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A brief introduction to abelian categories

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Introduction

Abelian categories are a class of categories that share some of the well known properties of the category of abelian groups **Ab**, such as the existence of kernels, cokernels and images as well as the validity of some isomorphism theorems. The category **Ab** is abelian itself. Therefore abelian categories can be thought as a generalization of abelian groups.

In the first chapter of this short introduction to abelian categories we go step by step towards the definition of abelian groups and show that **Ab** indeed satisfies it. In the secound chapter we develop some theory that aims to give further motivation for the study of abelian categories.

Chapter 1

Path towards the definition of abelian categories

The goal of this chapter is to define abelian categories step by step. We will demonstrate the single parts of the definition on the category of abelian groups **Ab**, hence proving that **Ab** is abelian on the fly. This chapter is based on chapter 7 of [Osb00], unless otherwise mentioned.

Definition 1.0.1. *Let $\mathcal{C}$ be a category, $A \in \mathrm{Obj}(\mathcal{C})$.*

1) *A is an initial object if for every $B \in \mathrm{Obj}(\mathcal{C})$ there is exactly one morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.*

2) *A is a final object if for every $B \in \mathrm{Obj}(\mathcal{C})$ there is exactly one morphism $f \in \mathrm{Hom}_{\mathcal{C}}(B, A)$.*

3) *A is a zero object if A is both an initial object and a final object.*

One easily checks that initial, final and zero objects are unique up to isomorphism, if they exist. In **Sets** the only initial object is $\emptyset$, final objects are exactly sets of cardinality one and there is no zero object. In **Ab** but also in the category of all groups **Gr** the only zero object is the trivial group $\{e\}$.

Definition 1.0.2. *Let $\mathcal{C}$ be a category, $A, B \in \mathrm{Obj}(\mathcal{C})$, $f : A \to B$ a morphism.*

1) *The morphism f is called left zero morphism if for any $X \in \mathrm{Obj}(\mathcal{C})$ and any morphisms $g, h : X \to A$ it holds that $f \circ g = f \circ h$.*

2) *The morphism f is called right zero morphism if for any $Y \in \mathrm{Obj}(\mathcal{C})$ and any morphisms $g, h : B \to Y$ it holds that $g \circ f = h \circ f$.*

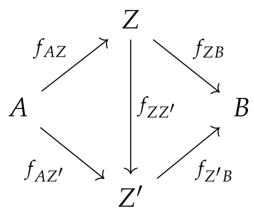
3) *The morphism f is a zero morphism if it is both a left zero morphism and a right zero morphism.*

Proposition 1.0.3. *Let $\mathcal{C}$ be a category in which a zero object $Z \in \mathrm{Obj}(\mathcal{C})$ exists. Then for any objects $A, B \in \mathrm{Obj}(\mathcal{C})$ with their unique morhpisms $f_{AZ} : A \to Z$ and $f_{ZB} : Z \to B$ the morphism $0_{AB} : A \to B$ defined by $0_{AB} := f_{ZB} \circ f_{AZ}$ is a zero morphism. Moreover, 0_{AB} is independent of the choice of Z.*

Proof. 0_{AB} is a left zero morphism: Let $X \in \text{Obj}(\mathcal{C})$, $g, h \in \text{Hom}_{\mathcal{C}}(X, A)$. Since Z is a final object, we get $f_{AZ} \circ g = f_{AZ} \circ h$. Therefore:

$$0_{AB} \circ g = f_{ZB} \circ f_{AZ} \circ g = f_{ZB} \circ f_{AZ} \circ h = 0_{AB} \circ h$$

0_{AB} is a right zero morphism follows silimarly by using that Z is an initial object, so 0_{AB} is indeed a zero morphism. Now let $Z' \in \text{Obj}(\mathcal{C})$ be another zero object with its unique morphisms $f_{AZ'} : A \to Z'$ and $f_{Z'B} : Z' \to B$ and define $0'_{AB} : A \to B$ in the same way as $0'_{AB} := f_{Z'B} \circ f_{AZ'}$. There is a (unique) morphism $f_{ZZ'} : Z \to Z'$. By using that Z is a final object and that Z' is an initial object the following diagram commutes:



Hence $0'_{AB} = f_{Z'B} \circ f_{AZ'} = f_{Z'B} \circ f_{ZZ'} \circ f_{AZ} = f_{ZB} \circ f_{AZ} = 0_{AB}$. □

In **Gr**, given two groups A and B, the morphism $0_{AB} : A \to B$ as in proposition 1.0.3 is the group morphism that maps everything to the identity element. The same holds in **Ab** of course. This coincides with our previous understanding of „zero morphisms“.

Definition 1.0.4. *A category $\mathcal{C}$ is called pre-additive if*

1) *There is a zero object in $\mathcal{C}$.*

2) *For any pair of objects $A, B \in \text{Obj}(\mathcal{C})$ there is a binary operation $+$ on $\text{Hom}_{\mathcal{C}}(A, B)$ so that $(\text{Hom}_{\mathcal{C}}(A, B), +)$ is an abelian group.*

3) *Composition of morphisms in $\mathcal{C}$ is compatible with the group structures from 2), that means the mapping*

$$\circ : \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(A, C) \qquad (f, g) \mapsto f \circ g$$

is $\mathbb{Z}$-bilinear with respect to the group structures on $\text{Hom}_{\mathcal{C}}(B, C)$, $\text{Hom}_{\mathcal{C}}(A, B)$, $\text{Hom}_{\mathcal{C}}(A, C)$.

Is is easy to see that in a pre-additive category $\mathcal{C}$ for $A, B \in \text{Obj}(\mathcal{C})$ the neutral element in $\text{Hom}_{\mathcal{C}}(A, B)$ is given by 0_{AB}, as defined in proposition 1.0.3: For a zero object $Z \in \text{Obj}(\mathcal{C})$ the morphism $f_{AZ} \in \text{Hom}_{\mathcal{C}}(A, Z)$ is the neutral element, because its group $\text{Hom}_{\mathcal{C}}(A, Z)$ is trivial, so by bilinearity of composition $0_{AB} = f_{ZB} \circ f_{AZ}$ is the neutral element of $\text{Hom}_{\mathcal{C}}(A, B)$. From now on, for pre-additive categories we will simply denote by $0 \in \text{Obj}(\mathcal{C})$ any zero object and by 0_{AB} or 0 the neutral element of $\text{Hom}_{\mathcal{C}}(A, B)$.

