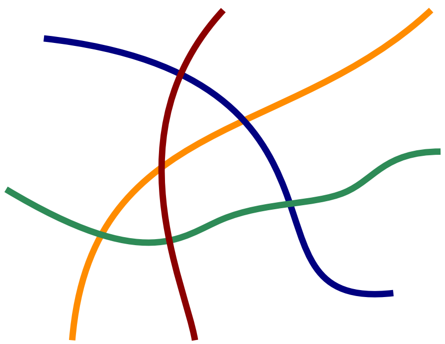
BMS Student Conference 2025

Random Generation of Pseudoline Arrangements



Sandro M. Roch

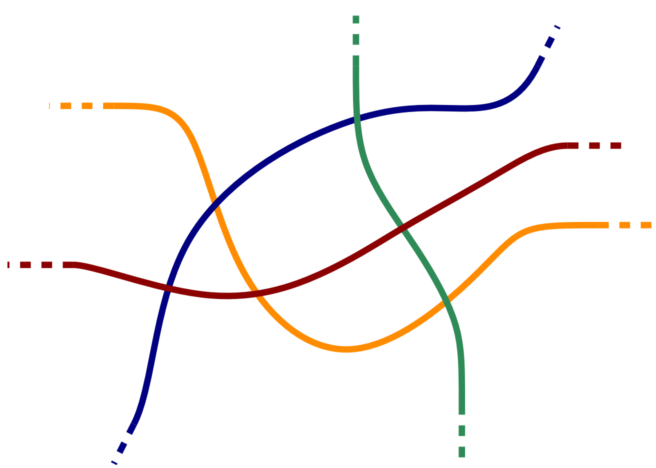
pseudoline arrangements

Def: *pseudoline arrangement*:

- Family of continuous curves $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}^2$ with

$$\lim_{t\to\infty} \|f_i(t)\| = \lim_{t\to-\infty} \|f_i(t)\| = \infty$$

- Each two cross in exactly one point.



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Def: *pseudoline arrangement*:

- Family of continuous curves $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}^2$ with

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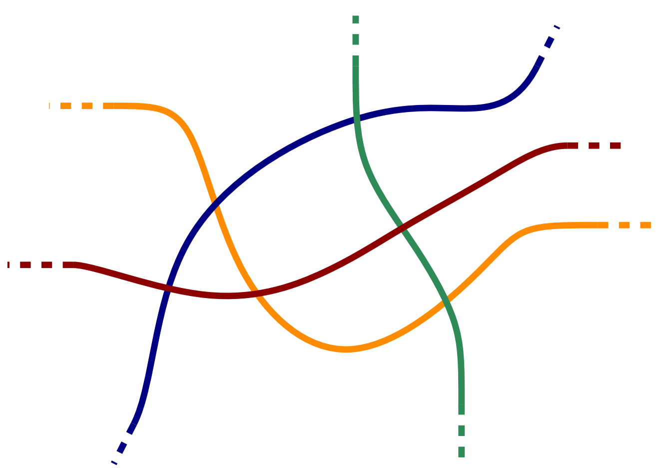
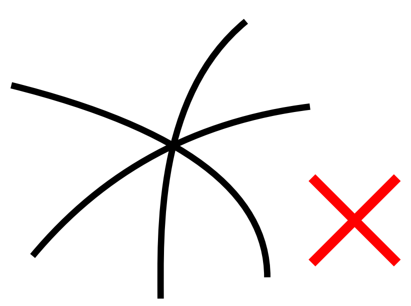
- Each two cross in exactly one point.
- No 3 pseudolines cross at a single point.





wiring diagrams

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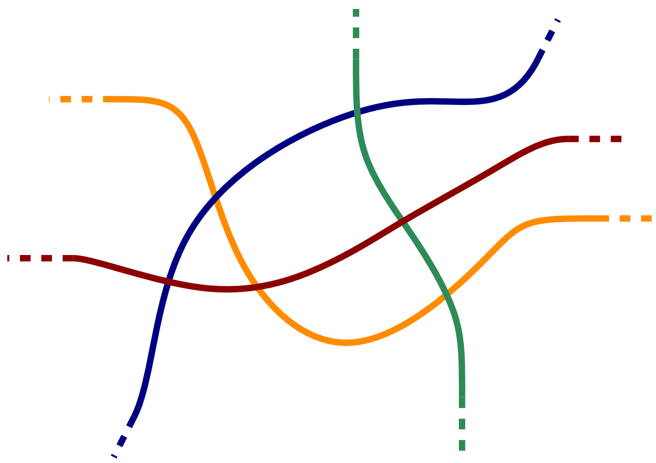

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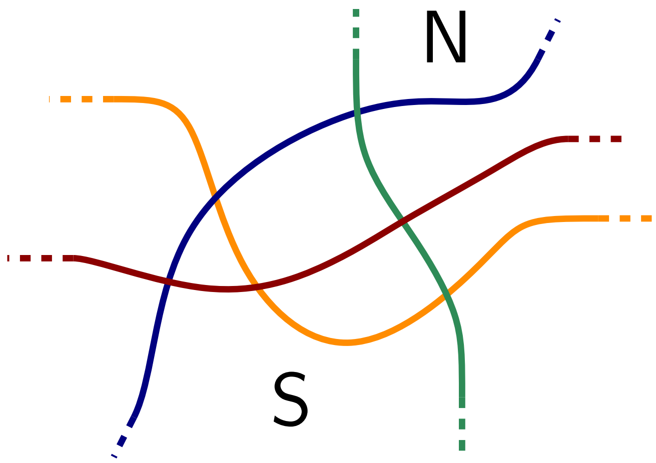

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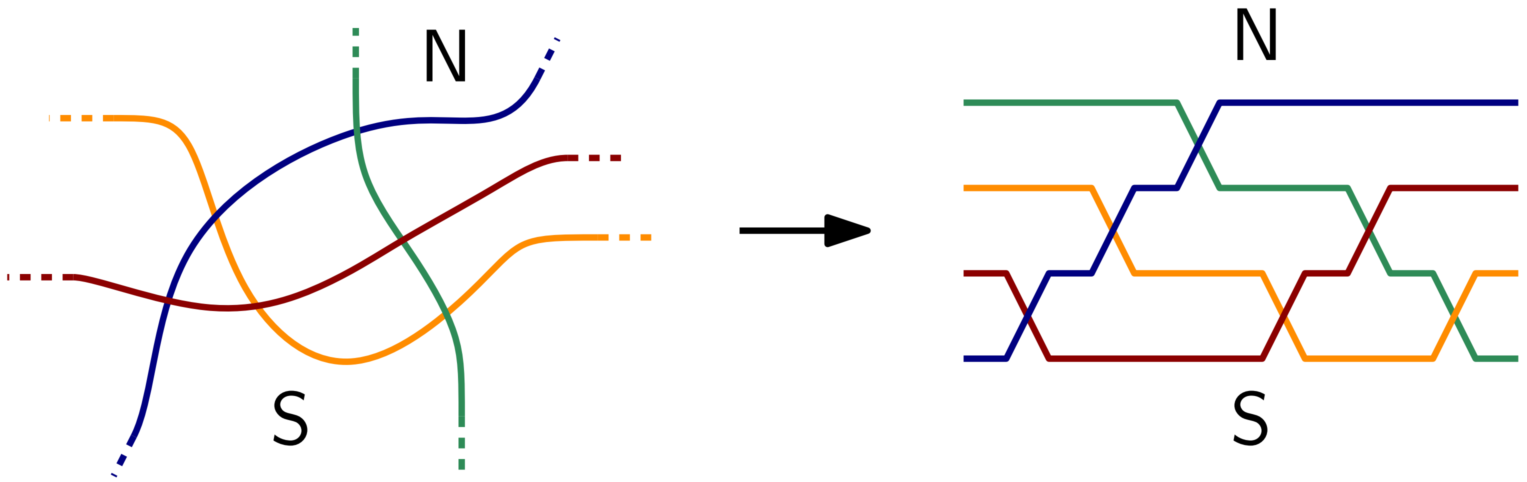

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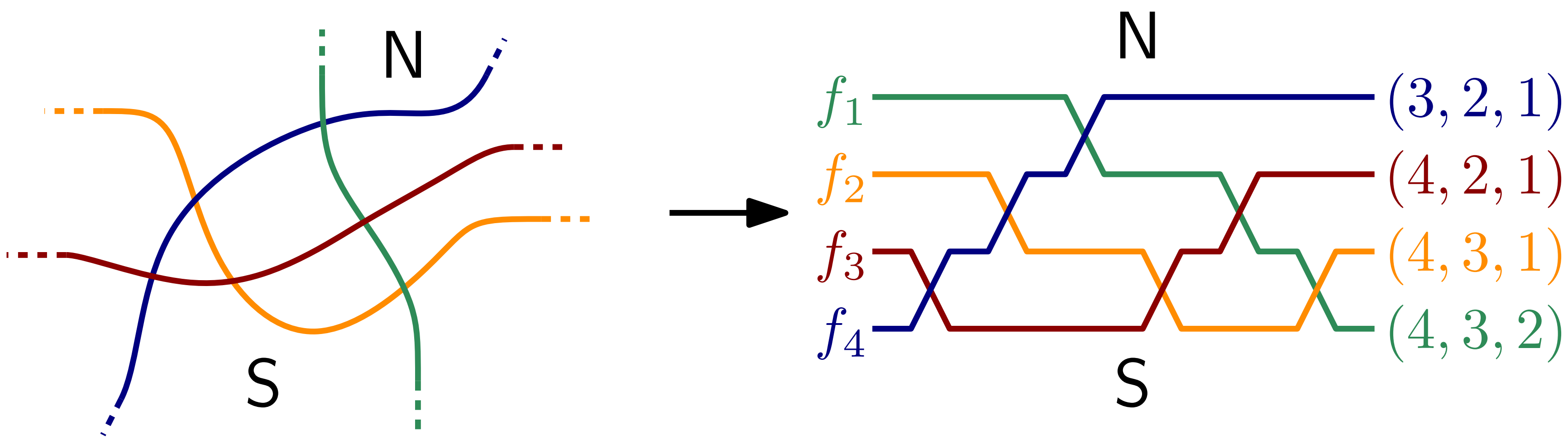
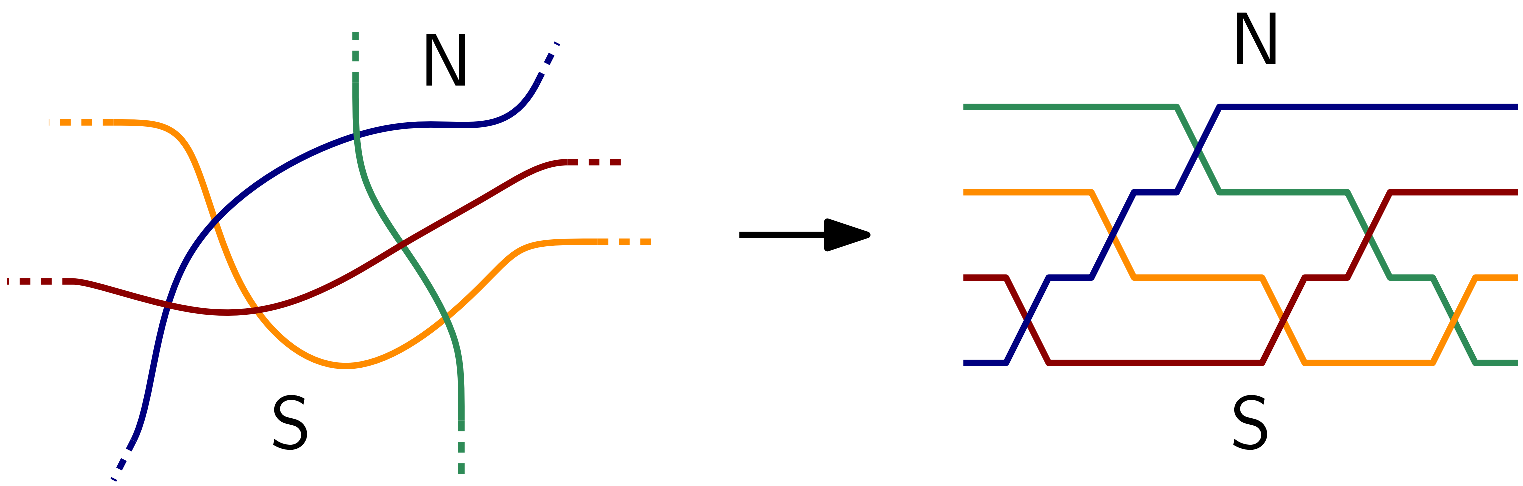
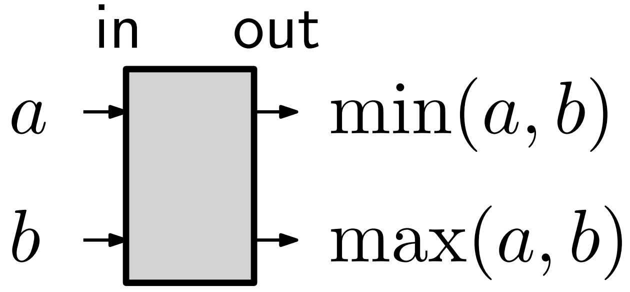


Encoding by permutations:
Permutation $\pi_i \in S_{n-1}$ encodes intersection order of f_i.

wiring diagrams



Wiring diagrams as sorting networks:



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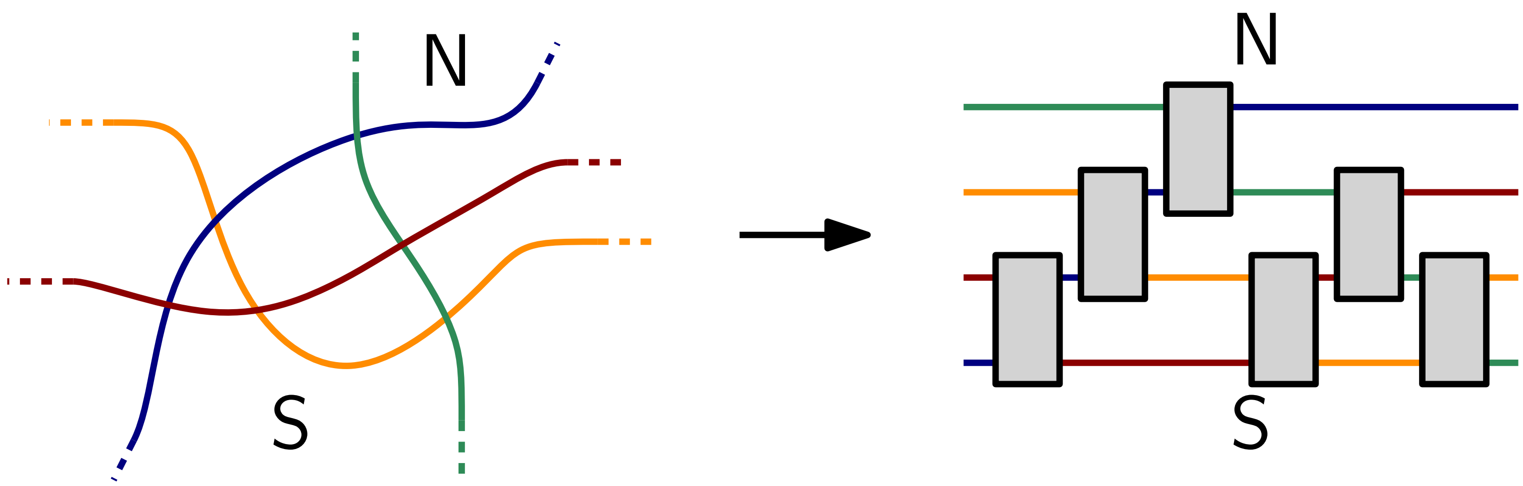

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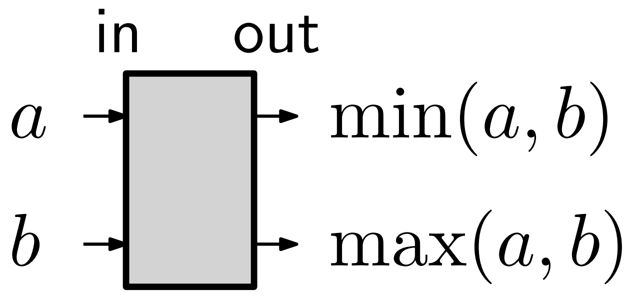

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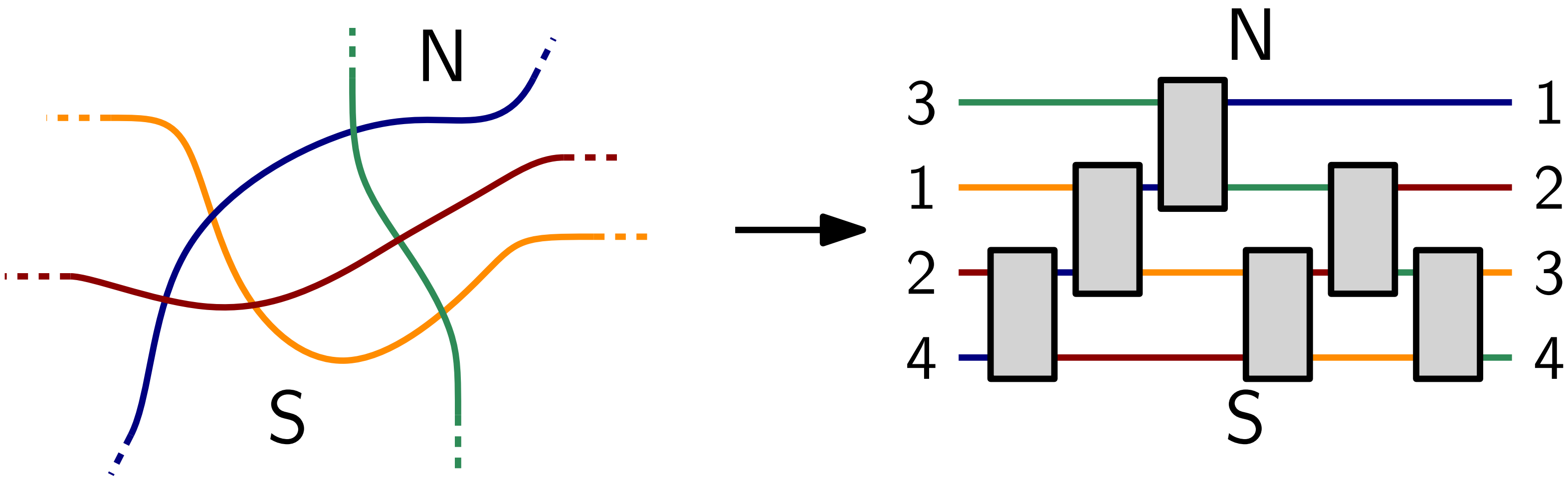

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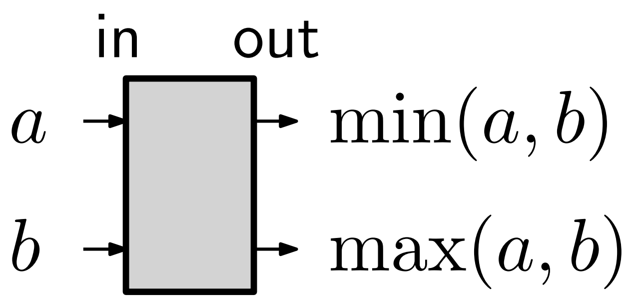

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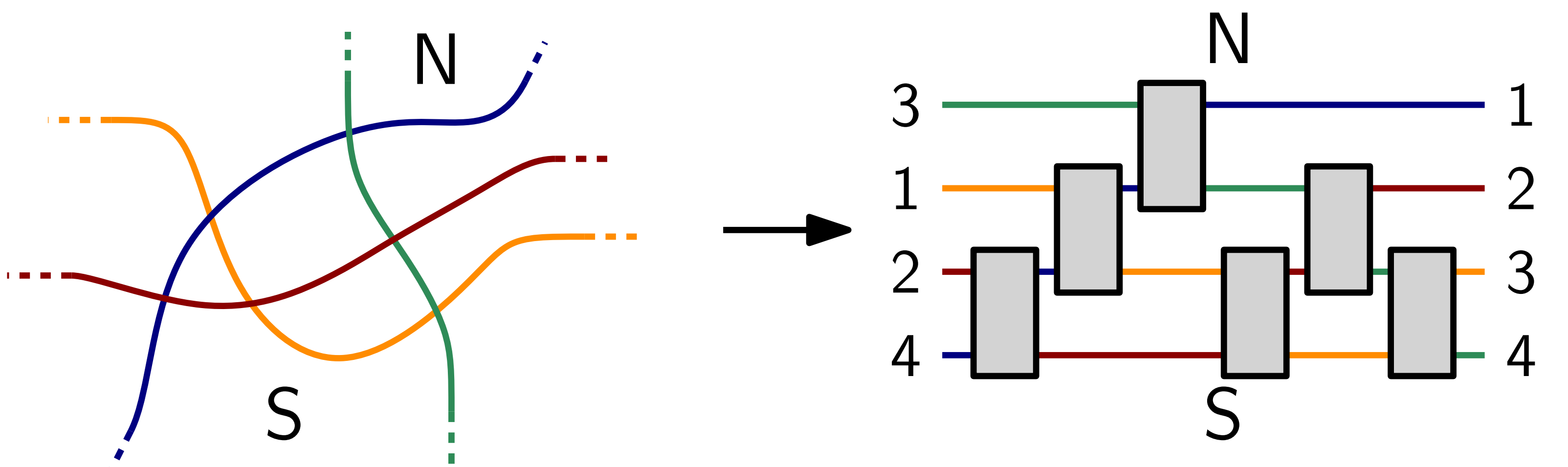

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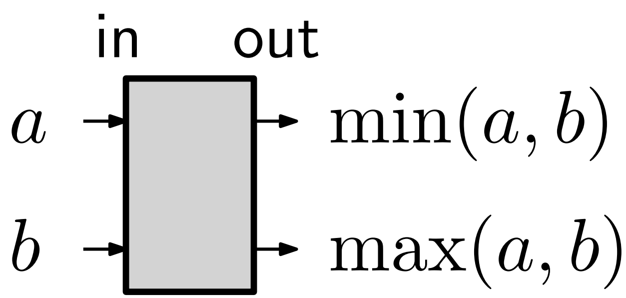

Sorting networks encode minimal sorting algorithms that are based on *comparison & exchange* of neighbor elements.

monotonic paths on permutahedron

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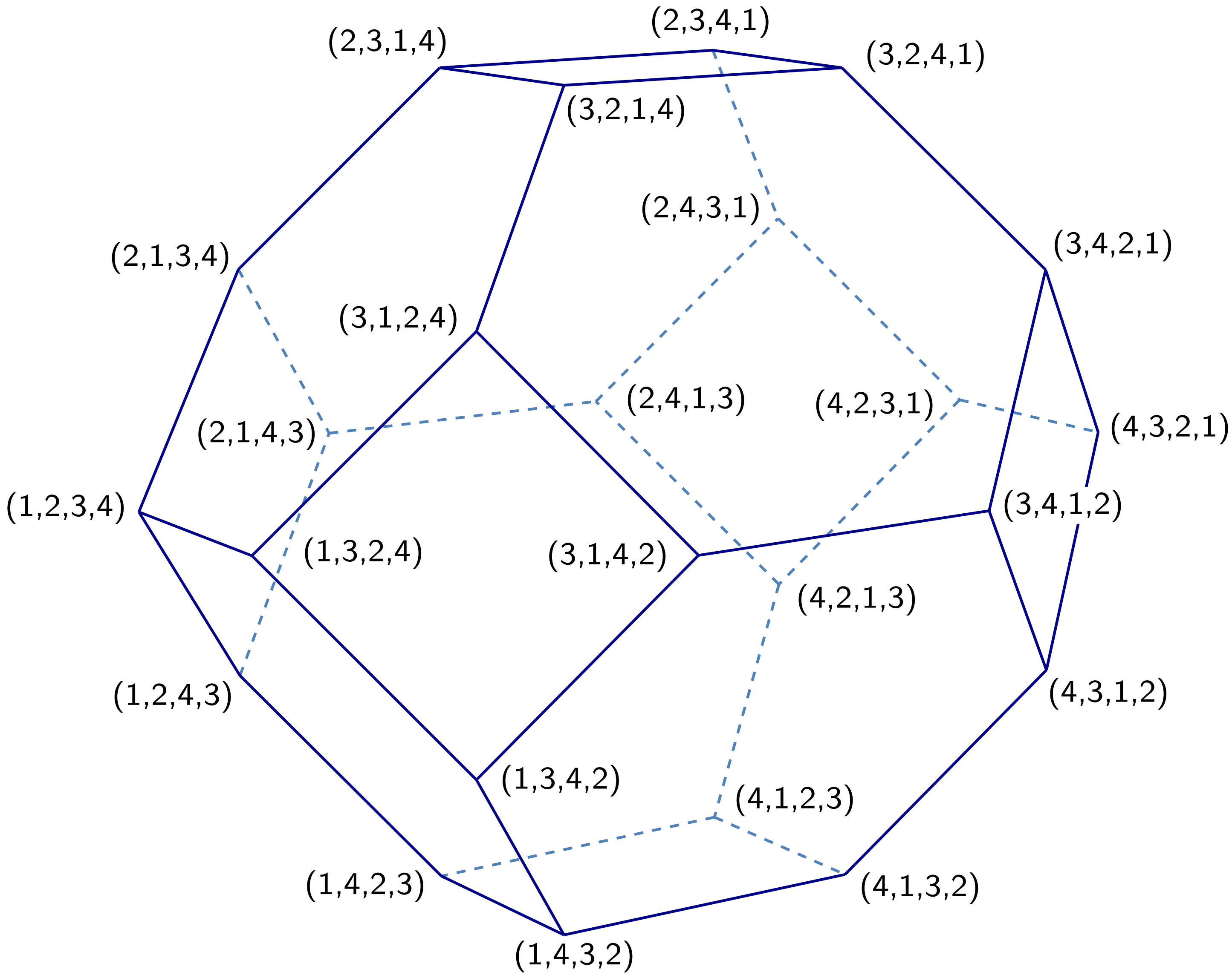

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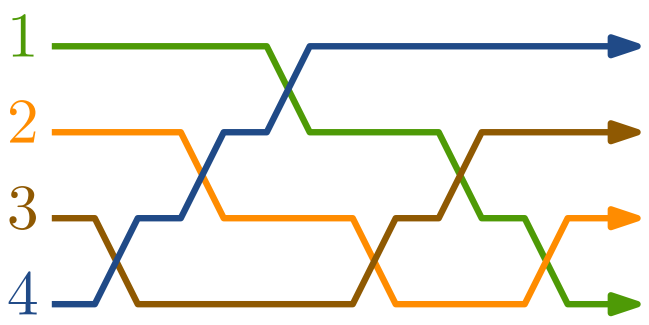

