Optimal Mollifiers for Reconstructing Spherical Images from Circular Means

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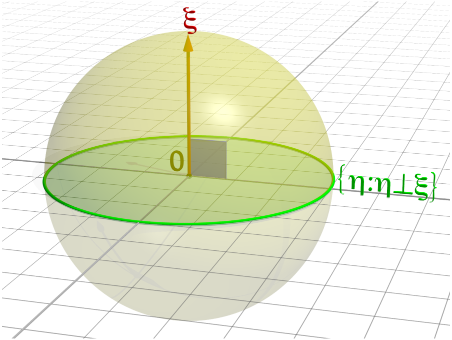
Recent Developments in Inverse Problems
September 18, 2015

Funk–Radon transform

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f : \mathbb{S}^2 \to \mathbb{C}$ on the sphere
- **Funk–Radon transform** computes the integrals along all great circles

$$\mathcal{R}f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}\cdot\boldsymbol{\eta}=0} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$
$$= \int_{\mathbb{S}^2} f(\boldsymbol{\eta})\delta(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}$$



- Generalization to **spherical convolutions**

$$h \star f(\boldsymbol{\xi}) = \int_{\mathbb{S}^2} f(\boldsymbol{\eta})h(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, \quad \boldsymbol{\xi} \in \mathbb{S}^2.$$

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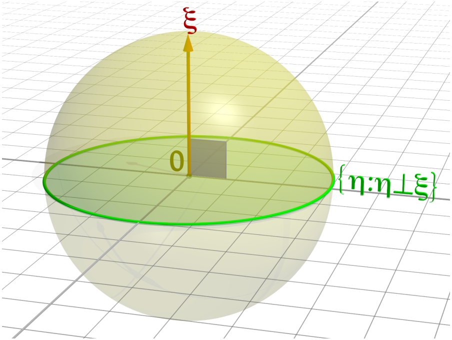

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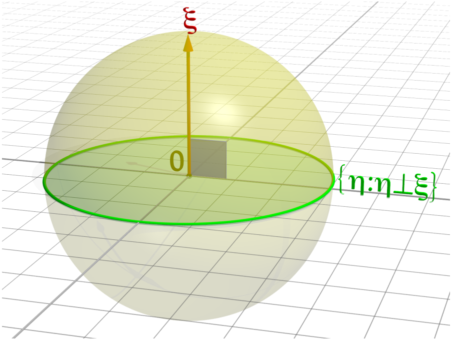

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What convolution operators are good for

- **Funk–Radon transform**
- Intersection bodies [Gardner, 2006]
- Q–ball imaging in medicine [Tuch, 2004]
- Surface wave models for earthquakes [Amirbekyan, Michel & Simons, 2008]
- Synthetic aperture radar (SAR) [Yarman & Yazici, 2011]

- **Hemispherical transfrom**
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$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}(n,k) Y_n^k$$

Eingenvalue decomposition [Minkowski, 1904]

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- The Funk–Radon transform is a smoothing operator [Strichartz, 1981]

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How to reconstruct f

- Given: $g(\boldsymbol{\xi}_m) = \mathcal{R}f(\boldsymbol{\xi}_m) + \varepsilon(\boldsymbol{\xi}_m), \quad m = 1, \ldots, M$
- ε is white stochastic noise
- **Eigenvalue decomposition** of $\mathcal{R}$ yields

$$\mathcal{R}^{-1} g = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{1}{\hat{\mathcal{R}}(n)} \hat{g}(n,k) Y_n^k$$

- **Discretization**: Use a quadrature formula to calculate

$$\hat{g}(n,k) = \int_{\mathbb{S}^2} g(\boldsymbol{\xi}) \overline{Y_n^k(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi} \approx \frac{1}{M} \sum_{m=1}^{M} g(\boldsymbol{\xi}_m) \overline{Y_n^k(\boldsymbol{\xi}_m)}$$

  and truncate the sum at degree N
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How to measure the error

