The cone-beam transform and spherical convolution operators

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(joint work with Ralf Hielscher and Alfred K. Louis)

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Content

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Funk–Radon transform

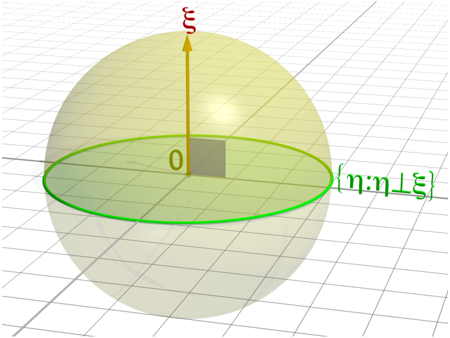
[Funk, 1911]

- Sphere $\mathbb{S}^{d-1} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- Funk–Radon transform

$$\mathcal{S}^{(0)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \delta(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta}=0} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})$$

(integrals of f along all great circles)

- Take derivatives of the delta function



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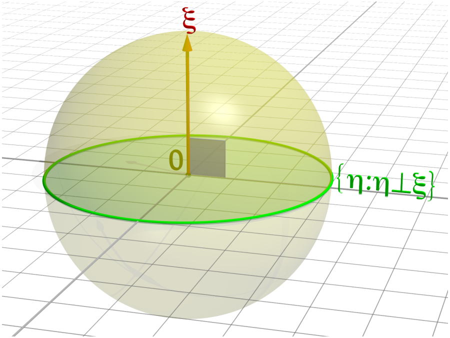

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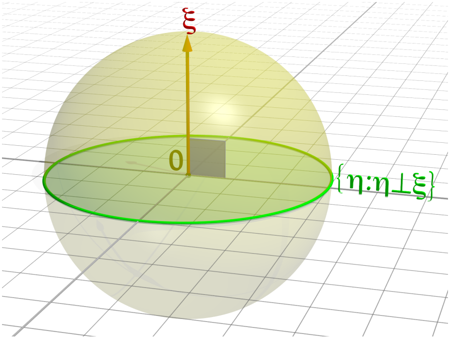

Generalized Funk–Radon transform

[Louis, 2016]

- **generalized Funk–Radon transform**

$$\mathcal{S}^{(j)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \delta^{(j)}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}$$

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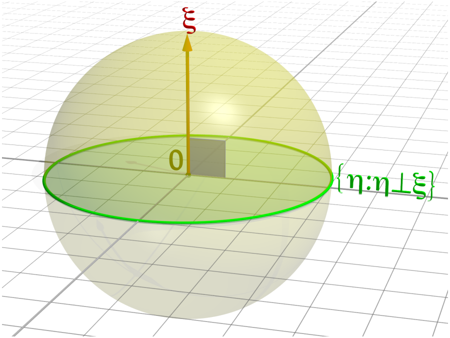
- $\frac{\partial}{\partial \boldsymbol{\xi}}$ ... directional derivative

- Similar definition for $j = 1$:

[Makai, Martini, Odor, 2000]



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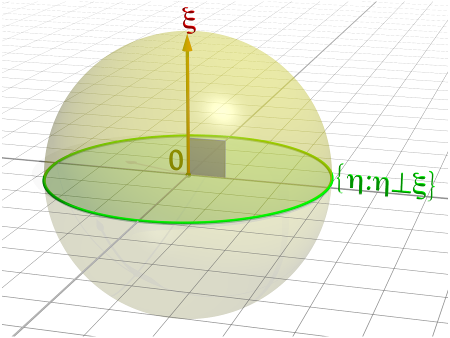

Spherical harmonics: An orthonormal basis on $\mathbb{S}^{d-1}$

The **spherical harmonics**

$$Y_n^k : \mathbb{S}^{d-1} \to \mathbb{C}, \qquad n \in \mathbb{N}_0,\ k = 1, \dots, N_{n,d}$$

form an orthonormal basis of $L^2(\mathbb{S}^{d-1})$.

