SFB F68
Tomography Across the Scales

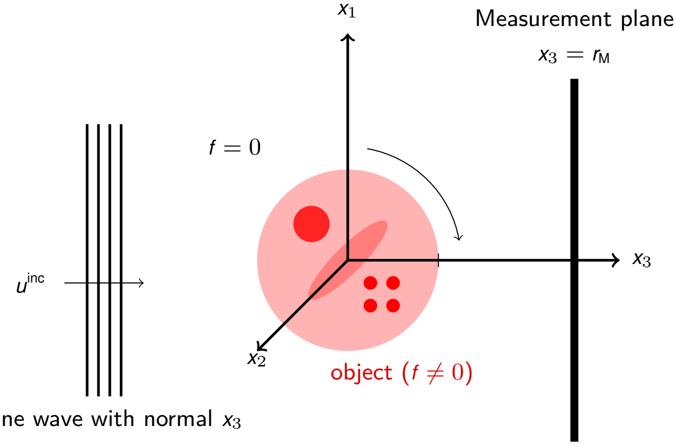
Technische Universität Berlin

Generalized Fourier Diffraction Theorem and Filtered Backpropagation for Tomographic Reconstruction

Michael Quellmalz | TU Berlin | GIP Meeting, Siegen, 25 September 2024
joint work with Clemens Kirisits, Eric Setterqvist

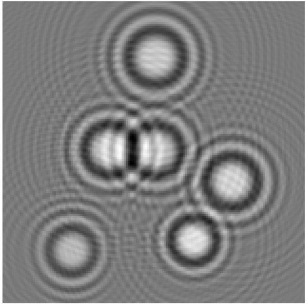
Outline

Optical Diffraction Tomography (ODT)



Optical Diffraction

Optical diffraction occurs when the wavelength of the incident wave is large $\approx$ the size of the object (μm scale)



Simulation of the scattered field from spherical particles (size $\approx$ wavelength)

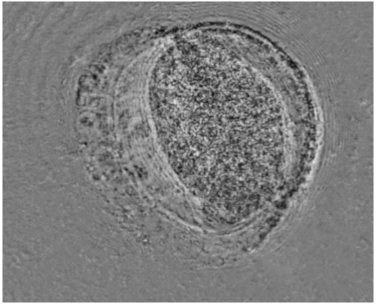


Image with diffraction

Model of Optical Diffraction Tomography (for one direction)

- We have: field $u^{\text{tot}}(\tilde{\boldsymbol{x}}, r_{\text{M}})$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^{d-1}$, at measurement plane $x_d = r_{\text{M}}$
- We want: scattering potential f on $\mathbb{R}^d$ with compact support
- Illumination by plane wave $u^{\text{inc}}(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}k_0 \boldsymbol{x}\cdot\boldsymbol{s}}$ with direction $\boldsymbol{s} \in \mathbb{S}^{d-1}$ and wave number k_0
- Total field $u^{\text{tot}}(\boldsymbol{x}) = u(\boldsymbol{x}) + u^{\text{inc}}(\boldsymbol{x})$ solves the wave equation

$$-\left(\Delta + f(\boldsymbol{x}) + k_0^2\right) u^{\text{tot}}(\boldsymbol{x}) = 0$$

- Rearranging yields

$$-\left(\Delta + k_0^2\right) u(\boldsymbol{x}) - \underbrace{\left(\Delta + k_0^2\right) u^{\text{inc}}(\boldsymbol{x})}_{=0} = f(\boldsymbol{x}) \left(u(\boldsymbol{x}) + u^{\text{inc}}(\boldsymbol{x})\right)$$

Born approximation

Assuming $|u| \ll |u^{\text{inc}}|$, we obtain the Helmholtz equation

$$-\left(\Delta + k_0^2\right) u(\boldsymbol{x}) = f(\boldsymbol{x}) u^{\text{inc}}(\boldsymbol{x})$$

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Outline

- Helmholtz equation

$$-(\Delta + k_0^2)u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d \tag{1}$$

- The outgoing solution satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u(\boldsymbol{x})}{\partial r} - \mathrm{i}k_0 u(\boldsymbol{x}) \right) = 0$$

uniformly in $\boldsymbol{s}$, where $\boldsymbol{x} = r\boldsymbol{s}$, $r = |\boldsymbol{x}|$, and $\partial/\partial r$ denotes the radial derivative

- It has the fundamental solution

$$G(\boldsymbol{x}) = \frac{\mathrm{i}}{4} \left(\frac{k_0}{2\pi|\boldsymbol{x}|} \right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(k_0\,|\boldsymbol{x}|),$$

where $H_a^{(1)}$ is the Hankel function of the first kind and order a.

