Approximation Properties of the Double Fourier Sphere Method

Michael Quellmalz | TU Berlin | Waves Conference, Karlsruhe, 15 February 2022
joint work with Sophie Mildenberger

Content

Spherical functions

Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$

Function $f\colon \mathbb{S}^2 \to \mathbb{C}$

Approximation of f

Write as finite sum of basis functions

$$f(\boldsymbol{\xi}) = \sum_{n=0}^{N} f_n b_n(\boldsymbol{\xi})$$

Choice of basis functions:

- Spherical harmonics
- Trigonometric polynomials (this talk)

Spherical harmonics

Use spherical coordinates

$$\phi(\lambda, \theta) = (\cos\lambda \, \sin\theta, \sin\lambda \, \sin\theta, \cos\theta)$$

Define the spherical harmonics of degree n

$$Y_n^k(\phi(\lambda,\theta)) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-k)!}{(n+k)!}}\, P_n^k(\cos\theta)\, \mathrm{e}^{\mathrm{i}k\lambda}$$

P_n^k .. associated Legendre function

- Y_n^k are orthogonal polynomials of degree n
- Y_n^k are eigenfunctions of the spherical Laplacian and Funk–Radon transforms
- fast numerical transforms available (NFSFT, SHT) [Driscoll, Healy 1994] [Kunis, Potts 2003] [Schaeffer 2013]
  However: Computations are still considerably slower than classical FFT on the torus

Fourier series on the torus

Torus $\mathbb{T}^2 := [-\pi, \pi)^2$.
Function $g\colon \mathbb{T}^2 \to \mathbb{C} \;\Leftrightarrow\; g\colon \mathbb{R}^2 \to \mathbb{C}$ is 2π-biperiodic

Fourier series on the torus

Let $g \in L_2(\mathbb{T}^2)$ and $\boldsymbol{n} \in \mathbb{Z}^2$. We define the $\boldsymbol{n}$-th **Fourier coefficient**

$$c_{\boldsymbol{n}}(g) := (2\pi)^{-2} \int_{\mathbb{T}^2} g(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{n}, \boldsymbol{x}\rangle}\, \mathrm{d}\boldsymbol{x},$$

the N-th **partial Fourier sum**

$$F_N g(\boldsymbol{x}) := \sum_{|\boldsymbol{n}|_\infty \le N} c_{\boldsymbol{n}}(g)\, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{n}, \boldsymbol{x}\rangle}, \qquad \boldsymbol{x} \in \mathbb{T}^2,$$

and the **Fourier series**

$$Fg := \lim_{N\to\infty} F_N g.$$

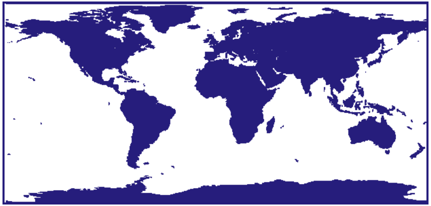
$F_N g$ can be computed efficiently with the fast Fourier transform (FFT) in $\mathcal{O}(N \log N)$ steps

The Double Fourier Sphere (DFS) method I

Spherical coordinates

$$\phi(\lambda, \theta) = (\cos\lambda \, \sin\theta, \sin\lambda \, \sin\theta, \cos\theta), \qquad \lambda \in [-\pi, \pi), \theta \in [0, \pi]$$

$$\xrightarrow[\text{transform}]{\text{longitude-latitude}}$$



2π periodic in λ,
but not periodic in θ

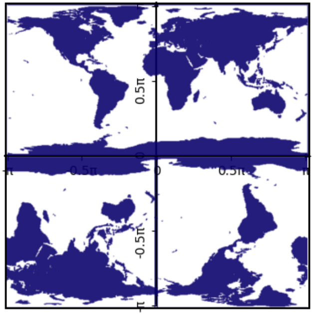
The Double Fourier Sphere (DFS) method II

Spherical coordinates

$$\phi(\lambda,\theta) = (\cos\lambda \, \sin\theta, \sin\lambda \, \sin\theta, \cos\theta), \qquad \lambda \in [-\pi,\pi), \theta \in [-\pi,\pi]$$