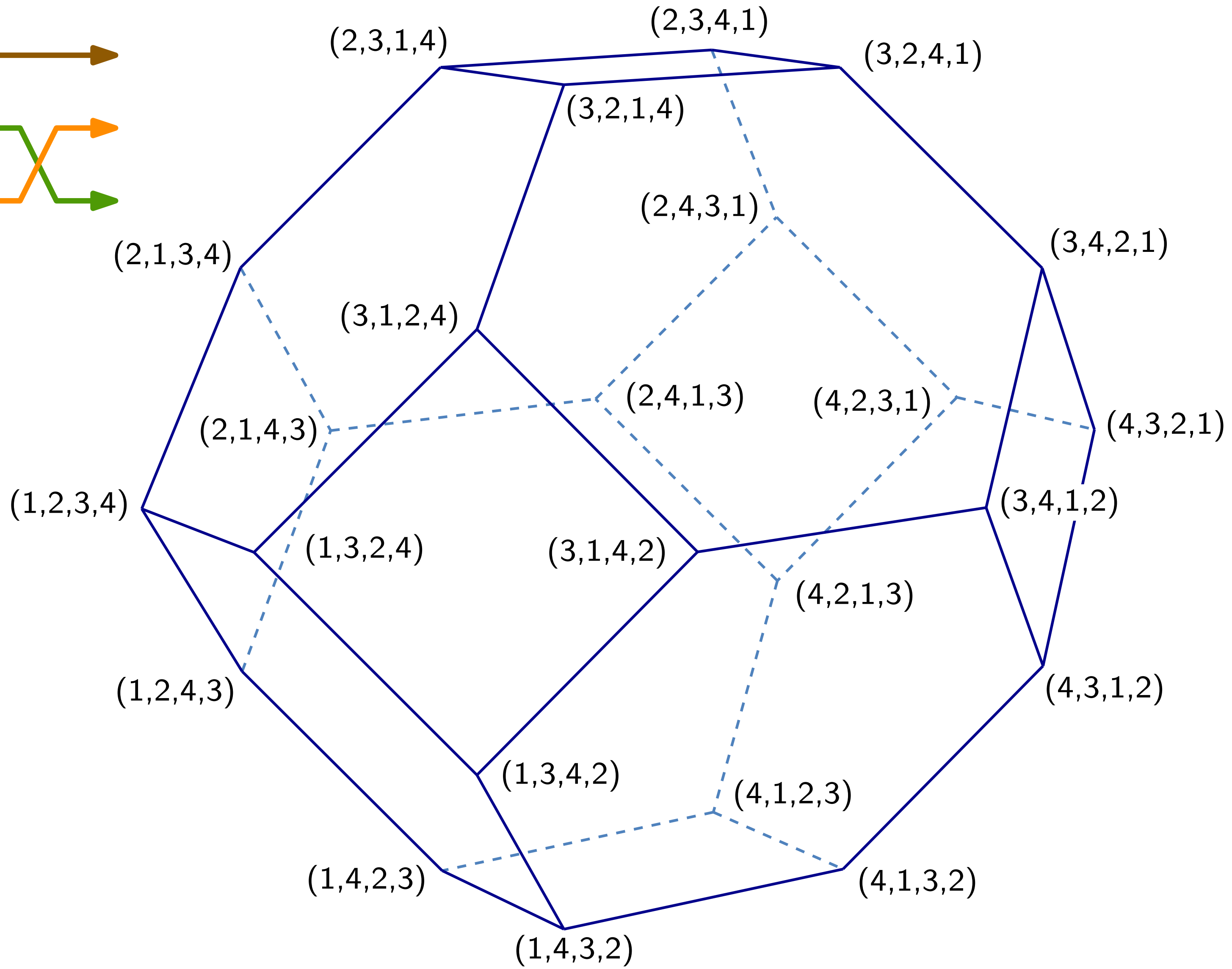

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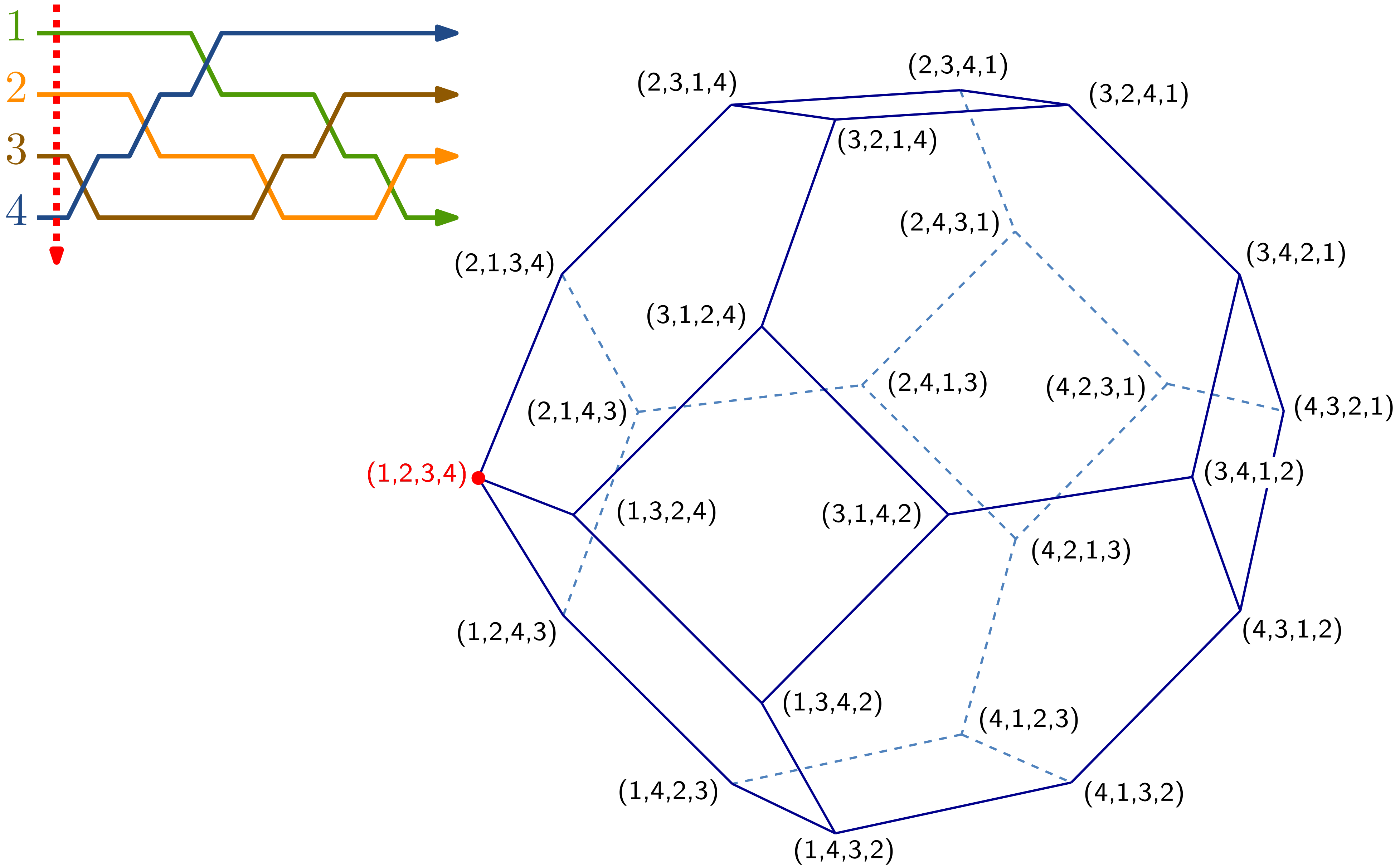

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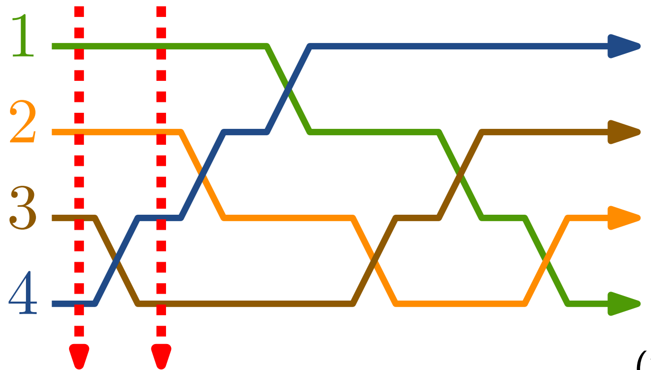

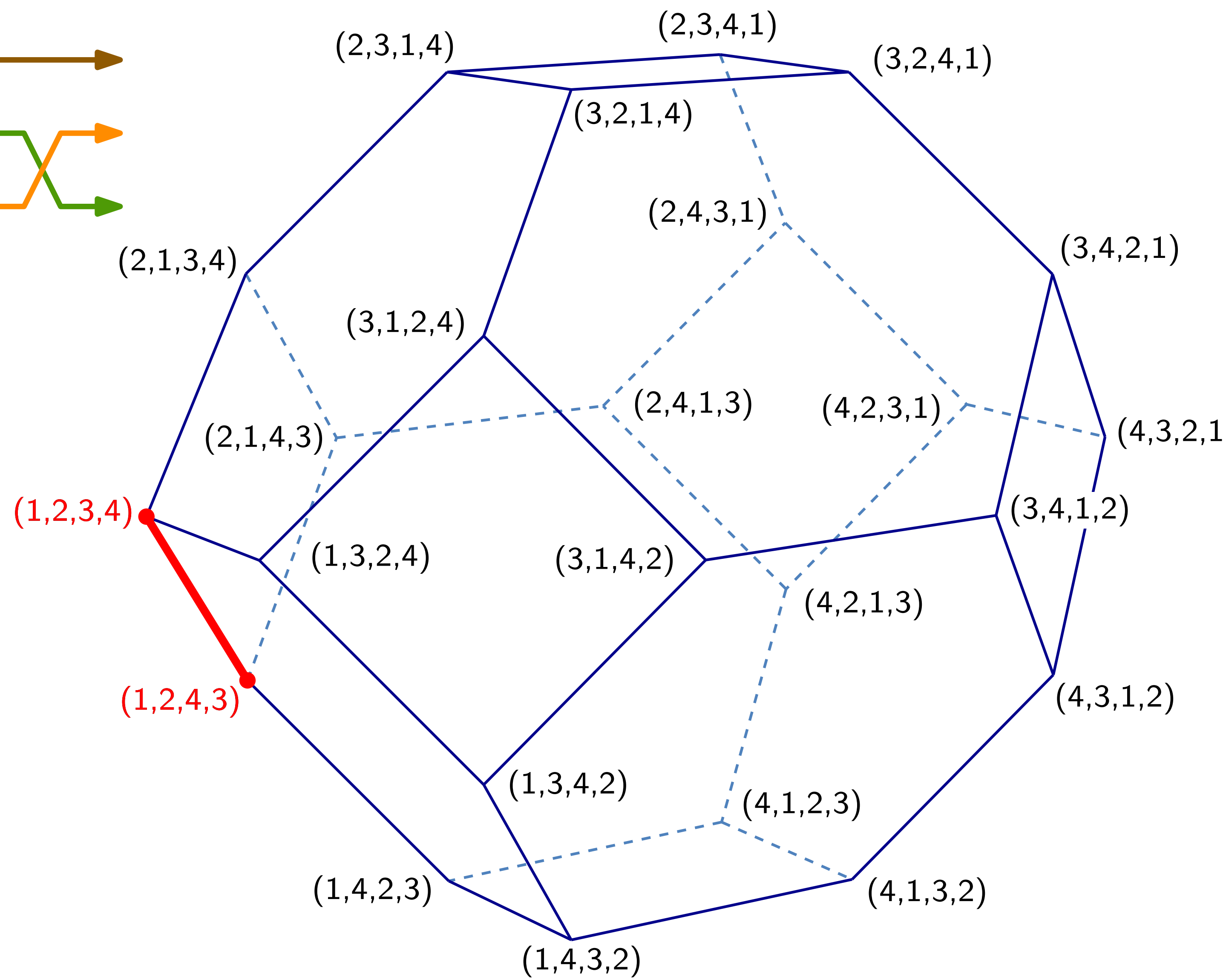

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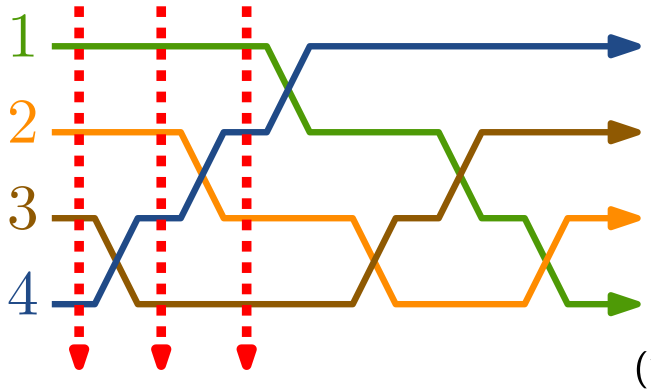

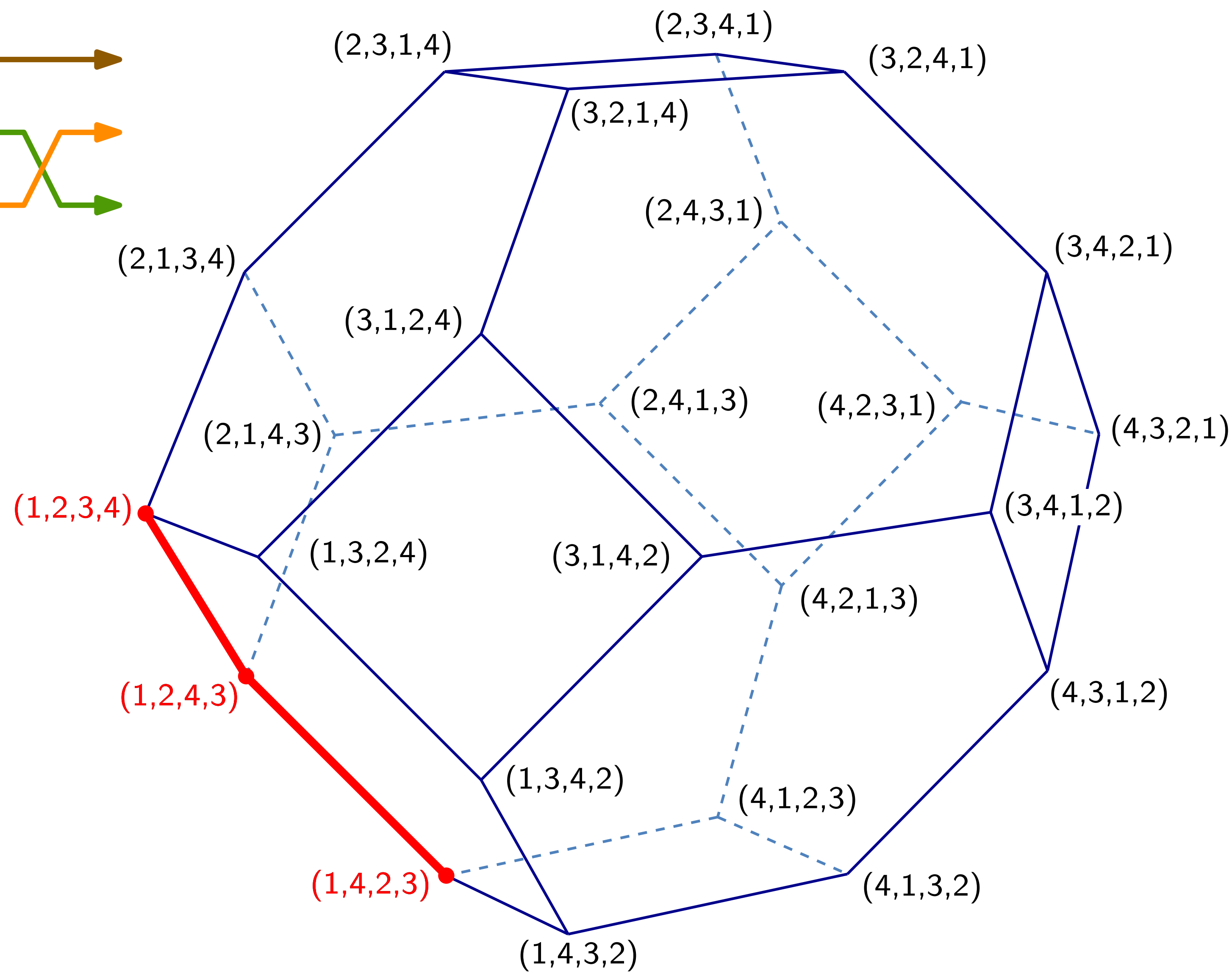

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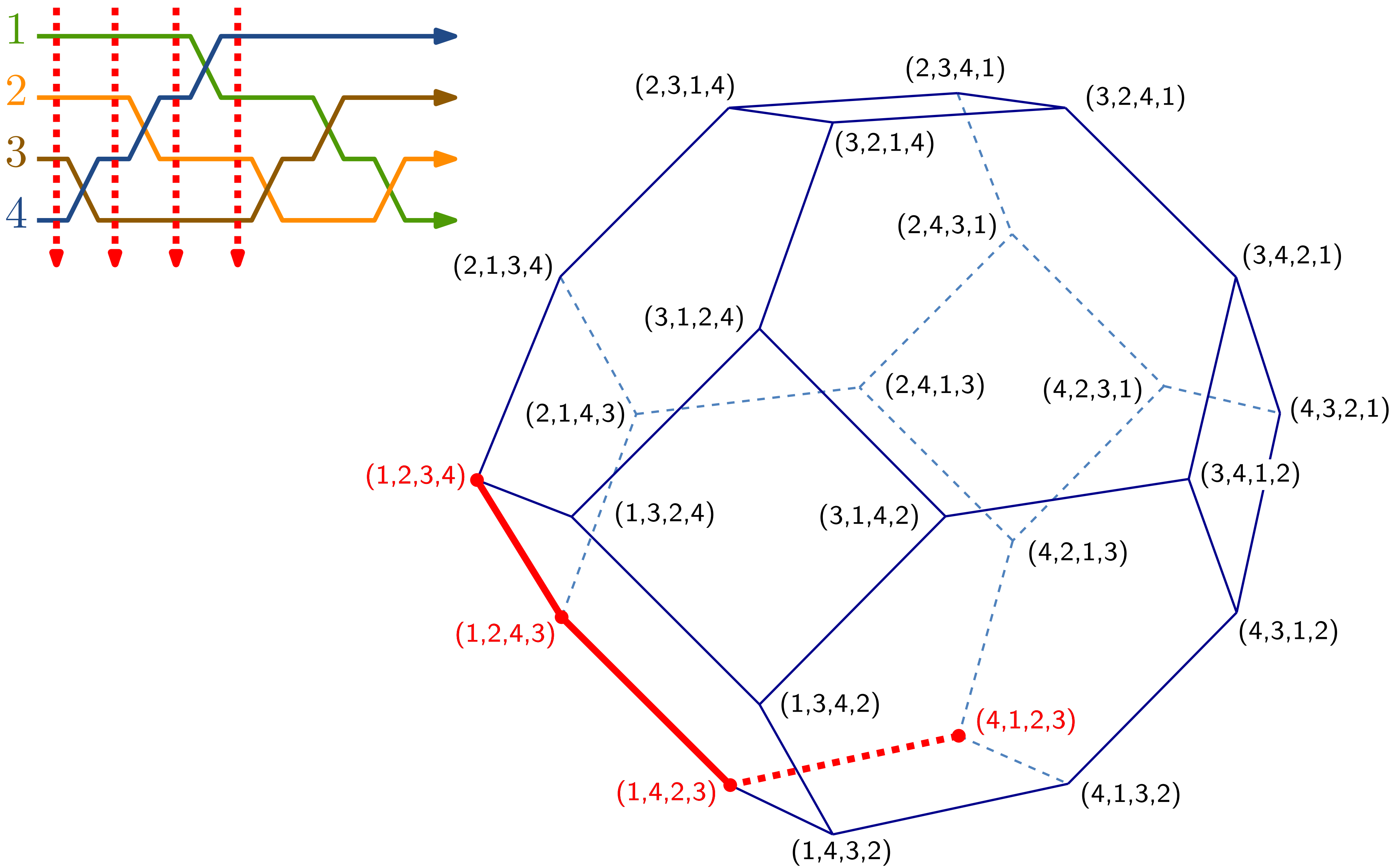

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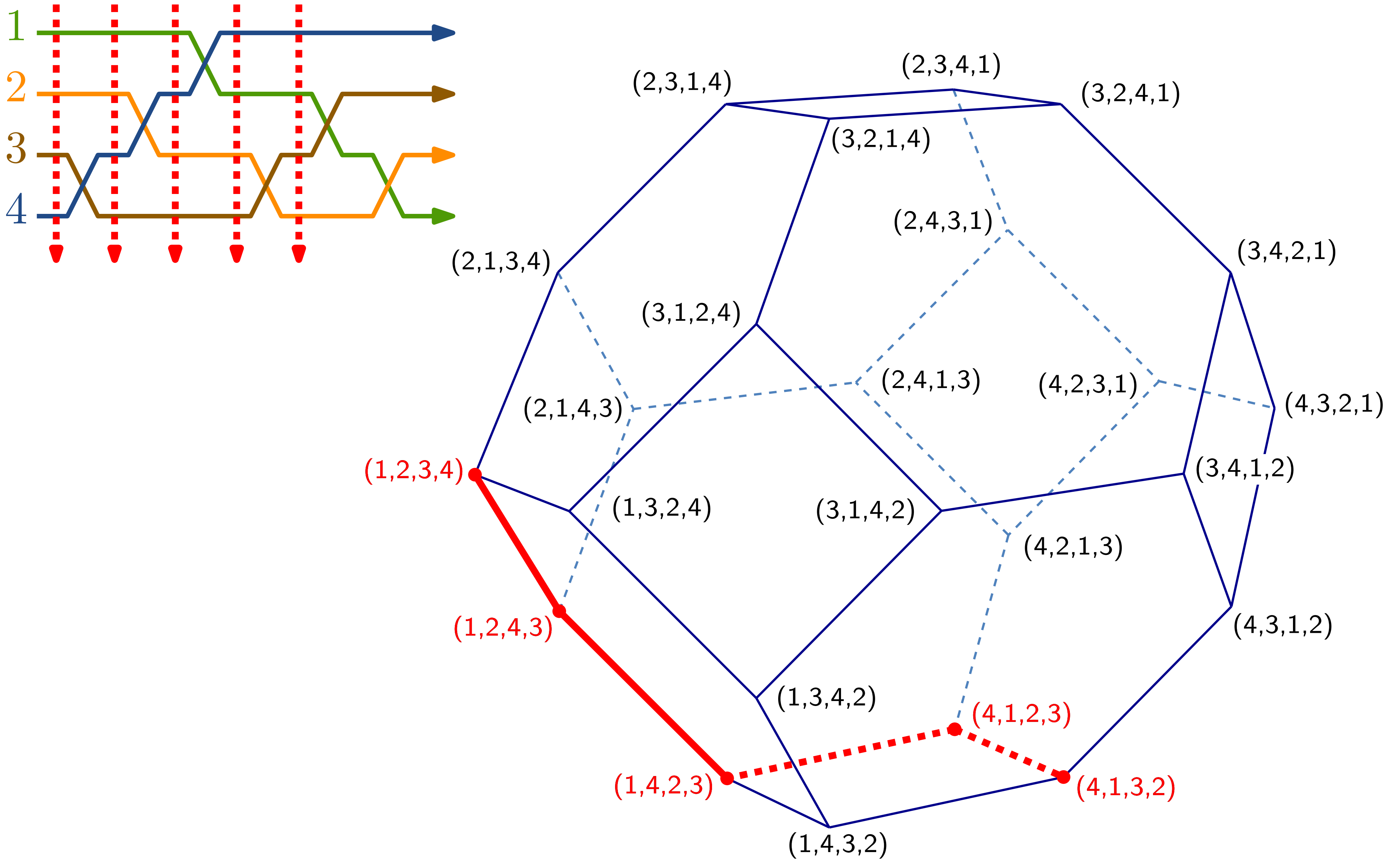

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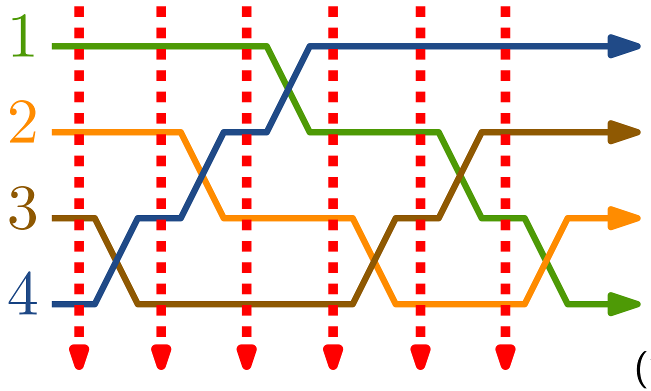