Lemma

For $g \in L^1(\mathbb{R}^d)$ with compact support,

$$u = g * G \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$$

is the unique outgoing solution of the Helmholtz equation (1).

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Some Auxiliary Lemmas

Lemma

If $g_n \to 0$ in $L^1(\mathbb{R}^d)$ and $\bigcup_n \operatorname{supp} g_n$ is bounded, then $g_n * G \to 0$ in $\mathcal{S}'(\mathbb{R}^d)$.

- Set $\tilde{\mathcal{F}}$ as partial Fourier transform in the first $d-1$ coordinates
- For $\boldsymbol{x} \in \mathbb{R}^d$, set $\tilde{\boldsymbol{x}} = (x_1, \dots, x_{d-1})$
- $\kappa(\boldsymbol{x}) := \sqrt{k_0^2 - |\boldsymbol{x}|^2}$, $\boldsymbol{x} \in \mathbb{R}^{d-1}$

Lemma

Let $d \geq 2$. Then $\tilde{\mathcal{F}}G$ is given by the locally integrable function

$$\tilde{\mathcal{F}}G(\boldsymbol{x}) = (2\pi)^{\frac{1-d}{2}} \frac{\mathrm{i}e^{\mathrm{i}\kappa(\tilde{\boldsymbol{r}})|x_d|}}{2\kappa(\tilde{\boldsymbol{x}})}.$$

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In contrast: $\mathcal{F}G$ exists as tempered distribution, but is not regular

Define $\boldsymbol{h}^{\pm}(\boldsymbol{x}) := \begin{pmatrix} \boldsymbol{x} \\ \pm\kappa(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \\ \pm\sqrt{k_0^2 - |\boldsymbol{x}|^2} \end{pmatrix}$, $x \in \mathbb{R}^{d-1}$ (hemisphere)

Generalized Fourier Diffraction Theorem [Kirisits, Q, Setterqvist 2024]

Let $d \geq 2$ and $g \in L^1(\mathbb{R}^d)$ has compact support. Then $\tilde{\mathcal{F}}u$, where $u = g * G$, is given by the locally integrable function

$$\tilde{\mathcal{F}}u(\boldsymbol{x}) = \sqrt{\frac{\pi}{2}} \frac{\mathrm{i}}{\kappa(\tilde{\boldsymbol{x}})} \left(\mathrm{e}^{\mathrm{i}\kappa(\tilde{\boldsymbol{x}})r_d} \mathcal{F}\left((1-\chi_{x_d})g\right)\left(\boldsymbol{h}^+(\tilde{\boldsymbol{x}})\right) + \mathrm{e}^{-\mathrm{i}\kappa(\tilde{\boldsymbol{x}})x_d} \mathcal{F}\left(\chi_{r_d} g\right)\left(\boldsymbol{h}^-(\tilde{\boldsymbol{x}})\right)\right),$$

where $\tilde{\boldsymbol{x}} = (x_1, \dots, x_{d-1})$, $\tilde{\mathcal{F}}$ is the $d-1$ dimensional partial Fourier transform and χ_{x_d} is the indicator function of $\{\boldsymbol{y} \in \mathbb{R}^d : y_d \geq x_d\}$.