Any $f \in L^2(\mathbb{S}^{d-1})$ can be written as series

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}(n,k) Y_n^k, \qquad \hat{f}(n,k) := \int_{\mathbb{S}^{d-1}} f(\xi) \overline{Y_n^k(\xi)} \, \mathrm{d}\xi$$

Fast algorithms for spherical Fourier transforms on $\mathbb{S}^2$

[Driscoll & Healy, 1994] [Potts, Steidl & Tasche, 1998] [Kunis & Potts, 2003] [Schaeffer, 2013]

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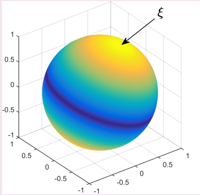
Spherical convolution

Funk–Hecke formula [Funk, 1915] [Hecke, 1917]

Let $g\colon [-1,1] \to \mathbb{C}$. Then

$$\int_{\mathbb{S}^{d-1}} Y_n^k(\boldsymbol{\eta})\, g(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = Y_n^k(\boldsymbol{\xi}) \int_{-1}^{1} g(t)\, P_{n,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t$$

$P_{n,d}$ – Legendre (ultraspherical) polynomial of degree n in dimension d
orthogonal polynomial on $[-1,1]$ w.r.t. the weight $(1-t^2)^{\frac{d-3}{2}}$



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For the generalized Funk–Radon transform: Insert $g(t) = \delta^{(j)}(t)$

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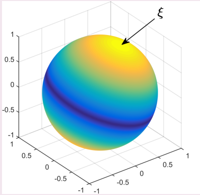

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Eigenvalue decomposition of the generalized Funk–Radon transform

Theorem [Q., Hielscher, Louis, 2018]

Let $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}^{(j)} \colon C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ satisfies the eigenvalue decomposition

$$\mathcal{S}^{(j)} Y_n^k = P_{n,d}^{(j)}(0)\, Y_n^k, \qquad n \in \mathbb{N}_0,\ k = -n, \dots, n$$

with eigenvalues

$$P_{n,d}^{(j)}(0) = \begin{cases} \left|\mathbb{S}^{d-2}\right| \, (-1)^{\frac{n+j}{2}} \dfrac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!}, & n+j \text{ even and } (n \geq j) \\ 0, & \text{otherwise} \end{cases}$$

Decay of the eigenvalues of $\mathcal{S}^{(j)}$

Lemma

Let $j \in \mathbb{N}_0$, $n + j$ even and $n \geq j$. The eigenvalues of $\mathcal{S}^{(j)}$ satisfy

$$\left|P_{n,d}^{(j)}(0)\right| \simeq n^{j-\frac{d-2}{2}}\, \pi^{\frac{d-1}{2}}\, 2^{\frac{d}{2}} \qquad \text{for } n \to \infty.$$

Theorem [Q., Hielscher, Louis, 2018]

Let $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$ extends to a continuous operator

$$\mathcal{S}^{(j)} \colon H^s(\mathbb{S}^{d-1}) \to H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1}).$$

The nullspace of $\mathcal{S}^{(j)}$ is the closed linear span

$$\overline{\operatorname{span}}\left\{Y_n^k : n + j \text{ odd or } (n \leq j - d + 1 \text{ and } d \text{ odd}),\ k = 1, \ldots, N_{n,d}\right\}.$$

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Special cases of j

$j = -1$: **modified hemispherical transform** [Ungar 1954] [Rubin, 1999]

$$\mathcal{S}^{(-1)} f(\boldsymbol{\xi}) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\boldsymbol{\xi}^\top \boldsymbol{\eta}) \; f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}.$$

$j = -2$: **spherical cosine transform** [Petty, 1961] [Schneider, 1967] [Groemer, 1996]

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If $d = 2j + 2$, the absolute value of the eigenvalues

$$P_{n,d}^{(j)}(0) = \left|\mathbb{S}^{2j}\right| \; (-1)^{\frac{n+j}{2}} \; (2j-1)!!, \qquad n + j \text{ even}$$

is constant. Hence $\mathcal{S}_{2j+2}^{(j)} \colon L^2(\mathbb{S}^{2j+1}) \to L^2(\mathbb{S}^{2j+1})$ is a partial isometry.

