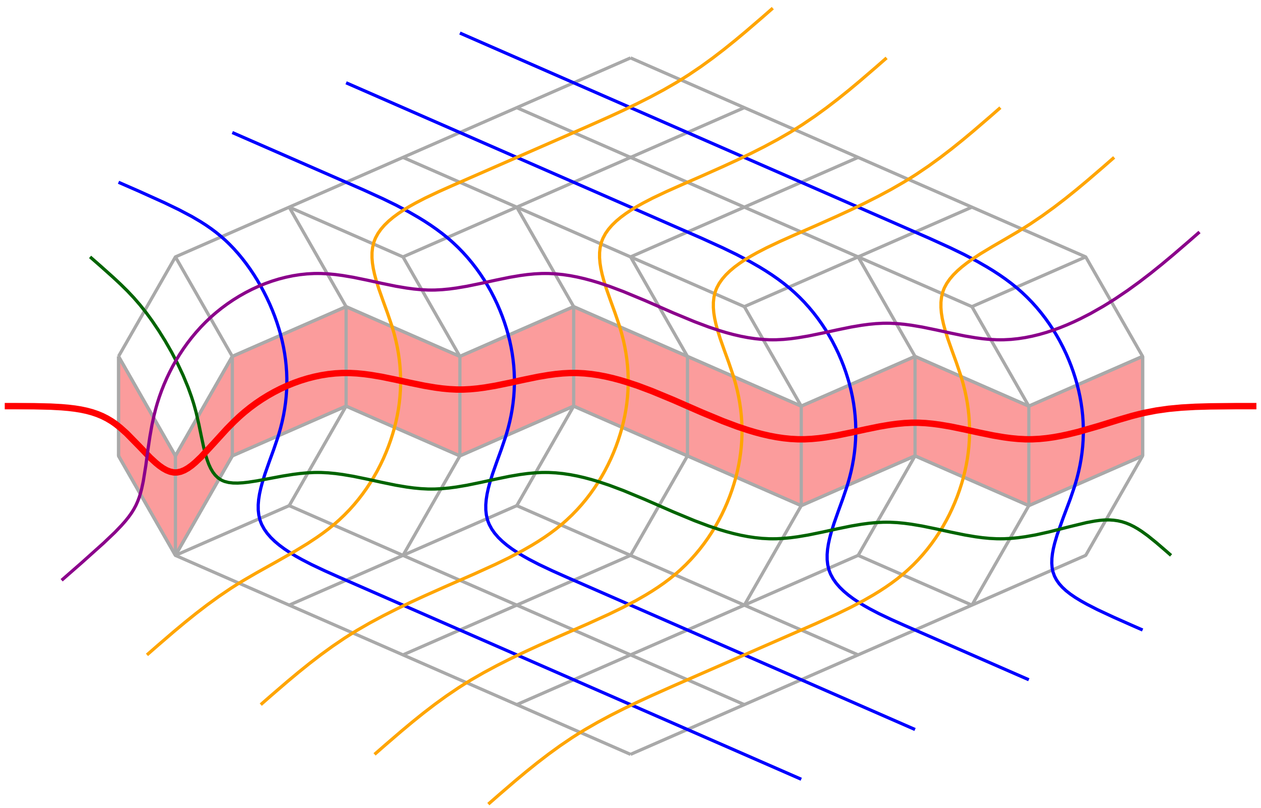

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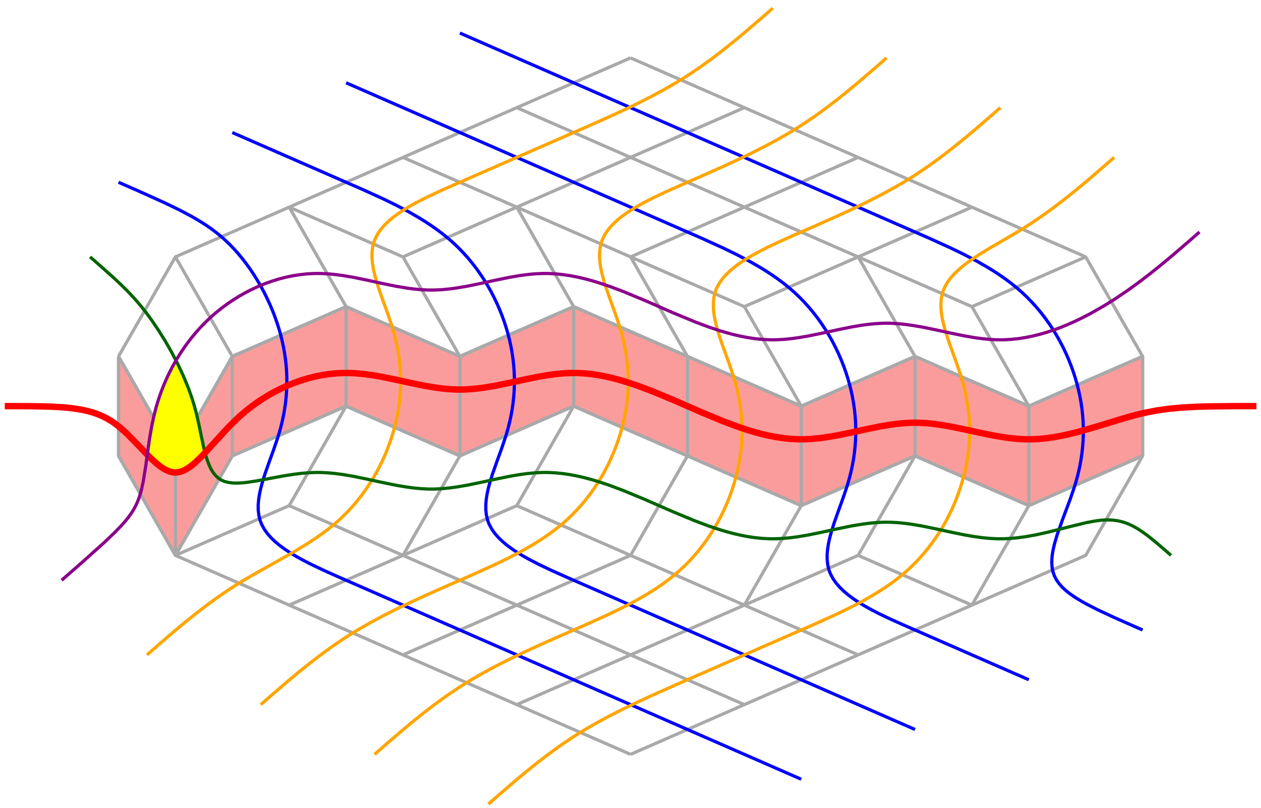