- Mean integrated squared error **MISE**

$$\mathbb{E}\,\|f - f^*_{N,\psi}\|^2_{L^2(\mathbb{S}^2)}$$

- For $s, S \geq 0$, define the class of functions with bounded Sobolev norm

$$\mathscr{F}(s,S) = \left\{ f \in H^s_{\text{even}}(\mathbb{S}^2) : \|f\|_{H^s(\mathbb{S}^2)} \leq S \right\}$$

- Want to minimize the **maximum risk**

$$R(N,\psi) = \sup_{f \in \mathscr{F}(s,S)} \mathbb{E}\,\|f - f^*_{N,\psi}\|^2_{L^2(\mathbb{S}^2)}$$

- **Minimax risk**

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How to choose the filter coefficients $\hat{\psi}(n)$

Decomposition of the MISE

$$\mathbb{E}\,\|f - f^*_{N,\psi}\|^2_{L^2(\mathbb{S}^2)} = \underbrace{\|f - \psi \star \mathcal{R}^{-1}\mathcal{L}_N\mathcal{R}f\|^2_{L^2(\mathbb{S}^2)}}_{\text{bias}} + \underbrace{\mathbb{E}\,\|\psi \star \mathcal{R}^{-1}\mathcal{L}_N\varepsilon\|^2_{L^2(\mathbb{S}^2)}}_{\text{variance}}$$

$\hat{\psi} = 1$ reduces bias error (caused by regularization)

$\hat{\psi} = 0$ reduces variance error (caused by noise)

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What we can expect

Theorem

- *Let $s > \frac{3}{2}$ and $S > 0$.*
- *For every $N \in \mathbb{N}$, let the quadrature be exact of degree $2N$ with $M \sim N^2$ nodes and constant weights.*
- *Let the white noise $\varepsilon(\boldsymbol{\xi}_m)$ be uncorrelated with expected value 0.*

There exists an asymptotically optimal family of mollifiers $\psi_L^s,\ L \in \mathbb{R}_+$ for the class $\mathscr{F}(s, S)$. For $N \to \infty$, there are parameters $L(N)$ such that

$$R\left(N, \psi^s_{L(N)}\right) \simeq \inf_{\psi} R(N, \psi) \simeq \text{const} \cdot N^{\frac{-2s}{s+\frac{3}{2}}}.$$

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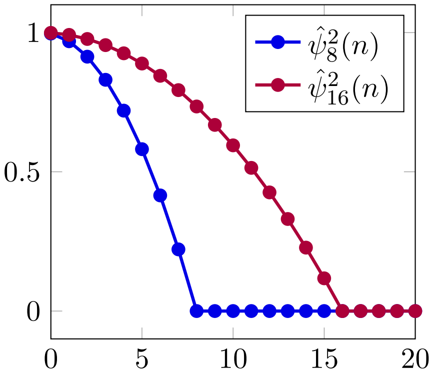
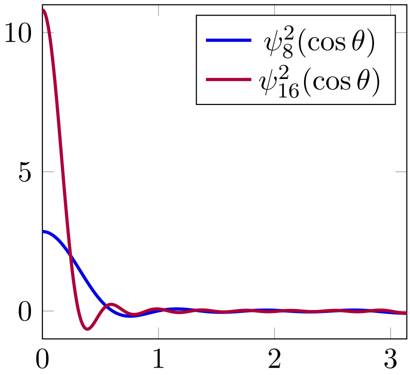
$$\hat{\psi}_L^s(n) = 1 - \left(\frac{n + \frac{1}{2}}{L + \frac{1}{2}} \right)^s$$

Filter coefficients $\hat{\psi}_L^s(n)$



Mollifier function ψ_L^s



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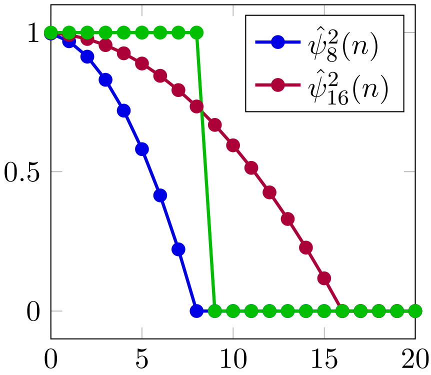