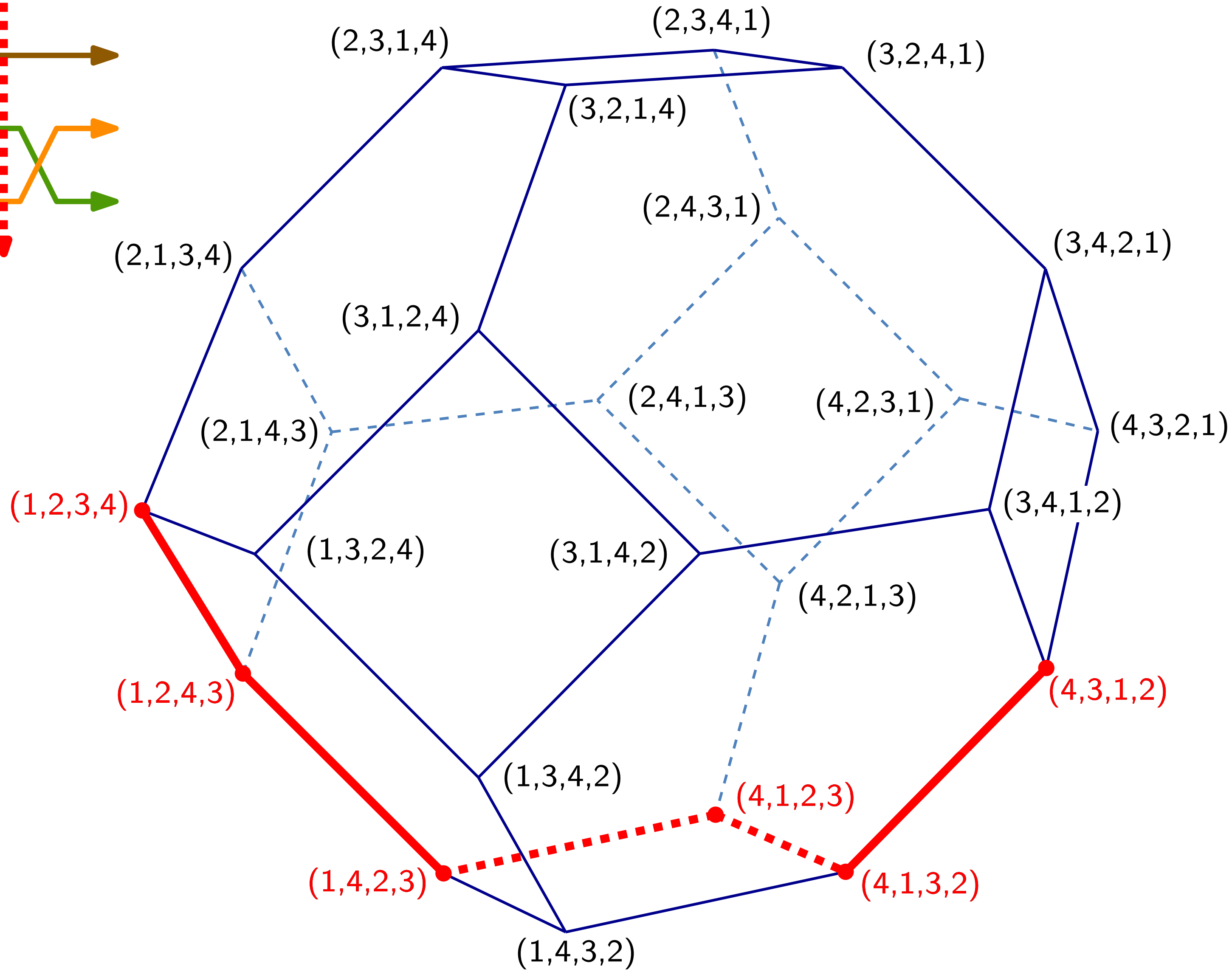

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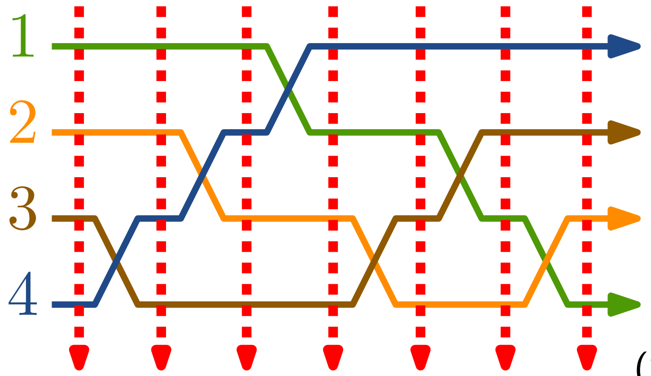

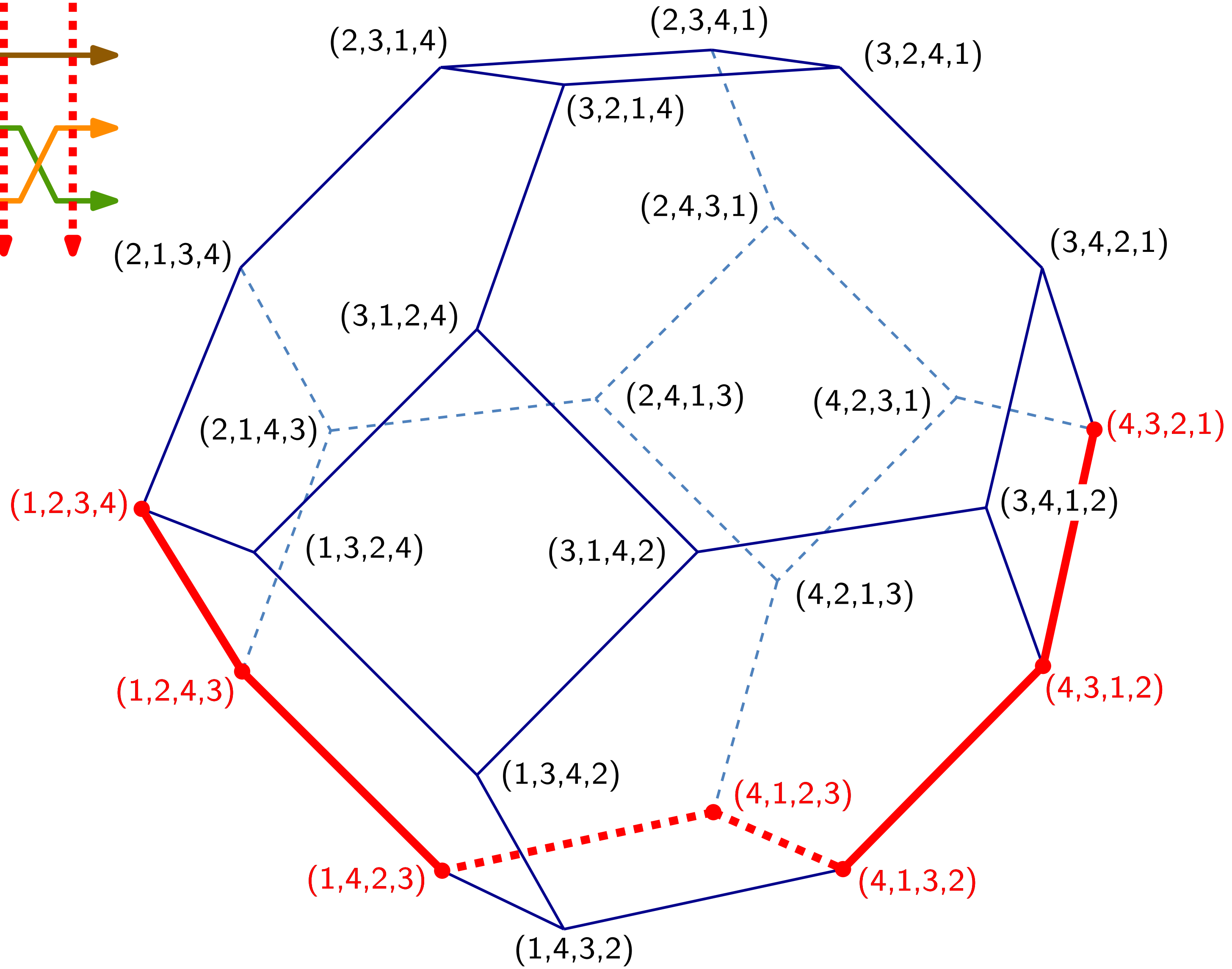

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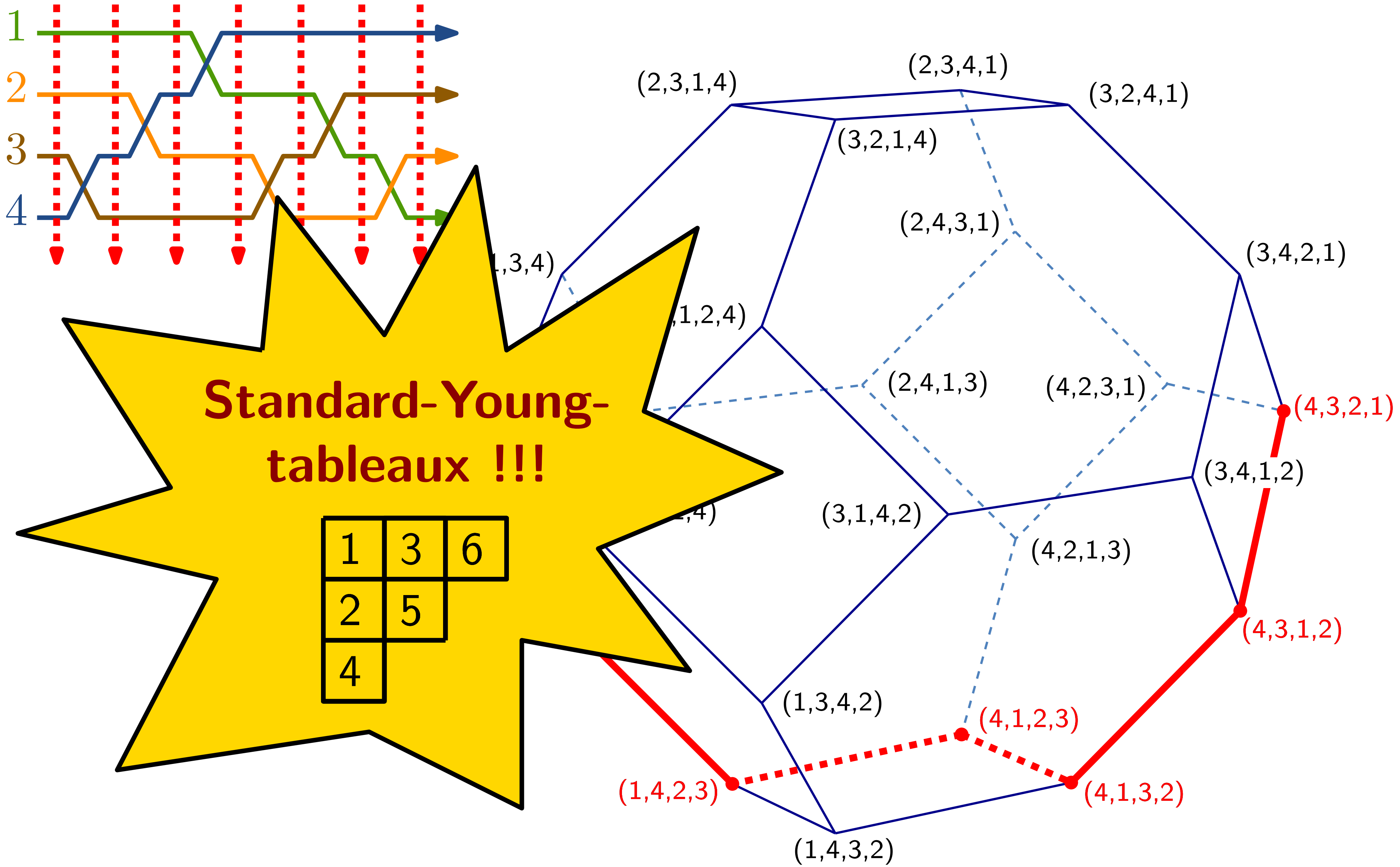
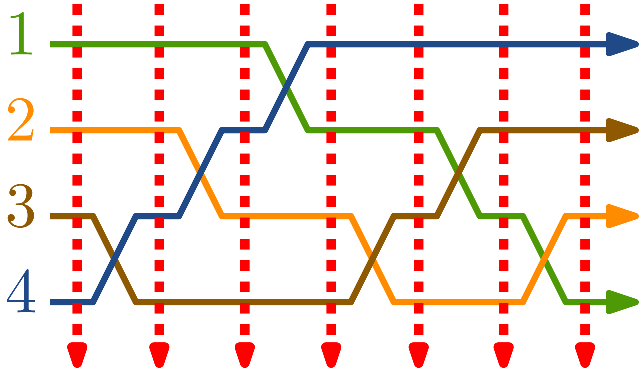


Edelman-Greene bijection



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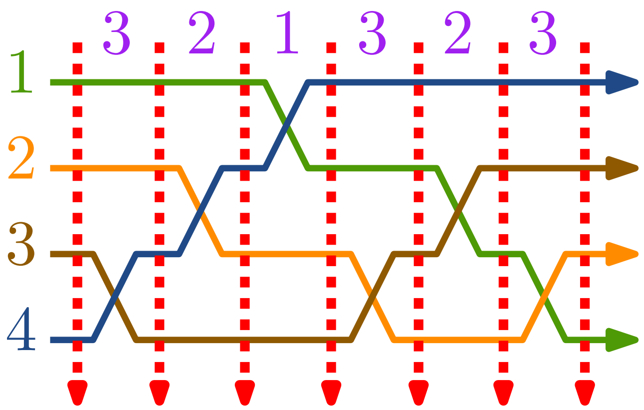

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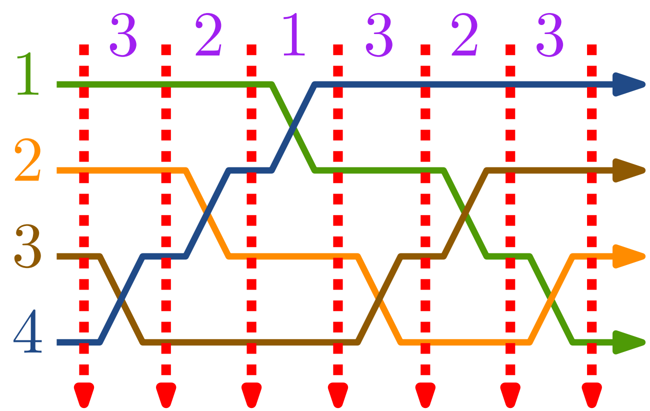

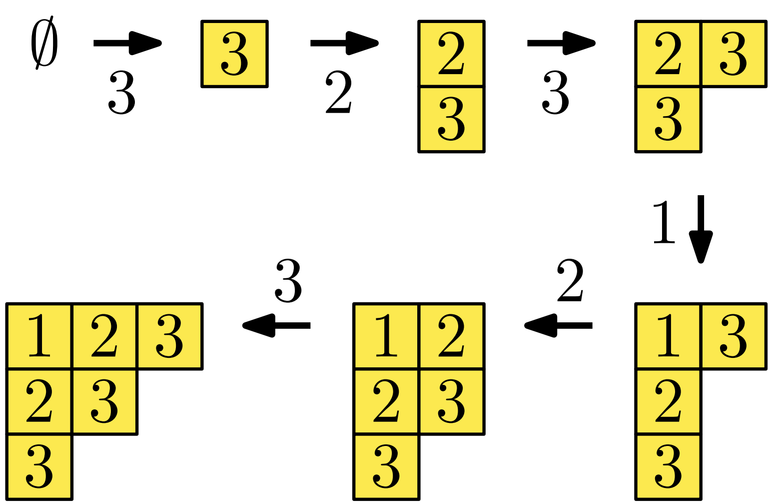

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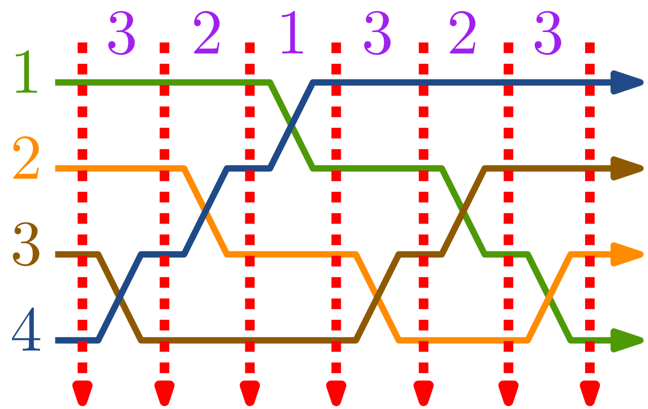
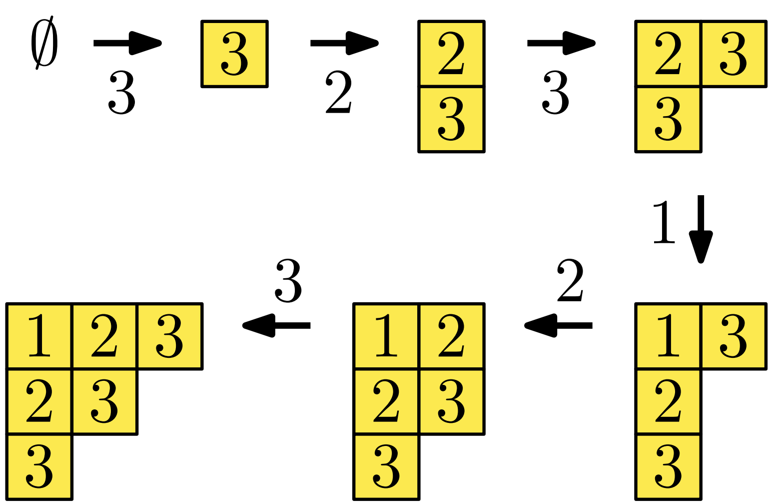
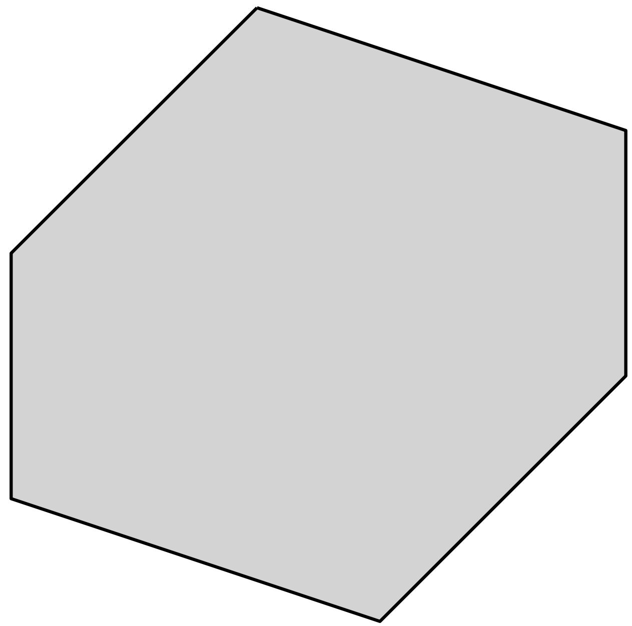




growth:

1	3	6
2	5	
4		

rhombic tilings



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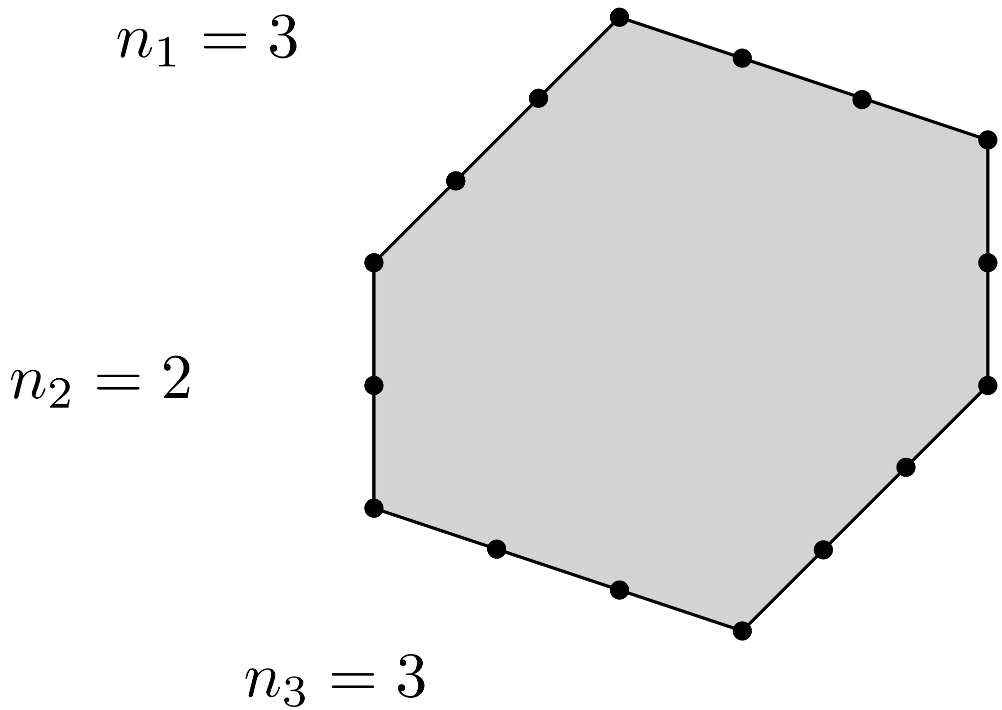
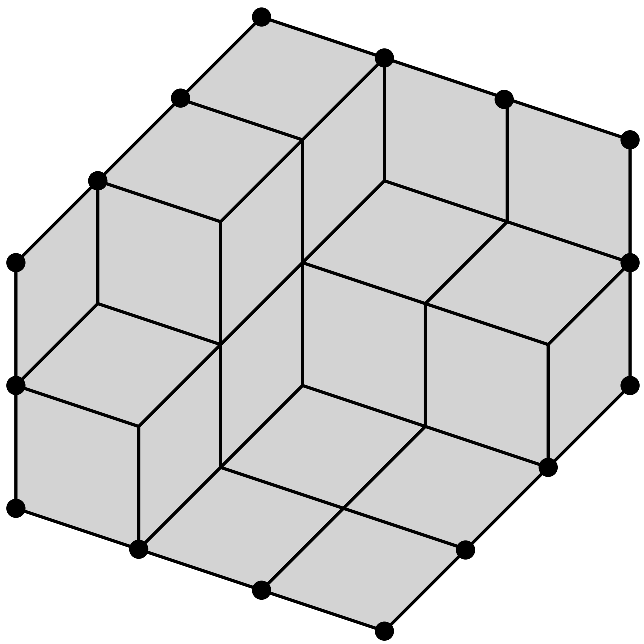


