Slicing of Radial Functions: a Dimension Walk in the Fourier Space

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Computations in high-dimensional spaces can often be realized only approximately, using a certain number of projections onto lower dimensional subspaces or sampling from distributions. In this paper, we are interested in pairs of real-valued functions (F, f) on $[0, \infty)$ that are related by the projection/slicing formula $F(\|x\|) = \mathbb{E}_\xi\big[f\big(|\langle x, \xi\rangle|\big)\big]$ for $x \in \mathbb{R}^d$, where the expectation value is taken over uniformly distributed direction in $\mathbb{R}^d$. While it is known that F can be obtained from f by an Abel-like integral formula, we construct conversely f from given F using their Fourier transforms. First, we consider the relation between F and f for radial functions $F(\|\cdot\|)$ that are Fourier transforms of L^1 functions. Besides d- and one-dimensional Fourier transforms, it relies on a rotation operator, an averaging operator and a multiplication operator to manage the walk from d to one dimension in the Fourier space. Then, we generalize the results to tempered distributions, where we are mainly interested in radial regular tempered distributions. Based on Bochner's theorem, this includes positive definite functions $F(\|\cdot\|)$.

1. Introduction

Radial functions play an important role in approximation theory [7, 50], kernel density estimation [30, 37], support vector machines [44, 45], kernelized principal component analysis [42, 43], simulation of optical scattering [12, 24], distance computations between probability measures [18, 47] as well as dithering [9, 15] in image processing, to mention only a few. Recently, they have found applications in machine learning in connection with Stein variational gradient descent flows [28] and Wasserstein gradient flows [1, 13, 19].

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A central issue in the above applications is the fast evaluation of radial functions, more precisely, the computation of "convolutions at nonequispaced knots"

$$\sum_{i=1}^{N} \alpha_i F(\|x_i - x_j\|), \quad j = 1, \ldots, N \tag{1}$$

for large $N \in \mathbb{N}$, where $x_i \in \mathbb{R}^d$ and $\alpha_j \in \mathbb{C}$. Throughout this paper, let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$, where the dimension becomes clear from the context. Certain methods for high-dimensional data $x_i \in \mathbb{R}^d$, $d \gg 1$, were proposed in the literature. A popular one, called random Fourier features [36], was analyzed, e.g., in [20, 46] and found recent applications for ANOVA approximation [33] and domain decomposition [27]. It relies on the linearity of the expectation value and Bochner's theorem, showing that a positive definite, radial function $F \circ \|\cdot\|$ on $\mathbb{R}^d$ is the Fourier transform of a positive measure μ, i.e., after proper scaling,

$$F(\|x\|) = \mathbb{E}_{v\sim\mu}\left[\mathrm{e}^{-2\pi \mathrm{i}\langle x,v\rangle}\right]. \tag{2}$$

Another technique, known as slicing [21], resembles the Radon transform and is well-known in the context of optimal transport [5, 6, 35]. It is based on the existence of a one-dimensional function f such that the slicing (projection) formula

$$F(\|\cdot\|) = \mathbb{E}_{\xi\sim\mathcal{U}_{\mathbb{S}^{d-1}}}\left[f\left(|\langle x, \xi\rangle|\right)\right], \tag{3}$$

holds true, where the expectation value is taken over the uniform distribution $\mathcal{U}_{\mathbb{S}^{d-1}}$ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. In other words, a radial function $F \circ \|\cdot\|$ fulfilling (3) can be evaluated at $x \in \mathbb{R}^d$ by projecting it onto lines of different directions ξ through the origin, see Fig. 1, followed by evaluating a one-dimensional function f at the projected points $\langle x, \xi\rangle$. For certain functions f, one-dimensional summations of the form (1) can be done in a very fast way, e.g., via sorting or fast Fourier transforms at nonequispaced knots [21, 26, 32], as done in various applications [3, 8, 22, 23].

The relation between F and its sliced version f in (3) is given by the Abel-type integral

$$F(s) = c_d \int_0^1 f(ts)(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t \tag{4}$$

with some constant c_d. Note that in [21], functions F having a power series were considered to determine their slicing functions f. In contrast to random Fourier features, slicing is not restricted to positive definite functions $F \circ \|\cdot\|$, and indeed it works also for other functions, which are of interest in applications, like Riesz kernels $\|\cdot\|^r$, $r \in (0, 2)$ or thin plate splines $\|\cdot\|^2 \log \|\cdot\|$. However, we see from its integral representation (4) that $F\colon [0,\infty) \to \mathbb{R}$ must have some smoothness properties. Indeed, (4) is closely related to Riemann–Liouville fractional integrals, and the injectivity of the transform (4) if $f \in L^1(\mathbb{R})$ as well as the inverse transform, which determines f from F, can be deduced via fractional derivatives, see Appendix D. However, the resulting integrals are often hard to evaluate, and we will follow another approach.