$f(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{S}^2$

$\xrightarrow{\text{DFS method}}$



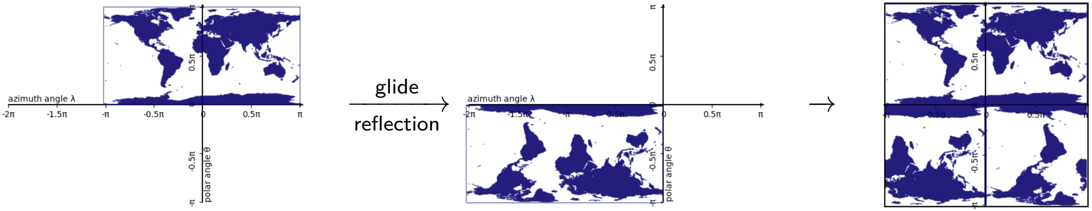
$\tilde{f}(\boldsymbol{x}) = f(\phi(\boldsymbol{x}))$ is 2π-periodic on $\mathbb{R}^2$

Properties of the DFS function $\tilde{f}$

1. $\tilde{f}$ is block-mirror centrosymmetric (BMC) $\tilde{f}(\lambda, \theta) = \tilde{f}(\lambda + \pi, -\theta)$
2. $\tilde{f}(\cdot, k\pi)$ is constant for all $k \in \mathbb{Z}$
3. $\tilde{f}$ is 2π-biperiodic (i.e. a function on $\mathbb{T}^2$)

Corollary

For every function $g \colon \mathbb{T}^2 \to \mathbb{C}$ that admints 1. and 2., there exists a unique $f \colon \mathbb{S}^2 \to \mathbb{C}$, such that $g = \tilde{f} = f \circ \phi$.



The DFS method and Fourier series

Expand DFS function $\tilde{f} = f \circ \phi$ as 2D Fourier series

$$\tilde{f}(\lambda, \theta) = \sum_{\boldsymbol{n} \in \mathbb{Z}^2} c_{\boldsymbol{n}}(\tilde{f})\, \mathrm{e}^{\mathrm{i}(n_1 \lambda + n_2 \phi)}.$$

Our contribution

- Show that $\tilde{f}$ has a “similar” smoothness as f and
- can be approximated by a certain Fourier series, which converges to f.

Selected References

P E Merilees.
"The pseudospectral approximation applied to the shallow water equations on a sphere."
Atmosphere 11.1 (1973)

S A Orszag.
"Fourier Series on Spheres."
Mon. Weather Rev. 108.5 (1980).

S Y K Yee.
"Studies on Fourier Series on Spheres."
Mon. Weather Rev. 102.1 (1974).

A Townsend, H Wilber, and G B Wright.
"Computing with functions in spherical and polar geometries I. The sphere."
SIAM J. Sci. Comput. 38.4 (2016)

S Mildenberger and M Quellmalz.
"Approximation properties of the double Fourier sphere method."
Arxiv prerpint, 2108.05662 (2021).

Function spaces

Continuously differentiable functions. Let $U \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0$. We define

$$\mathcal{C}^k(U) := \{f \in \mathcal{C}(U) \mid D^\beta f \in \mathcal{C}(U) \text{ for all } \beta \in \mathbb{N}_0^d,\ |\beta| \le k\}.$$

with the norm

$$\|f\|_{\mathcal{C}^k(U)} := \sum_{|\beta|\le k} \sup_{\boldsymbol{x}\in U} \left| D^\beta f(\boldsymbol{x}) \right|.$$

On $\mathbb{S}^2$: Definition via $\mathcal{C}^k$ extension to $\mathbb{R}^3$.

Hölder space. For $f \in \mathcal{C}^k(U)$ and $\alpha \in (0,1)$, we define the $\mathcal{C}^{k,\alpha}$**-seminorm**

$$|f|_{\mathcal{C}^{k,\alpha}(U)} := \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in U,\, \boldsymbol{x}\ne\boldsymbol{y} \\ \beta\in\mathbb{N}_0^d,\, |\beta|=k}} \frac{\left\|D^\beta f(\boldsymbol{x}) - D^\beta(\boldsymbol{y})\right\|}{\|\boldsymbol{x}-\boldsymbol{y}\|^\alpha}.$$

The (k,α)-**Hölder space**

$$\mathcal{C}^{k,\alpha}(U) := \{f \in \mathcal{C}^k(U) \mid |f|_{\mathcal{C}^{k,\alpha}(U)} < \infty\}$$

is a Banach space equipped with the norm $\|\cdot\|_{\mathcal{C}^{k,\alpha}(U)} := \|\cdot\|_{\mathcal{C}^k(U)} + |\cdot|_{\mathcal{C}^{k,\alpha}(U)}$.