In **Ab**, given two abelian groups $A, B \in \text{Obj}(\mathcal{C})$, there is the obvious group structure on $\text{Hom}_{\mathcal{C}}(A, B)$: For $f, f' \in \text{Hom}_{\mathcal{C}}(A, B)$ one sets $(f + f')(a) := f(a) + f'(a)$ for $a \in A$. Clearly, for $f, f' \in \text{Hom}_{\mathcal{C}}(A, B)$, $g, g' \in \text{Hom}_{\mathcal{C}}(B, C)$ we have $g \circ (f + f') = (g \circ f) + (g \circ f')$ and $(g + g') \circ f = (g \circ f) + (g' \circ f)$, which means that the group structures are compatible with composition. Therefore **Ab** is pre-additive. However, the same construction cannot be applied to **Gr**. Without commutativity $f + f'$ does not need to be a group homomorphism. The following theorem aims to give motivation for pre-additive categories. The idea of its proof is taken from [Sta].

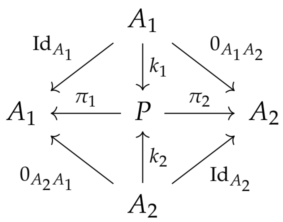
Theorem 1.0.5. *Let $\mathcal{C}$ be a pre-additive category. Then finite products and finite coproducts coincide. That means:*

1) *If $\{\pi_i : P \to A_i : i \in I\}$ is a product in $\mathcal{C}$ with $|I| < \infty$, then there exist $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, P)$ for $i \in I$ so that $\{\phi_i : A_i \to P : i \in I\}$ is a coproduct in $\mathcal{C}$.*

2) *If $\{\phi_i : A_i \to P : i \in I\}$ is a coproduct in $\mathcal{C}$ with $|I| < \infty$, then there exist $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(P, A_i)$ for $i \in I$ so that $\{\pi_i : P \to A_i : i \in I\}$ is a product in $\mathcal{C}$.*

Proof. For 1) let us first assume $|I| = 2$, so let $A_1, A_2 \in \mathrm{Obj}(\mathcal{C})$ so that the product $P := A_1 \times A_2$ with projections $\pi_1 : P \to A_1$ and $\pi_2 : P \to A_2$ exists.

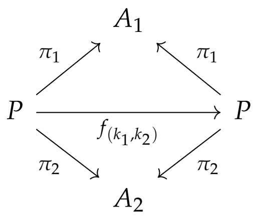
We apply the universal property of P on $\mathrm{Id}_{A_1} : A_1 \to A_1$ and $0_{A_1 A_2} : A_1 \to A_2$ and obtain a morphism $k_1 : A_1 \to P$ so that $\pi_1 \circ k_1 = \mathrm{Id}_{A_1}$ and $\pi_2 \circ k_1 = 0_{A_1 A_2}$. We apply the universal property of P also on $\mathrm{Id}_{A_2} : A_2 \to A_2$ and $0_{A_2 A_1} : A_2 \to A_1$ and obtain a morphism $k_2 : A_2 \to P$ so that $\pi_1 \circ k_2 = 0_{A_2 A_1}$ and $\pi_2 \circ k_2 = \mathrm{Id}_{A_2}$. Alltogether the following diagram commutes:



For any $X \in \mathrm{Obj}(\mathcal{C})$ and morphisms $j_1 \in \mathrm{Hom}_{\mathcal{C}}(A_1, X)$, $j_2 \in \mathrm{Hom}_{\mathcal{C}}(A_2, X)$ we define $f_{(j_1,j_2)} : P \to P'$ by $f_{(j_1,j_2)} := j_1 \circ \pi_1 + j_2 \circ \pi_2$. By the commutativity of the diagram above we have:

$$\pi_1 \circ f_{(k_1,k_2)} = \pi_1 \circ k_1 \circ \pi_1 + \pi_1 \circ k_2 \circ \pi_2 = \mathrm{Id}_{A_1} \circ \pi_1 + 0_{A_2 A_1} \circ \pi_2 = \pi_1$$

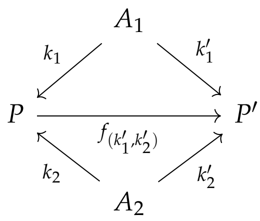
Similarly, we get $\pi_2 \circ f_{(k_1,k_2)} = \pi_2$, hence the following diagram commutes:



Instead of $f_{(k_1,k_2)}$ the morphism Id_P also lets this diagram commute. Therefore, by the universal property of the product P (uniqueness), we have $f_{(k_1,k_2)} = \mathrm{Id}_P$. Now, with this observation, we claim that $\{k_i : A_i \to P : i = 1, 2\}$ is a coproduct of A_1, A_2. Suppose we have $P' \in \text{Obj}(\mathcal{C})$ and $k_1' \in \text{Hom}_{\mathcal{C}}(A_1, P')$, $k_2' \in \text{Hom}_{\mathcal{C}}(A_2, P')$. Then we show that $f_{(k_1',k_2')} : P \to P'$ is a connecting morphism. By the commutativity of the first diagram:

$$f_{(k_1',k_2')} \circ k_1 = k_1' \circ \pi_1 \circ k_1 + k_2' \circ \pi_2 \circ k_1 = k_1' \circ \text{Id}_{A_1} + k_2' \circ 0_{A_1 A_2} = k_1'$$

Similarly we get $f_{(k_1',k_2')} \circ k_2 = k_2'$. So indeed the following diagram commutes:



In order to show that the connecting morphism is unique, suppose $f'_{(k_1',k_2')} : P \to P'$ is another morphism so that the last diagram commutes. Then we have:

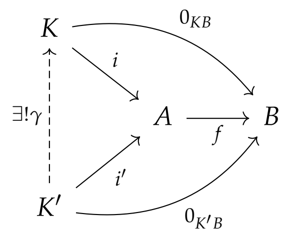
$$\begin{aligned}
(f_{(k_1',k_2')} - f'_{(k_1',k_2')}) &= (f_{(k_1',k_2')} - f'_{(k_1',k_2')}) \circ \text{Id}_P \\
&= (f_{(k_1',k_2')} - f'_{(k_1',k_2')}) \circ f_{(k_1,k_2)} \\
&= (f_{(k_1',k_2')} - f'_{(k_1',k_2')}) \circ (k_1 \circ \pi_1 + k_2 \circ \pi_2) \\
&= (f_{(k_1',k_2')} \circ k_1 - f'_{(k_1',k_2')} \circ k_1) \circ \pi_1 + (f_{(k_1',k_2')} \circ k_2 - f'_{(k_1',k_2')} \circ k_2) \circ \pi_2 \\
&= (k_1' - k_1') \circ \pi_1 + (k_2' - k_2') \circ \pi_2 = 0_{PP'}
\end{aligned}$$