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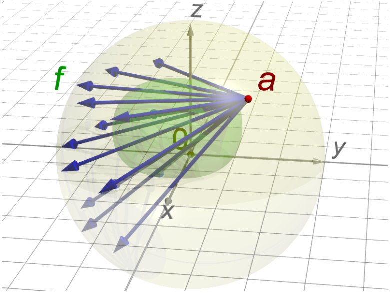
Cone-beam transform

- $f\colon \mathbb{R}^d \to \mathbb{R}$
- $\boldsymbol{a} \in \mathbb{R}^d$ ... source of the ray
- $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ ... direction of the ray

- **Cone-beam transform**
  (a.k.a. divergent beam X-ray transform)

$$\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a}+t\boldsymbol{\omega})\,\mathrm{d}t$$



[Hamaker et al. 1980] [Tuy, 1983] [Finch, 1985] [Feldkamp, Davis, Kress, 1984]

Radon transform

- $f\colon \mathbb{R}^d \to \mathbb{R}$
- **Radon transform**

$$\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\boldsymbol{x}^\top \boldsymbol{\omega} = s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$$

- Integral along (hyper-)plane with normal $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$

Cone-beam and Radon transform (in 3D)

[Grangeat, 1991]

Consider a “fan” of ray integrals orthogonal to $\boldsymbol{\omega}$

$$\int_{\boldsymbol{\xi}\in\mathbb{S}^2,\, \boldsymbol{\xi}^\top\boldsymbol{\omega}=0} \mathcal{D}f(\boldsymbol{a},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

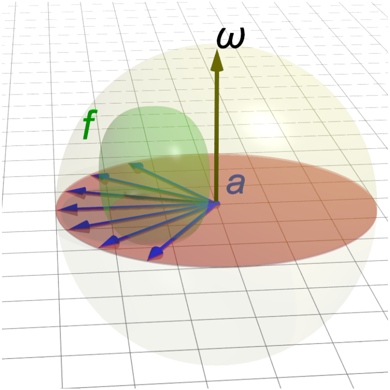
Want the plane integral

$$\mathcal{R}f(\boldsymbol{\omega},\boldsymbol{a}^\top\boldsymbol{\omega}) = \int_{\boldsymbol{x}^\top\boldsymbol{\omega}=\boldsymbol{a}^\top\boldsymbol{\omega}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

Grangeat's formula

$$\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega},\boldsymbol{a}^\top\boldsymbol{\omega}) = \int_{\boldsymbol{\xi}\in\mathbb{S}^2,\boldsymbol{\xi}^\top\boldsymbol{\omega}=0} \frac{\partial}{\partial\boldsymbol{\omega}}\mathcal{D}f(\boldsymbol{a},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

$\frac{\partial}{\partial\boldsymbol{\omega}}$ ... directional derivative w.r.t. $\boldsymbol{\xi}$



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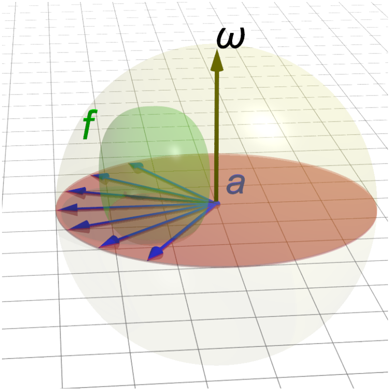

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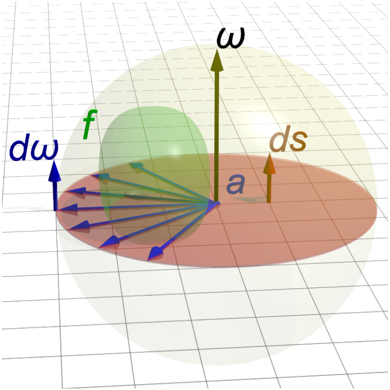

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Recall the generalized Funk–Radon transform

$$\mathcal{S}^{(j)}f(\omega) = \int_{\mathbb{S}^2} \delta^{(j)}(\xi^\top\omega)\, f(\xi)\,\mathrm{d}\xi, \qquad \omega \in \mathbb{S}^2.$$