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Transfer of Hölder smoothness

Theorem [Mildenberger, Q. 2021]

Let $k \in \mathbb{N}_0$ and $f \in \mathcal{C}^{k+1}(\mathbb{S}^2)$. Then for all $0 < \alpha < 1$, the DFS function $\tilde{f}$ of f is in $\mathcal{C}^{k,\alpha}(\mathbb{T}^2)$ with

$$\left|\tilde{f}\right|_{\mathcal{C}^{k,\alpha}(\mathbb{T}^2)} \leq (k+3)! \, \|f\|_{\mathcal{C}^{k+1}(\mathbb{S}^2)},$$

and

$$\left\|\tilde{f}\right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T}^2)} \leq \frac{7}{4}(k+3)! \, \|f\|_{\mathcal{C}^{k+1}(\mathbb{S}^2)}.$$

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Sobolev smoothness

Does the result for Hölder spaces also hold for Sobolev spaces?

Spherical Sobolev space

$$H^s(\mathbb{S}^2) := \left\{ f \in L^2(\mathbb{S}^2) : \|f\|_{H^s(\mathbb{S}^2)} = \|(-\Delta + \tfrac{1}{4})^{s/2} f\|_{L^2(\mathbb{S}^2)} < \infty \right\}.$$

Theorem (Sobolev space) [Mildenberger, Q. 2021]

Let

$$f \colon \mathbb{S}^2 \to \mathbb{R}, \quad f(\xi) = \begin{cases} \ln\left(\ln \frac{1}{\sqrt{1-\xi_3^2}}\right), & |\xi_3| \neq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in H^1(\mathbb{S}^2)$, but $\tilde{f} \notin H^1(\mathbb{T}^2)$.

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DFS Fourier series

- Represent $f\colon \mathbb{S}^2 \to \mathbb{C}$ via the Fourier series of its DFS function $\tilde{f}\colon \mathbb{T}^2 \to \mathbb{C}$,

$$F_N \tilde{f}(\boldsymbol{x}) = \sum_{|\boldsymbol{n}|_\infty \le N} c_{\boldsymbol{n}}(\tilde{f})\, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{n}, \boldsymbol{x}\rangle}, \qquad \boldsymbol{x} \in \mathbb{T}^2.$$

- $\tilde{f}$ is BMC-symmetric, but the basis functions $\mathrm{e}^{\mathrm{i}(n_1\lambda + n_2\theta)}$ are not BMC

- Define BMC-symmetric function

$$e_{\boldsymbol{n}}(\lambda, \theta) := \begin{cases} \mathrm{e}^{\mathrm{i}(n_1\lambda + n_2\theta)} + (-1)^{n_1}\mathrm{e}^{\mathrm{i}(n_1\lambda - n_2\theta)}, & n_2 \neq 0, \\ \mathrm{e}^{\mathrm{i}(n_1\lambda)}, & n_2 = 0. \end{cases}$$

- Define the basis functions

$$b_{\boldsymbol{n}}(\boldsymbol{\xi}) := e_{\boldsymbol{n}}(\phi^{-1}(\boldsymbol{\xi})), \qquad \boldsymbol{n} \in \mathbb{Z} \times \mathbb{N}_0,\ \boldsymbol{\xi} \in \mathbb{S}^2.$$

(ϕ is bijective on $[-\pi, \pi) \times (0, \pi) \cup \{(0,0), (0,\pi)\}$)

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DFS Fourier series II

Set

$$\tilde{L}_2(\mathbb{S}^2) := \left\{ f\colon \mathbb{S}^2 \to \mathbb{C} : \|f\|^2_{\tilde{L}_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} |f(\boldsymbol{\xi})|^2 \, \frac{1}{\sqrt{1-\xi_3^2}} \, \mathrm{d}\boldsymbol{\xi} < \infty \right\}.$$

Theorem [Mildenberger, Q. 2021]

Let $f \in \tilde{L}_2(\mathbb{S}^2)$. Then the Fourier coefficients of its DFS function $\tilde{f} \in L_2(\mathbb{T}^2)$ satisfy

$$c_{n_1,n_2}(\tilde{f}) = (-1)^{n_1} c_{n_1,-n_2}(\tilde{f}), \qquad \boldsymbol{n} \in \mathbb{Z}^2.$$

Then we have

$$S_N f(\boldsymbol{\xi}) := \sum_{n_1=-N}^{N} \sum_{n_2=0}^{N} c_{\boldsymbol{n}}(\tilde{f})\, b_{\boldsymbol{n}}(\boldsymbol{\xi}) = F_N \tilde{f}(\phi^{-1}(\boldsymbol{\xi})), \qquad \boldsymbol{\xi} \in \mathbb{S}^2.$$

The set $\{b_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{Z} \times \mathbb{N}_0\}$ is an orthogonal basis of $\tilde{L}_2(\mathbb{S}^2)$.