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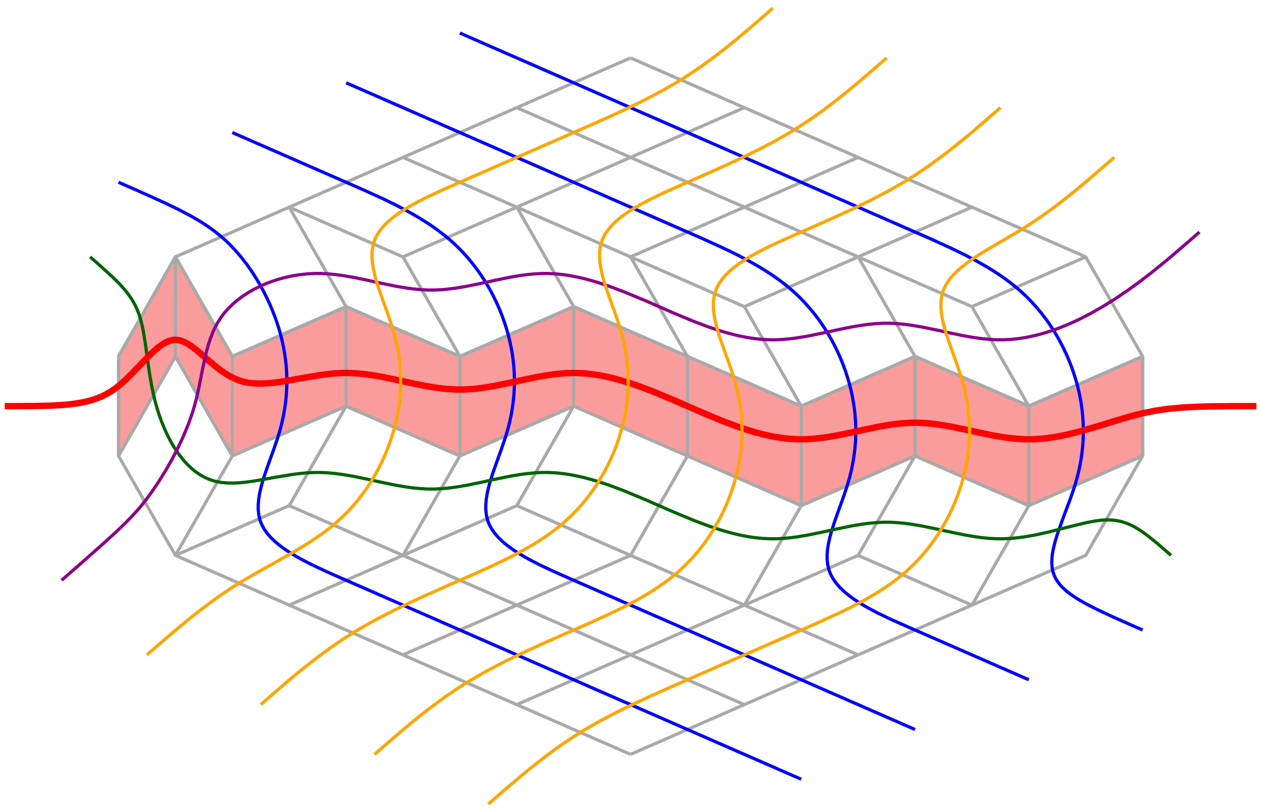