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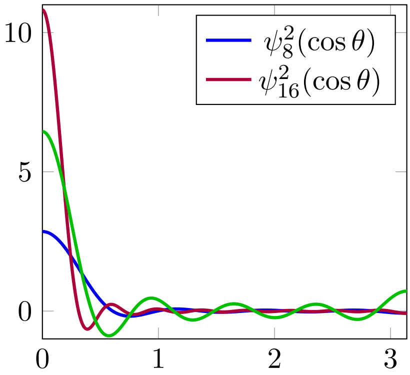

Algorithm to compute the estimator

Given: $g(\boldsymbol{\xi}_m),\ m = 1, \ldots, M$

1. Compute the Fourier coefficients $\widehat{\mathcal{L}_N g}(n,k) = \frac{1}{M} \sum_{m=1}^{M} g(\boldsymbol{\xi}_m) \overline{Y_n^k(\boldsymbol{\xi}_m)}$
2. Compute the regularization $\widehat{f_{N,\psi}^*}(n,k) = \frac{\hat{\psi}(n)}{\hat{\mathcal{R}}(n)} \widehat{\mathcal{L}_N g}(n,k)$
3. Compute the estimator $f_{N,\psi}^* = \sum_{n=0}^{N} \sum_{k=-n}^{n} \widehat{f_{N,\psi}^*}(n,k) Y_n^k$

- Complexity: $\mathcal{O}(N^2 \log^2 N)$ with fast spherical Fourier transform (NFSFT) [Driscoll & Healy, 1994] [Potts, Steidl & Tasche, 1998] [Kunis & Potts, 2003] [Keiner & Potts, 2008]

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1. Compute the Fourier coefficients $\widehat{\mathcal{L}_N g}(n,k) = \frac{1}{M} \sum_{m=1}^{M} g(\boldsymbol{\xi}_m) \overline{Y_n^k(\boldsymbol{\xi}_m)}$
2. Compute the regularization $\widehat{f_{N,\psi}^*}(n,k) = \frac{\hat{\psi}(n)}{\hat{\mathcal{R}}(n)} \widehat{\mathcal{L}_N g}(n,k)$
3. Compute the estimator $f_{N,\psi}^* = \sum_{n=0}^{N} \sum_{k=-n}^{n} \widehat{f_{N,\psi}^*}(n,k) Y_n^k$

- Complexity: $\mathcal{O}(N^2 \log^2 N)$ with fast spherical Fourier transform (NFSFT) [Driscoll & Healy, 1994] [Potts, Steidl & Tasche, 1998] [Kunis & Potts, 2003] [Keiner & Potts, 2008]

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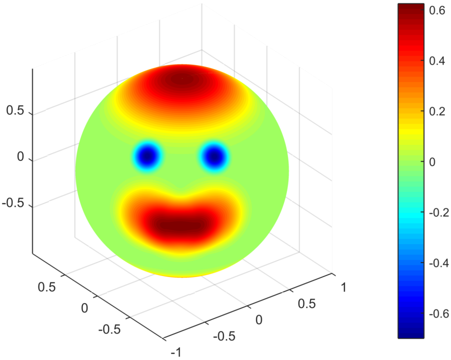
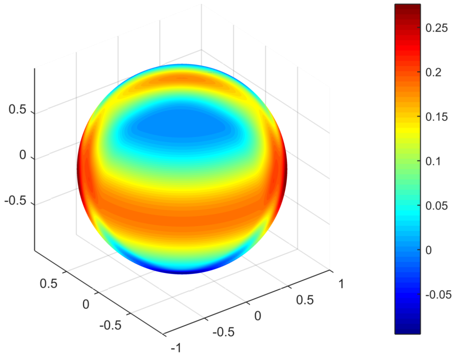
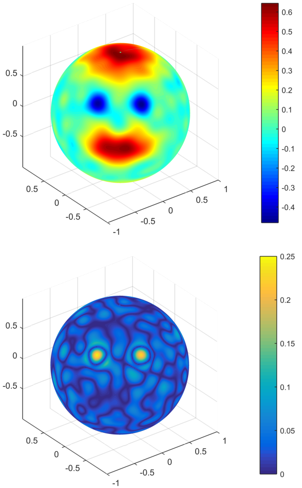
A test function