rhombic tilings

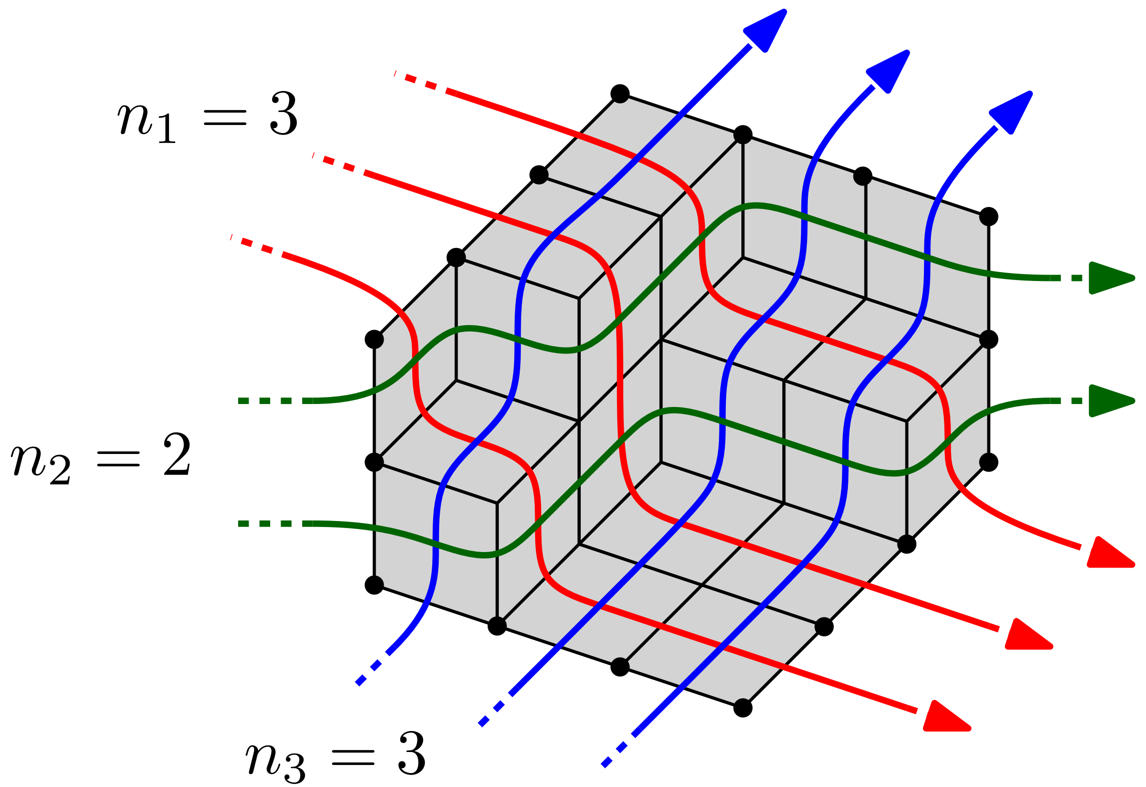
$n_1 = 3$



$n_2 = 2$

$n_3 = 3$

rhombic tilings



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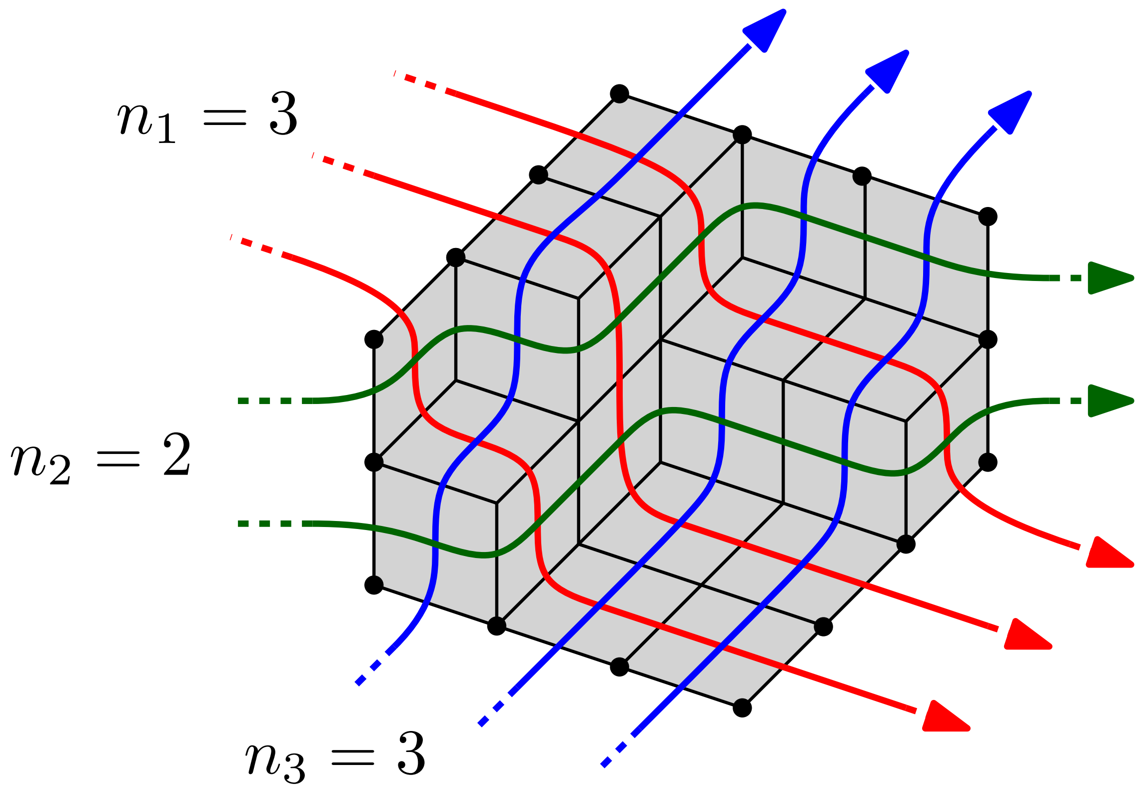
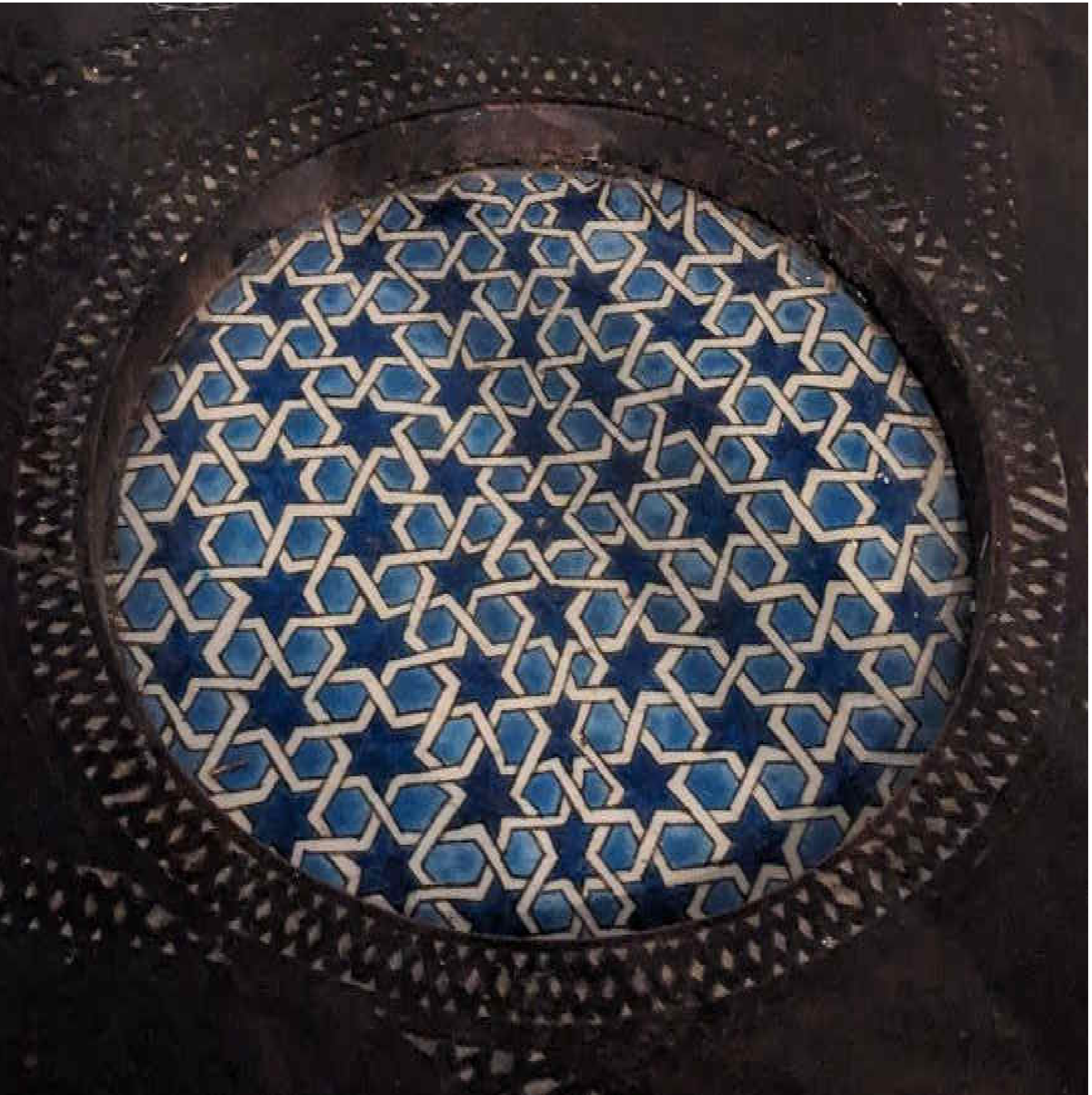
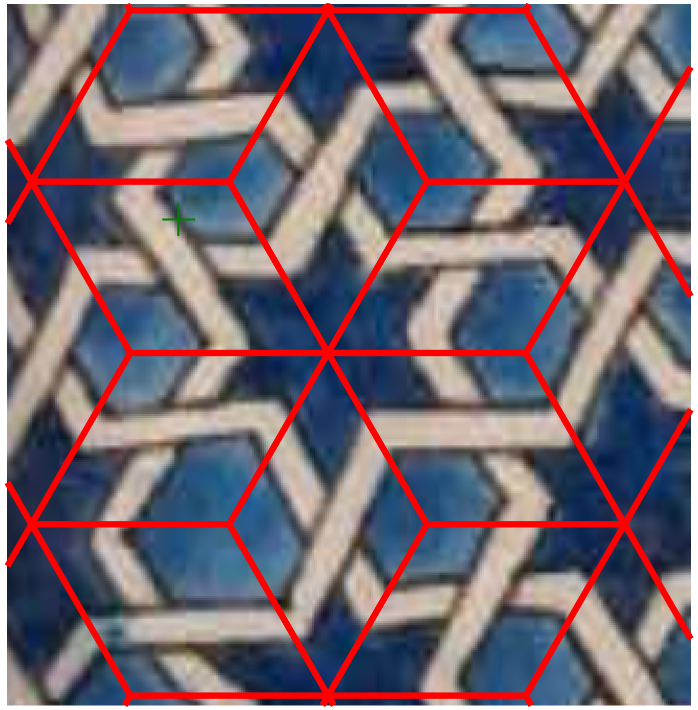


$\Rightarrow$ *generalized pseudoline arrangement*:

- *parallel class* of $n_1, \ldots, n_r$ pseudolines
- (Only) pseudolines of different classes cross

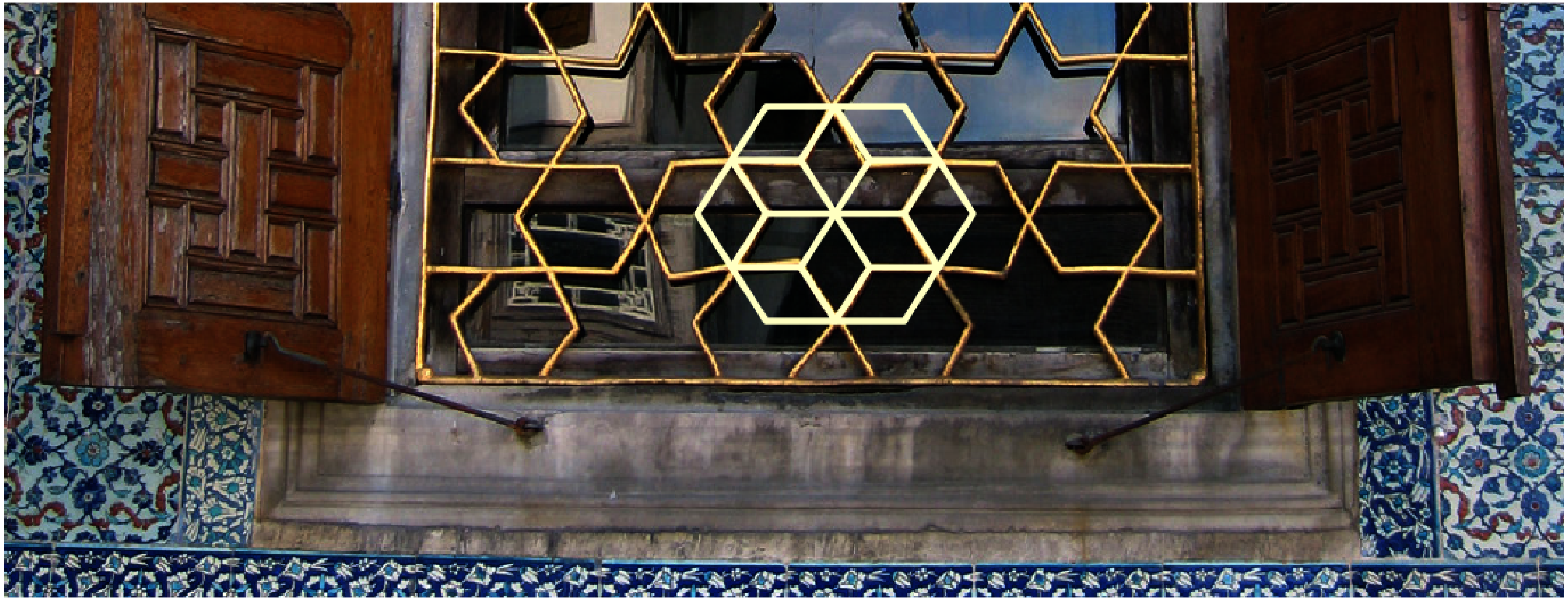
rhombic tilings





Aslan Pasha Mosque
Ioannina, Greece

rhombic tilings



Topkapı Palace, Istanbul, Turkey

pseudoline arrangements

wiring diagrams

signotopes

plane partitions

permutations

rhombic
tilings

sorting networks

higher Bruhat
orders

Standard Young tableaux

families of monotonic
non-crossing paths

oriented matroid of rank 3

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Problem:
How can pseudoline arrangements be efficiently generated uniformly at random?

rhombic tilings

sorting networks

higher Bruhat orders

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families of monotonic non-crossing paths

oriented matroid of rank 3

rapidly mixing Markov chains

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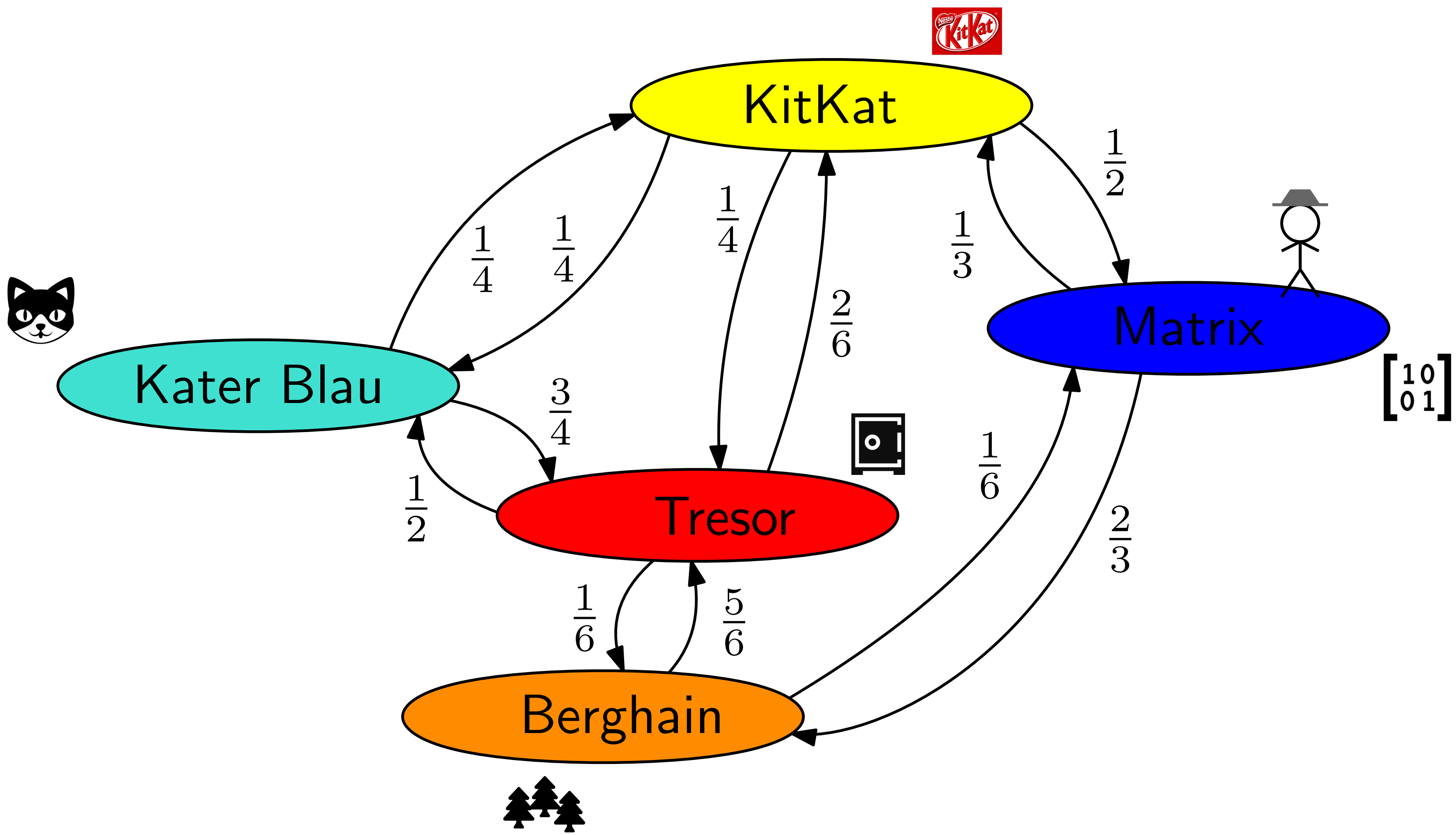
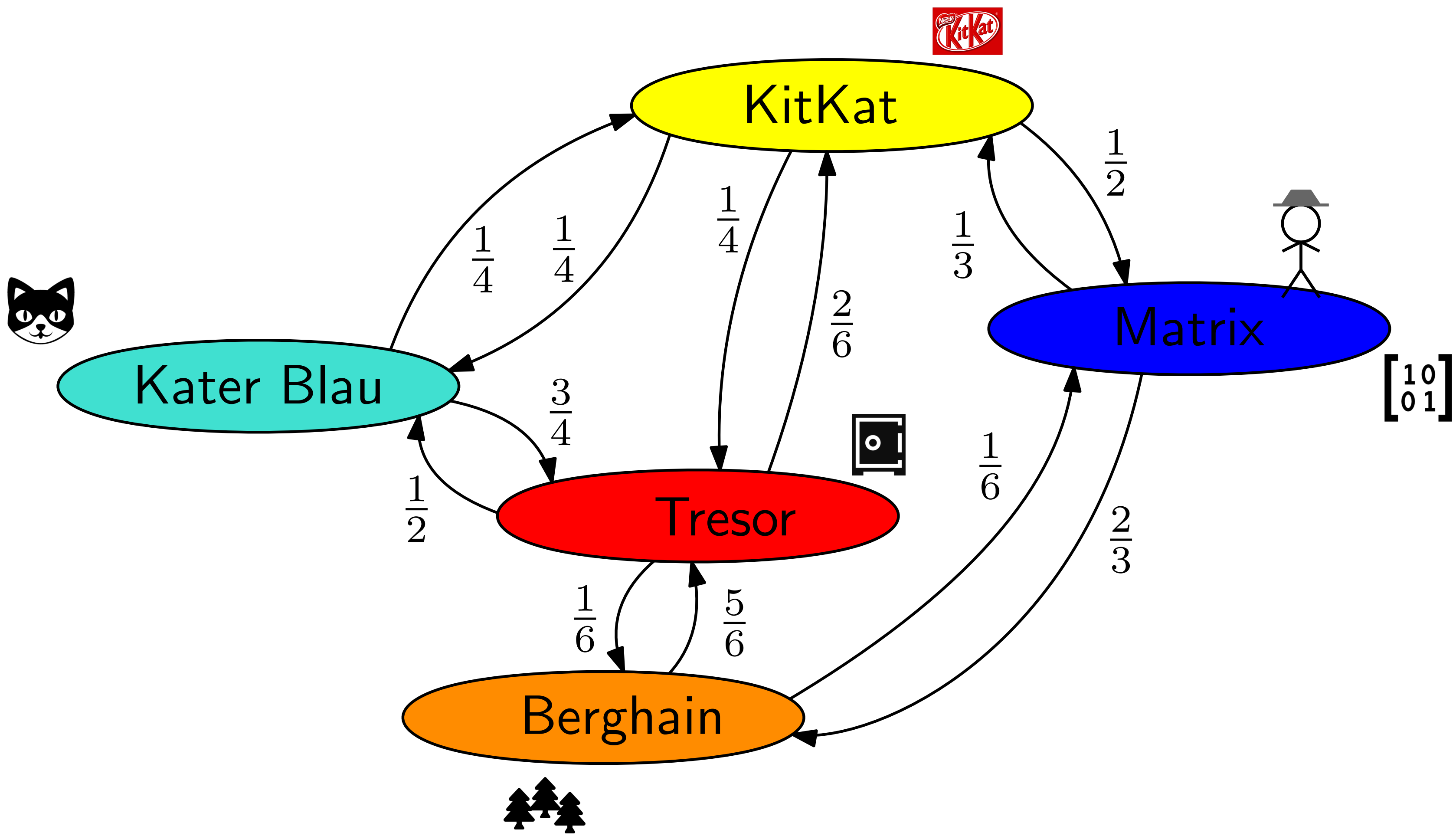


rapidly mixing Markov chains

Markov chain (X_t), state space $\mathscr{X}$, transition prob. $P : \mathscr{X} \times \mathscr{X} \to [0,1]$



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$$\mathbb{P}[X_t = x] \to \pi(x)$$

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- *Total variation distance* of distributions μ, μ' on $\mathscr{X}$:

$$\|\mu - \mu'\|_{\mathsf{TV}} := \sup_{M \subseteq \mathscr{X}} |\mu(M) - \mu'(M)|$$

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- Define $d(t) := \max_{x \in \mathcal{X}} \|P^t(x, \cdot) - \pi\|_{\mathsf{TV}}$

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Markov chain (X_t), state space $\mathscr{X}$, transition prob. $P : \mathscr{X} \times \mathscr{X} \to [0,1]$

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Def: A class of Markov chains is *rapidly mixing* if for each of them

$$\tau(\varepsilon) \in \mathcal{O}\left(p\left(\log \frac{|\mathscr{X}|}{\varepsilon}\right)\right) \qquad \text{for some } p \in \mathbb{R}[X].$$

random generation using Markov chains

Idea:

- States $\mathscr{X} = \{\text{arrangements of fixed size}\}$
- Symmetric transition probabilities

$\implies$ After many steps get almost uniform arrangement

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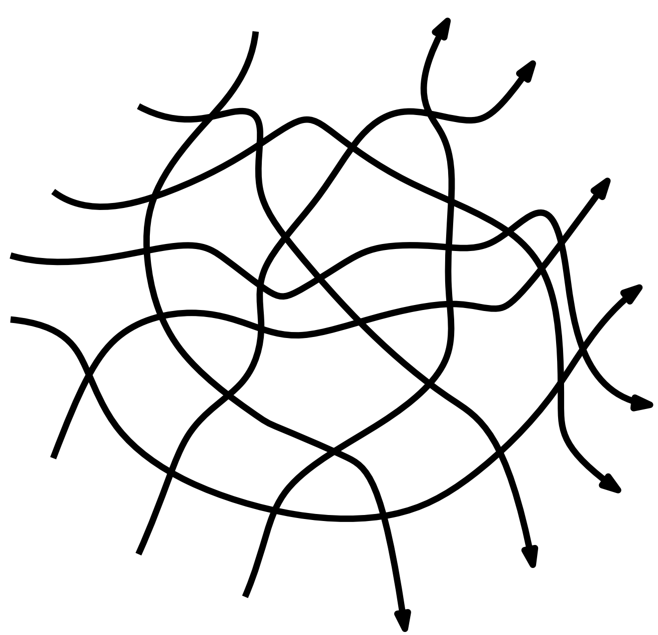
Markov chain I: random reinsertion of pseudoline



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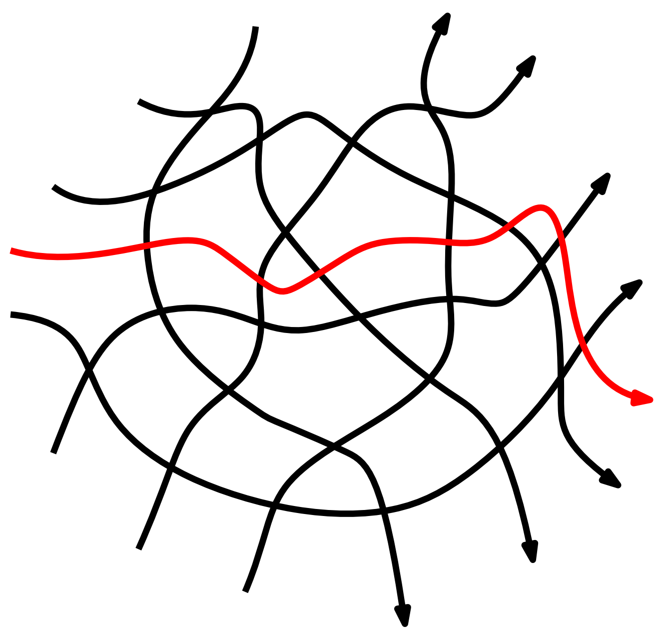

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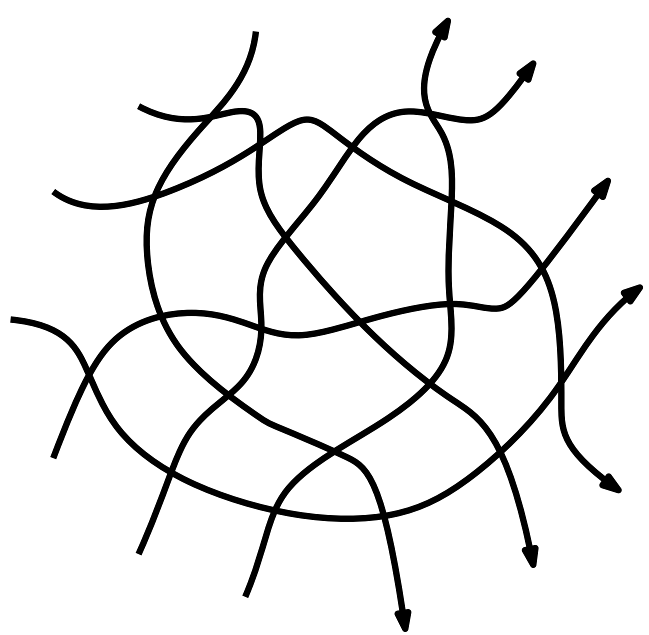

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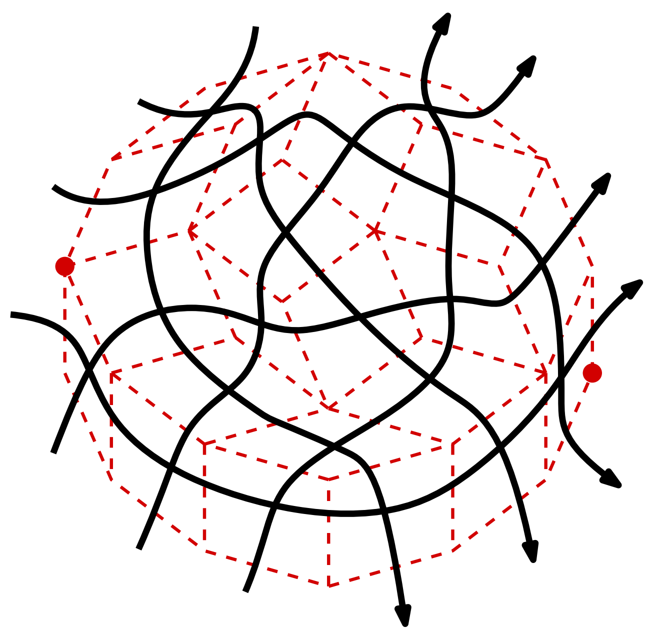

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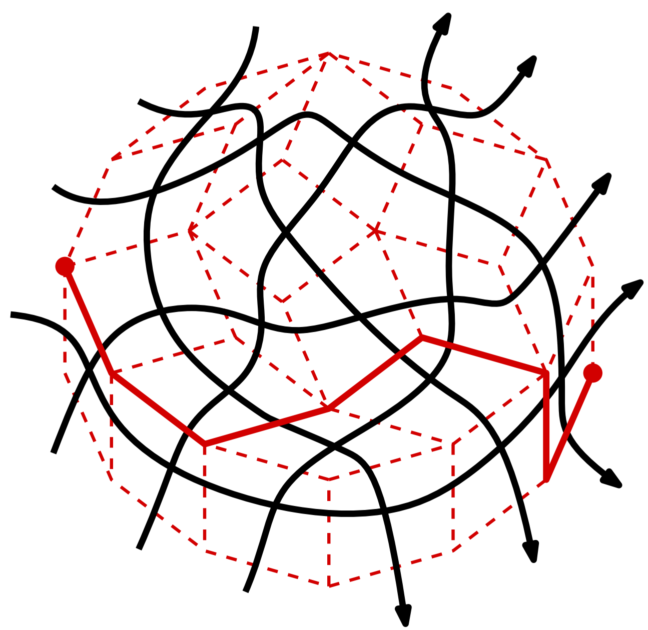

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$\Longrightarrow$ After many steps get almost uniform arrangement

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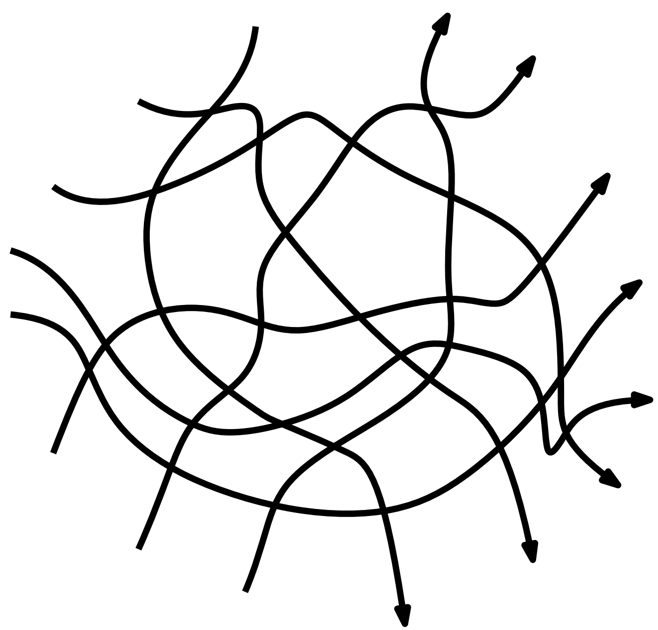