If x_d is sufficiently large or sufficiently small such that

$$\pm(x_d - y_d) > 0 \quad \text{for all } \boldsymbol{y} \in \operatorname{supp} g,$$

then

$$\tilde{\mathcal{F}}u(\boldsymbol{x}) = \sqrt{\frac{\pi}{2}} \frac{\mathrm{i}\mathrm{e}^{\pm\mathrm{i}\kappa(\tilde{\boldsymbol{x}})x_d}}{\kappa(\tilde{\boldsymbol{x}})} \hat{g}(\boldsymbol{h}^{\pm}(\tilde{\boldsymbol{x}})).$$

(Classical) Fourier Diffraction Theorem

Additional assumptions:

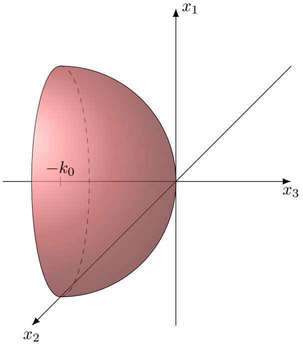
- the incident field is a plane wave $u^{\mathrm{inc}}(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}k_0 \boldsymbol{x}\cdot\boldsymbol{s}}$
- the measurement plane $x_d = r_{\mathrm{M}}$ not intersect $\operatorname{supp} f$

Then

$$\sqrt{\frac{2}{\pi}}\kappa \mathrm{i} \mathrm{e}^{-\mathrm{i}\kappa r_{\mathrm{M}}} \tilde{\mathcal{F}} \underbrace{u(\tilde{\boldsymbol{x}}, r_{\mathrm{M}})}_{\text{measured}} = \mathcal{F} f(\boldsymbol{h}(\tilde{\boldsymbol{x}}) - k_0 \boldsymbol{s}), \quad \tilde{\boldsymbol{x}} \in \mathbb{R}^{d-1},$$

where $\boldsymbol{h}(\tilde{\boldsymbol{x}}) := \begin{pmatrix} \tilde{\boldsymbol{x}} \\ \kappa \end{pmatrix}$ and $\kappa = \sqrt{k_0^2 - |\tilde{\boldsymbol{x}}|^2}$.

Formula well-known from [Wolf 1969] [Natterer Wuebbeling 2001] [Kak Slaney 2001]



Semisphere $\boldsymbol{h}(\boldsymbol{k})$ of available data in Fourier space

Focused Beams

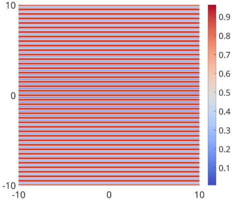
- The generalized version allows for other incidence fields
- For an incident Herglotz wave

$$u^{\mathrm{inc}}(\boldsymbol{r}) = \int_{\mathbb{S}^{d-1}} a(\boldsymbol{s}) e^{\mathrm{i} k_0 \boldsymbol{s}\cdot\boldsymbol{r}} \,\mathrm{d}s(\boldsymbol{s}), \quad \text{where } a \in L^2(\mathbb{S}^{d-1}),$$

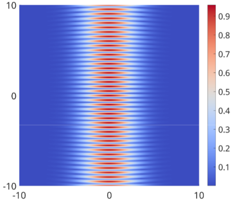
it yields

$$\tilde{\mathcal{F}} u(\boldsymbol{x}, r_{\mathrm{M}}) = \sqrt{\frac{\pi}{2}} \frac{\mathrm{i} e^{\pm \mathrm{i}\kappa(\boldsymbol{x}) r_{\mathrm{M}}} k_0^2}{\kappa(\boldsymbol{x})} \int_{\mathbb{S}^{d-1}} a(\boldsymbol{s}) \hat{f}(\boldsymbol{h}^{\pm}(\boldsymbol{x}) - k_0 \boldsymbol{s}) \,\mathrm{d}s(\boldsymbol{s}).$$

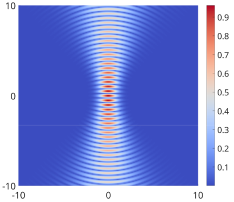
- Reconstruction algorithms for focused beams (see images below) in [Kirisits Naujoks Scherzer 2024]



Plane wave $\mathrm{Re}(u^{\mathrm{inc}})$



Focused wave



Tightly focused wave

Outline

Moving the Object and Incidence Direction

- Rigid motion of the object causes scattering potential $f(R_t(\boldsymbol{x} - \boldsymbol{d}_t))$ with
  - Rotation $R_t \in \mathrm{SO}(d)$ (with $R_0 := \mathrm{id}$)
  - Translation $\boldsymbol{d}_t \in \mathbb{R}^d$ (with $\boldsymbol{d}_0 := \mathbf{0}$)
- Incidence direction $\boldsymbol{s}_t \in \mathbb{S}^{d-1}$
- Position of the measurement plane is kept fixed (equivalent to moving object and incidence simultaneously)