Convergence of DFS Fourier series

Known on the torus $\mathbb{T}^2$

If $g \in \mathcal{C}^{k,\alpha}(\mathbb{T}^2)$, its Fourier series converges with $\|f - F_N f\|_{\mathcal{C}(\mathbb{T}^2)} \leq M_{k,\alpha} \|f\|_{\mathcal{C}^{k,\alpha}(\mathbb{T}^2)} N^{1-k-\alpha}$.

Theorem [Mildenberger, Q. 2021]

Let $k \in \mathbb{N}$, $0 < \alpha \leq 1$, and $f \in \mathcal{C}^{k,\alpha}(\mathbb{S}^2)$. Then there is a constant $M_{k,\alpha}$ depending only on k and α such that

$$\left\| f - \sum_{n_1=-N}^{N} \sum_{n_2=0}^{N} c_n(\tilde{f}) b_n \right\|_{\mathcal{C}(\mathbb{S}^2)} \leq M_{k,\alpha} \|f\|_{\mathcal{C}^{k,\alpha}(\mathbb{S}^2)} N^{1-k-\alpha}.$$

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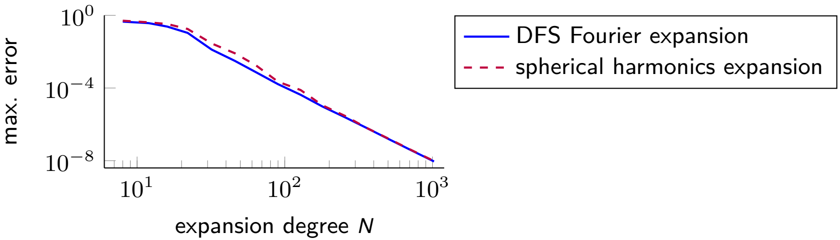
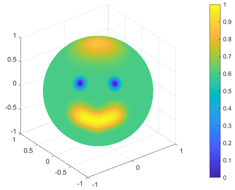
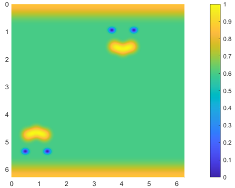
Numerical example

Compare DFS Fourier expansion with spherical harmonics expansion





$f \in \mathcal{C}^{3,\alpha}(\mathbb{S}^2)$, $\alpha < 1$



DFS function $\tilde{f} \in \mathcal{C}^{3,\alpha}(\mathbb{T}^2)$

Logarithmic plot of the maximal error $\|f - S_N f\|_{\mathcal{C}(\mathbb{S}^2)}$

Computation time for expansion of degree $N = 1024$:

- DFS Fourier expansion 0.06 s
- spherical harmonic expansion 0.67 s with NFFT (+ precomputations 3.05 s)

Fourier expansion $S_N f$ consists of $(N+1)(2N+1)$ summands, while the spherical harmonics expansion consists of $(N+1)^2$ summands.

Comparison

	DFS Fourier expansion	Spherical harmonics expansion
Basis functions	$\mathrm{e}^{\mathrm{i}(n_1\lambda+n_2\theta)} + (-1)^{n_1}\mathrm{e}^{\mathrm{i}(n_1\lambda-n_2\theta)}$, $n_1 \in \mathbb{Z}$, $n_2 \in \mathbb{N}_0$	$Y_n^k = P_n^k(\cos\theta)\mathrm{e}^{\mathrm{i}k\lambda}$, $n \in \mathbb{N}_0$, $k = -n, \dots, n$
Orthogonality	$\tilde{L}^2(\mathbb{S}^2)$ with weight $w(\lambda,\theta) = \frac{1}{\sin\theta}$	$L^2(\mathbb{S}^2)$
Rotational invariance	✗	✓ (for fixed degree n)
Eigenfunctions	✗	of the Laplace–Beltrami operator Δ
Computation	via 2D-FFT	via fast Legendre transform and (no tensor product structure, slower than 2D-FFT)
Convergence	✓	✓

Conclusion

- DFS method maps the Hölder space $\mathcal{C}^{k,\alpha}(\mathbb{S}^2)$ continuously into the respective spaces on the torus $\mathbb{T}^2$.
- We provide a series expansion of the spherical function in terms of basis functions that are orthogonal with respect to a weight on the sphere.
- The DFS Fourier series of Hölder functions converges with the expected rate.
- Numerical tests indicate a comparable approximation quality as a spherical harmonics expansion.
- DFS method is faster than spherical harmonics expansion.

- DFS method appends to other geometries.

Thank you for your attention!

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