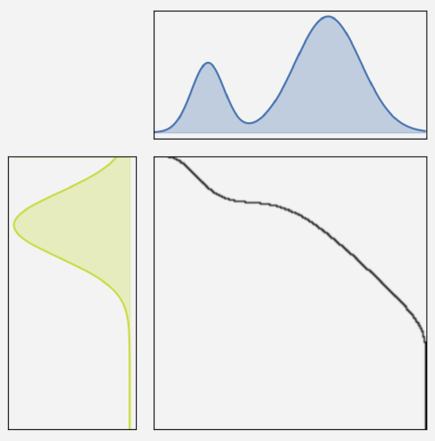
SLICED OPTIMAL TRANSPORT ON THE SPHERE

Robert Beinert[1], Michael Quellmalz[1], Léo Buecher[1,2], Gabriele Steidl[1]
[1] Technische Universität Berlin, [2] Université Paris-Saclay

Introduction

Optimal Transport



- $\mathcal{P}(\mathbb{X})$ probability measures on compact manifold $\mathbb{X}$
- Pushforward of $f\colon \mathbb{X} \to \mathbb{Y}$ and $\mu \in \mathcal{P}(\mathbb{X})$ is

$$f_\# \mu := \mu \circ f^{-1} \in \mathcal{P}(\mathbb{Y})$$

- $\mu \in \mathcal{P}(\mathbb{X})$, $\nu \in \mathcal{P}(\mathbb{Y})$
- Wasserstein-p distance

$$W_p^p(\mu,\nu) := \min_{\pi \in \mathcal{P}(\mathbb{X}\times\mathbb{Y})} \left\{ \int_{\mathbb{X}\times\mathbb{Y}} \|\boldsymbol{x}-\boldsymbol{y}\|_2^p \mathrm{d}\pi(\boldsymbol{x},\boldsymbol{y}) : P_{1\#}\pi = \mu, P_{2\#}\pi = \nu \right\}$$

where $p \in [1,\infty)$ and P_i is the projection to the i-th component

Optimal Transport on the Real Line $\mathbb{X} = \mathbb{R}$

- Cumulative distribution function $F_\mu(x) := \mu((-\infty, x])$
- By [5] the Wasserstein distance is easy to compute

$$W_p^p(\mu,\nu) = \int_0^1 |F_\mu^{-1}(x) - F_\nu^{-1}(x)|^p \, \mathrm{d}x$$

Sliced Optimal Transport on $\mathbb{R}^d$

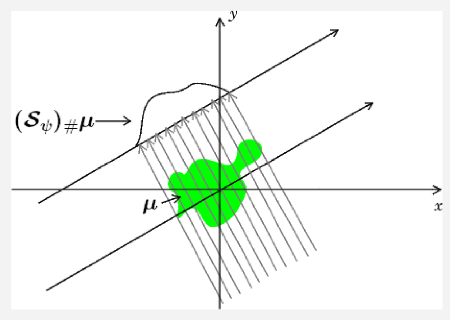
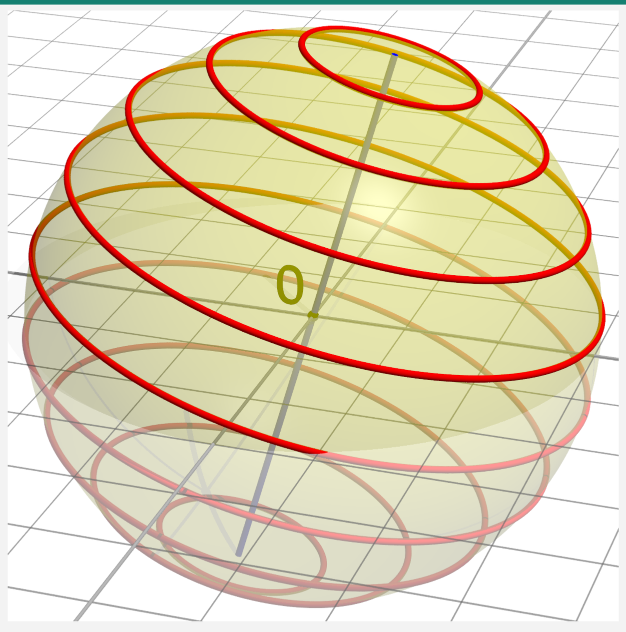


