TECHNISCHE UNIVERSITÄT CHEMNITZ

Faculty of Mathematics
Applied Functional Analysis
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Reconstruction of Functions on the Sphere from Spherical Means

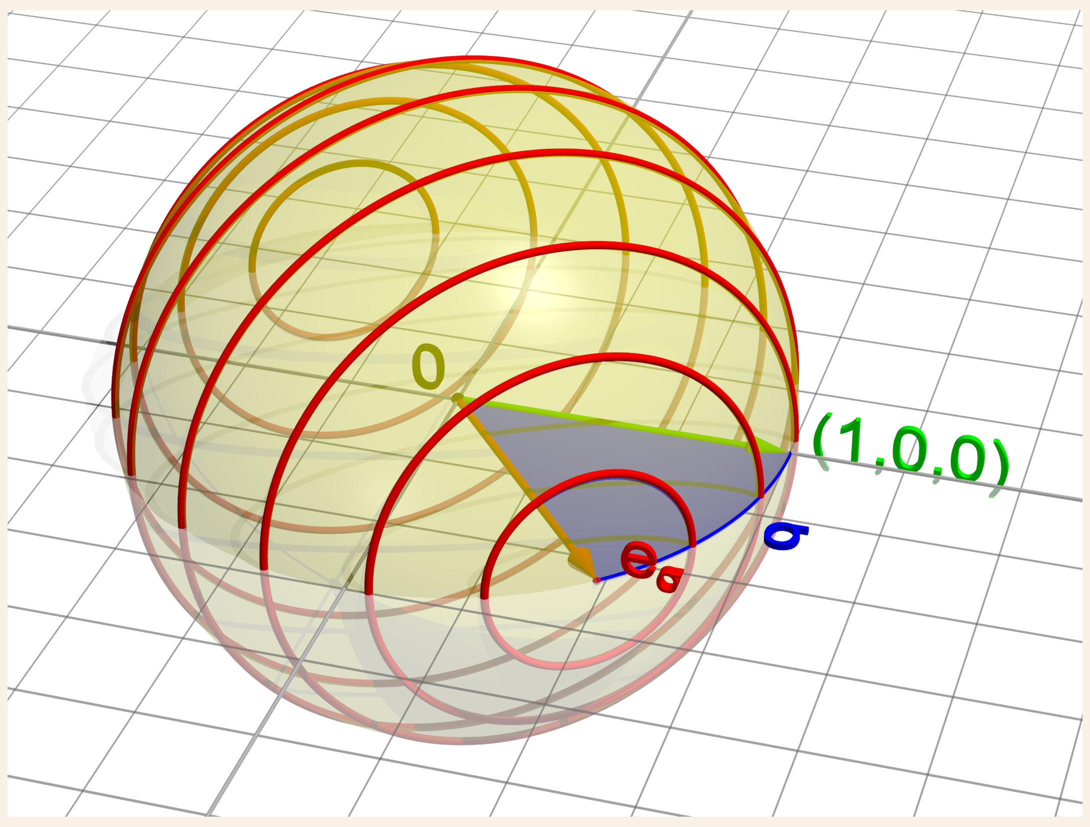
Vertical slice transform

Definition:

$$\mathcal{T}\colon L^2(\mathbb{S}^2) \to L^2([0,2\pi)\times[-1,1]),$$
$$\mathcal{T}f(\sigma,t) = \frac{1}{2\pi\sqrt{1-t^2}} \int_{\boldsymbol{\xi}\cdot\boldsymbol{e}_\sigma = t} f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

where

$$\boldsymbol{e}_\sigma = (\cos\sigma,\,\sin\sigma,\,0)^\top.$$



Symmetry property

$\mathcal{T}f$ vanishes for functions f that are odd in the third coordinate ξ_3. Hence, only the even part of f can be reconstructed.

Task

We have the discrete noisy data

$$g(\sigma_m,t_m) = \mathcal{T}f(\sigma_m,t_m) + \varepsilon_m, \quad m=1,\ldots,M,$$

where ε is a Gaussian random vector.
We want to reconstruct f.