Hence $f_{(k_1',k_2')} = f'_{(k_1',k_2')}$, so the connecting morphism is unique and therefore the universal property of the coproduct is satisfied. For $|I| > 2$, apply induction. We will not prove 2), because it is very similar. □

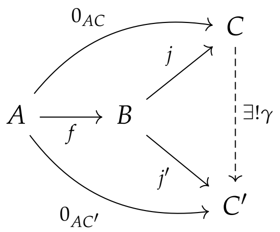
Another concept that we are used to in **Ab** and which is very important for homological algebra is the one of kernels, images and cokernels of morphisms. The problem with defining them in the language of category theory is that we are on a very abstract level. We cannot assume that morphisms are actually sets of mappings (as in concrete categories), and we cannot assume that objects contain something like a zero element. But pre-additive categories have some additional structure that allows us to define at least kernels and cokernels in such a way that it will coincide in **Ab** with what we already know to be a kernel or a cokernel.

Definition 1.0.6. *Let $\mathcal{C}$ be a category with a zero object with the zero morphisms 0_{XY} for any two objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ as in proposition 1.0.3. Let $f : A \to B$ be a morphism in $\mathcal{C}$.*

1) *A pair $(K, i : K \to A)$ where $K \in \mathrm{Obj}(\mathcal{C})$ is a kernel of f if $f \circ i = 0_{KB}$ and whenever there is another such object $K' \in \mathrm{Obj}(\mathcal{C})$ with a morphism $i' : K' \to A$ and $f \circ i' = 0_{K'B}$ then there is a unique morphism $\gamma : K' \to K$ such that $i \circ \gamma = i'$.*



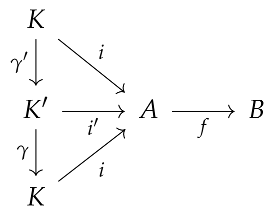
2) *A pair $(C, j : B \to C)$ where $C \in \mathrm{Obj}(\mathcal{C})$ is a cokernel of f if $j \circ f = 0_{AC}$ and whenever there is another such object $C' \in \mathrm{Obj}(\mathcal{C})$ with a morphism $j' : B \to C'$ and $j' \circ f = 0_{AC'}$ then there is a unique morphism $\gamma : C \to C'$ such that $\gamma \circ j = j'$.*



Having defined kernels, one could also define a cokernel as a pair $(C, j : B \to C)$ so that $j \in \mathrm{Hom}_{\mathcal{C}^\circ}(C, B)$ is a kernel in the opposite category of $f \in \mathrm{Hom}_{\mathcal{C}^\circ}(B, A)$. This is equivalent to the definition above.

Proposition 1.0.7. *Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ in a category $\mathcal{C}$ with zero objects. Then kernels of f are unique up to isomorphism: Given two kernels $(K, i : K \to A)$ and $(K', i' : K' \to A)$ of f then there exists an isomorphism $\gamma : K' \to K$. Moreover, $i \circ \gamma = i'$. A similar statement holds for cokernels.*

Proof. Let $(K, i : K \to A)$ and $(K', i' : K' \to A)$ be two kernels of f. Applying the definition of K yields a morphism $\gamma : K' \to K$ such that $i \circ \gamma = i'$. Applying the definition of K' yields a morphism $\gamma' : K \to K'$ such that $i' \circ \gamma' = i$. So the following diagram commutes:



Applying the uniqueness part of the definition of K on K itself gives that there can only be one morphism $\phi : K \to K$ such that $i \circ \phi = i$. But Id_K and also $\gamma \circ \gamma'$ satisfy this property, so $\gamma \circ \gamma' = \mathrm{Id}_K$. By applying analogously the definition of K' one gets $\gamma' \circ \gamma = \mathrm{Id}_{K'}$. Hence $\gamma : K' \to K$ is an isomorphism. The statement can be proven similarly about cokernels. □

In the categories **Gr** and **Ab** kernels and cokernels always exist. For example, let $f : A \to B$ be a group morphism. Then we know that $\mathrm{Ker}(f) := \{a \in A : f(a) = 0\}$ is again a group. Any group $K \cong \mathrm{Ker}(f)$ that is isomorphic to the set theoretical kernel is together with the inclusion morphism $i : K \to A$ a kernel in the new category theoretical sense.

We want to note that there is a slightly more general way to define kernels and cokernels that also applies to categories without zero objects. One has to replace in definition 1.0.6 the concrete zero morphisms 0_{XY}, which are defined via zero objects, by a collection of zero morphisms, i.e. one morphism for each pair of objects, that satisfies some more general commutativity condition. A category with such a collection of zero morphisms is then called *category with zero morphisms*. In the following we will only cover pre-additive categories, in which by definition zero objects exist, so we will not need this generalization.

Before going further in the definitions towards abelian categories, we will show that kernels and cokernels in the category theoretical sense have indeed some of the properties that we are familiar with.

Proposition 1.0.8. *Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ in a pre-additive category and let $(K, i : K \to A)$ a kernel of f and $(C, j : B \to C)$ a cokernel of f.*

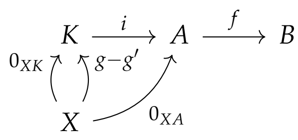
1) i is a monomorphism and j is an epimorphism.

2) If $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, Y)$ is a monomorphism for some $Y \in \mathrm{Obj}(\mathcal{C})$, then $(K, i : K \to A)$ is also a kernel of $\phi \circ f$. Similarly, if $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ is an epimorphism for some $X \in \mathrm{Obj}(\mathcal{C})$, then $(C, j : B \to C)$ is also a cokernel of $f \circ \phi$.

3) f is a monomorphism if and only if K is a zero object. f is an epimorphism if and only if C is a zero object.

Proof. We only show the first parts of 1), 2), 3), the secound parts are the dual in the opposite category.