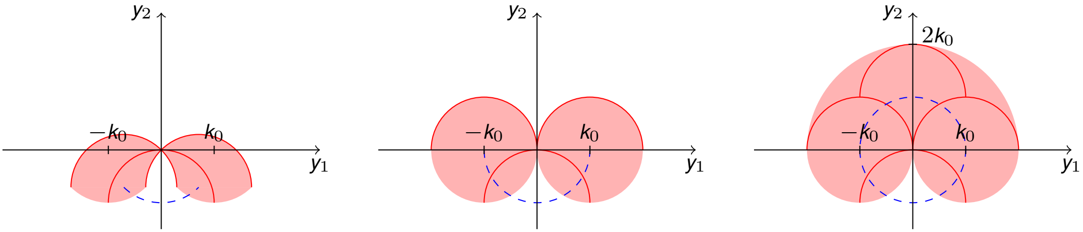
Fourier diffraction theorem (with motion)

The quantity

$$\mu_t(\boldsymbol{x}) := \sqrt{\frac{2}{\pi}}\kappa \mathrm{i} \mathrm{e}^{-\mathrm{i}\kappa r_M} \tilde{\mathcal{F}} u(\boldsymbol{k}, r_M) = \mathcal{F} f(\underbrace{R_t \boldsymbol{h}(\boldsymbol{x}) - k_0 \boldsymbol{s}_t}_{\text{Fourier cover}})\, \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{d}_t, \boldsymbol{h}(\boldsymbol{x})\rangle}, \quad \|\boldsymbol{x}\| < k_0,$$

depends only on the measurements of u.

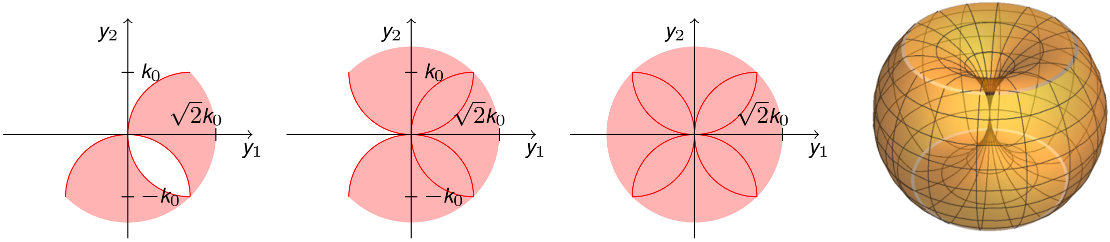
Fourier Cover: Angle Scan



Quarter turn $t \in [\pi/4, 3\pi/4]$ — Half turn $t \in [0, \pi]$ — Full turn $t \in [0, 2\pi]$

2D Fourier coverage for incidence direction $\boldsymbol{s}(t) = (\cos t, \sin t)$. Measurements are taken at $r_2 = r_M$. The Fourier coverage (light red) is a union of infinitely many semicircles, whose centers lie on the dashed blue curve.

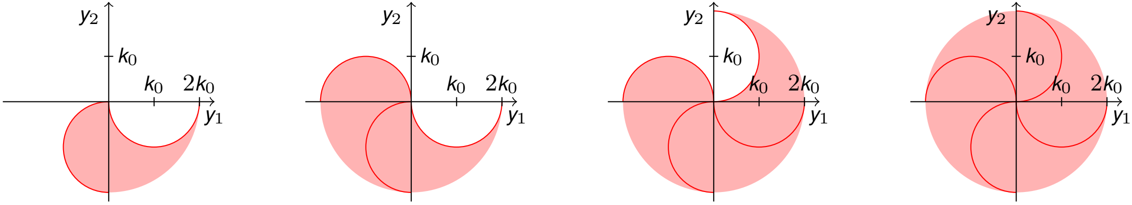
Fourier coverage: Object Rotation (Incidence Parallel to Measurement Plane)



(a) Quarter turn (b) Half turn (c) Full turn (d) 3D cover, full turn

Figure: 2D Fourier coverage for a rotating object, incidence direction $\boldsymbol{s} = (0, 1)$ and measurements taken at $r_2 = r_M$. The Fourier coverage (light red) is a union of infinitely many semicircles, some of which are depicted in red.
(d): 3D Fourier coverage for incidence $\boldsymbol{s} = (0, 0, 1)$ and rotation around first axis.