Image by Brendan F. Hayden

- Slicing operator $\mathcal{S}_\psi\colon \mathbb{R}^d \to \mathbb{R}$, $\boldsymbol{x} \mapsto \boldsymbol{x}\cdot\boldsymbol{\psi}$
- Sliced Wasserstein distance [4]

$$\mathrm{SW}_p^p(\mu,\nu) := \int_{\mathbb{S}^{d-1}} \mathrm{W}_p^p((\mathcal{S}_\psi)_\#\mu, (\mathcal{S}_\psi)_\#\nu) \, \mathrm{d}\boldsymbol{\psi}$$

Parallel Slicing of the Sphere $\mathbb{S}^{d-1}$



- Slicing operator for fixed $\boldsymbol{\psi} \in \mathbb{S}^{d-1}$:

$$\mathcal{S}_\psi\colon \mathbb{S}^{d-1} \to [-1,1], \quad \boldsymbol{\xi} \mapsto \boldsymbol{\xi}\cdot\boldsymbol{\psi}$$

- Slice $\mathcal{S}_\psi^{-1}(t)$ is intersection of $\mathbb{S}^{d-1}$ and plane with normal $\boldsymbol{\psi}$ and distance t from the origin

Parallel Slice Transform

- For functions $f \in L^1(\mathbb{S}^{d-1})$:

$$\mathcal{U}f(\boldsymbol{\psi},t) := \frac{1}{|\mathbb{S}^{d-1}|\sqrt{1-t^2}} \int_{\mathcal{S}_\psi^{-1}(t)} f(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}, \quad (\boldsymbol{\psi},t) \in \mathbb{S}^{d-1}\times(-1,1)$$

- For measures $\mu \in \mathcal{P}(\mathbb{S}^{d-1})$:

$$\mathcal{U}_\psi\mu := (\mathcal{S}_\psi)_\#\mu \in \mathcal{P}([-1,1]),$$
$$\mathcal{U}\mu := T_\#(u_{\mathbb{S}^{d-1}} \times \mu) \in \mathcal{P}(\mathbb{S}^{d-1}\times[-1,1]) \text{ with } T(\boldsymbol{\psi},\boldsymbol{\xi}) = (\boldsymbol{\psi}, \mathcal{S}_\psi(\boldsymbol{\xi})),$$

where $u_{\mathbb{S}^{d-1}}$ is the normalized Lebesgue measure on $\mathbb{S}^{d-1}$

Parallelly Sliced Wasserstein Distance

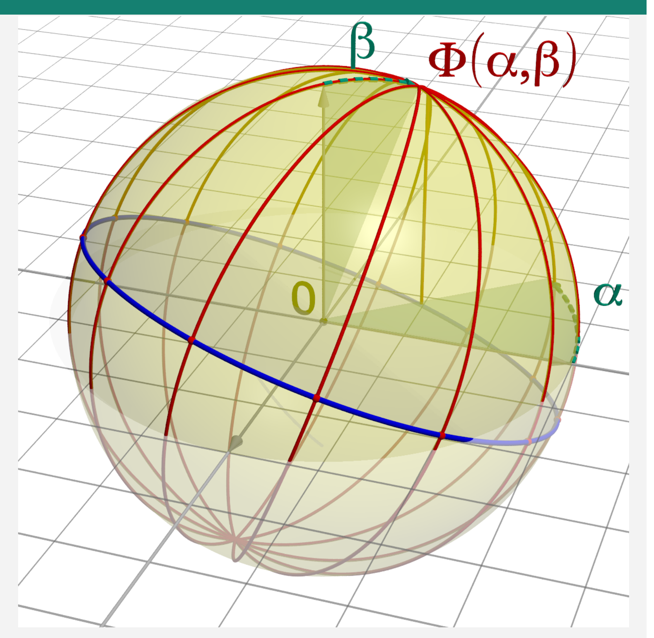
$$\mathrm{PSW}_p^p(\mu,\nu) := \int_{\mathbb{S}^{d-1}} \mathrm{W}_p^p(\mathcal{U}_\psi\mu, \mathcal{U}_\psi\nu) \, \mathrm{d}u_{\mathbb{S}^{d-1}}(\boldsymbol{\psi})$$

Theorem (Metric Properties) [3]

- PSW_p is a rotationally invariant metric on $\mathcal{P}(\mathbb{S}^{d-1})$ for all $p \in [1,\infty)$.
- There exist constants $c_{d,p}, C_{d,p} > 0$ such that

$$c_{d,p}\mathrm{PSW}_p(\mu,\nu) \le \mathrm{W}_p(\mu,\nu) \le C_{d,p}\mathrm{PSW}_p(\mu,\nu)^{\frac{1}{p(d+1)}} \quad \forall \mu,\nu \in \mathcal{P}(\mathbb{S}^{d-1}).$$