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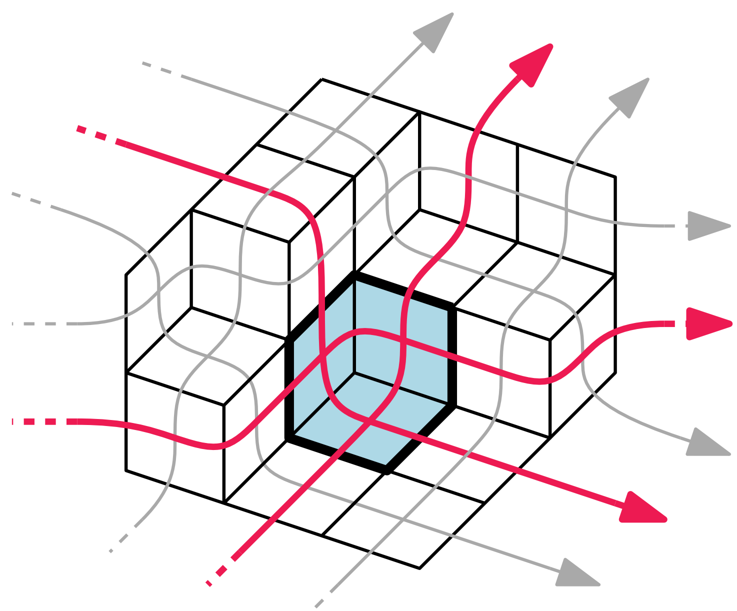
Markov chain II: random triangle flip



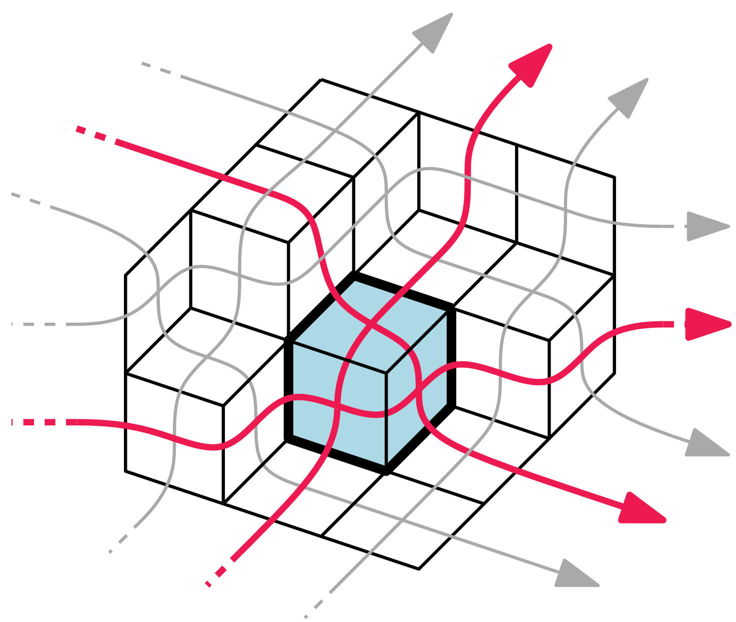
random generation using Markov chains

Idea:

- States $\mathscr{X} = \{\text{arrangements of fixed size}\}$
- Symmetric transition probabilities

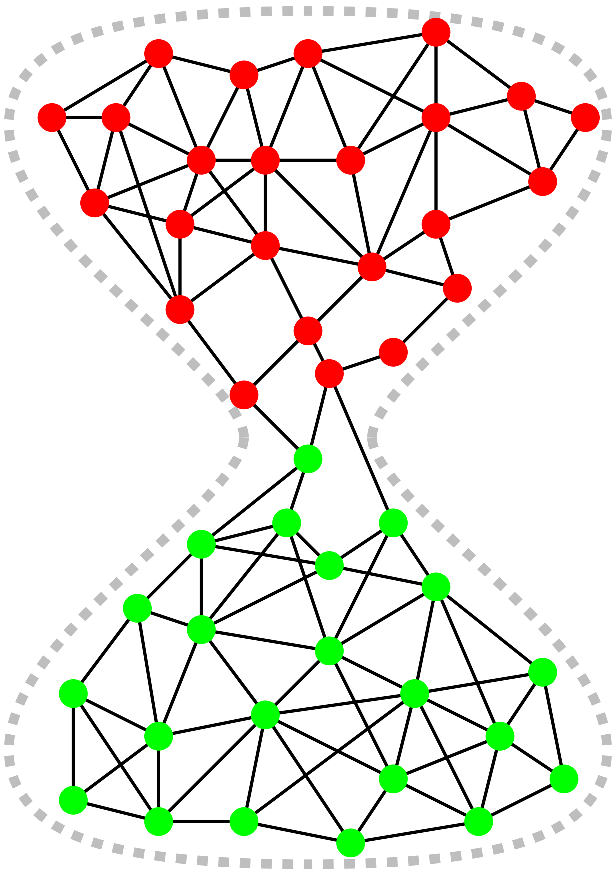
  $\implies$ After many steps get almost uniform arrangement

Markov chain II: random triangle flip

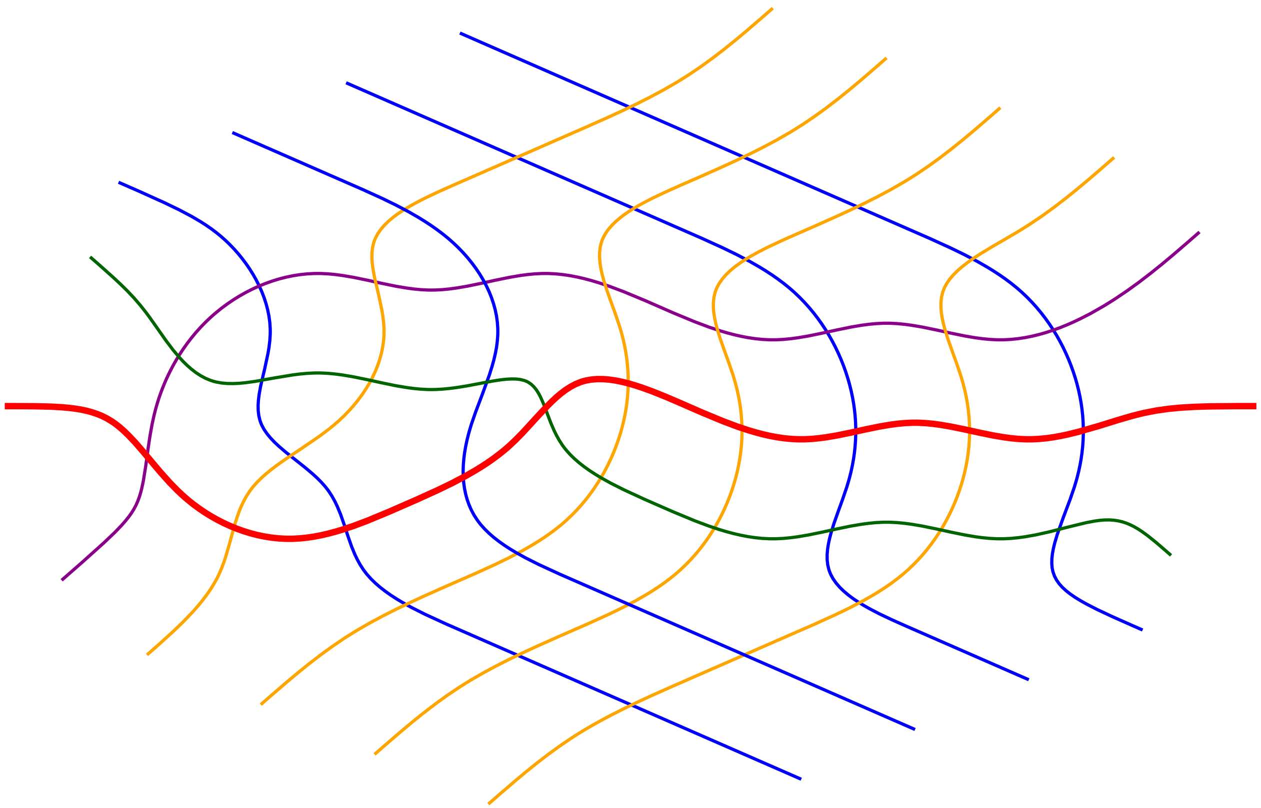


bottleneck

Markov chain having a „bottleneck“:



flips on single pseudoline



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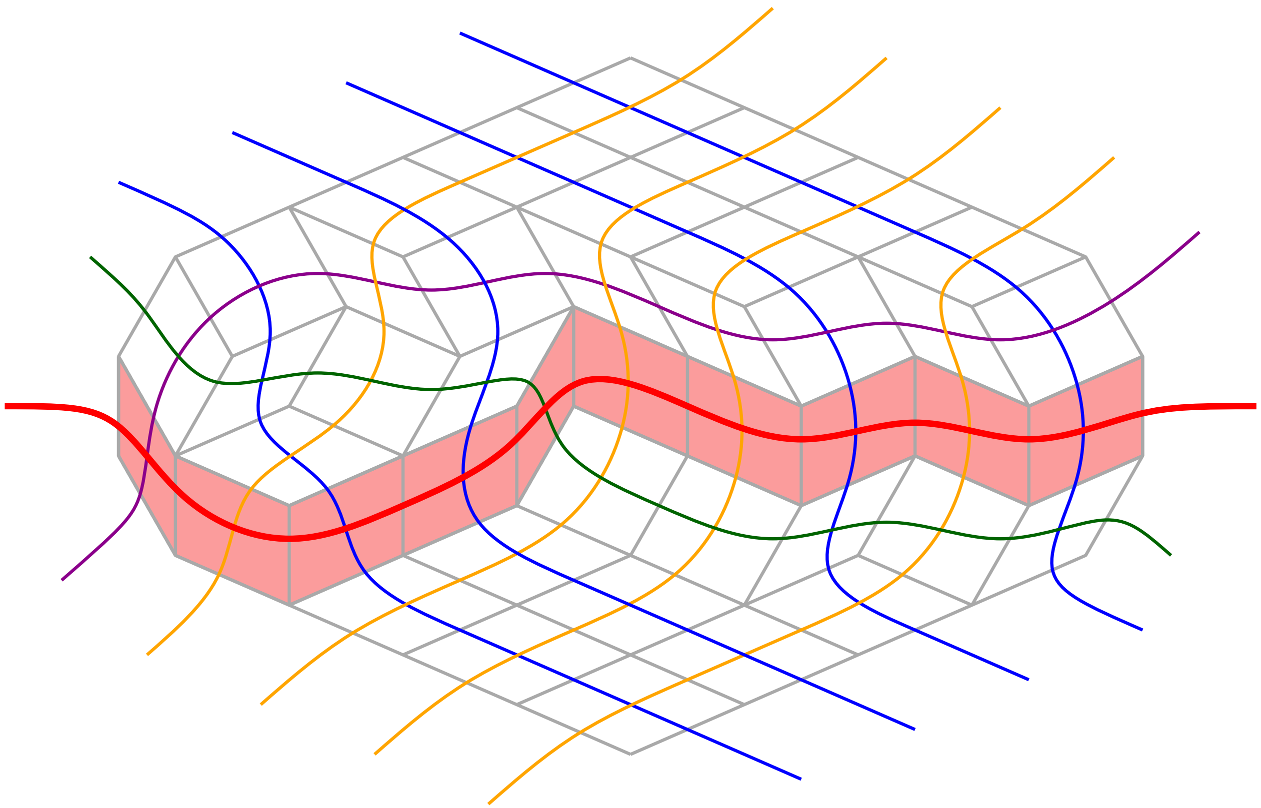

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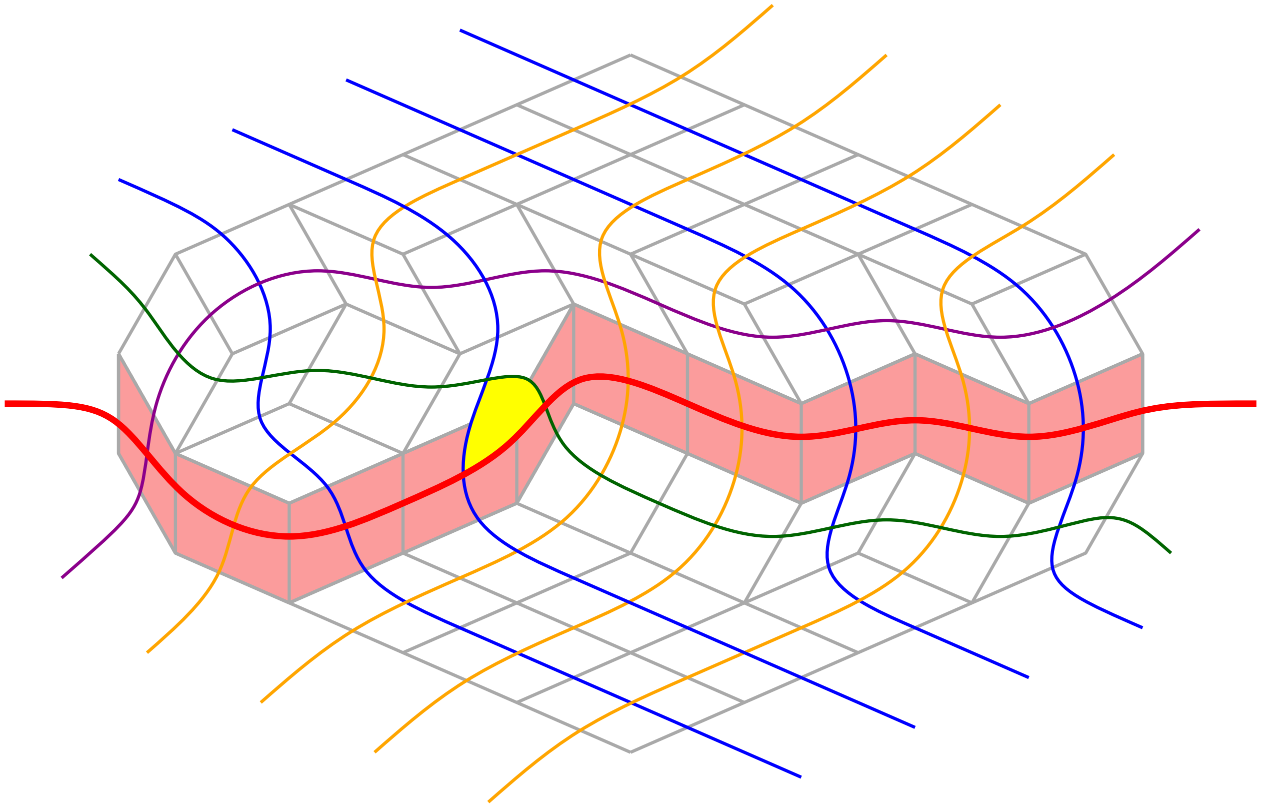

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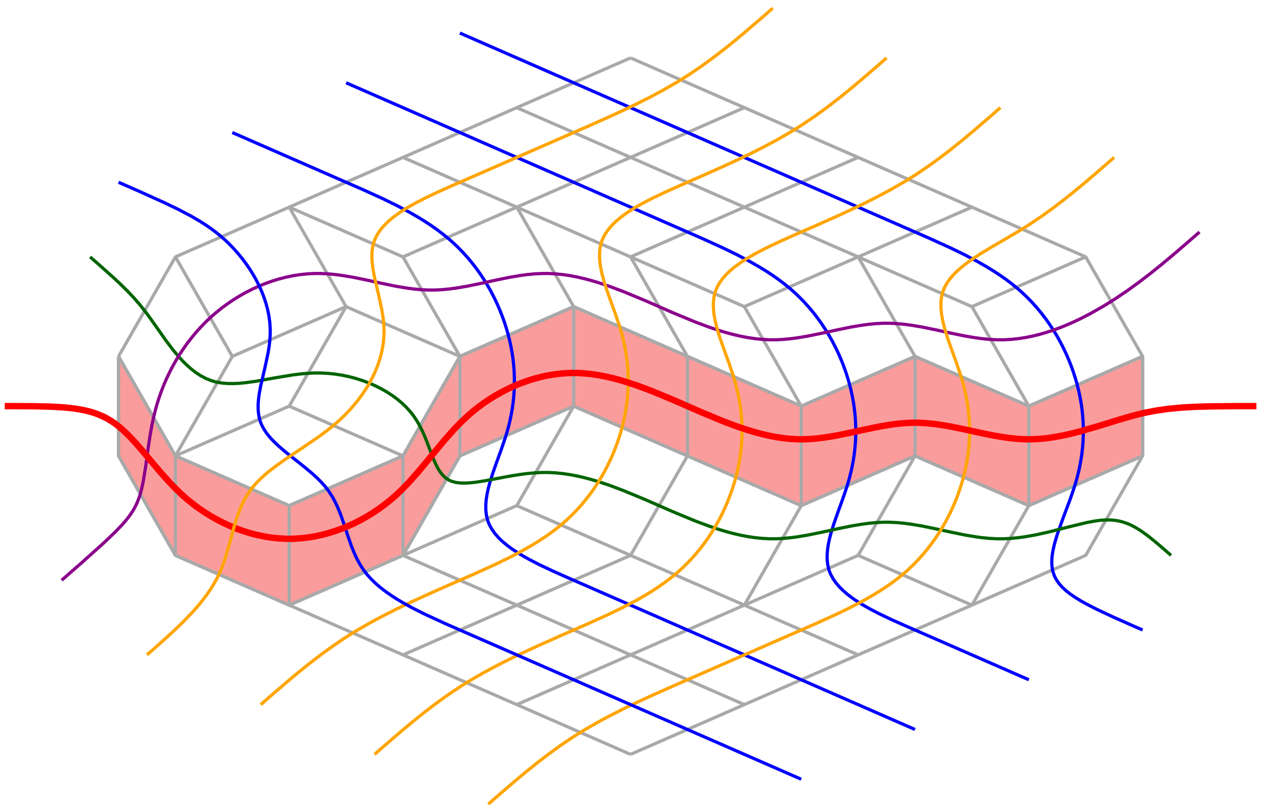

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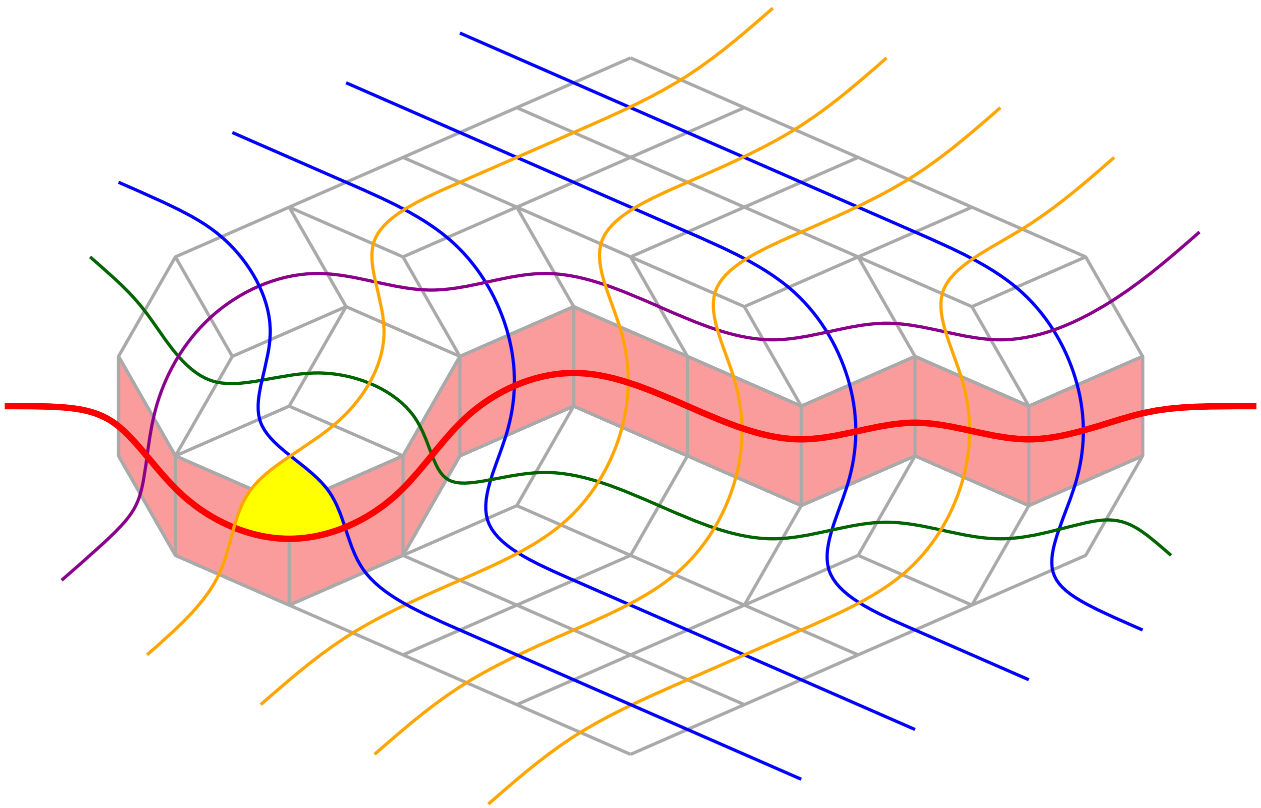

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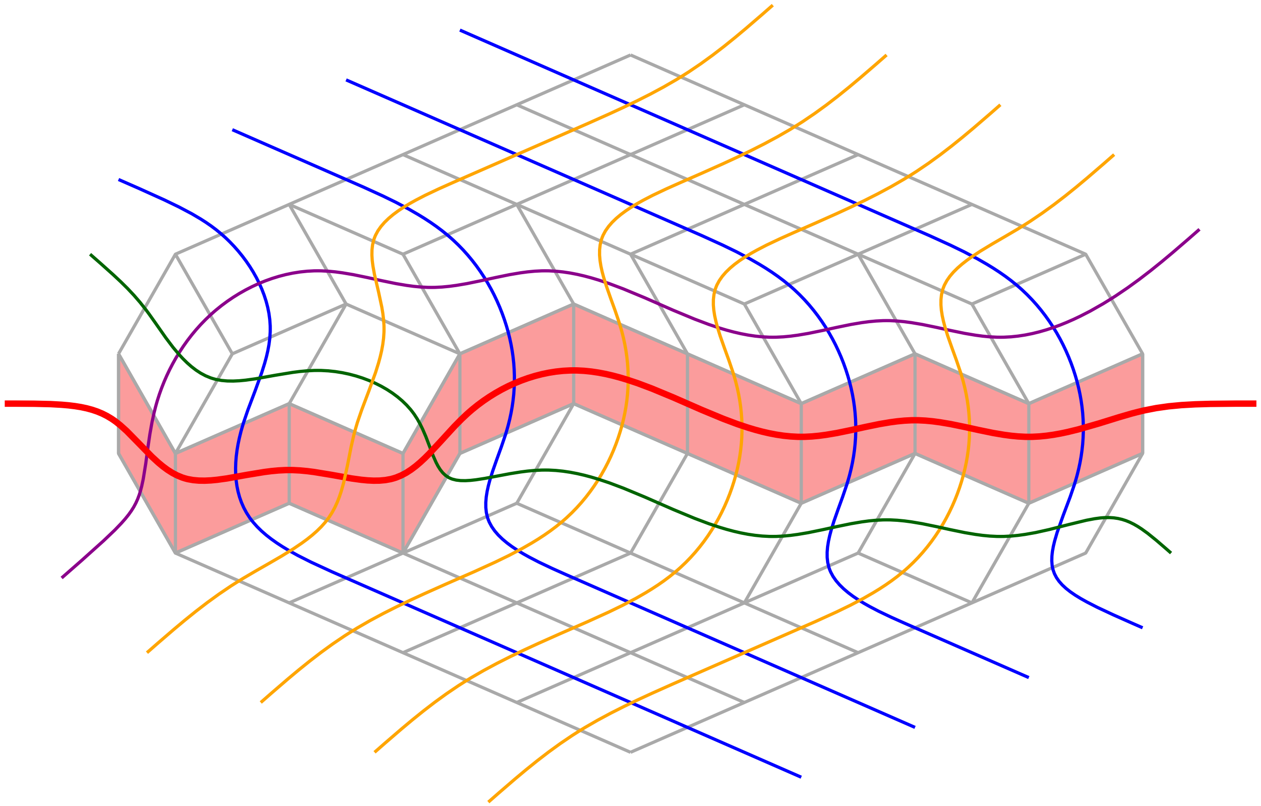