Write Grangeat's formula as

[Louis, 2016]

$$\frac{\partial}{\partial s}\mathcal{R}f(\omega, s)\Big|_{s=\boldsymbol{a}^\top\omega} = -\mathcal{S}^{(1)}_\omega\mathcal{D}f(\boldsymbol{a},\omega)$$

In general dimension d

$$(-1)^d\left(\frac{\partial}{\partial s}\right)^{d-2}\mathcal{R}f(\omega, s)\Big|_{s=\boldsymbol{a}^\top\omega} = \mathcal{S}^{(d-2)}_\omega\mathcal{D}f(\boldsymbol{a},\omega)$$

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Singular Value decomposition of the Radon transform $\mathcal{R}$

Theorem [Louis, 1984]

For $m \in \mathbb{N}_0$, $l = 0, \dots, m$ with $m + l$ even and $k = 1, \dots, N_{l,d}$

$$\mathcal{R} V_{m,l,k}(\boldsymbol{\omega}, s) = \frac{\sqrt{2m+d}\,\Gamma(\frac{d}{2})\, m!}{2^{1-d}\,\pi^{1-\frac{d}{2}}\,(m+d-1)!}\,(1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s)\, Y_l^k(\boldsymbol{\omega}),$$

where

$$V_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+d}\, s^l\, P_{\frac{m-l}{2}}^{(0,l+\frac{d-2}{2})}(2s^2-1)\, Y_l^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$

and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n and orders $\alpha, \beta > -1$.

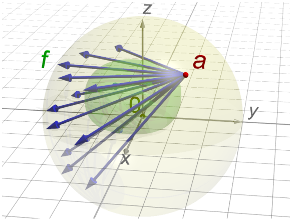
Cone-beam transform

- Let $f\colon \mathbb{B}^d = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \le 1\} \to \mathbb{R}$
- Consider the cone-beam transform

$$\mathcal{D}\colon L^2(\mathbb{B}^d) \to L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$$

$$\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a} + t\boldsymbol{\omega})\,\mathrm{d}t, \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$

with sources $\boldsymbol{a} \in \mathbb{S}^{d-1}$ on the sphere



Cone-beam transform

Singular value decomposition [Q., Hielscher, Louis, 2018]

The cone-beam transform $\mathcal{D}$ with sources $\boldsymbol{a}$ on the sphere $\mathbb{S}^{d-1}$ and d odd has the SVD

$$\mathcal{D}V_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = \mu_{m,d} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^{j}(\boldsymbol{a})} \sum_{n=m+1-l}^{l+m+1}{}' \nu_{n,d} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^{i}(\boldsymbol{\omega}),$$

where $\sum'$ denotes the summation over odd indices and

$$\mu_{m,d} = \sqrt{\frac{2^{d+1}\,\pi^{d-1}}{2m+d}}, \qquad \nu_{n,d} = \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{(n+d-3)!!}$$

and the Gaunt coefficients $G_{n_1,k_1,n_2,k_2}^{n,k} = \int_{\mathbb{S}^{d-1}} Y_{n_1}^{k_1}(\boldsymbol{\xi})\, Y_{n_2}^{k_2}(\boldsymbol{\xi})\, \overline{Y_n^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}$.

Remarks

- dimension $d = 3$ [Kazantsev, 2015]
- d odd [Q., Hielscher, Louis, 2018]
- Similar result for full lines (X-ray transform) [Maaß, 1987]

Decay of the singular values $\lambda_{m,l,d}$ of the cone-beam transform $\mathcal{D}$

- Lower bound: $\lambda_{m,l,d} \geq c_d\, m^{-1/2}$ for $m \to \infty$
- Upper bound: $\lambda_{m,l,d} \leq C_d$
- for dimension $d = 3$: $\lambda_{m,l,3} \leq C\, m^{-1/4}$ [Kazantsev, 2015]
- Conjecture: $\lambda_{m,l,d} \leq C_d\, m^{-1/2}$

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- Investigated the **generalized Funk–Radon transform**
- Characterization of its nullspace and range
- Grangeat's formula connects the cone-beam transform, Radon transform and generalized Funk–Radon transform
- SVD of the **cone-beam transform** for sources on the sphere
- **Future:** Limited data, algorithm for the cone-beam transform, ...

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\endinput

Happy Birthday, Professor Tasche