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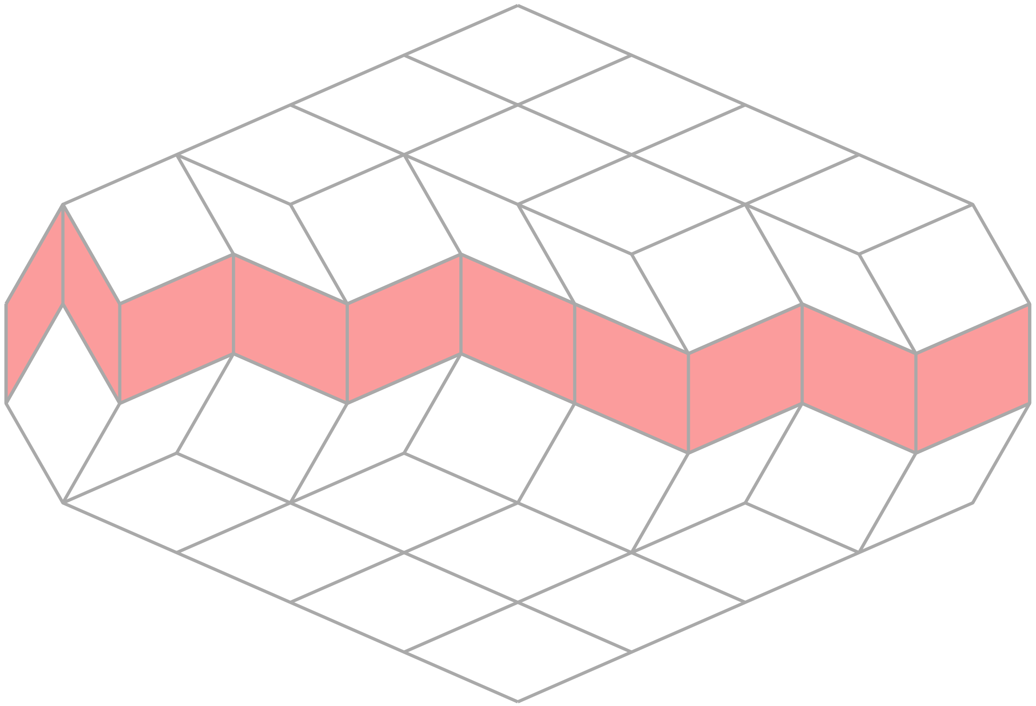

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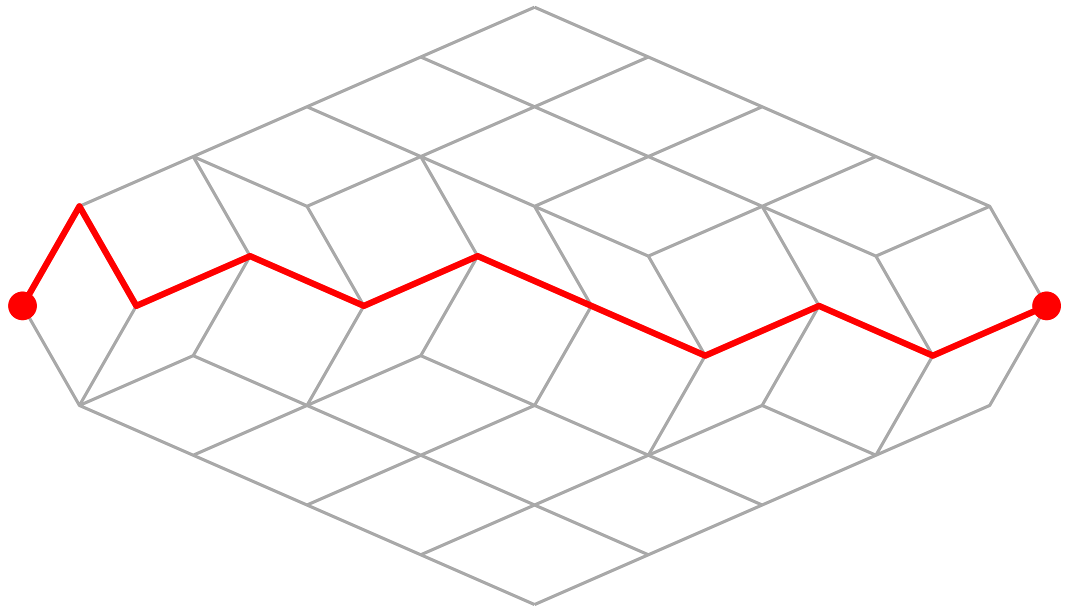