Ad 1): Suppose $g, g' \in \mathrm{Hom}_{\mathcal{C}}(X, K)$ for some $X \in \mathrm{Obj}(\mathcal{C})$ with $i \circ g = i \circ g'$. Then $i \circ (g - g') = 0_{XA}$. Because of $f \circ 0_{XA} = 0_{XB}$ there is by the kernel property of K only one morphism $\gamma : X \to K$ with $i \circ \gamma = 0_{XA}$. But both morphisms 0_{XK} and $(g - g')$ satisfy this, so $g - g' = 0_{XK}$. Hence $g = g'$.



Ad 2): Suppose $K' \in \mathrm{Obj}(\mathcal{C})$, $i' : K' \to A$ with $\phi \circ f \circ i' = 0_{K'Y}$. Then, because ϕ is a monomorphism, $f \circ i' = 0_{K'B}$. Then by the kernel property of K, there is indeed a unique morphism $\gamma : K' \to K$ so that $i \circ \gamma = i'$.

Ad 3): Let $Z \in \mathrm{Obj}(\mathcal{C})$ be a zero object. Assume that f is a monomorphism. Suppose there is an object $K' \in \mathrm{Obj}(\mathcal{C})$ and $i' : K' \to A$ with $f \circ i' = 0$. Then, since f is a monomorphism, $i' = 0$. The morphism $\gamma = 0_{K'Z}$ is the only morphism $\gamma : K' \to Z$, and it satisfies $0_{ZA} \circ \gamma = i'$. Hence $(Z, 0_{ZA})$ is a kernel of f. So $K \cong Z$ is also a zero object.

For the other direction: Assume K is a zero object, so $i = 0_{KA}$. Suppose there are $g, g' \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ for some $X \in \mathrm{Obj}(\mathcal{C})$ with $f \circ g = f \circ g'$. Then $f \circ (g - g') = 0_{XB}$ and by the kernel property of K there is unique morphism $\gamma : X \to K$ with $0_{KA} \circ \gamma = g - g'$. Hence $g - g' = 0_{XA}$ and $g = g'$, so f is a monomorphism. □

Definition 1.0.9. *A category $\mathcal{C}$ is pre-abelian if:*

1) *The category $\mathcal{C}$ is pre-additive.*

2) *For any finite family of objects $\{A_i\}_{i \in I} \subset \mathrm{Obj}(\mathcal{C})$, $|I| < \infty$ the product $\prod_{i \in I} A_i$ exists.*

3) *For any morhpism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ both kernel and cokernel of f exist.*

Of course by theorem 1.0.5 2) is equivalent to the existence of finite coproducts. For abelian groups it is well-known that finite products and coproducts of abelian groups are given by their direct sums together with the natural projections resp. injections, so **Ab** is a pre-abelian category. Now the final step to abelian categories comes.

Definition 1.0.10. *Let $\mathcal{C}$ be a pre-abelian category.*

1) *Category $\mathcal{C}$ is Ab-monic, if for every monomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for any $A, B \in \mathrm{Obj}(\mathcal{C})$ there is an object $C \in \mathrm{Obj}(\mathcal{Y})$ and a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, Y)$ so that (A, f) is a kernel of g.*

2) *Category $\mathcal{C}$ is Ab-epic, if for every epimorphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for any $A, B \in \mathrm{Obj}(\mathcal{C})$ there is an object $X \in \mathrm{Obj}(\mathcal{C})$ and a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ so that (B, f) is a cokernel of g.*

3) *Category $\mathcal{C}$ is abelian, if it is both Ab-monic and Ab-epic.*

In other words, abelian categories are pre-additive categories in which finite products (and coproducts) exist and in which morphisms have kernels and cokernels and also appear as kernels and cokernels.

Proposition 1.0.11. *The category **Ab** is abelian.*

Proof. We have already seen that **Ab** is a pre-abelian category.

Ab is Ab-monic: Let $f : A \to B$ be a monomorphism between abelian groups. Consider the projection $\pi : B \to {}^{B}\!/_{\mathrm{Im}\, f}$. Then (A, f) is a kernel of π: Suppose there are $A' \in \mathrm{Obj}(\mathcal{C})$ and $f' : A' \to B$ with $\pi \circ f' = 0$, so $\mathrm{Im}\, f' \subset \mathrm{Ker}\, \pi = \mathrm{Im}\, f$. The mapping f is injective, so there is an inverse morphism $f^{-1} : \mathrm{Im}\, f \to A$. Then the morphism $\gamma := f^{-1} \circ f'$ is the unique morphism with $f \circ \gamma = f'$.

Ab is Ab-epic: Let $f : A \to B$ be an epimorphism between abelian groups. Consider the inclusion $i : \mathrm{Ker}\, f \to A$. Then (B, f) is a cokernel of i: Suppose there are $B' \in \mathrm{Obj}(\mathcal{C})$ and $f' : A \to B'$ with $f' \circ i = 0$. The mapping f is surjective and because of $\mathrm{Ker}\, f \subset \mathrm{Ker}\, f'$ we can properly define the morphism $\gamma : B \to B'$ so that $\gamma \circ f = f'$, which is unique with this property. □

Chapter 2

Some theory about abelian categories

This chapter is based on chapter 7 of [Osb00] and its exercises. We will learn more examples of abelian categories and we will investigate further convenient properties.

2.1 More analogies to abelian groups

In the category **Ab** of abelian groups, a morphism is an isomorphism iff it is injective and surjective. In cagetory theory, the concept of monomorphisms and epimorphisms generalize injective and surjective morphisms such as in **Ab**. Therefore it is natural to expect that a morphism in category theory is an isomorphism if and only if it is a monomorphism and an epimorphism. Indeed isomorphisms are monomorphisms and epimorphisms. However in general the reverse does not hold. But we will see that in abelian categories, or slightly more general, in Ab-monic pre-abelian categories, it does.

Definition 2.1.1. *Let $\mathcal{C}$ be a category.*

1) A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is a bimorphism, if it is both monic and epic.

2) The category $\mathcal{C}$ is a balanced category, if every bimorphism is an isomorphism.