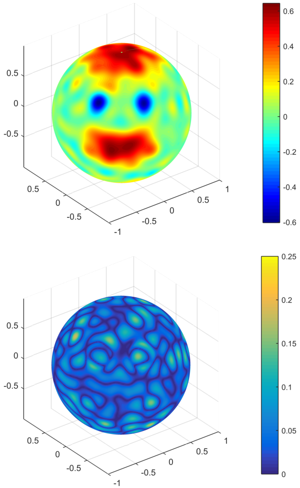
Test function f (quadratic spline)



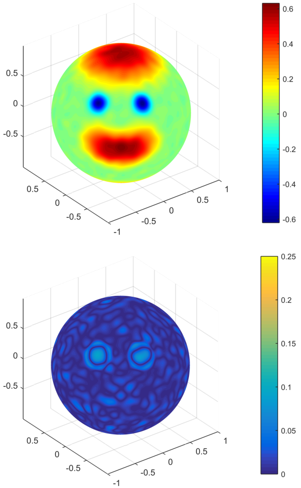
Funk–Radon transform $\mathcal{R}f$



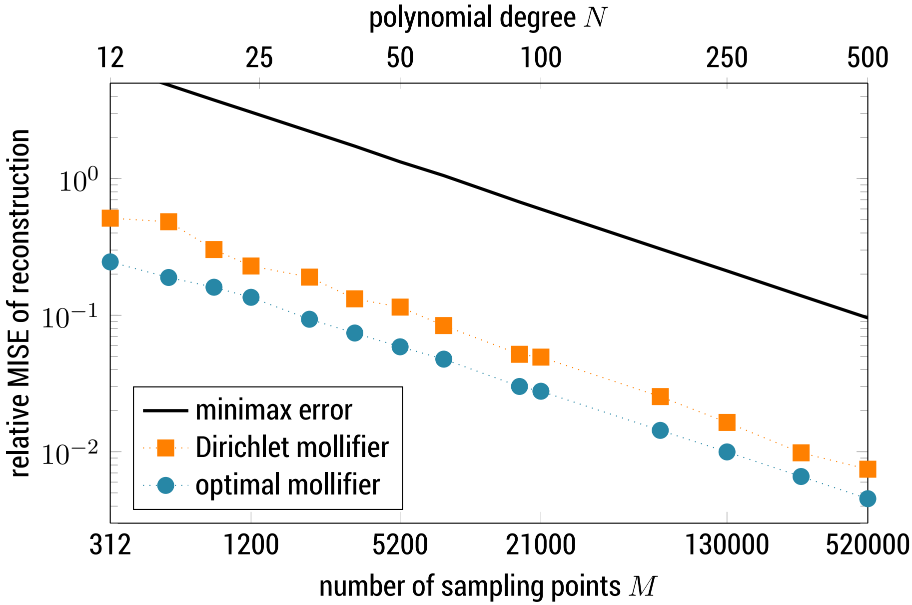
optimal mollifier
$N = 100$



Dirichlet mollifier
$N = 100$



optimal mollifier
$N = 500$


polynomial degree N
12
25
50
100
250
500
relative MISE of reconstruction
10^0
10^{-1}
10^{-2}
minimax error
Dirichlet mollifier
optimal mollifier
312
1200
5200
21000
130000
520000
number of sampling points M

```
\endinput
```