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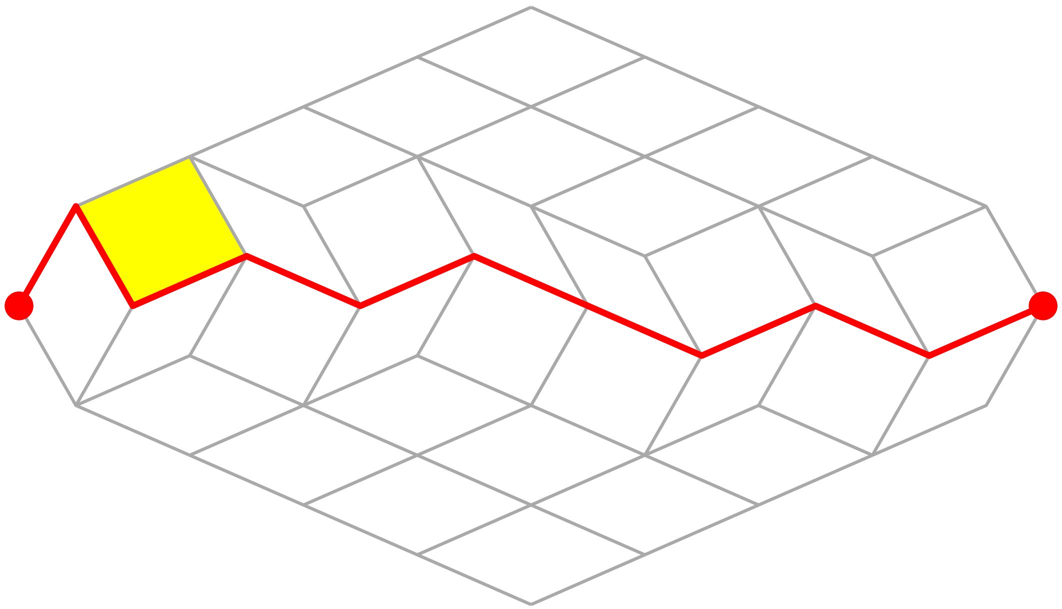

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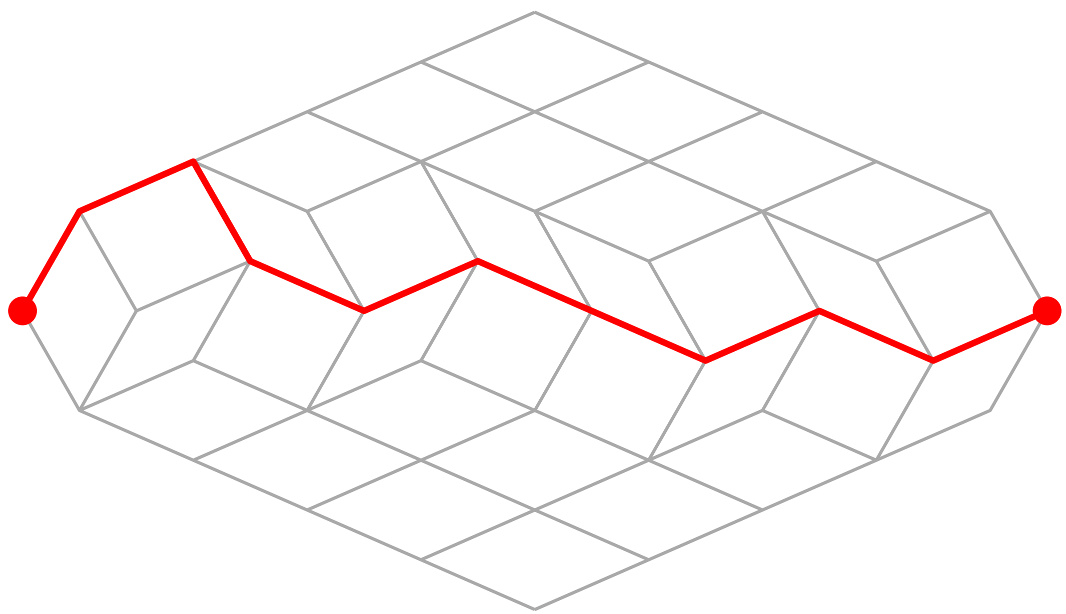

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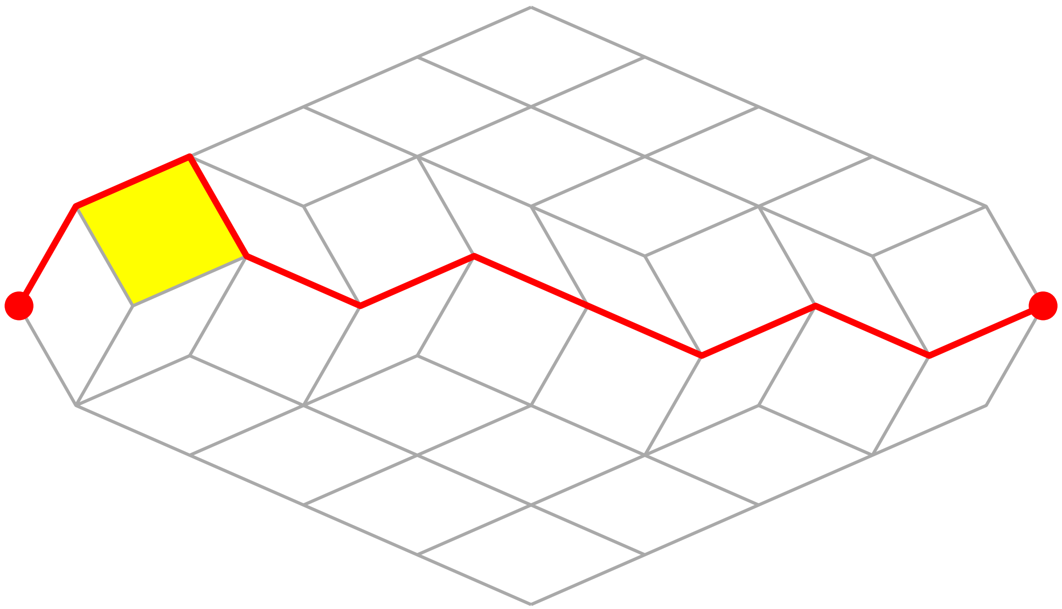