Theorem 2.1.2. *Every Ab-monic pre-abelian category is balanced.*

Proof. Let $f : A \to B$ be a bimorphism, where $A, B \in \mathrm{Obj}(\mathcal{C})$. Because f is a monomorphism, there is an object $C \in \mathrm{Obj}(\mathcal{C})$ and a morphism $g : B \to C$ so that (A, f) is a kernel of g. Then by definition $g \circ f = 0_{AC}$, hence $g = 0_{BC}$, because f is an epimorphism. But then $(B, \mathrm{Id}_B : B \to B)$ is also a kernel of g. According to proposition 1.0.7 there is an isomorphism $p : A \to B$ with $p = \mathrm{Id}_B \circ p = f$:

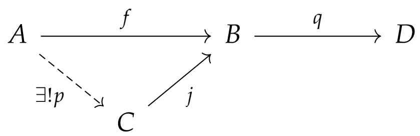
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g=0_{BC}} & C \\ & \overset{\sim p}{\searrow} & \uparrow{\scriptstyle \mathrm{Id}_B} & & \\ & & B & & \end{array}$$

Hence f is an isomorphism. □

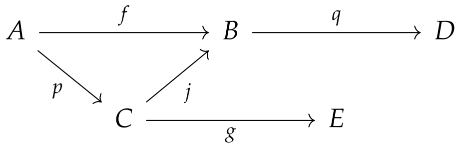
Let $f : A \to B$ be a group morphism between abelian groups. Then $\mathrm{Im}(f) \subset B$ is also an abelian group. We directly get a surjective morphism $p : A \to \mathrm{Im}(f)$, $a \mapsto f(a)$ and an (injective) embedding $j : \mathrm{Im}(f) \to B$, $f(a) \mapsto f(a)$ so that $f = j \circ p$. This means that in the category **Ab** every morphism can be decomposed into a monomorphism and an epimorphism. This property holds for abelian categories in general:

Theorem 2.1.3. *Let $\mathcal{C}$ be an Ab-monic pre-abelian category and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for some $A, B \in \mathrm{Obj}(\mathcal{C})$. Then there is an object $C \in \mathrm{Obj}(\mathcal{C})$, an epimorphism $p : A \to C$ and a monomorphism $j : C \to B$ so that $f = j \circ p$.*

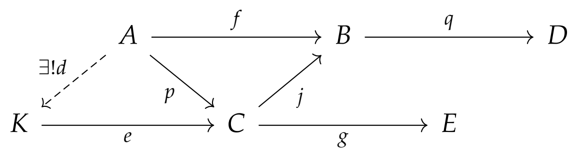
Proof. Let $f : A \to B$. Let $(D, q : B \to D)$ be a cokernel of f and $(C, j : C \to B)$ be a kernel of q. Since by the definition of cokernel we have $q \circ f = 0_{AD}$ and by the definition of kernel, there is a unique morphism $p : A \to C$ so that the following diagram commutes:



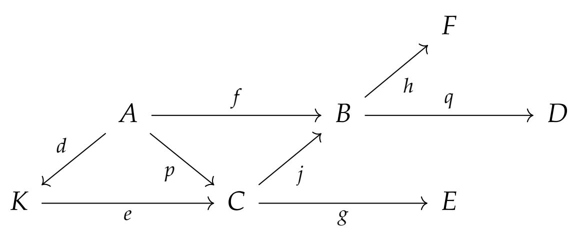
This is already the desired decomposition of f. By proposition 1.0.8 1) the morphism j is indeed a monomorphism, so all that remains to show is that p is an epimorphism. To that end, assume there are morphisms $g_1, g_2 : C \to E$ for some $E \in \mathrm{Obj}(\mathcal{C})$ with $g_1 \circ p = g_2 \circ p$. Put $g := g_1 - g_2$, so $g \circ p = (g_1 - g_2) \circ p = g_1 \circ p - g_2 \circ p = 0_{AE}$.



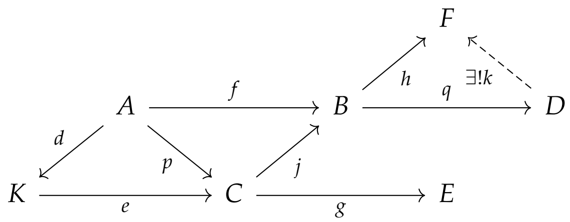
Let $(K, e : K \to C)$ be a kernel of g. By definition and because of $g \circ p = 0_{AE}$ there is a unique morphism $d : A \to K$ so that the following diagram commutes:



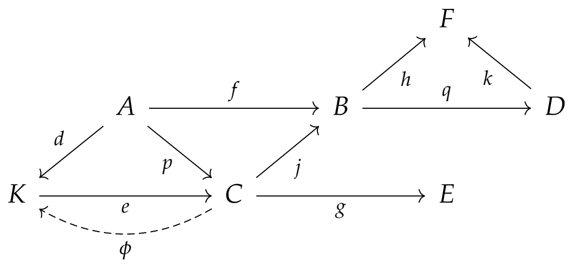
The morphisms j and e are monomorphisms, because both of them come from kernels (prop. 1.0.8). Then $j \circ e$ is also a monomorphism and because $\mathcal{C}$ is Ab-monic there is an object $F \in \mathrm{Obj}(\mathcal{C})$ and a morphism $h : B \to F$ so that $(K, j \circ e : K \to B)$ is a kernel of h.



From $h \circ j \circ e = 0_{KF}$ we get $h \circ j \circ e \circ d = 0_{AF}$ and by the commutativity of the diagram $h \circ f = 0_{AF}$. Because $(D, q : B \to D)$ is a cokernel of f, there is a unique morphism $k : D \to F$ so that the following diagram commutes:



We have $q \circ j = 0_{CD}$, because (C, j) was chosen as a kernel of q, so $h \circ j = k \circ q \circ j = 0_{CF}$. Since $(K, j \circ e : K \to B)$ is a kernel of h, there is a morphism $\phi : C \to K$ with $j \circ e \circ \phi = j$:



Since j is a monomorphism, $e \circ \phi = \mathrm{Id}_C$. Therefore $g = g \circ \mathrm{Id}_C = g \circ e \circ \phi = 0_{KE} \circ \phi = 0_{CE}$, since (K, e) was chosen as a kernel of g. So $g_1 - g_2 = g = 0_{CE}$ and $g_1 = g_2$. So p is indeed an epimorphism. $\square$

In the next section we will see that the intermediate object C in theorem 2.1.3 is unique up to isomorphism. For this reason it can be thought as a generalization of the image of a group morphism in **Ab**.