Fourier Cover: Object Rotation (Incidence Perpendicular to Measurement Plane)



(a) Quarter turn (b) Half turn (c) Three-quarter turn (d) Full turn

Figure: 2D Fourier coverage for a rotating object, incidence direction $\boldsymbol{s} = (1, 0)$ and measurements taken at $r_2 = r_M$.

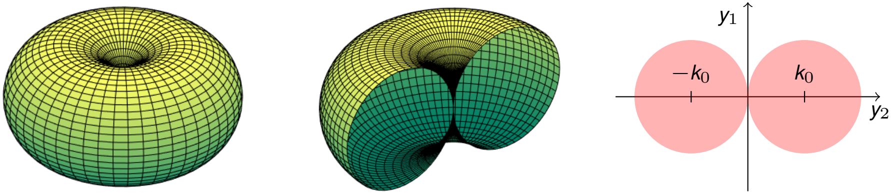
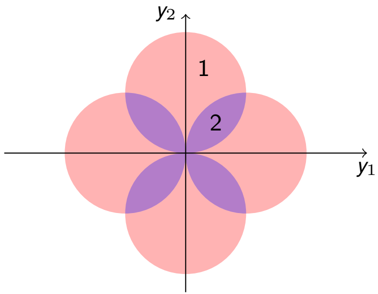
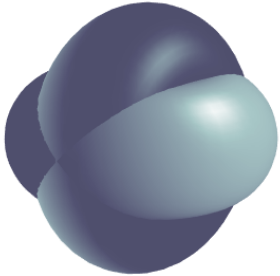


Figure: 3D Fourier coverage for a full rotation of the object about the r_1-axis with incidence direction $\boldsymbol{s} = (0, 1, 0)$.

Fourier cover: Angle scan & Rotation



2D Fourier cover
(colors represent Banch indicatrix)



3D Fourier cover

- Incidence is rotated along a half circle
- and the experiment is repeated with the object rotated by 90°.

Outline

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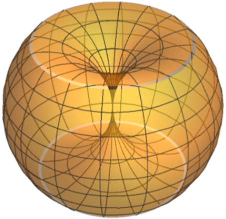
Apprach 1: Filtered Backpropagation



Idea: Inverse Fourier transform of $\mathcal{F}f$ restricted to the set of available data $\mathcal{Y}$,

$$f_{\mathrm{bp}}(\boldsymbol{r}) := (2\pi)^{-\frac{d}{2}} \int_{\mathcal{Y}} \mathcal{F}f(\boldsymbol{y})\, \mathrm{e}^{\mathrm{i}\boldsymbol{y}\cdot\boldsymbol{r}}\, \mathrm{d}\boldsymbol{y}$$

with the transformation $T(\boldsymbol{x}, t) := R_t \boldsymbol{h}(\boldsymbol{x})$

Theorem [Kirisits, Q, Setterqvist 2024]

Let the rotation $R_t \in SO(d)$, translation $\boldsymbol{d}_t$ and incidence $\boldsymbol{s}_t \in \mathbb{S}^{d-1}$ be piecewice C^1. Then

$$f_{\mathrm{bp}}(\boldsymbol{r}) = (2\pi)^{-\frac{d}{2}} \int_0^T \int_{\mathcal{B}_{k_0}} \mathcal{F}f(T(\boldsymbol{x}, t))\, \mathrm{e}^{\mathrm{i}\, T(\boldsymbol{x},t)\cdot(\boldsymbol{r}+\boldsymbol{d}_t)}\, \frac{|\det \nabla T(\boldsymbol{x}, t)|}{\operatorname{Card} T^{-1}(T(\boldsymbol{x}, t))}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t,$$

where $\det \nabla T(\boldsymbol{x}, t) = \dfrac{k_0(t) k_0'(t) - R_t \boldsymbol{h}(\boldsymbol{x}, t)\, (k_0(t) R_t \boldsymbol{s}_t)'}{\kappa}$.