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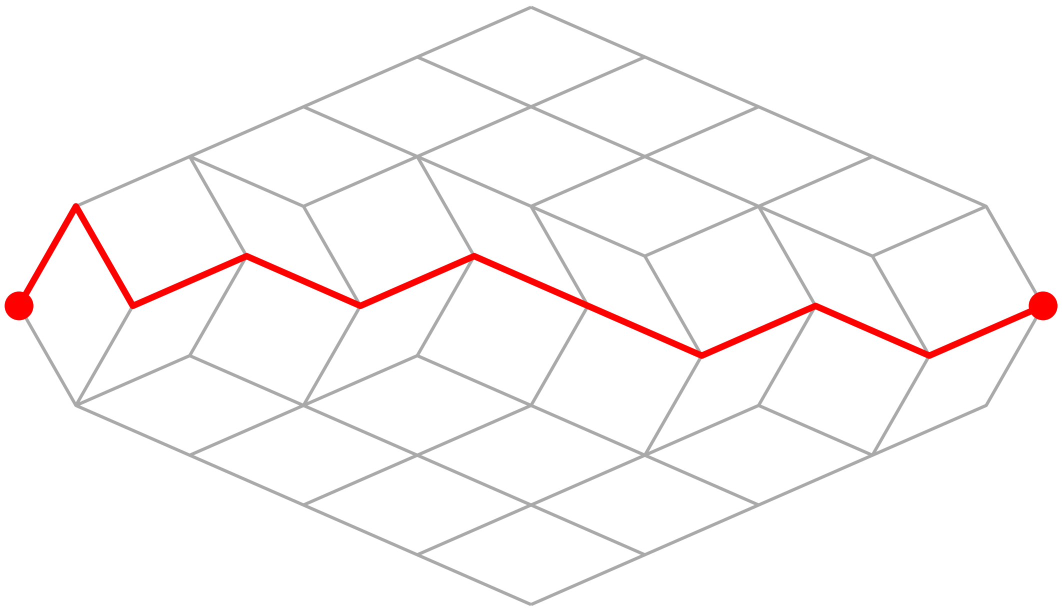