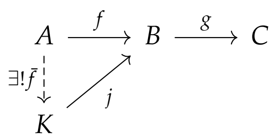
2.2 Homology on abelian categories

Homological algebra deals with exact sequences. We would like to have a notion of exactness on the level of abelian categories. There are the following two concepts.

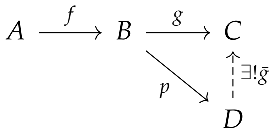
Definition 2.2.1. *Let $\mathcal{C}$ be a pre-abelian category, $A, B, C \in \mathrm{Obj}(\mathcal{C})$ objects and $f : A \to B$, $g : B \to C$ morphisms with $g \circ f = 0_{AC}$, so we have the following diagram:*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

1) *This diagram is called kernel-exact, if for every kernel $(K, j : K \to B)$ of g the unique morphism $\bar{f} : A \to K$ with $j \circ \bar{f} = f$ is an epimorphism.*



2) *This diagram is called cokernel-exact, if for every cokernel $(D, p : B \to D)$ of f the unique morphism $\bar{g} : D \to C$ with $\bar{g} \circ p = g$ is a monomorphism.*



Lemma 2.2.2. *Consider the following diagram:*

$$0 \xrightarrow{f} A \xrightarrow{g} B$$

Then the following are equivalent:

1) *The diagram is kernel-exact.*

2) *g is a monomorphism.*

3) *The diagram is cokernel-exact.*

Proof. 1) $\implies$ 2): Let $(K, j : K \to A)$ be a kernel of g. Then, since the diagram is kernel exact, the unique $\bar{f} : 0 \to K$ with $j \circ \bar{f} = f = 0_{0A}$ is an epimorphism, therefore $j = 0_{KA}$. Now let $C \in \mathrm{Obj}(\mathcal{C})$ and $\gamma_1, \gamma_2 : C \to A$ arbitrary with $g \circ \gamma_1 = g \circ \gamma_2$. This means $g \circ (\gamma_1 - \gamma_2) = 0_{CB}$. Then by definition of the kernel there is a morphism $\phi : C \to K$ so that $\gamma_1 - \gamma_2 = j \circ \phi = 0_{KA} \circ \phi = 0_{CA}$. Hence $\gamma_1 = \gamma_2$ and g is a monomorphism.

2) $\implies$ 1): Let $(K, j : K \to A)$ be a kernel of g and $\bar{f} : 0 \to K$ the unique morphism with $j \circ \bar{f} = f = 0_{0A}$. To show that $\bar{f}$ is an epimorphism, suppose for any object $C \in \mathrm{Obj}(\mathcal{C})$ there are morphisms $\gamma_1, \gamma_2 : K \to C$ with $\gamma_1 \circ \bar{f} = \gamma_2 \circ \bar{f}$. According to proposition 1.0.8 3) K is a zero object, which means $\gamma_1 - \gamma_2 = 0_{KC}$. Therefore, $\gamma_1 = \gamma_2$ and $\bar{f}$ is an epimorphism.

2) $\iff$ 3): Up to isomorphism, $(A, \mathrm{Id}_{\mathrm{A}} : \mathrm{A} \to \mathrm{A})$ is the cokernel of g and g itself is the unique morphism so that $g \circ \mathrm{Id}_A = g$. Then the diagram satisfies the definition of cokernel-exactness for this kernel if and only if g is a monomorphism. One easily ckecks that this is a property which does not depend on the choice of the kernel. □

Theorem 2.2.3. *Let $\mathcal{C}$ be a pre-abelian category. Then the following are equivalent:*

1) *$\mathcal{C}$ is an Ab-monic category.*

2) *$\mathcal{C}$ is a balanced category and cokernel-exact sequences are kernel-exact sequences.*

3) *Let $0 \to A \to B \to C$ be a cokernel-exact sequence, then $A \to B$ is a kernel of $B \to C$.*

4) *If $A \to B$ is a monomorphism with cokernel $B \to C$, then $A \to B$ is a kernel of $B \to C$.*

Proof. 1) $\implies$ 2): Since $\mathcal{C}$ is Ab-monic, by proposition 2.1.2 $\mathcal{C}$ is balanced. Assume we have a cokernel-exact sequence:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Let $(K, j : K \to B)$ be a kernel of g and $(D, q : B \to D)$ be a cokernel of f, and let $\bar{f} \in \mathrm{Hom}_{\mathcal{C}}(A, K)$ and $\bar{g} \in \mathrm{Hom}_{\mathcal{C}}(D, C)$ the unique morphisms so that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle \exists ! \bar{f}} \downarrow & \nearrow_{j} & & \searrow^{q} & \uparrow {\scriptstyle \exists ! \bar{g}} \\ K & & & & D \end{array}$$

By definition of cokernel-exactness, $\bar{g}$ is a monomorphism. By proposition 1.0.8 2) a kernel of q is also a kernel of $\bar{g} \circ q$. Then, because of the uniqueness of kernels up to isomorphism, $(K, j : K \to B)$, which is a kernel of $g = \bar{g} \circ q$, is also a kernel of q. Now we are in the situation of the proof of theorem 2.1.3 a), in which j was also defined as a kernel of a cokernel, and in which it was shown that $\bar{f}$ (denoted in the proof as p) is an epimorphism. Hence the sequence is kernel-exact.

2) $\implies$ 3): Assume we have a cokernel-exact sequence:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

Then, if we assume that cokernel-exact sequences are kernel-exact, this sequence is also kernel exact. Let $(K, j : K \to B)$ a kernel of g and let $\bar{f} \in \mathrm{Hom}_{\mathcal{C}}(A, K)$ be the homomorphism so that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & & {\scriptstyle \exists ! \bar{f}} \downarrow & \nearrow_{j} & & & \\ & & K & & & & \end{array}$$

By kernel-exactness, $\bar{f}$ is an epimorphism. By lemma 2.2.2 2) we have that $f = j \circ \bar{f}$ is a monomorphism, hence $\bar{f}$ is a monomorphism as well, so $\bar{f}$ is a bimorphism. Thus, by assuming that $\mathcal{C}$ is balanced, $\bar{f}$ is an isomorphism, and therefore $(A, f : A \to B)$ is a kernel of g.