Banach indicatrix $\operatorname{Card}(T^{-1}(\boldsymbol{y}))$ needs to be estimated (except for special cases).
Well-known for rotation around coordinate axis [Devaney 1982]

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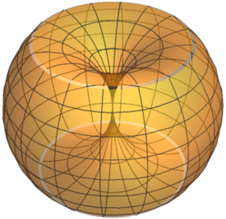

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Discretization

- Object $f(\boldsymbol{x_k})$ with $\boldsymbol{x_k} = \boldsymbol{k}\frac{2L_s}{K}$, $\boldsymbol{k} \in \mathcal{I}_K^d := \{-K/2, \dots, K/2-1\}^d$
- Measurements $u_{t_m}^{\mathrm{tot}}(\boldsymbol{y_n}, r_{\mathrm{M}})$ with $\boldsymbol{y_n} = \boldsymbol{n}\frac{2L_{\mathrm{M}}}{N}$, $\boldsymbol{n} \in \mathcal{I}_N^{d-1}$
- discrete Fourier transform (DFT)

$$[\boldsymbol{F}_{\mathrm{DFT}}\, u_{t_m}]_{\boldsymbol{\ell}} := \sum_{\boldsymbol{n}\in\mathcal{I}_N^{d-1}} u_{t_m}(\boldsymbol{y_n}, r_{\mathrm{M}})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{n}\cdot\boldsymbol{\ell}/N}, \qquad \boldsymbol{\ell} \in \mathcal{I}_N^{d-1},$$

- Non-uniform discrete Fourier transform (NDFT)

$$[\boldsymbol{F}_{\mathrm{NDFT}}\boldsymbol{f}]_{m,\boldsymbol{\ell}} := \sum_{\boldsymbol{k}\in\mathcal{I}_K^d} f_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{x_k}\cdot\left(R_{t_m}\boldsymbol{h}(\boldsymbol{y_\ell})\right)}, \qquad m \in \mathcal{J}_M,\ \boldsymbol{\ell} \in \mathcal{I}_N^{d-1}$$

Discretized forward operator

$$\boldsymbol{D}^{\mathrm{tot}}\boldsymbol{f} := \boldsymbol{F}_{\mathrm{DFT}}^{-1}(\boldsymbol{c} \odot \boldsymbol{F}_{\mathrm{NDFT}}\boldsymbol{f}) + \mathrm{e}^{\mathrm{i}k_0 r_{\mathrm{M}}}, \qquad \boldsymbol{f} \in \mathbb{R}^{K^d},$$

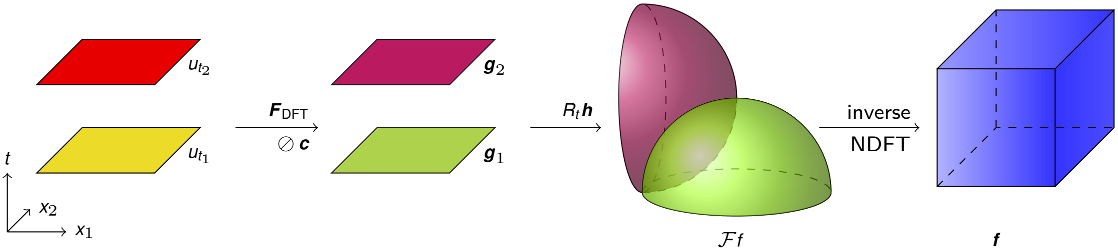
where $\boldsymbol{c} = \left[\frac{\mathrm{i}}{\kappa(\boldsymbol{y_\ell})}\, \mathrm{e}^{\mathrm{i}\,\kappa(\boldsymbol{y_\ell})\, r_{\mathrm{M}}} \left(\frac{N}{L_{\mathrm{M}}}\right)^{d-1} \left(\frac{L_s}{K}\right)^d\right]_{\boldsymbol{\ell}\in\mathcal{I}_N^{d-1}}$