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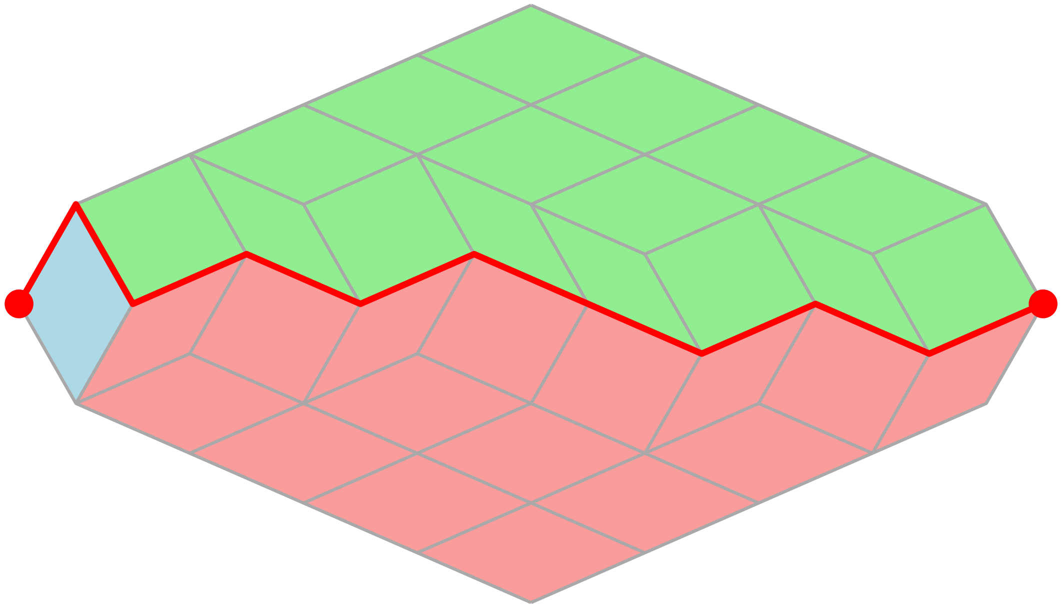
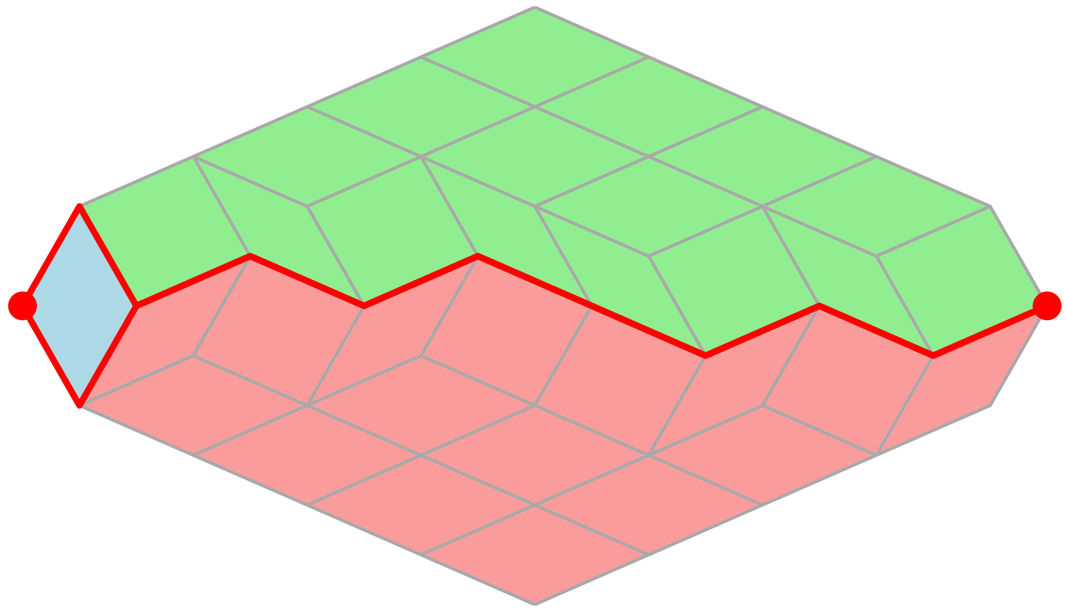


- Partition of states into two classes:
  - paths **above the blue rhombus**
  - paths **below the blue rhombus**

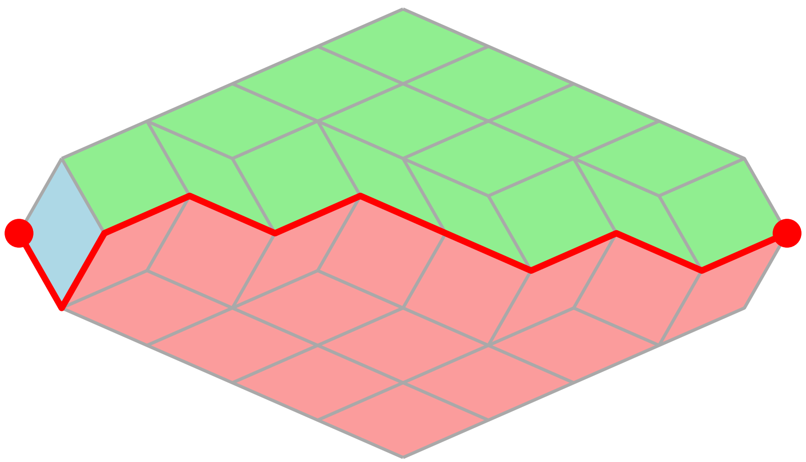
flips on single pseudoline



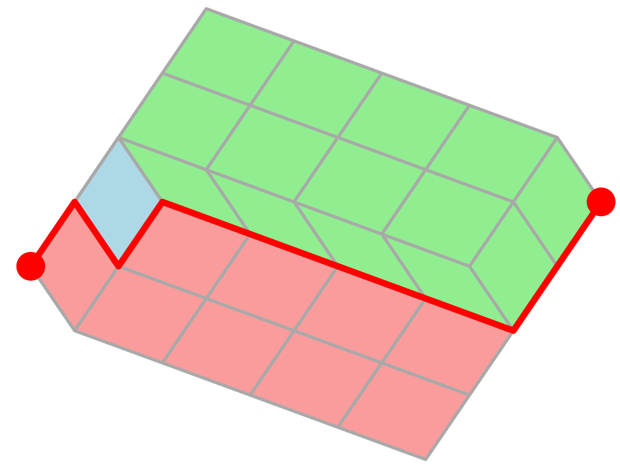
- Partition of states into two classes:
  - paths **above the blue rhombus**
  - paths **below the blue rhombus**
- Only a flip on the blue rhombus connects both classes!

flips on single pseudoline

$r = 5$ parallel classes:
(generalizable to more)



$r = 4$ parallel classes:



- Partition of states into two classes:
  - paths **above the blue rhombus**
  - paths **below the blue rhombus**
- Only a flip on the blue rhombus connects both classes!