Methods

Singular value decomposition

$$\mathcal{T}Y_n^k = \lambda_n^k B_n^k, \quad n\in\mathbb{N}_0,\ |k|\le n.$$

- Y_n^k ... spherical harmonics of degree n
- λ_n^k ... singular values of $\mathcal{T}$
- B_n^k ... orthonormal basis on $[0,2\pi)\times[-1,1]$

Smoothing the inverse $\mathcal{T}^\dagger g$ with filter coefficients $\hat\psi(n)$

$$\mathcal{T}^\dagger g = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} \frac{1}{\lambda_n^k}\left\langle g, B_n^k\right\rangle Y_n^k$$
$$\rightsquigarrow \psi\star\mathcal{T}^\dagger g = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} \hat\psi(n)\frac{1}{\lambda_n^k}\left\langle g, B_n^k\right\rangle Y_n^k.$$

Use **numerical quadrature** for the discretized inner product

$$\left\langle g, B_n^k\right\rangle_M = \sum_{m=1}^{M}\omega_m g(\sigma_m,t_m)\overline{B_n^k(\sigma_m,t_m)}.$$

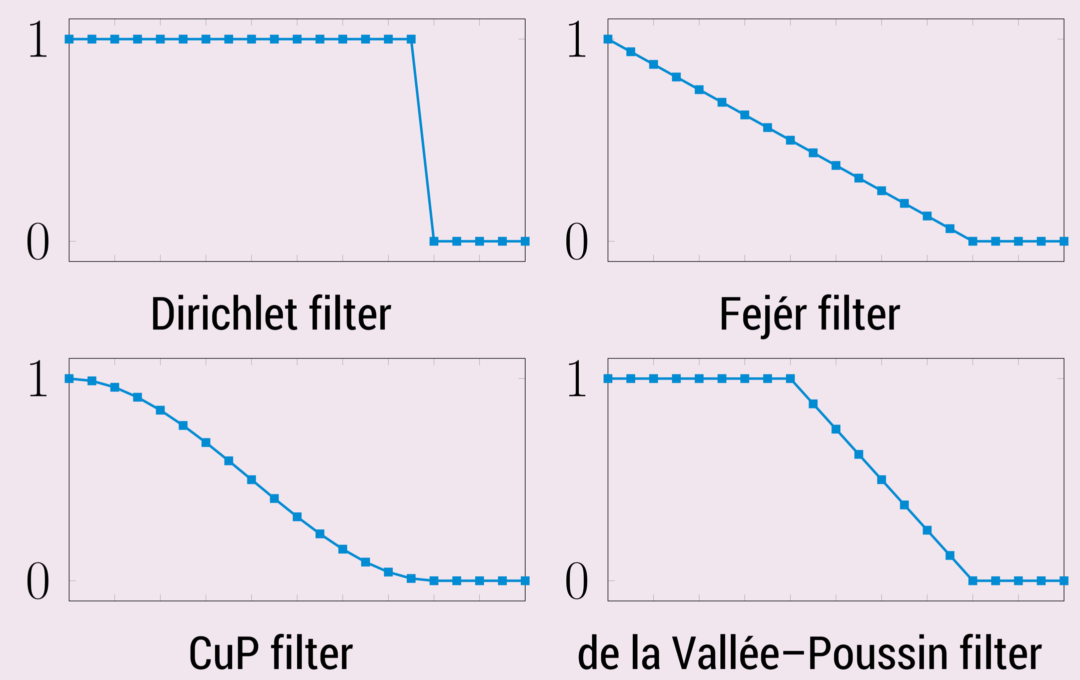
Truncation at degree N to define the estimator

$$\mathcal{E}_{M,\psi}g = \sum_{n=0}^{N}\sum_{k=-n}^{n}\hat\psi(n)\frac{1}{\lambda_n^k}\left\langle g, B_n^k\right\rangle_M Y_n^k.$$

Choice of the filter coefficients

The filter coefficients $\hat\psi(n)$ should be

- almost one for small n, and
- zero for large n.