3) $\implies$ 4): If $(C, q : B \to C)$ is a cokernel of a morphism $f : A \to B$, then the sequence $A \xrightarrow{f} B \xrightarrow{q} C$ is cokernel-exact (The connecting morphism is the isomorphism $\mathrm{Id}_C : C \to C$). By lemma 2.2.2 the sequence $0 \to A \xrightarrow{f} B$ is also cokernel-exact, because f is a monomorphism. Then the total sequence $0 \to A \xrightarrow{f} B \xrightarrow{q} C$ is cokernel-exact. Then by assuming 3) $(A, f : A \to B)$ is a kernel of $q : B \to C$.

4) $\implies$ 3): For an arbitrary monomorphism $f : A \to B$, a cokernel $(C, q : B \to C)$ of f has f as its kernel assuming 3). □

By applying theorem 2.2.3 on the opposite category $\mathcal{C}^o$ one receives the following corollary.

Corollary 2.2.4. *Let $\mathcal{C}$ be a pre-abelian category. Then the following are equivalent:*

1) *$\mathcal{C}$ is an Ab-epic category.*

2) *$\mathcal{C}$ is a balanced category and kernel-exact sequences are cokernel-exact sequences.*

3) *Let $A \to B \to C \to 0$ be a kernel-exact sequence, then $B \to C$ is a cokernel of $A \to B$.*

4) *If $A \to B$ is an epimorphism with kernel $K \to A$, then $A \to B$ is a cokernel of $K \to A$.*

By using the equivalence between the first and secound parts of both theorem 2.2.3 and its corollary 2.2.4 we know that in abelian categories, which are by definition Ab-monic and Ab-epic, the notions of kernel-exactness and cokernel-exactness coincide, which is one more nice feature of abelian categories. The following theorem extends theorem 2.1.3.

Theorem 2.2.5. *Let $\mathcal{C}$ be an Ab-monic pre-abelian category and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for some $A, B \in \mathrm{Obj}(\mathcal{C})$. Then f can be decomposed into $f = j \circ p$, where $p : A \to C$ is an epimorphism and $j : C \to B$ is a monomorphism, for some $C \in \mathrm{Obj}(\mathcal{C})$ which is unique up to isomorphism.*

Proof. We are only missing the part that the intermediate object C is unique up to isomorphism.

Let $(D, q : B \to D)$ be a cokernel of f, and let the decomposition from theorem 2.1.3 given by an object $C \in \mathrm{Obj}(\mathcal{C})$, a monomorphism $j \in \mathrm{Hom}_{\mathcal{C}}(C, B)$ and an epimorphism $p \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ with $j \circ p = f$.

Now suppose $C' \in \mathrm{Obj}(\mathcal{C})$, a monomorphism $j' \in \mathrm{Hom}_{\mathcal{C}}(C, B)$ and an epimorphism $p' \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ are also a decomposition of f, i.e. $j' \circ p' = f'$. By proposition 1.0.8 2), a cokernel of j' is also a cokernel of $j' \circ p' = f$. Then, because of the uniqueness of cokernels up to isomorphism, $(D, q : B \to D)$ is a cokernel of j'. By the part 4) of the previous theorem 2.2.3, $(C', j' : C' \to B)$ is a kernel of q. Remember that in the construction of theorem 2.1.3 $(C, j : C \to B)$ was also chosen as a kernel of q. By the uniquness of kernels we get $C \cong C'$. □

2.3 More abelian categories

So far, the only example of an abelian category was **Ab** itself. For some ring R, another example of an abelian category is **R-Mod**, the category of (left-) modules over R. This is quite clear, since modules are also abelian groups and all the proofs of the first chapter remain valid for R-modules.

There is also a simple way of constructing new abelian categories from already known ones. Let $\mathcal{C}$ be an arbitrary category for which $\mathrm{Obj}(\mathcal{C})$ is a proper set. Such a category is called small category[1]. Let $\mathcal{D}$ be an abelian category. Then one can consider the so called functor category $\mathcal{D}^{\mathcal{C}}$. Its objects are the covariant functors $F : \mathcal{C} \to \mathcal{D}$ and the morphisms between any two such functors F, G are the natural transformations from F to G. In the following we will sketch the proof that $\mathcal{D}^{\mathcal{C}}$ is abelian. We will focus rather on the constructions than on proving everything in detail.

[1]As in the lecture we assume for all categories that for $A, B \in \mathrm{Obj}(\mathcal{C})$ the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a proper set.

Lemma 2.3.1. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ an abelian category. Then the functor category $\mathcal{D}^{\mathcal{C}}$ is pre-additive.*

Proof. $\mathcal{D}^{\mathcal{C}}$ has a zero object: Let $F : \mathcal{C} \to \mathcal{D}$ be the functor that maps every $X \in \mathrm{Obj}(\mathcal{C})$ to the zero object $0 \in \mathrm{Obj}(\mathcal{D})$ and maps every morphism in $\mathcal{D}$ to Id_0. Then for any covariant functor $G : \mathcal{C} \to \mathcal{D}$ there is exactly one natural transformation $\eta : F \to G$, $\eta_X : 0 \to G(X)$ for $X \in \mathrm{Obj}(\mathcal{C})$, and there is exactly one natural transformation $\eta' : G \to F$, $\eta'_X : G(X) \to 0$ for $X \in \mathrm{Obj}(\mathcal{C})$. Therefore, F is a zero object in $\mathcal{D}^{\mathcal{C}}$.

Given two covariant functors $F, G : \mathcal{C} \to \mathcal{D}$ and natural transformations $\eta^1, \eta^2 : F \to G$ one defines $\eta^1 + \eta^2$ by $(\eta^1 + \eta^2)_X = \eta^1_X + \eta^2_X$, where the addition on the right hand side is the addition on $\mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$. One checks that this defines a group structure and is compatible with composition of natural transformations. □

Lemma 2.3.2. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ an abelian category. Then the functor category $\mathcal{D}^{\mathcal{C}}$ is pre-abelian.*

Proof. By lemma 2.3.1 we know that $\mathcal{D}^{\mathcal{C}}$ is pre-additive.