Semicircular Slicing of the Sphere $\mathbb{S}^2$



- Spherical coordinates on $\mathbb{S}^2$:

$$\boldsymbol{\Phi}(\varphi,\theta) := (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$$

- Euler angles on $\mathrm{SO}(3)$:

$$\boldsymbol{\Psi}(\alpha,\beta,\gamma) := R_3(\alpha)\, R_2(\beta)\, R_3(\gamma)$$

($R_i(\alpha)$ rotation around i-th axis with angle α)

Normalized Semicircle Transform

- For functions $f \in L^1(\mathbb{S}^2)$:

$$\mathcal{W}f(\alpha,\beta,\gamma) := \frac{1}{4\pi}\int_0^\pi f\left(\boldsymbol{\Psi}(\alpha,\beta,0)\boldsymbol{\Phi}(\gamma,\theta)\right) \sin\theta \, \mathrm{d}\theta$$

- For measures $\mu \in \mathcal{P}(\mathbb{S}^2)$:

$$\mathcal{W}_{\alpha,\beta}\mu := (\mathcal{A}_{\alpha,\beta})_\#\mu \in \mathcal{P}(\mathbb{T})$$
$$\mathcal{W}\mu := T_\#(u_{\mathbb{S}^2} \times \mu) \in \mathcal{P}(\mathrm{SO}(3)) \text{ with } T(\boldsymbol{\Phi}(\alpha,\beta),\boldsymbol{\xi}) = \boldsymbol{\Psi}(\alpha,\beta,\mathcal{A}_{\alpha,\beta}(\boldsymbol{\xi}))$$

($\mathcal{A}_{\alpha,\beta}\colon \mathbb{S}^2 \to \mathbb{T} \simeq [0,2\pi)$, $\mathcal{A}_{\alpha,\beta}(\boldsymbol{\xi}) := \mathrm{azi}(\boldsymbol{\Psi}(\alpha,\beta,0)^\top \boldsymbol{\xi})$, and $\mathrm{azi}(\Phi(\varphi,\theta)) = \varphi$ for all φ,θ)

Semicircular Sliced Wasserstein Distance [2]

$$\mathrm{SSW}_p^p(\mu,\nu) := \int_{\mathbb{S}^2} \mathrm{W}_p^p(\mathcal{W}_{\alpha,\beta}\mu, \mathcal{W}_{\alpha,\beta}\nu) \, \mathrm{d}u_{\mathbb{S}^2}(\boldsymbol{\Phi}(\alpha,\beta))$$

Theorem (Metric Properties) [1]

- The normalized semicircle transform $\mathcal{W}\colon \mathcal{P}(\mathbb{S}^2) \to \mathcal{P}(\mathrm{SO}(3))$ is injective.
- SSW_p is a rotationally invariant metric on $\mathcal{P}(\mathbb{S}^2)$ for all $p \in [1,\infty)$.

- Definition can be generalized to $\mathbb{S}^{d-1}$

Wasserstein Barycenters

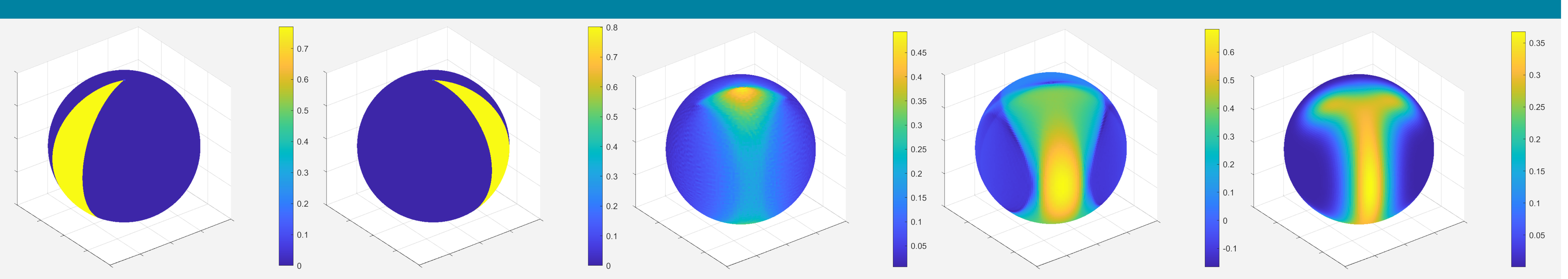
- Measures $\mu_i \in \mathcal{P}(\mathbb{X})$, $i = 1,\dots,M$, weights $\lambda_i \ge 0$ with $\sum_{i=1}^M \lambda_i = 1$
- Wasserstein barycenter