Results

Source condition. We assume that f is in the Sobolev space $H^s(\mathbb{S}^2)$ with bounded norm

$$\|f\|_{H^s(\mathbb{S}^2)} \le S.$$

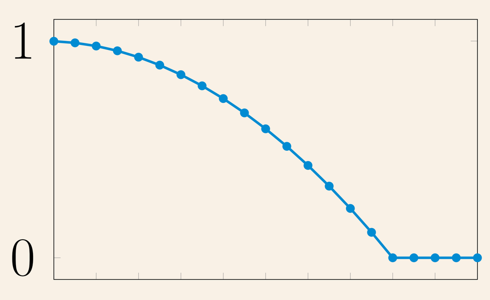
Theorem. There exists a family of optimal filters $\psi_{L(M)}^s$ such that for $M\to\infty$

$$\min_{\psi}\max_{\|f\|_{H^s}\le S}\mathbb{E}\left\|f-\mathcal{E}_{M,\psi}g\right\|_2^2$$
$$\simeq \max_{\|f\|_{H^s}\le S}\mathbb{E}\left\|f-\mathcal{E}_{M,\psi_{L(M)}^s}g\right\|_2^2$$
$$\simeq \mathrm{const}\cdot M^{\frac{-2s}{2s+3}}.$$

They have the coefficients

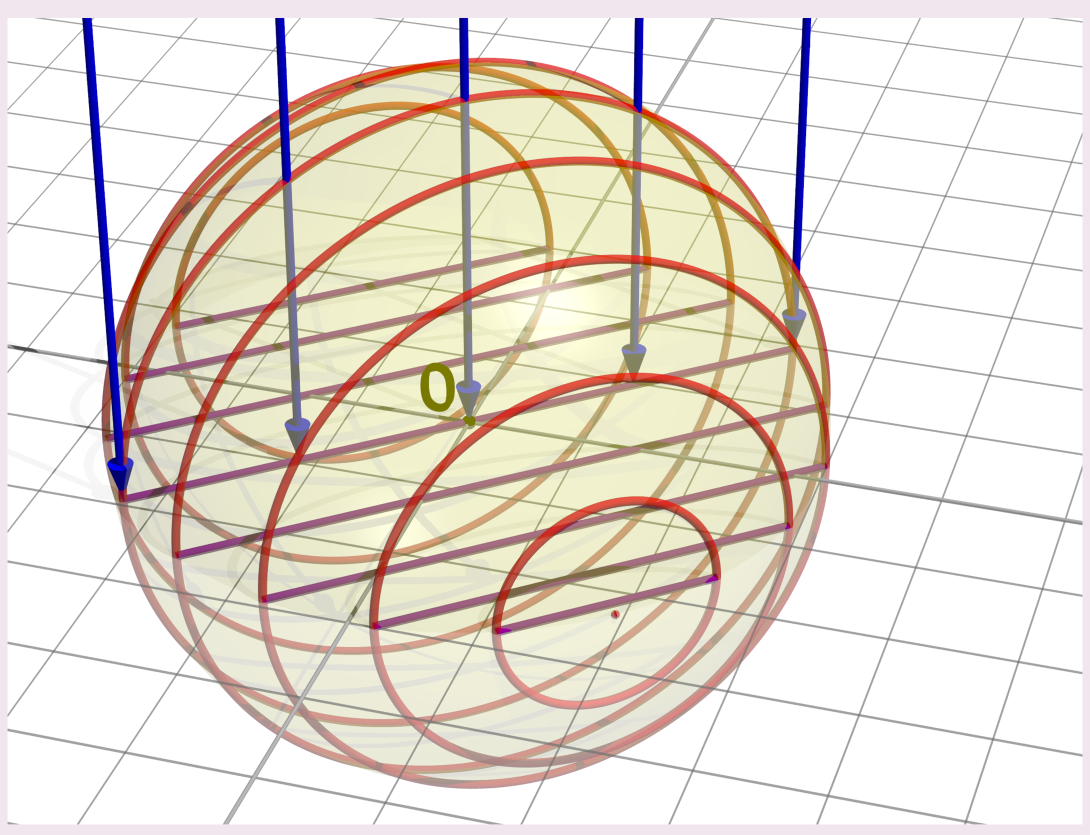
$$\hat\psi_L^s(n) = 1-\left(\frac{n+1/2}{L+1/2}\right)^s$$