flips on single pseudoline

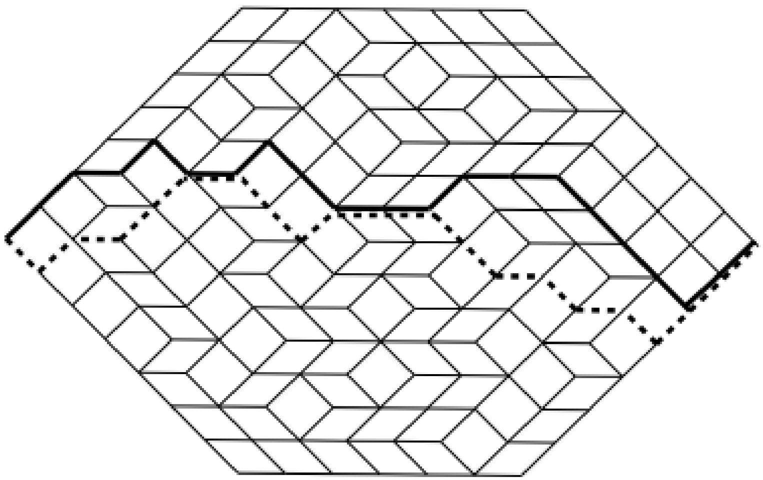
Theorem (R., 2021):
The Markov chain which operates on generalized pseudoline arrangements and flips random triangles with involvement of a distinguished parallel class is

- ... **rapidly-mixing** on 3 parallel classes, and...
- ... in general **not rapidly-mixing** on 4 or more parallel classes.

Statement for 3 classes follows from (Luby, Randall & Sinclair, 1995)

flips on single pseudoline

Destainville, 2001: *Mixing times of plane rhombus tilings*

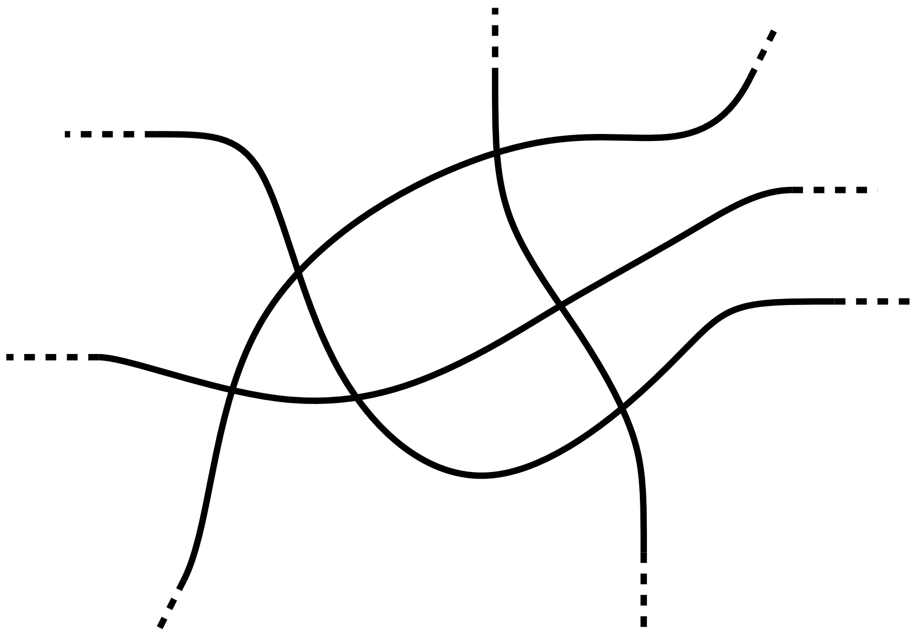


„Nevertheless, the above arguments do not exclude definitively the existence of rare slow fibers, [...]“

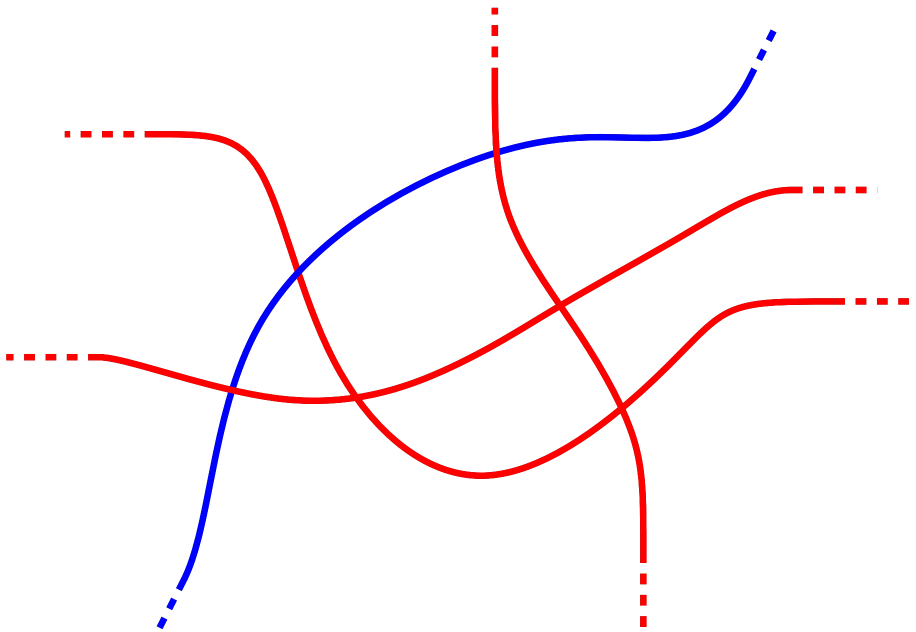
Now we know: „slow fibers“ do exist!