$$\mathrm{Bary}_{\mathbb{X}}(\mu_i,\lambda_i)_{i=1}^M := \operatorname*{arg\,min}_{\nu\in\mathcal{P}(\mathbb{X})} \sum_{i=1}^M \lambda_i \mathrm{W}_2^2(\nu,\mu_i)$$

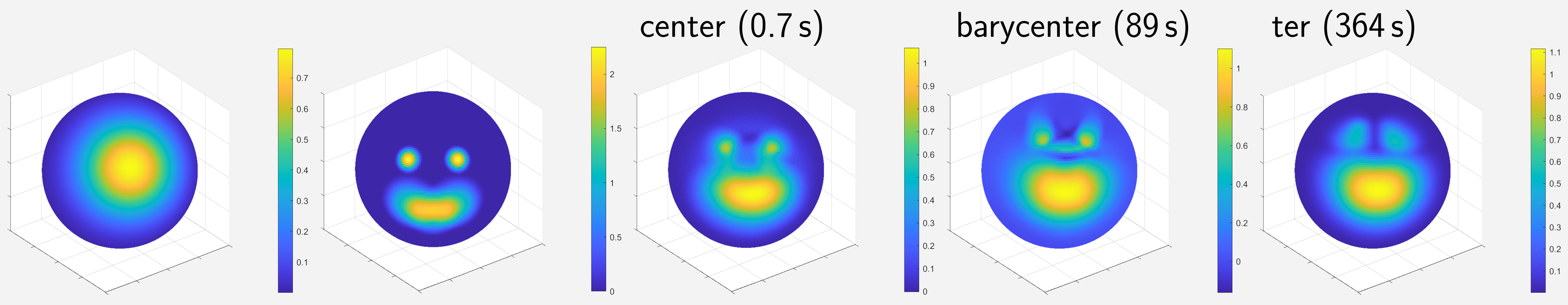
- Parallely sliced Radon barycenter of $\mu_i \in \mathcal{P}_{\mathrm{ac}}(\mathbb{S}^{d-1})$:

$$\mathrm{Bary}_{\mathbb{S}^{d-1}}^{\mathcal{U}}(\mu_i,\lambda_i)_{i=1}^M := \mathcal{U}^\dagger\left(\left(\mathrm{Bary}_{\mathbb{R}}(\lambda_i, \mathcal{U}_\psi\mu_i)_{i=1}^M\right)_{\psi\in\mathbb{S}^{d-1}}\right),$$

where $\mathcal{U}^\dagger$ is the pseudoinverse whose argument is viewed as a density function on $\mathbb{S}^{d-1}\times[-1,1]$



First input μ_1 | Second input μ_2 | Parallelly sliced barycenter (0.7 s) | Semicircular sliced barycenter (89 s) | Wasserstein barycenter (364 s)



First input μ_1 | Second input μ_2 | Parallelly sliced barycenter | Semicircular sliced barycenter | Wasserstein barycenter

Conclusions

- Both parallel slicing and semicircular slicing provide metrics on $\mathcal{P}(\mathbb{S}^2)$
- Parallel slicing is computationally more efficient
- Application for barycenters and image classification

References

[1] R. Beinert, M. Quellmalz, G. Steidl: Sliced Optimal Transport on the Sphere. *Inverse Problems* 39 (2023).
[2] C. Bonet, P. Berg, N. Courty, F. Septier, L. Drumetz, M.-T. Pham: Spherical sliced-Wasserstein. *ICLR* (2023).
[3] M. Quellmalz, L. Buecher, G. Steidl: Parallelly Sliced Optimal Transport on Spheres and on the Rotation Group. *Preprint* 2401.16896, 2024.
[4] J. Rabin, G. Peyré, J. Delon, and M. Bernot: Wasserstein barycenter and its application to texture mixing. *SSVM* (2012).
[5] C. Villani: *Topics in Optimal Transportation*. AMS, Providence, 2004.

{beinert,quellmalz,buecher,steidl}@math.tu-berlin.de