Finite products exist: Let $\{F_i : \mathcal{C} \to \mathcal{D} : i \in I\}$ a family of covariant functors for an index set I with $|I| < \infty$. Let $F : \mathcal{C} \to \mathcal{D}$ be the covariant functor that maps an object $X \in \mathrm{Obj}(\mathcal{C})$ to the product $\prod_{i\in I} F_i(X)$, which exists because $\mathcal{D}$ is pre-abelian. Let $\pi^i_X : \prod_{i\in I} F_i(X) \to F_i(X)$ denote the i-th projection of $F(X)$. For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ let $F(f) : F(X) \to F(Y)$ be the unique morphism that lets the following diagram for all $i \in I$ commute:

$$\begin{array}{ccc} F(X) = \prod_{i\in I} F_i(X) & \xrightarrow{\exists! F(f)} & \prod_{i\in I} F_i(Y) = F(Y) \\ \downarrow \pi^i_X & & \downarrow \pi^i_Y \\ F_i(X) & \xrightarrow[F_i(f)]{} & F_i(Y) \end{array}$$

Then F is a covariant functor. We claim that F is the product of $\{F_i : i \in I\}$. What are its projections? These have to be natural transformations $\eta^i : F \to F_i$. As the diagram suggests, we can simply set $\eta^i_X := \pi^i_X$. We skip the part of showing that $(F, \eta^i : F \to F_i)$ indeed satisfies the universal property of products, which is straight forward.

Kernels exist: Let $A, B : \mathcal{C} \to \mathcal{D}$ be covariant functors and $\eta : A \to B$ a natural transformation. For $X \in \mathrm{Obj}(\mathcal{C})$ let $(K(X), \kappa_X : K(X) \to A(X))$ with $K(X) \in \mathrm{Obj}(\mathcal{D})$ be a kernel of $\eta_X : A(X) \to B(X)$. Now take a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We claim that there is a unique morphism $K(f) : K(X) \to K(Y)$ that lets the following diagram commute:

$$\begin{array}{ccccc} K(X) & \xrightarrow{\kappa_X} & A(X) & \xrightarrow{\eta_X} & B(X) \\ \exists! K(f) \downarrow & & \downarrow A(f) & & \downarrow B(f) \\ K(Y) & \xrightarrow[\kappa_Y]{} & A(Y) & \xrightarrow[\eta_y]{} & B(Y) \end{array}$$

This is because $\eta_Y \circ A(f) \circ \kappa_X = B(f) \circ \eta_X \circ \kappa_X = B(f) \circ 0 = 0$ and by the property of $(K(Y), \kappa_Y : K(Y) \to A(Y))$ to be a kernel of η_Y. By this construction, $K : \mathcal{C} \to \mathcal{D}$ becomes a covariant functor and $\kappa : K \to A$ a natural transformation. We claim that $(K, \kappa : K \to A)$

is a kernel of $\eta : A \to B$. Clearly, $\eta \circ \kappa$ is the natural transformation whose components $(\eta \circ \kappa)_X : K(X) \to B(X)$ are zero morphisms. This is indeed the zero morphism in $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(K, B)$, as one can easily check. Again, we skip the part of showing that (K, κ) satisfies the universal property of kernels.

Using a dual argument one can show that cokernels exist as well. Therefore, $\mathcal{D}^{\mathcal{C}}$ is pre-abelian. □

The idea of the following proof is taken from [Fre66].

Theorem 2.3.3. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ an abelian category. Then the functor category $\mathcal{D}^{\mathcal{C}}$ is abelian.*

Proof. By lemma 2.3.2 we know that $\mathcal{D}^{\mathcal{C}}$ is a pre-abelian category. In here, we only show that $\mathcal{D}^{\mathcal{C}}$ is Ab-monic. Let $\eta \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(A, B)$ be a monomorphism, so $\eta : A \to B$ is a natural transformation between two covariant functors $A, B : \mathcal{C} \to \mathcal{D}$.

Since η is a monomorphism, by part 3 of 1.0.8 the kernel $(K, \kappa : K \to A)$ is a zero object in $\mathcal{D}^{\mathcal{C}}$, so $K(X) = 0$ for all $X \in \mathrm{Obj}(\mathcal{C})$. Remember how the kernels in $\mathcal{D}^{\mathcal{C}}$ were constructed componentwise in the previous lemma 2.3.2. Then again by part 3 of 1.0.8 we know that η_X is a monomorphism for all $X \in \mathcal{C}$. Now for $X \in \mathcal{C}$ let $(C(X), \tau : B(X) \to C(X))$ be a cokernel of η_X. Since $\mathcal{D}$ is Ab-monic, by part 4 of theorem 2.2.3 we know that $(A(X), \eta_X : A(X) \to B(X))$ is a kernel of $\tau_X : B(X) \to C(X)$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be a morphism, then we claim that there is a unique morphism $C(f) : C(X) \to C(Y)$ so that the following diagram commutes:

$$\begin{array}{ccccc} A(X) & \xrightarrow{\eta_X} & B(X) & \xrightarrow{\tau_X} & C(X) \\ {\scriptstyle A(f)}\downarrow & & \downarrow{\scriptstyle B(f)} & & \downarrow{\scriptstyle \exists! C(f)} \\ A(Y) & \xrightarrow[\eta_Y]{} & B(Y) & \xrightarrow[\tau_Y]{} & C(Y) \end{array}$$

This is because $\tau_Y \circ B(f) \circ \eta_X = \tau_Y \circ \eta_Y \circ A(f) = 0 \circ A(f) = 0$ and by the universal property of the cokernel $(C(X), \tau_X)$ of η_X. By this construction, $C : \mathcal{C} \to \mathcal{D}$ becomes a covariant functor and $\tau : B \to C$ a natural transformation. By the way we constructed the kernels in $\mathcal{D}^{\mathcal{C}}$ componentwise in the previous lemma 2.3.2, $(A, \eta : A \to B)$ is indeed a kernel of τ. □

2.4 Mitchell's embedding theorem

We now state the most famous result about abelian categories. The proof is too complex for this short introduction and we will skip it.

Theorem 2.4.1 (Mitchell's Embedding Theorem)**.** *Let $\mathcal{C}$ be a small abelian category. Then there is a ring R and a full and faithful functor $F : \mathcal{C} \to R$-Mod.*

Proof. The proof can be found in [Wei94]. □

Informally, Mitchell's Theorem states that every abelian category can be embedded into a category of modules over a ring. For the category **Ab** this is trivial. The astonishing point of Mitchell's theorem is that abelian categories are in some sense contained in categories whose objects are sets with some extra structure and whose morphisms are mappings between those (So called concrete categories). This is something very pleasant, because abelian categories in the first hand might be very abstract, whereas categories of modules are considered to be well understood.

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