bichromatic triangle conjecture

bichromatic triangle conjecture



bichromatic triangle conjecture



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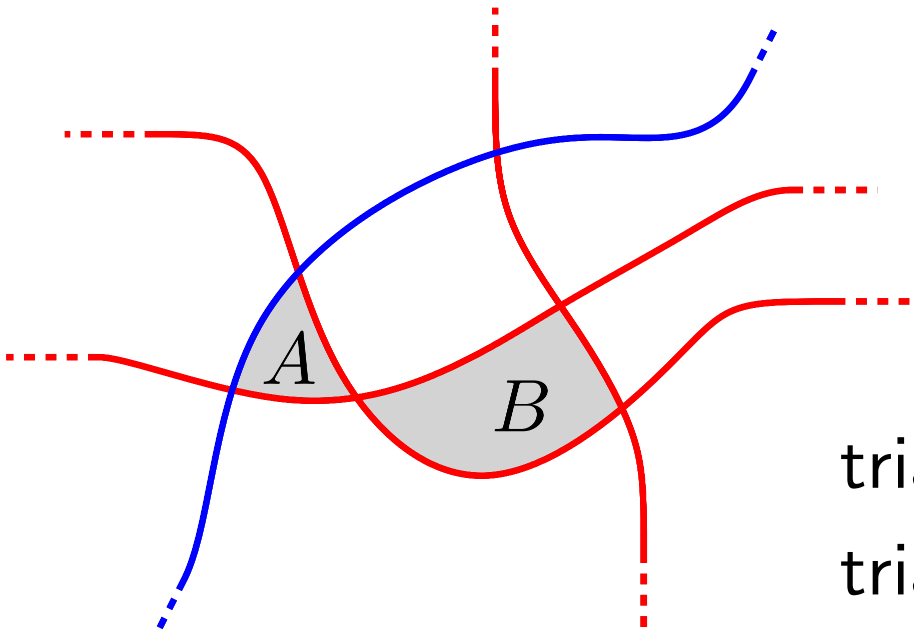
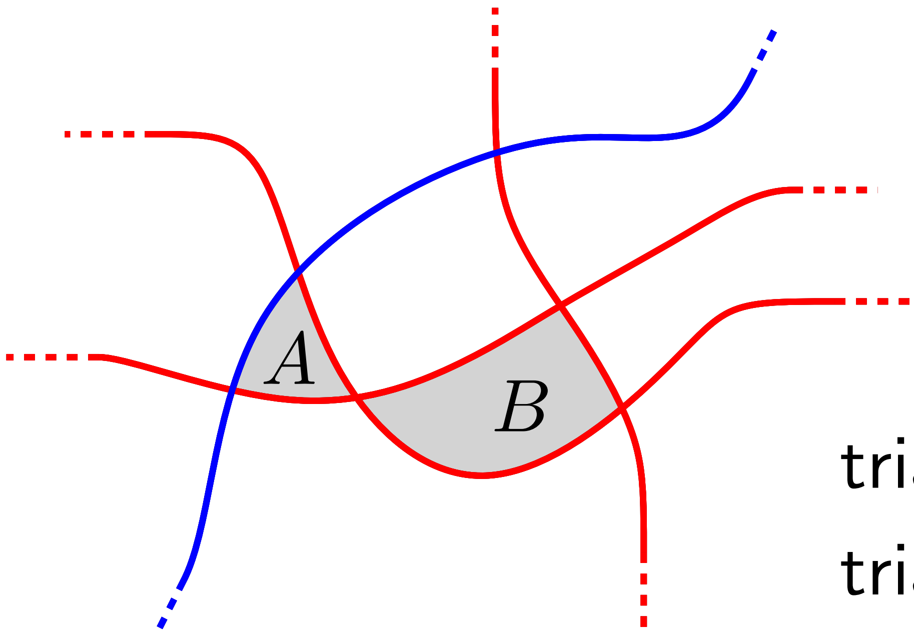


triangle A bichromatic

triangle B monochromatic

bichromatic triangle conjecture



triangle A bichromatic

triangle B monochromatic

Conjecture:
(Björner, Las Vergnas, Sturmfels, White, Ziegler, 1999)

Every truly two-colored arrangement of at least three pseudolines contains a bichromatic triangle.

er

¿P~~re~~guntas?