Reconstruction of f

Inverse

$$\boldsymbol{f} \approx \boldsymbol{F}_{\mathrm{NDFT}}^{-1}\Big(\big(\boldsymbol{F}_{\mathrm{DFT}}\boldsymbol{u}^{\mathrm{tot}} - \mathrm{e}^{ik_0 r_{\mathrm{M}}}\big) \oslash \boldsymbol{c}\Big)$$

Crucial part: inversion of NDFT $\boldsymbol{F}_{\mathrm{NDFT}}^{-1}$



Approach 2: Conjugate Gradient (CG) Method

- Conjugate Gradients (CG) on the normal equations

$$\mathop{\arg\min}_{\boldsymbol{f}\in\mathbb{R}^{K^3}} \quad \|\boldsymbol{F}_{\mathrm{NDFT}}(\boldsymbol{f}) - \boldsymbol{g}\|_2^2$$

- NFFT (Non-uniform fast Fourier transform) for computing $\boldsymbol{F}_{\mathrm{NDFT}}(\boldsymbol{f})$ in $\mathcal{O}\left(N^3 \log N\right)$ steps

[Dutt Rokhlin 93], [Beylkin 95], [Potts Steidl Tasche 01], [Potts Kunis Keiner 04+]

Approach 3: TV (Total Variation) Regularization

- Regularized inverse

$$\mathop{\arg\min}_{\boldsymbol{f}\in\mathbb{R}^{K^3}} \quad \chi_{\mathbb{R}^{K^3}_{\geq 0}}(\boldsymbol{f}) + \tfrac{1}{2}\|\boldsymbol{F}_{\mathrm{NDFT}}(\boldsymbol{f}) - \boldsymbol{g}\|_2^2 + \lambda \mathrm{TV}(\boldsymbol{f}),$$

- Primal-dual (PD) iteration [Chambolle & Pock 2010]
- Adaptive selection of step sizes [Yokota & Hontani 2017]

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- Conjugate Gradients (CG) on the normal equations

$$\operatorname*{arg\,min}_{\boldsymbol{f}\in\mathbb{R}^{K^3}} \quad \|\boldsymbol{F}_{\mathsf{NDFT}}(\boldsymbol{f}) - \boldsymbol{g}\|_2^2$$

- NFFT (Non-uniform fast Fourier transform) for computing $\boldsymbol{F}_{\mathsf{NDFT}}(\boldsymbol{f})$ in $\mathcal{O}\left(N^3 \log N\right)$ steps

[Dutt Rokhlin 93], [Beylkin 95], [Potts Steidl Tasche 01], [Potts Kunis Keiner 04+]

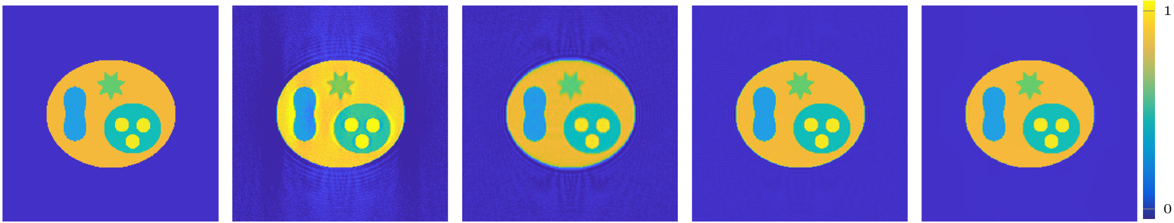
Approach 3: TV (Total Variation) Regularization

- Regularized inverse

$$\operatorname*{arg\,min}_{\boldsymbol{f}\in\mathbb{R}^{K^3}} \quad \chi_{\mathbb{R}^{K^3}_{\geq 0}}(\boldsymbol{f}) + \tfrac{1}{2}\|\boldsymbol{F}_{\mathsf{NDFT}}(\boldsymbol{f}) - \boldsymbol{g}\|_2^2 + \lambda \mathrm{TV}(\boldsymbol{f}),$$

- Primal-dual (PD) iteration [Chambolle & Pock 2010]
- Adaptive selection of step sizes [Yokota & Hontani 2017]

3D Reconstruction: Moving Rotation Axis



Ground truth $\boldsymbol{f}$
($240 \times 240 \times 240$ grid)