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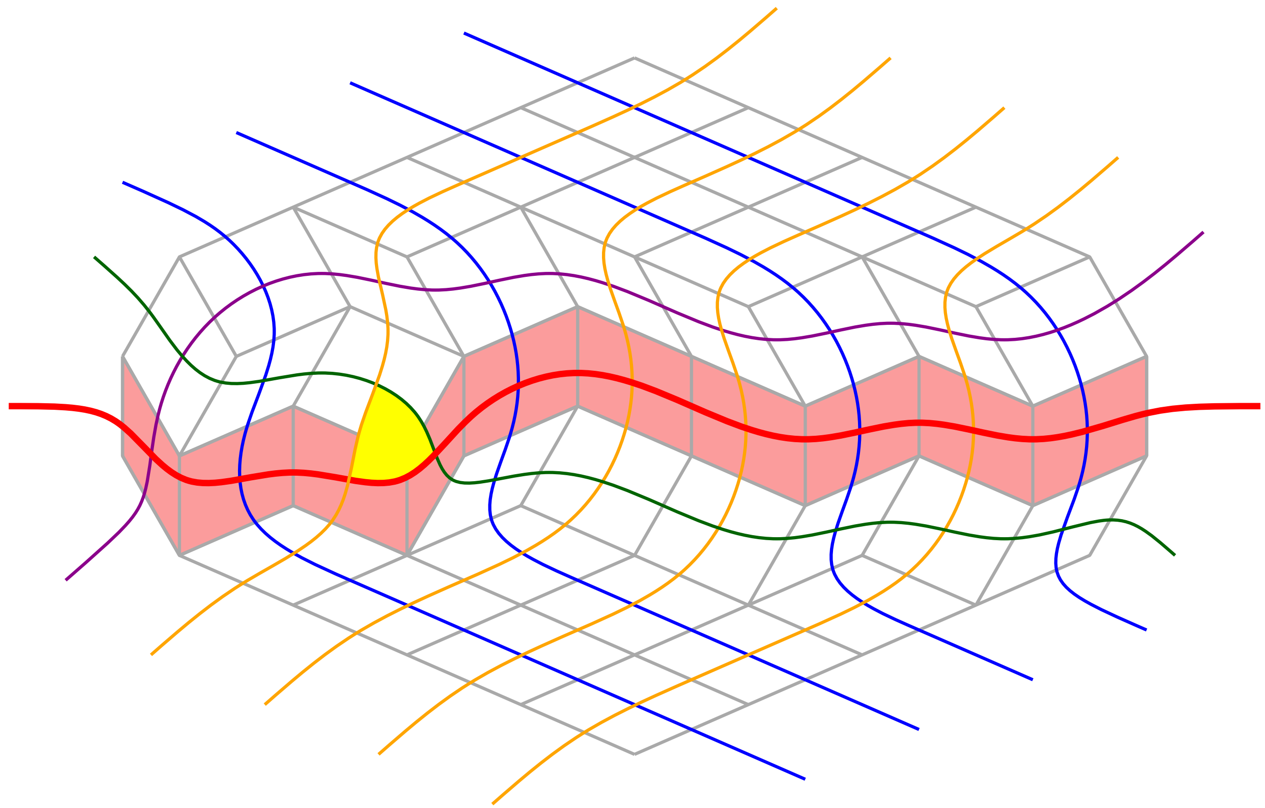

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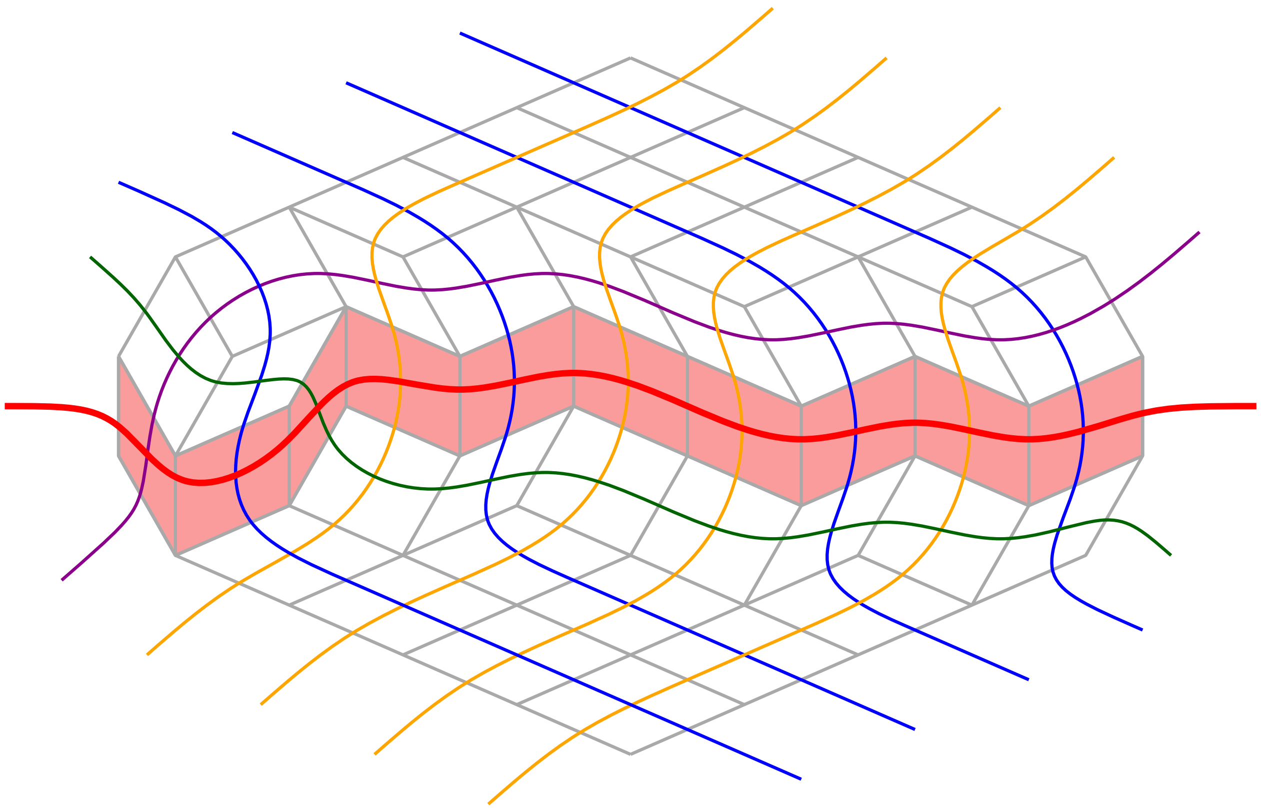

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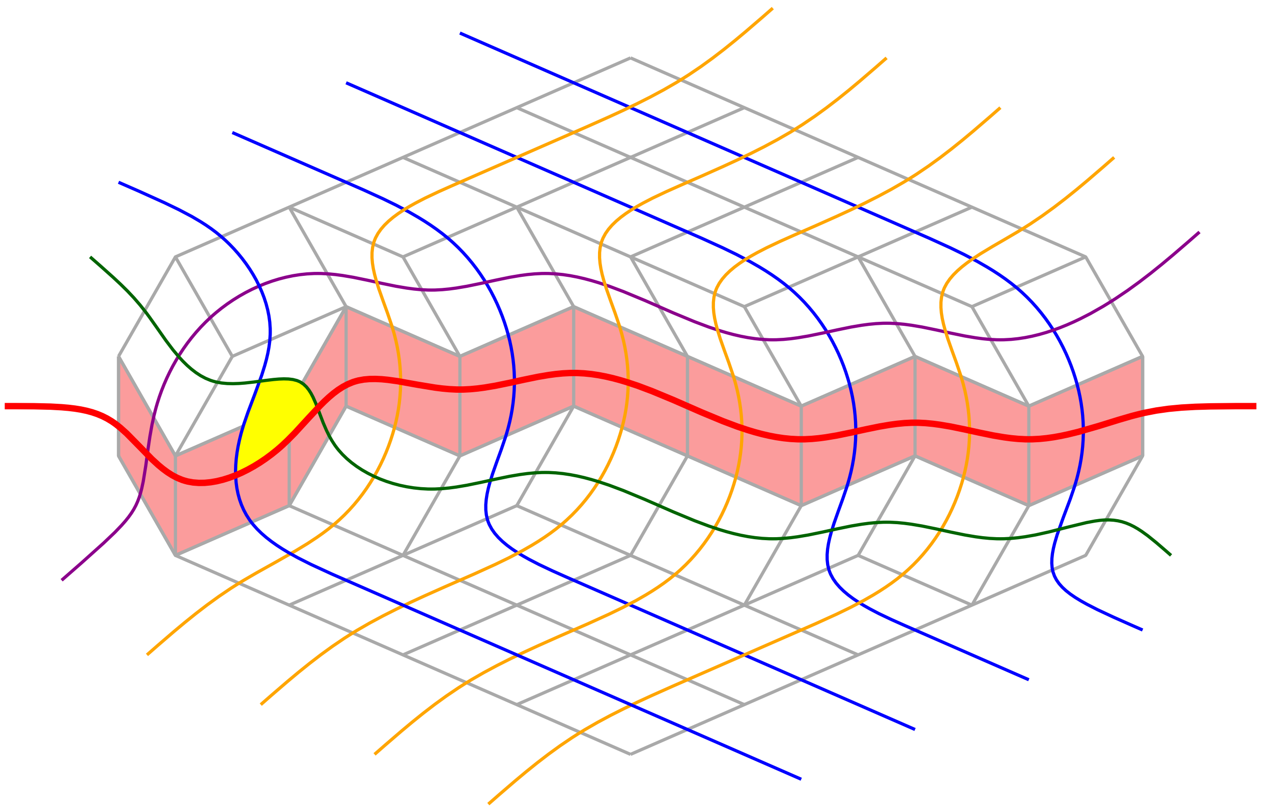

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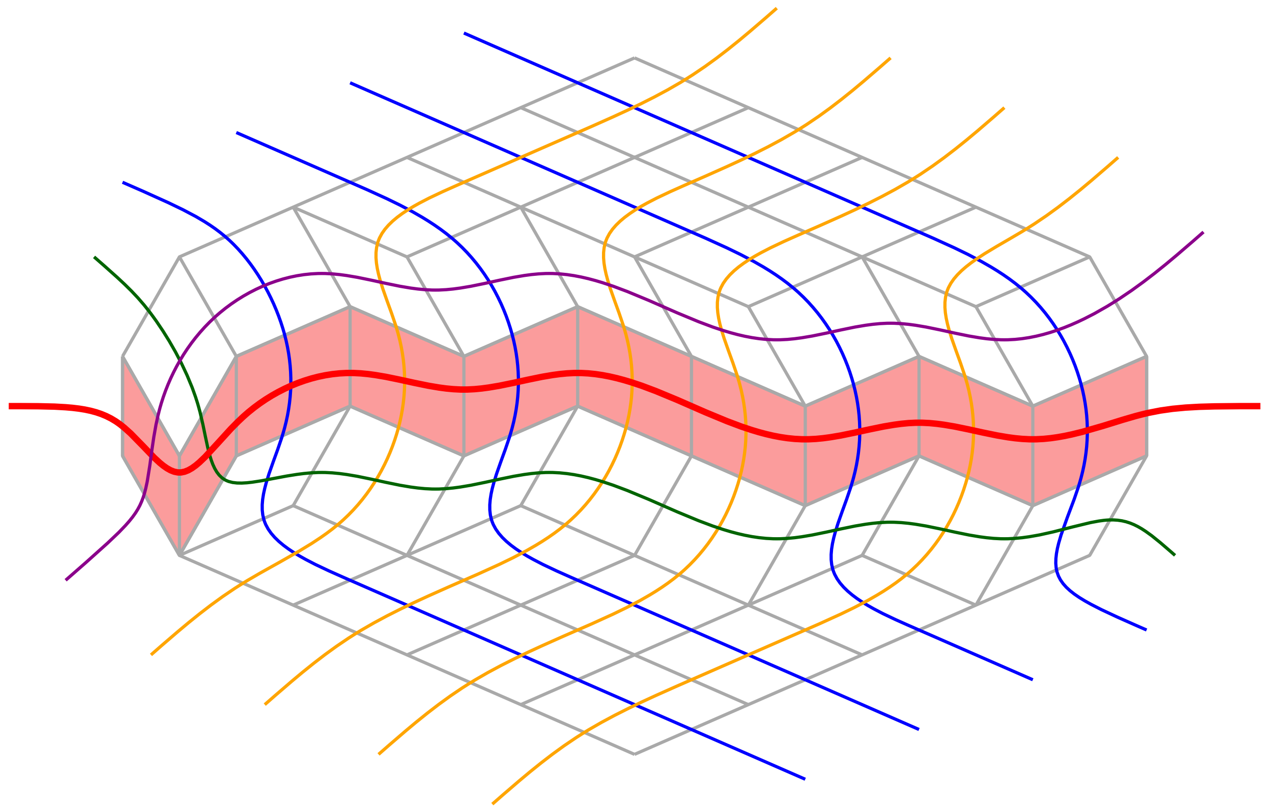

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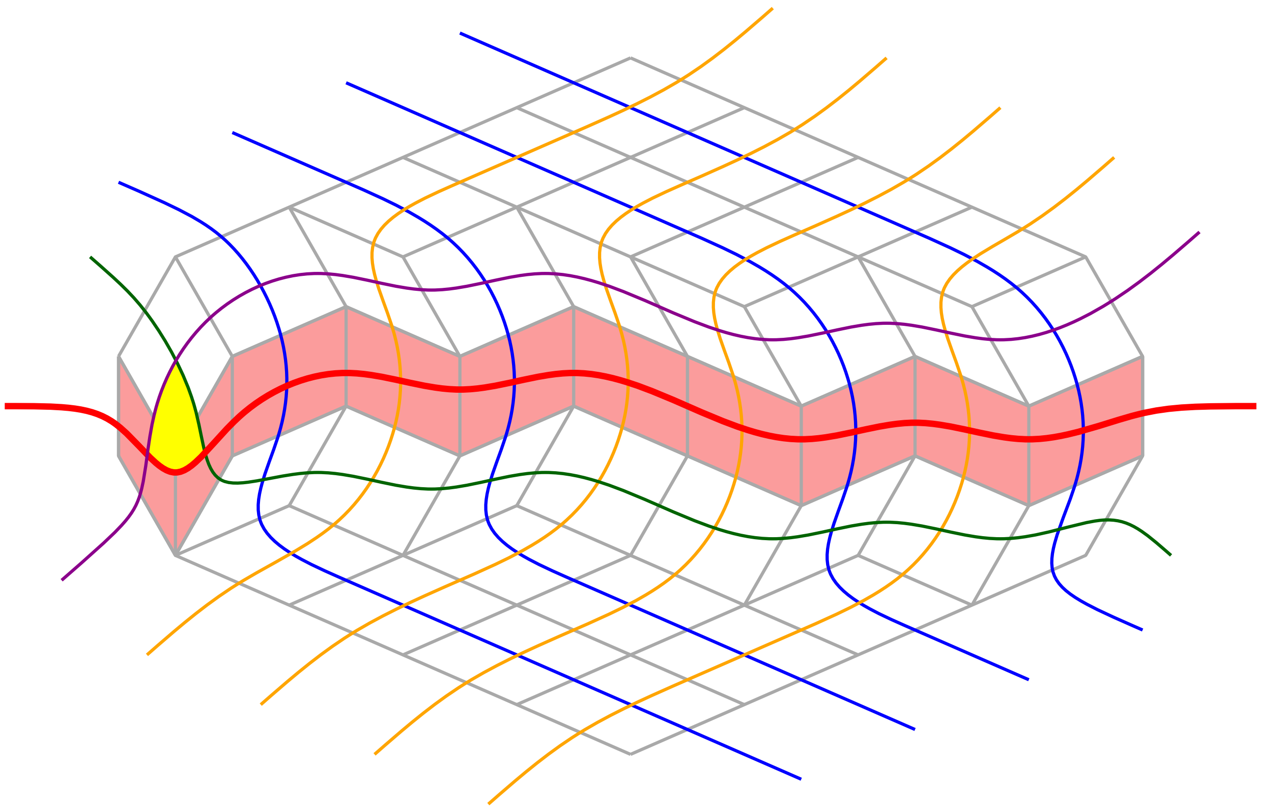

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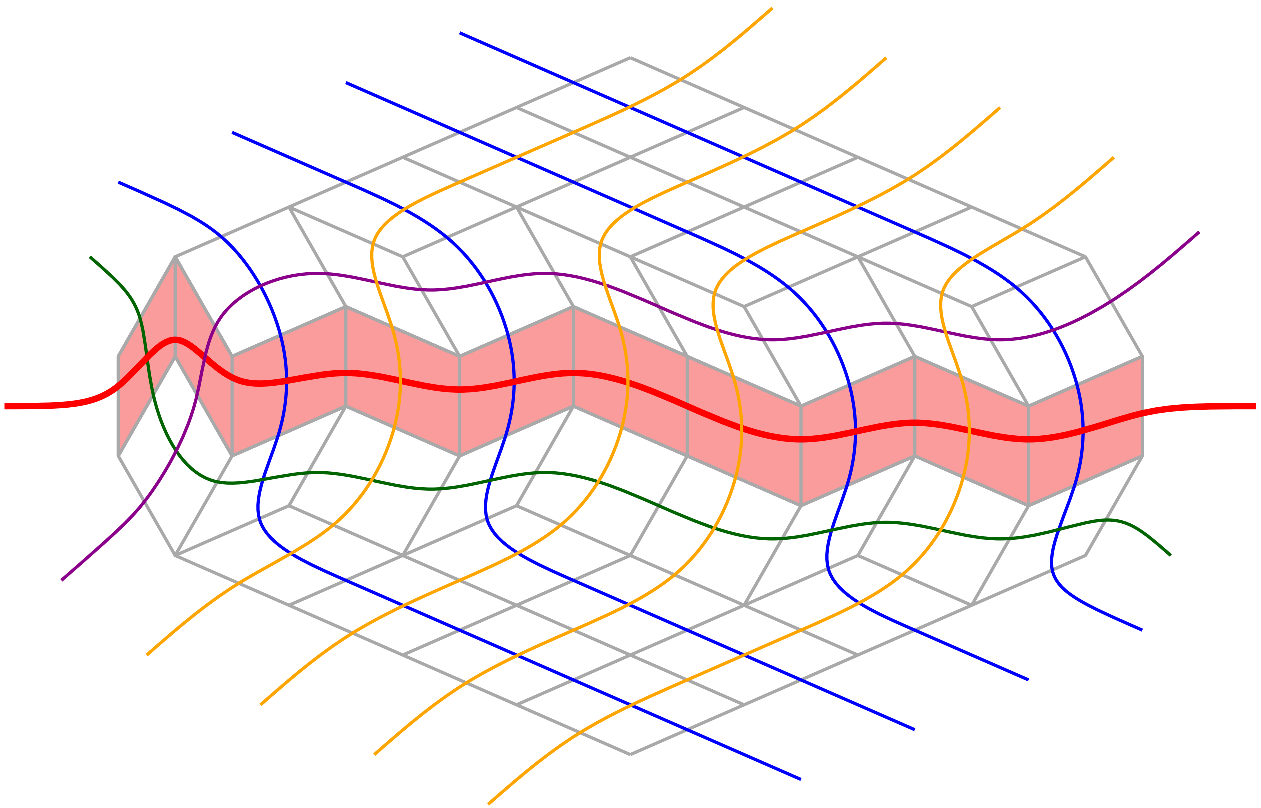

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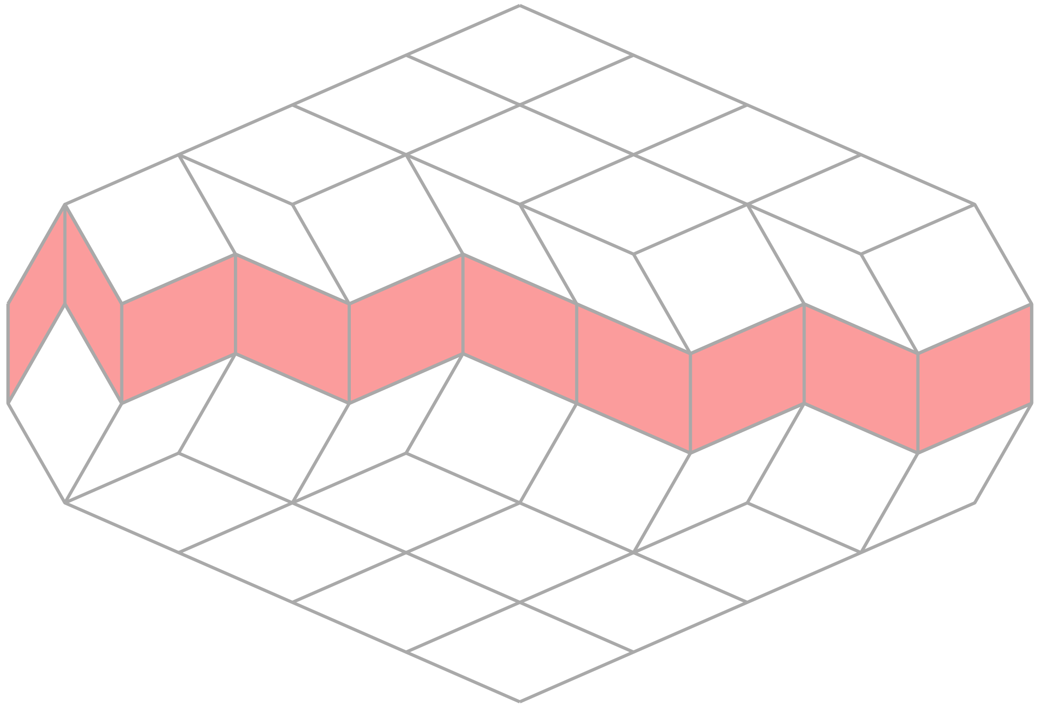

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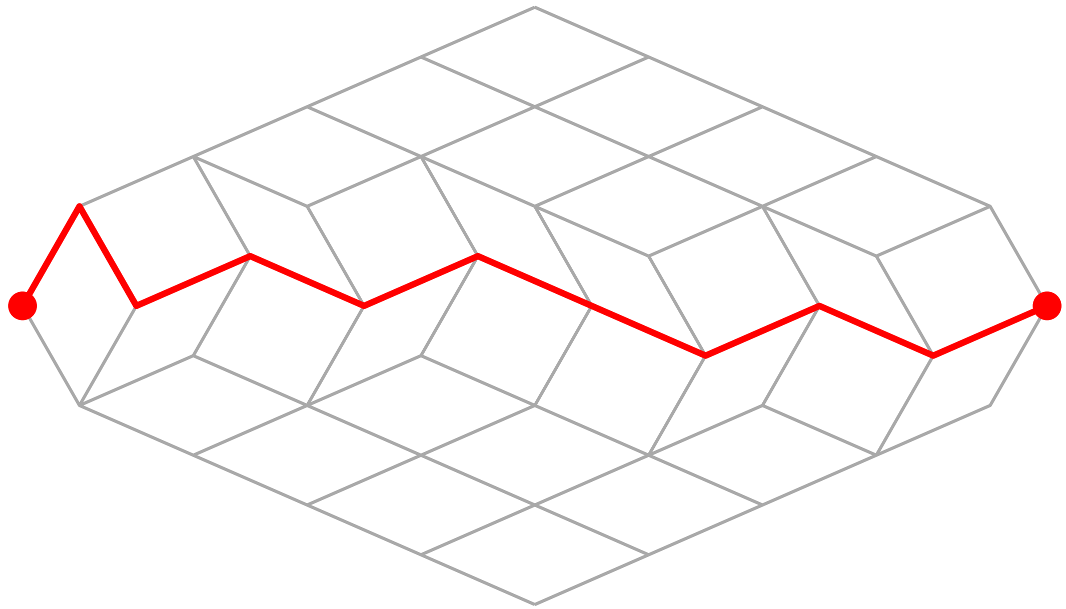

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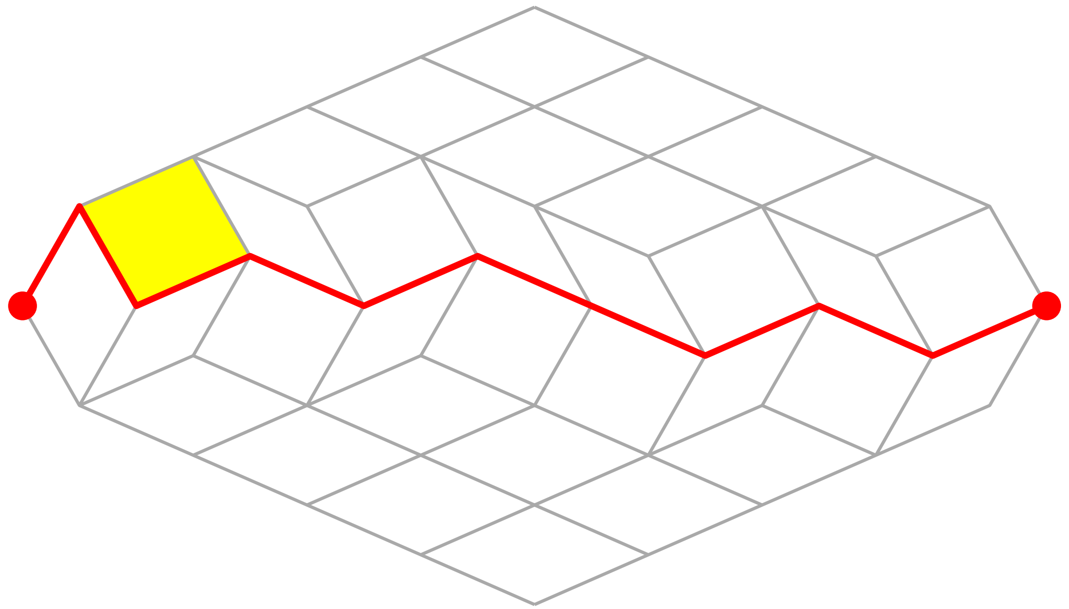

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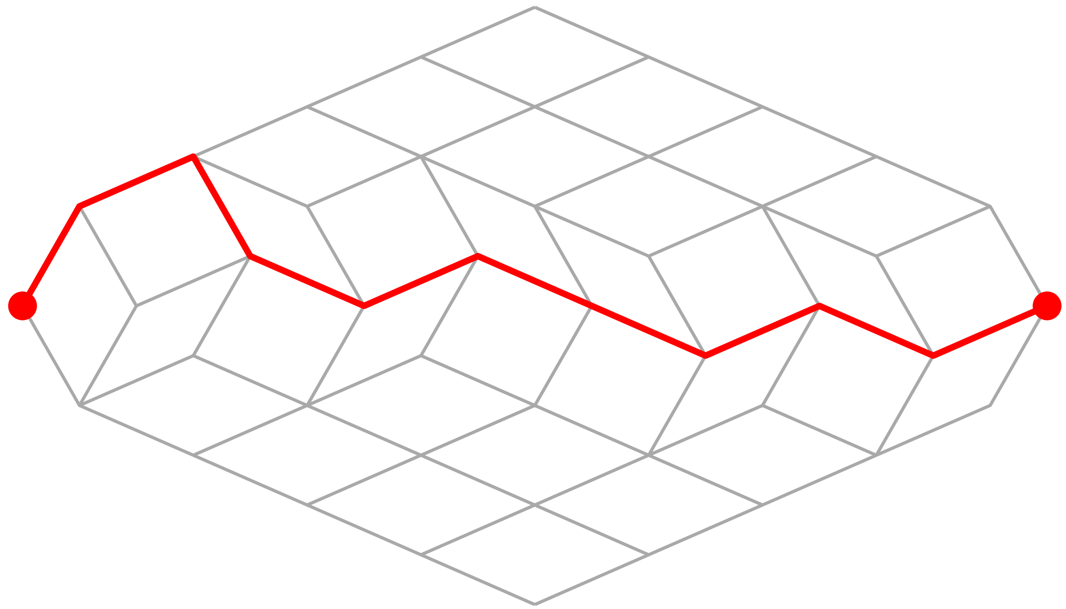