for $n\le L$.
(right image with $s=2$)



optimal filter coefficients

Alternative reconstruction approach



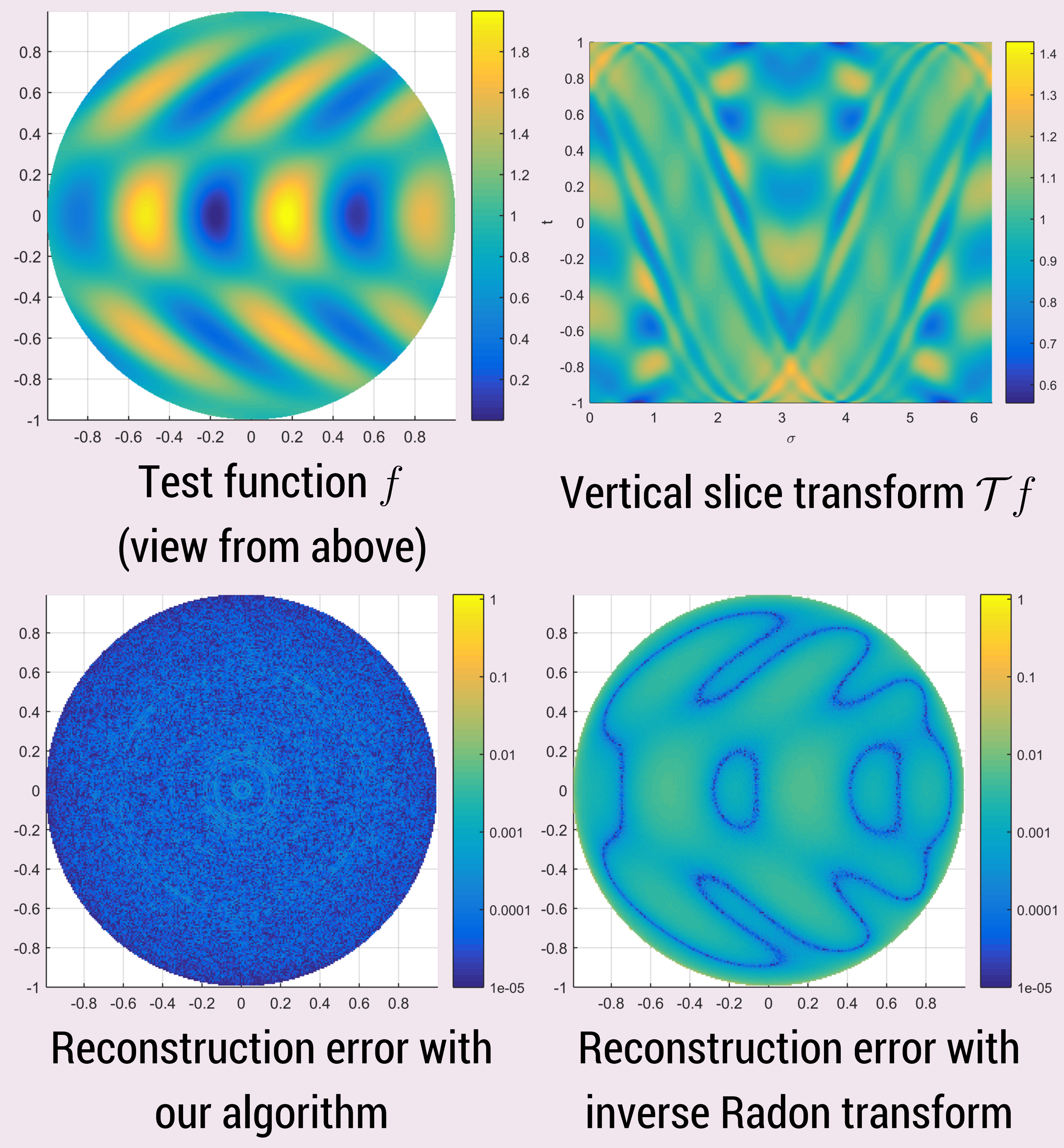
Orthogonal projection along the third coordinate turns the circular average transform $\mathcal{T}$ into the Radon transform $\mathcal{R}$ on the unit disc via