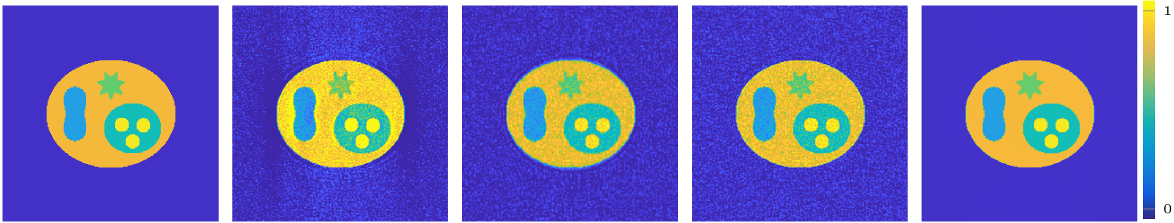
Backpropagation
PSNR 24.17
SSIM 0.171
5 s

Backpropagation
(with inticatrix estimation)
PSNR 31.84
SSIM 0.350
5 s +22 s precopmpute

CG Reconstruction
PSNR 35.84
SSIM 0.962
82 s

TV ($\lambda = 0.02$)
PSNR 40.95
SSIM 0.972
1395 s

3D Reconstruction: Moving Rotation Axis and 5 % Gaussian Noise



Ground truth $\boldsymbol{f}$

Backpropagation
PSNR 21.19
SSIM 0.075

Backpropagation (with indicatrix estimation)
PSNR 25.50
SSIM 0.157

CG Reconstruction
PSNR 24.10
SSIM 0.193

TV ($\lambda = 0.05$)
PSNR 38.01
SSIM 0.772

Conclusions

- Generalized Fourier diffraction theorem on $L^1(\mathbb{R}^d)$
- Filtered backpropagation formula for a wide range of experimental setups
- Compared image reconstruction methods

Conclusions

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Thank you for your attention!

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Estimation of the Banach indicatrix

- Banach indicatrix $\mathrm{Card}(T_{\pm}^{-1}(\boldsymbol{y}))$ counts how often a point $\boldsymbol{y} \in \mathbb{B}^d_{2k_0}$ is "hit" by T_+.
- In a discrete setting, it is unlikely that a point $\boldsymbol{y}$ is exactly hit by any grid point $T_+(\tilde{\boldsymbol{x}}_m, t_n)$.
- The coverage for fixed time t is the hemisphere

$$\{T_+(\tilde{\boldsymbol{x}}, t) : \tilde{\boldsymbol{x}} \in \mathbb{B}^{d-1}_{k_0}\} = \left\{\boldsymbol{z} \in \mathbb{R}^d : \|\boldsymbol{z} + k_0(t) R_t \boldsymbol{s}_t\| = k_0(t),\ \boldsymbol{z} \cdot R_t \boldsymbol{e}^d > -k_0(t) \boldsymbol{s}_t \cdot \boldsymbol{e}^d\right\},$$

  which moves continuously with t.
- Idea: a point $\boldsymbol{y}$ is hit by T_+ between the time steps t_{n-1} and t_n, if the sign of $\|\boldsymbol{y} - R_t \boldsymbol{s}_t\| - k_0(t)$ changes between these time steps. We use the approximation

$$\mathrm{Card}(T_+^{-1}(\boldsymbol{y})) \approx \sum_{n=1}^{N} \frac{\chi(n)}{2} \left|\mathrm{sgn}(\|\boldsymbol{y} + k_0(t_{n-1}) R_{t_{n-1}} \boldsymbol{s}_{t_{n-1}}\| - k_0(t_{n-1})) - \mathrm{sgn}(\|\boldsymbol{y} + k_0(t_n) R_{t_n} \boldsymbol{s}_{t_n}\| - k_0(t_n))\right|,$$

  where

$$\chi(n) := \begin{cases} 1, & \text{if } \boldsymbol{y} \cdot (R_t \boldsymbol{e}^d) > -k_0(t_n) \boldsymbol{s}_{t_n} \cdot \boldsymbol{e}^d, \\ 0, & \text{otherwise.} \end{cases}$$

  with the sign function $\mathrm{sgn}(x) := \frac{x}{|x|}$ for $x \neq 0$ and $\mathrm{sgn}(0) := 0$.