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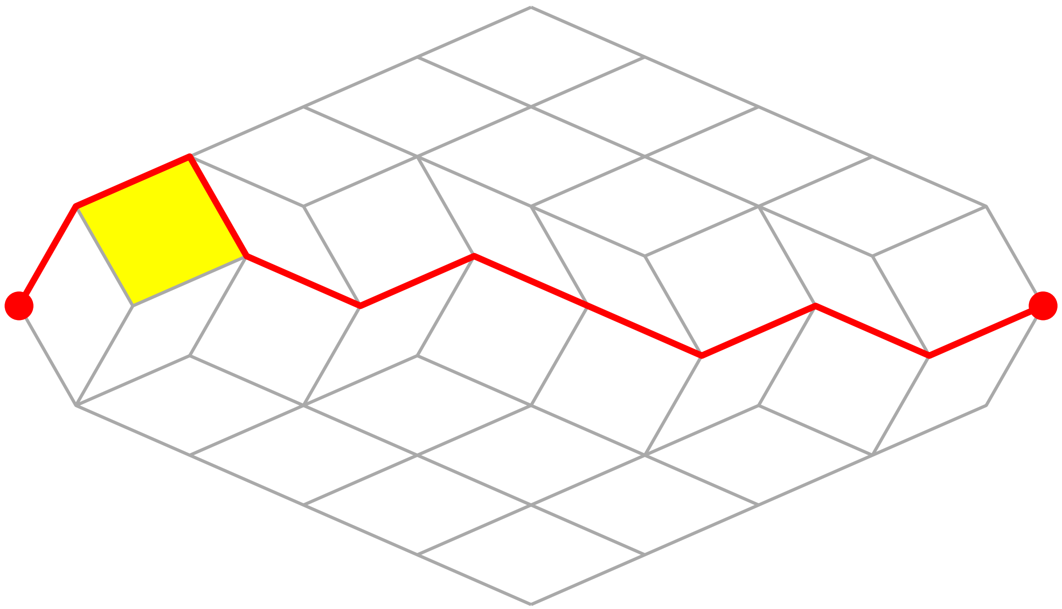

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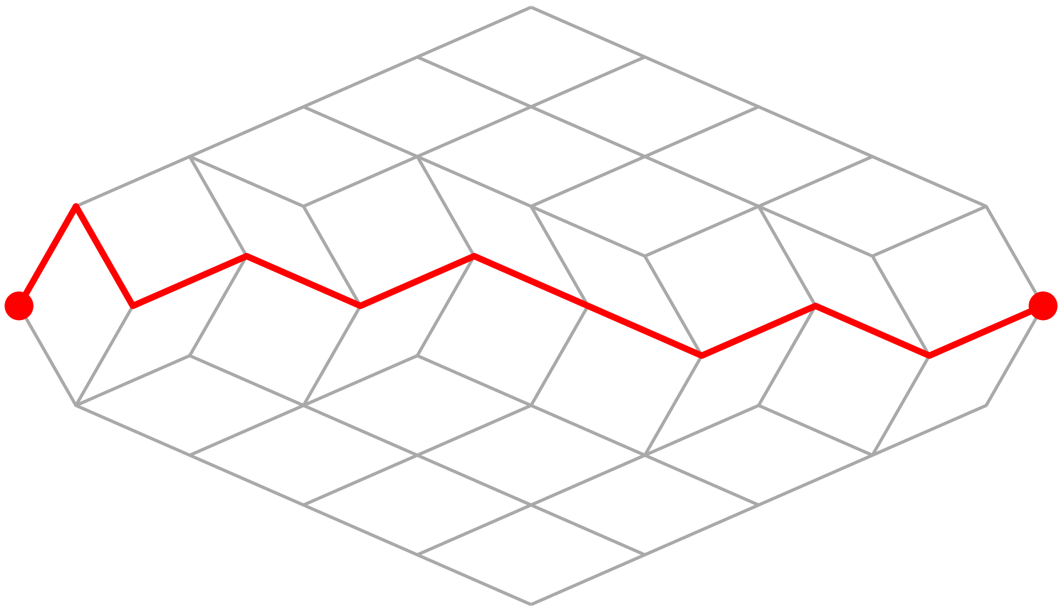

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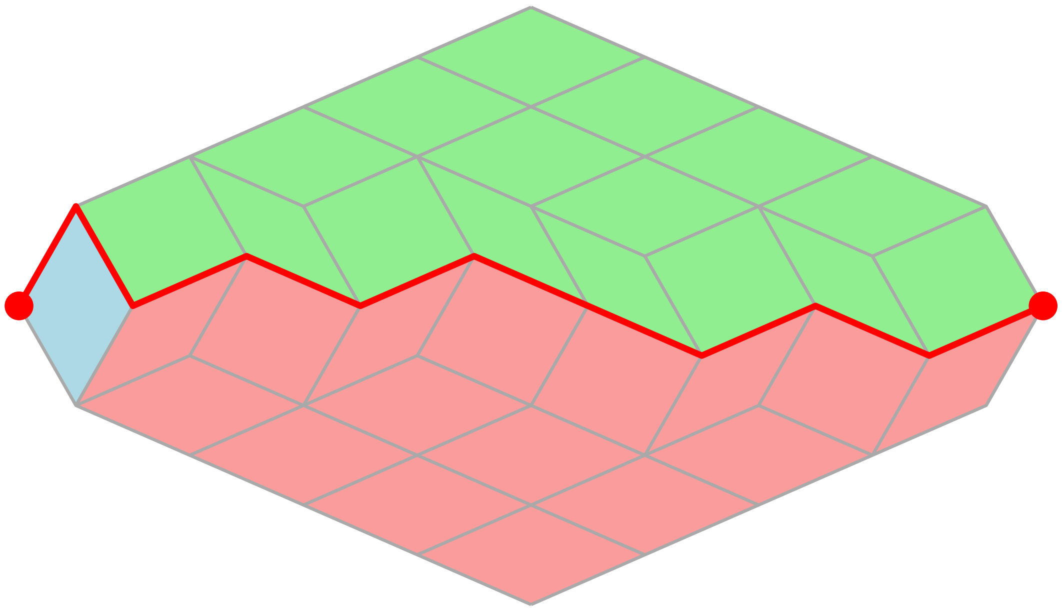
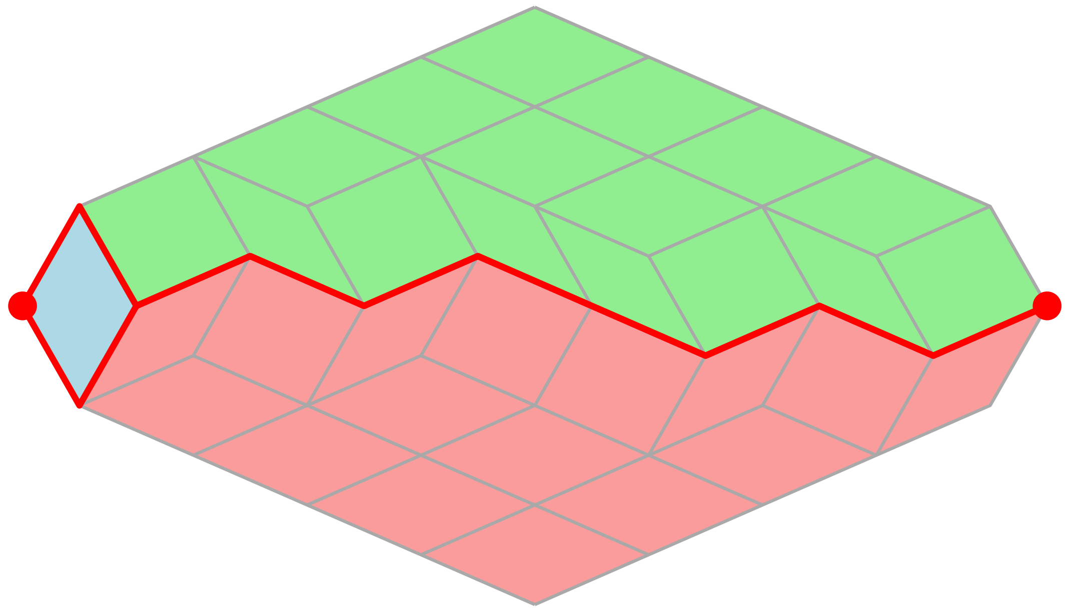


- Partition of states into two classes:
  - paths **above the blue rhombus**
  - paths **below the blue rhombus**

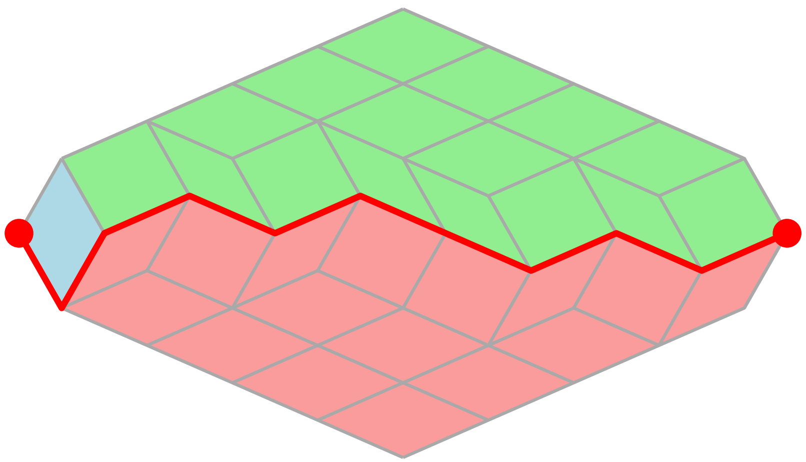
flips on single pseudoline



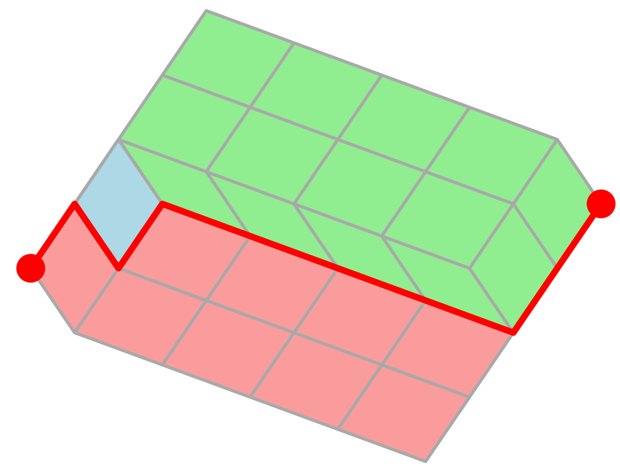
- Partition of states into two classes:
  - paths **above the blue rhombus**
  - paths **below the blue rhombus**
- Only a flip on the blue rhombus connects both classes!

flips on single pseudoline

$r = 5$ parallel classes:
(generalizable to more)



$r = 4$ parallel classes:



- Partition of states into two classes:
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  - paths **below the blue rhombus**
- Only a flip on the blue rhombus connects both classes!

flips on single pseudoline

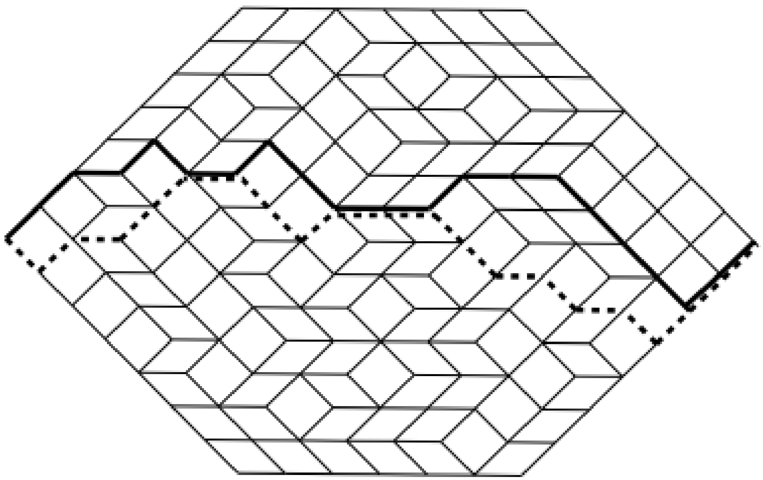
Theorem (R., 2021):
The Markov chain which operates on generalized pseudoline arrangements and flips random triangles with involvement of a distinguished parallel class is

- ... **rapidly-mixing** on 3 parallel classes, and...
- ... in general **not rapidly-mixing** on 4 or more parallel classes.

Statement for 3 classes follows from (Luby, Randall & Sinclair, 1995)

flips on single pseudoline

Destainville, 2001: *Mixing times of plane rhombus tilings*



„Nevertheless, the above arguments do not exclude definitively the existence of rare slow fibers, [...]“

Now we know: „slow fibers“ do exist!

three parallel classes

three parallel classes

Def: matrix $[h_{i,j}] \in {\mathbb{N}_0}^{r \times s}$ is called *plane partition*, if rows and columns are monotonically increasing.

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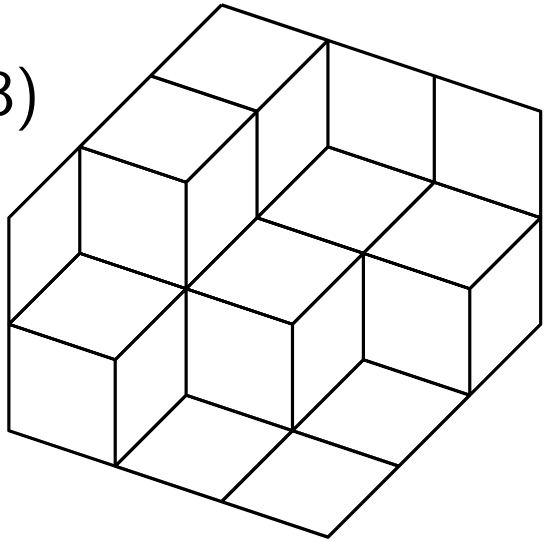

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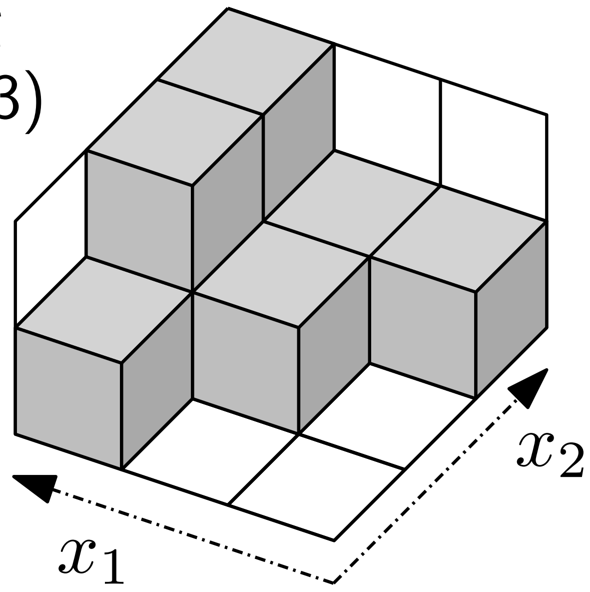

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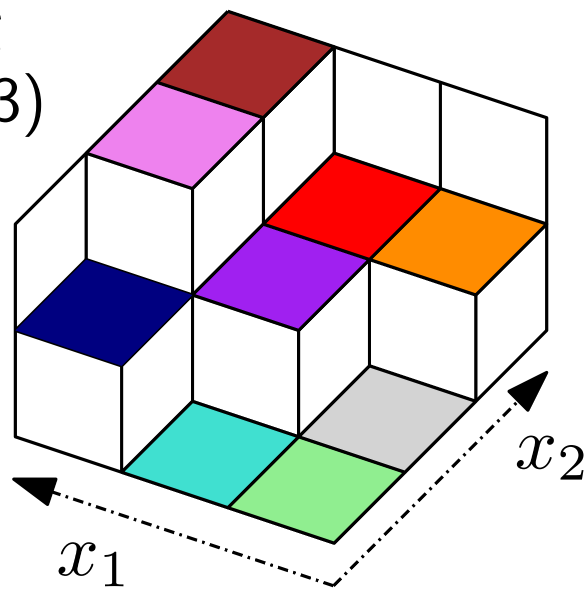

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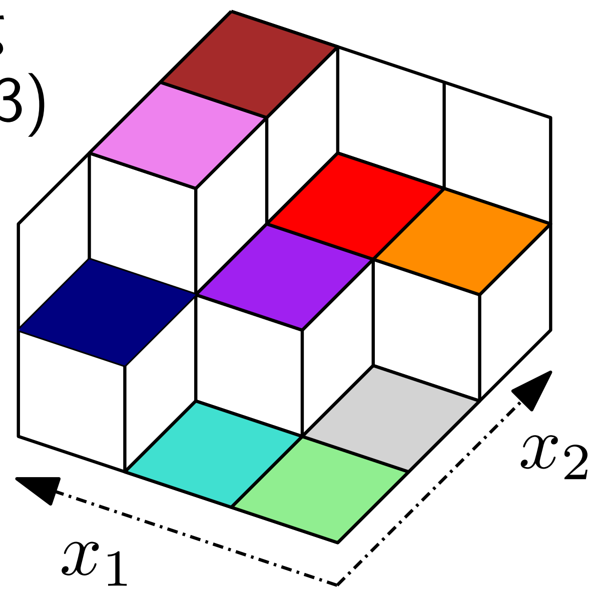

$$\begin{array}{cc} & x_2 \\ x_1 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \end{array}$$

plane partition with entries $h_{i,j} \leq 2$

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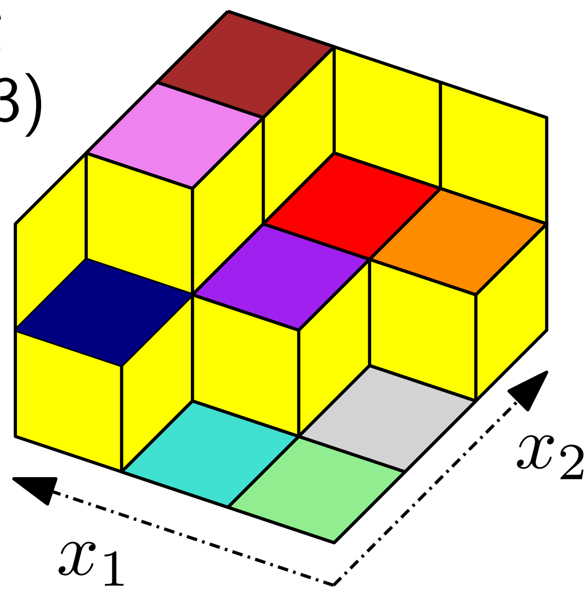

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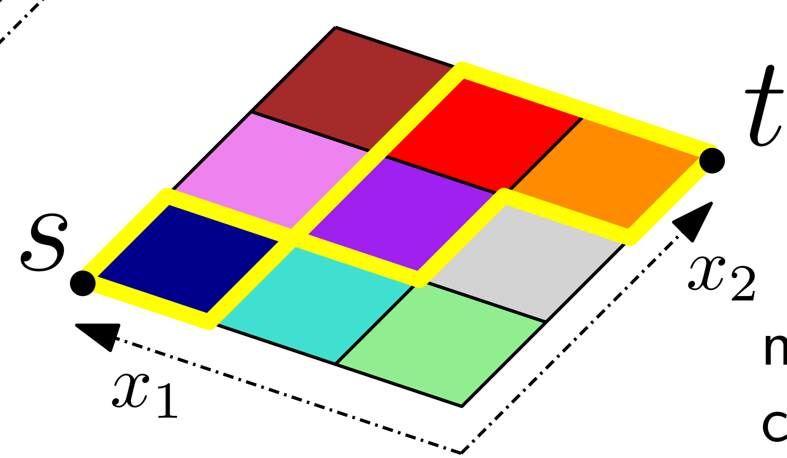

monotonic, non-crossing grid paths

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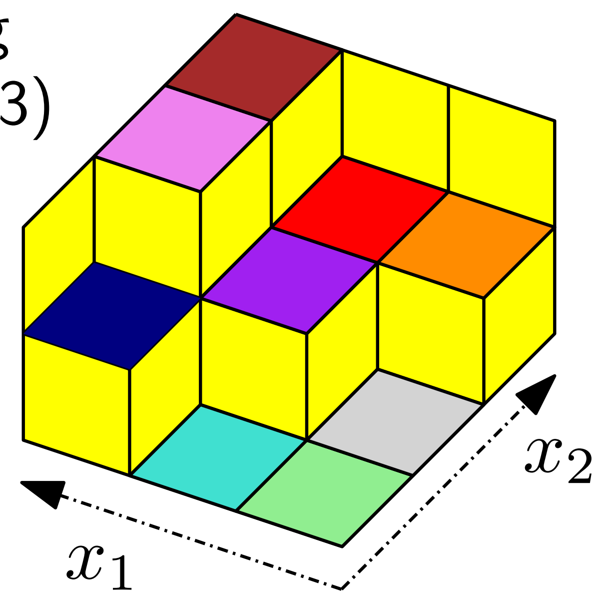

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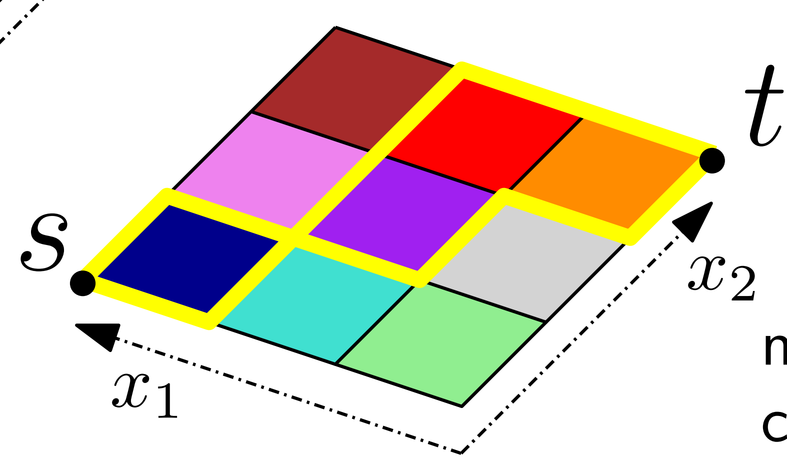

monotonic, non-crossing grid paths

min $\sum f_{i,j}(A_{i,j})$
s.t. A p.p., $A_{i,j} \leq h$

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Theorem: (Luby, Randall & Sinclair, 1995)
The Markov chain that flips triangles in generalized pseudoline arrangements of 3 parallel classes is rapidly mixing.