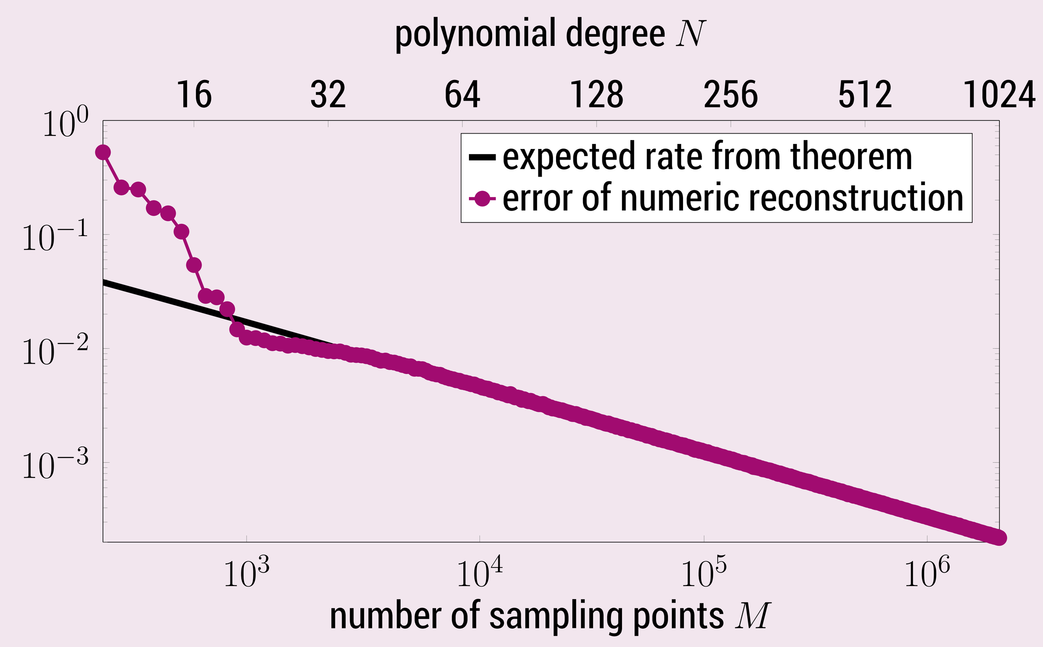
$$\mathcal{T}f = \mathcal{R}\left[\frac{f(\xi_1,\xi_2)}{\pi\sqrt{1-\xi_1^2-\xi_2^2}}\right],$$

provided f is even in ξ_3 and thus independent of ξ_3.

Numerical experiments

The computation of the estimator $\mathcal{E}_{M,\psi}g$ can be done with the help of the fast spherical Fourier transform in only $\mathcal{O}(M\log^2 M)$ steps.





Conclusion

We have introduced a new algorithm for inverting the vertical slice transform. We discovered that the filter coefficients of the type ψ_L^s are optimal. Our error estimates were confirmed in numerical tests.

References

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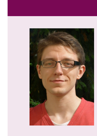
Contact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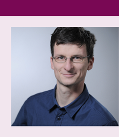
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