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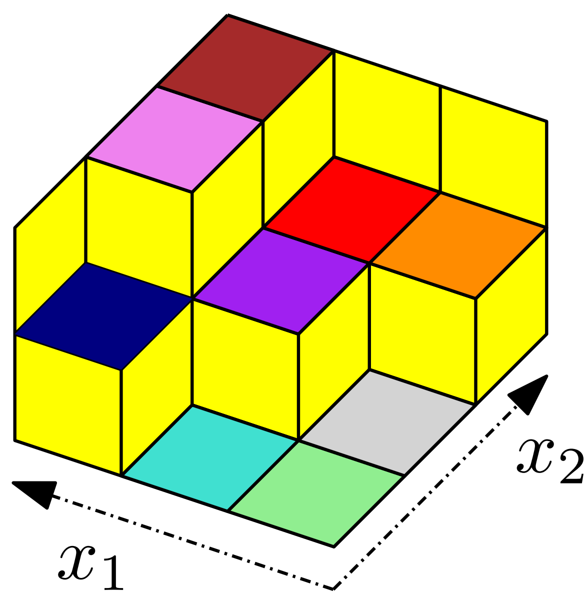

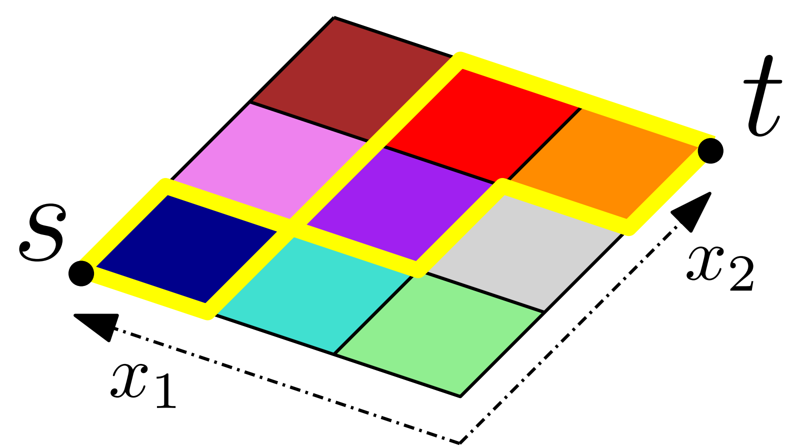

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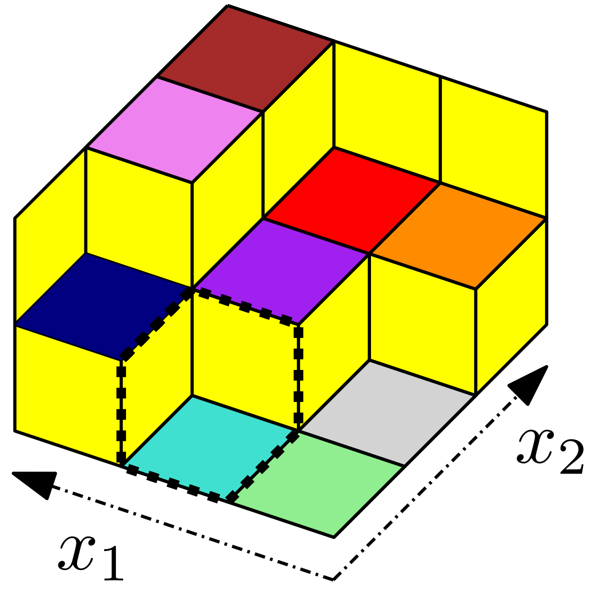

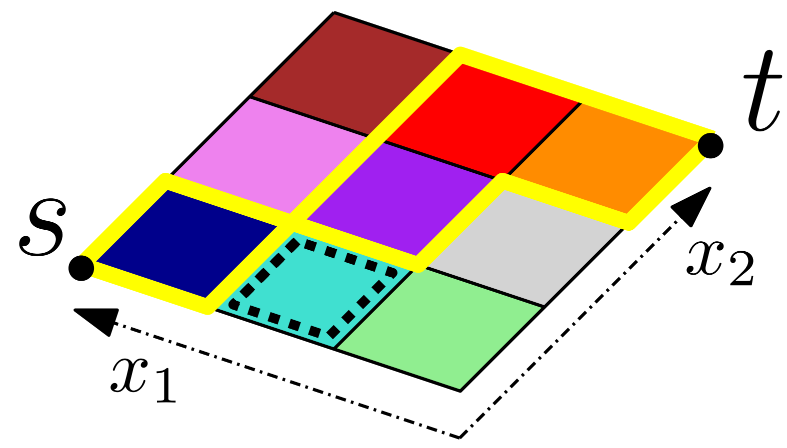

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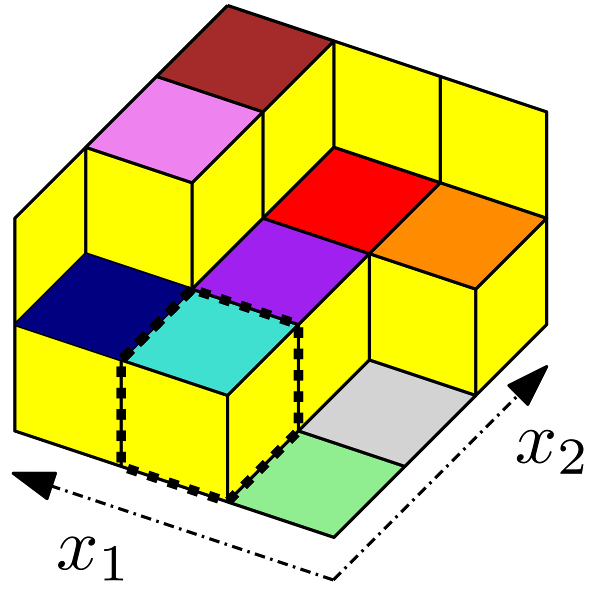

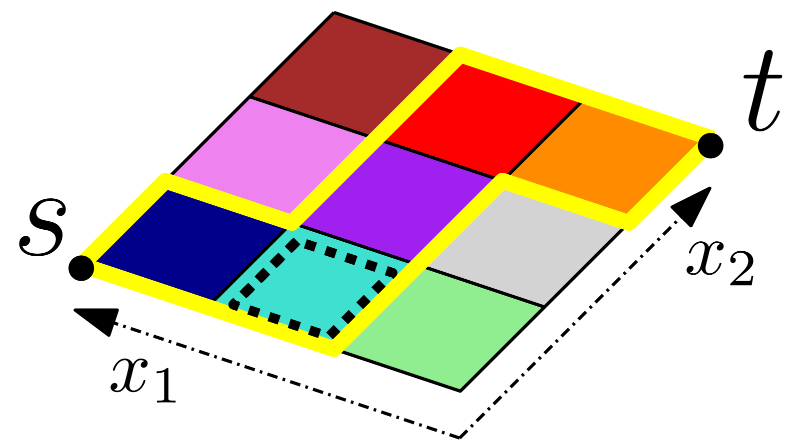

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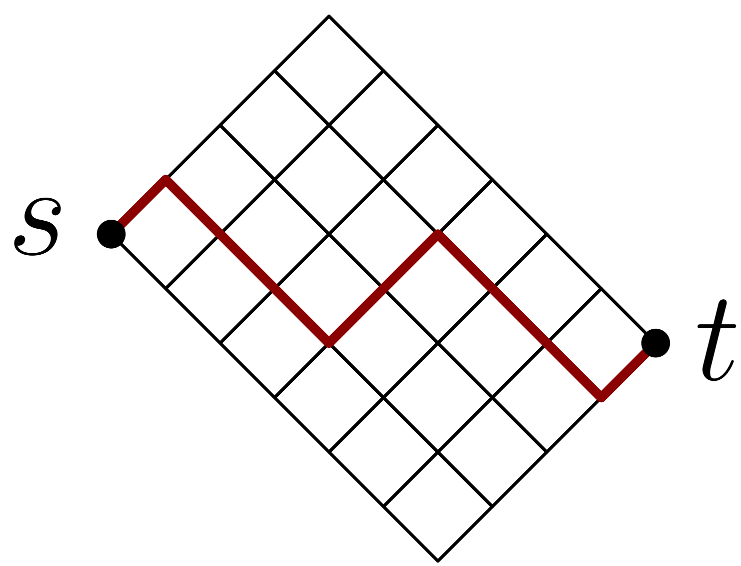
Simple case: Only one s-t-path



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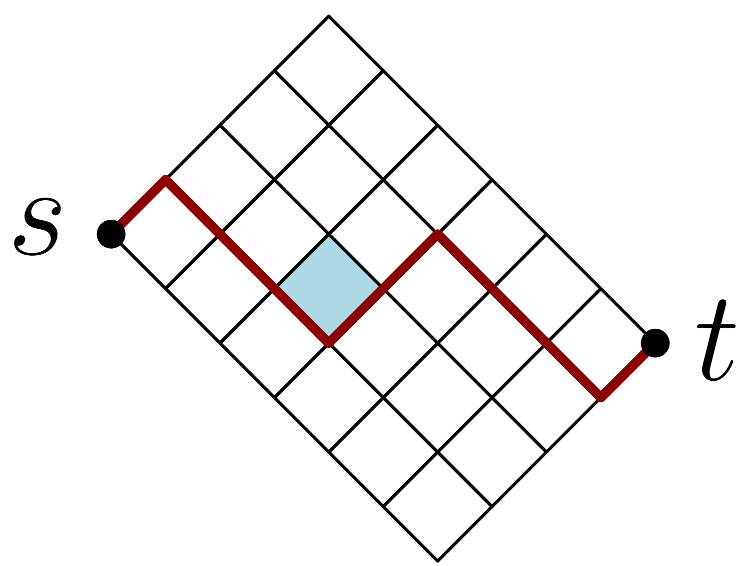

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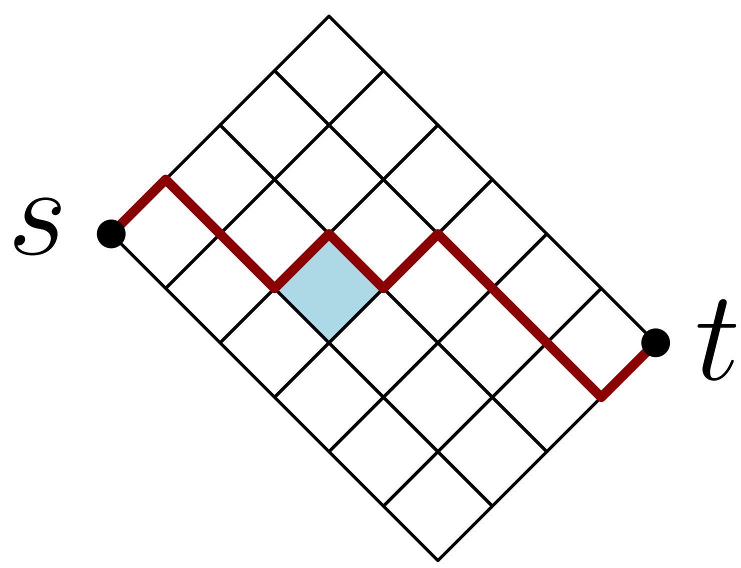

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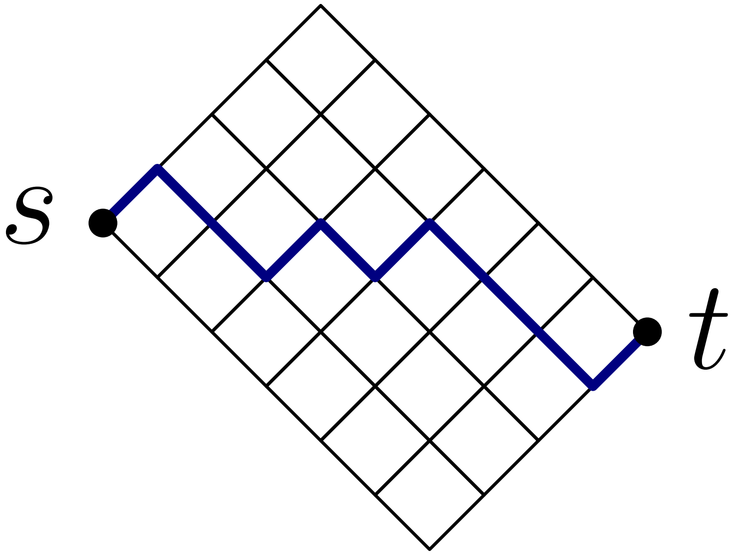
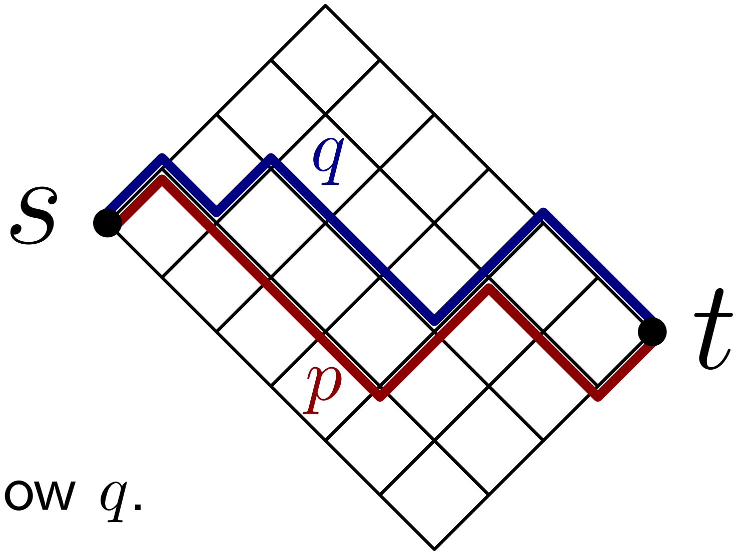


Technique: Monotone coupling

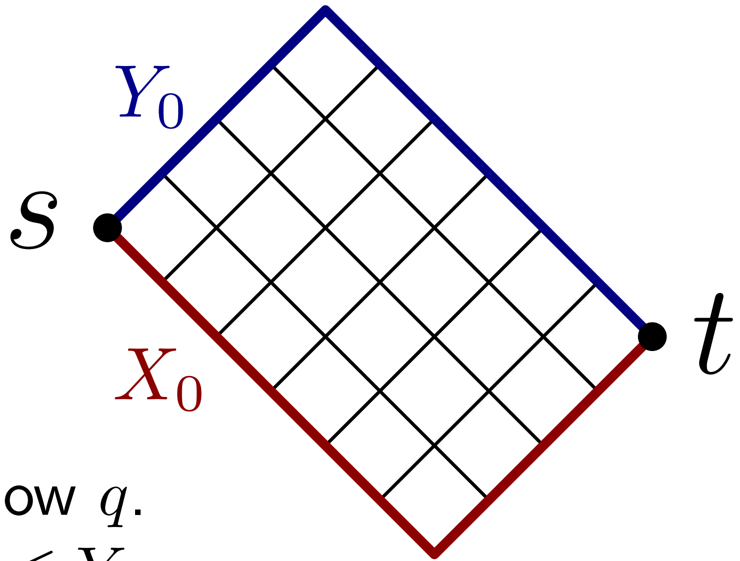
three parallel classes



Technique: Monotone coupling

- Partial order on paths: $p \leq q$ iff p below q.

three parallel classes



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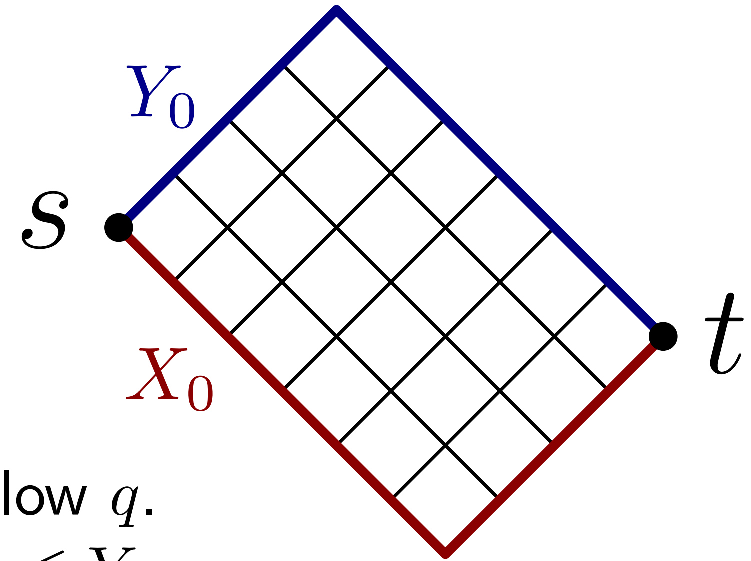
three parallel classes



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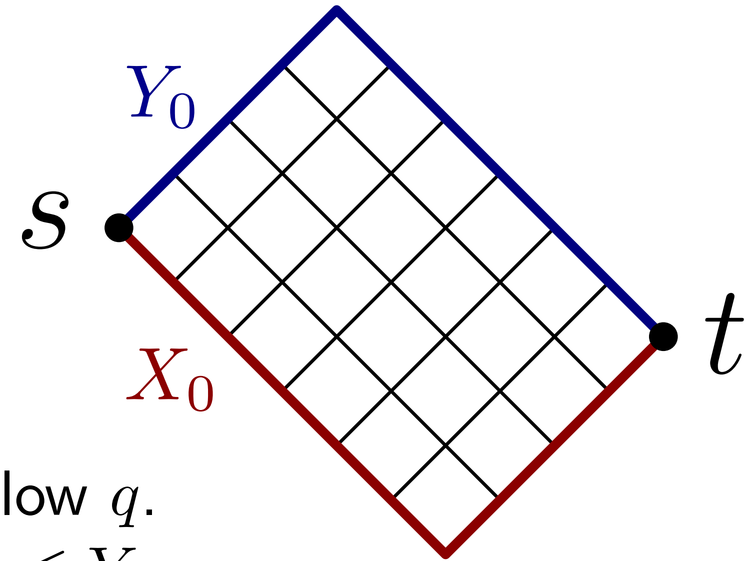
three parallel classes



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- Preserves $X_t \leq Y_t$ for all t.

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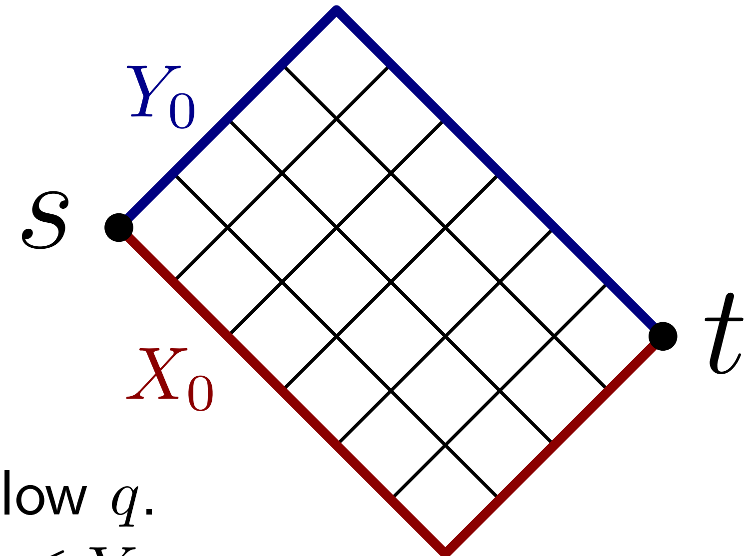

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- Random variable $\tau_C := \min\{t \ : \ X_t = Y_t\}$

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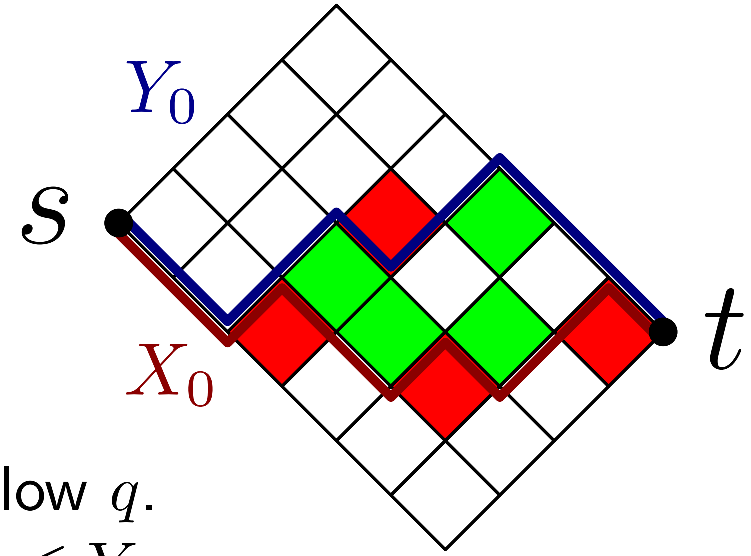

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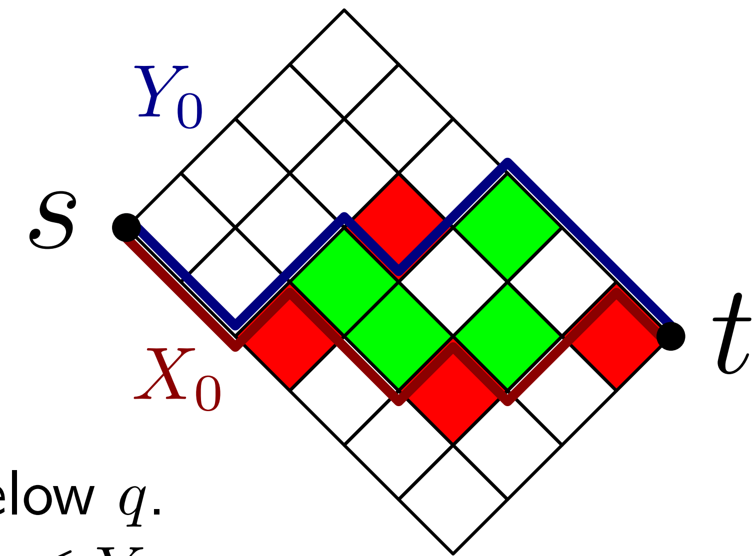
Get upper bound on $\mathbb{E}[\tau_c]$ by upper bounding expected change of area between X_t and Y_t: $\mathbb{E}[\Delta d(X_t, Y_t)] \leq 0$.

three parallel classes



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Theory $\implies \tau(\varepsilon) \leq 6 \cdot \mathbb{E}[\tau_C] \left(1 + \log\left(\frac{1}{\varepsilon}\right)\right)$

bichromatic triangle conjecture

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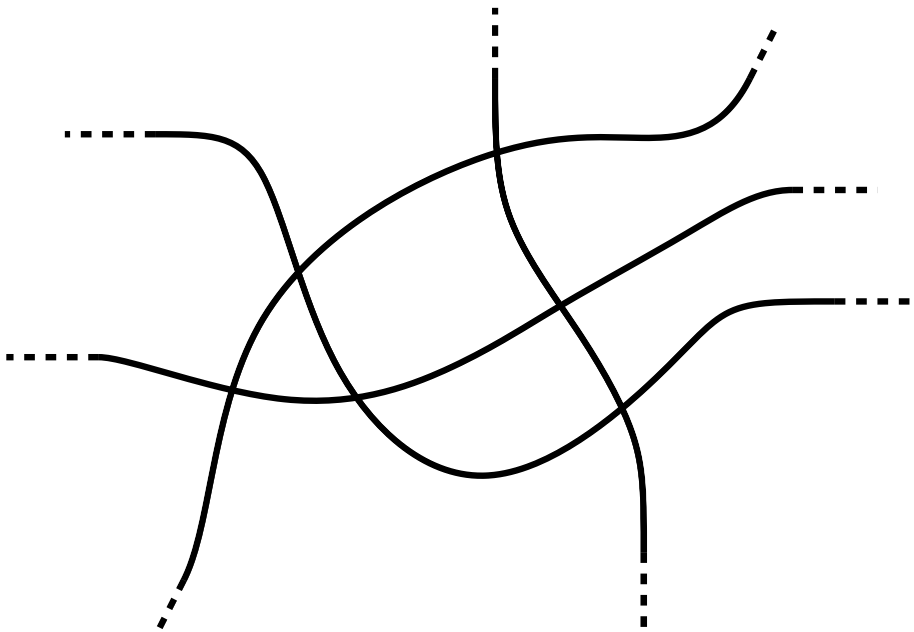
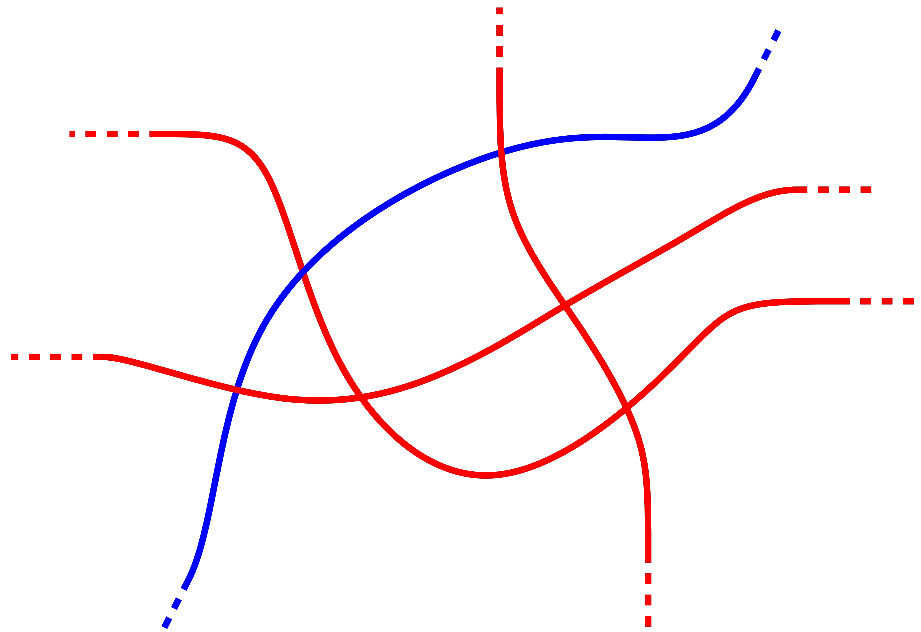


bichromatic triangle conjecture



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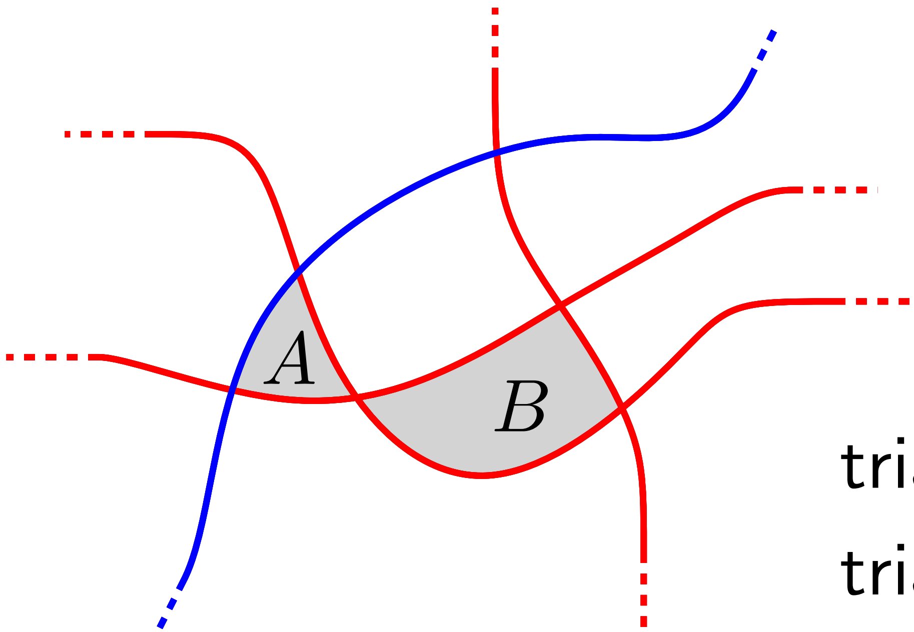
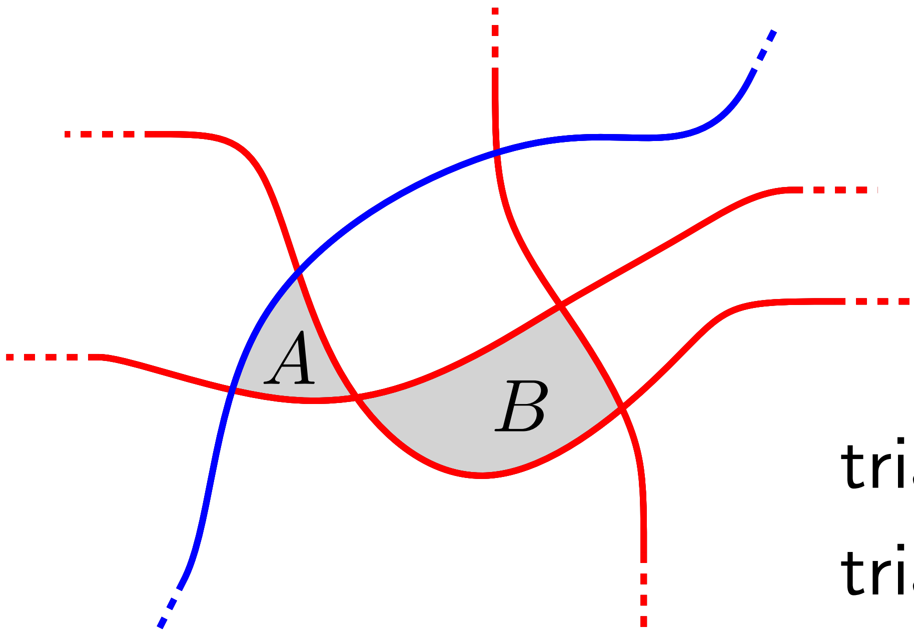


triangle A bichromatic

triangle B monochromatic

bichromatic triangle conjecture



triangle A bichromatic

triangle B monochromatic

Conjecture:
(Björner, Las Vergnas, Sturmfels, White, Ziegler, 1999)
Every truly two-colored arrangement of at least three pseudolines contains a bichromatic triangle.

Thank you!