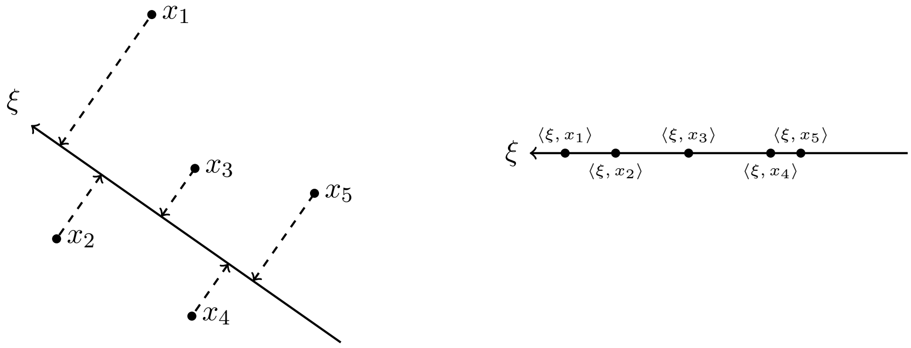


Figure 1: Projection of points $x_1, \dots, x_5 \in \mathbb{R}^2$ onto the line in direction ξ.

In this paper, we are interested in the relation between F and f from a Fourier analytic point of view. More precisely, we show how f can be obtained from a radial function $F \circ \|\cdot\|$ that is the Fourier transform of a radial L^1 function. Then we will see that this is a special case of a recovery formula for radial regular tempered distributions. Since measures can be considered as tempered distributions, the later one also includes positive definite functions appearing in Bochner's relation (2). Radial tempered distributions were already considered in the literature, e.g., in [11, 17]. However, to the best of our knowledge, our rigorous proofs of certain properties needed for our approach are novel. The dimension reduction from a multivariate, radial function $F \circ \|\cdot\|$ to a univariate one $f \circ |\cdot|$ in the Fourier space can be easily realized by applying a multiplication operator arising from a variable transform, and is actually what we call "dimension walk", a notation borrowed from Wendland [50, Chap. 9.2]. We are completely aware that also projections onto larger than one-dimensional subspaces may be of interest, but are out of the scope of this paper.

Outline of the paper: in Section 2 we introduce our two main players, namely the rotation operator and its inverse, the averaging operator. Then we recall the relation between the slicing formula (3) and the Abel-like integral (4). The "dimension walk" is realized by a multiplication operator. Moreover, we determine the smoothness of functions F determined by the Abel-like integral. Then, in Section 3, we show as a starting point, how the function f in (4) can be computed from a radial function $F \circ \|\cdot\|$ that is the Fourier transform of a radial L^1 function. As a by-product of the smoothness result for the Abel-like integral, we will see that the Fourier transform of a radial function in $\mathbb{R}^d$ is $\lfloor d-2/2 \rfloor$ times continuously differentiable, a result that should be known in the literature, although we did not find a direct reference. In Section 4, we first recall the definition of radial Schwartz functions and prove that the averaging and rotation operator are continuous operators on these spaces. This allows to generalize the reconstruction to radial regular tempered distributions. Clearly, this is more general than the approach in the previous section and we provide in particular two examples. Since measures can be treated as special tempered distributions, we obtain a result for positive definite radial functions $F \circ \|\cdot\|$ based on Bochner's theorem. Auxiliary technical results are postponed to the appendix.

2. Rotating, Averaging and Slicing

We denote by $\mathcal{C}(\mathbb{R}^d)$ the space of complex-valued continuous functions, by $\mathcal{C}_b(\mathbb{R}^d)$ the Banach space of bounded continuous functions, and by $\mathcal{C}_0(\mathbb{R}^d)$ the Banach space of continuous functions vanishing for $\|x\| \to \infty$ with the norm

$$\|\Phi\|_\infty \coloneqq \sup_{x \in \mathbb{R}^d} |\Phi(x)|.$$

Let $\mathcal{C}^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions. Further, let $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $p \in [1, \infty)$, denote the space of locally p-integrable functions and $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ the space of locally bounded functions.

We are interested in radial functions $\Phi \colon \mathbb{R}^d \to \mathbb{R}$, which are characterized by the property that for all $x \in \mathbb{R}^d$,

$$\Phi(x) = \Phi(Qx) \quad \text{for all} \quad Q \in \mathrm{O}(d),$$

where $\mathrm{O}(d)$ denotes the set of orthogonal $d \times d$ matrices. We need two operators. The *rotation operator* $\mathcal{R}_d$ associates to $F \colon [0, \infty) \to \mathbb{R}$ the radial function $\mathcal{R}_d F \colon \mathbb{R}^d \to \mathbb{R}$ given by

$$\mathcal{R}_d F \coloneqq F \circ \|\cdot\|. \tag{5}$$

Since every function $F \colon [0, \infty) \to \mathbb{R}$ can be identified with its *even* continuation $F : \mathbb{R} \to \mathbb{R}$, we define $\mathcal{R}_d$ alternatively for all even functions on $\mathbb{R}$. Every radial function is of the form (5). The *spherical averaging operator* $\mathcal{A}_d$ assigns to a function $\Phi \colon \mathbb{R}^d \to \mathbb{R}$, which is integrable on every sphere $r\mathbb{S}^{d-1}$, $r > 0$, the function $\mathcal{A}_d \Phi \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{A}_d \Phi(r) \coloneqq \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \Phi(r\,\xi) \,\mathrm{d}\xi = \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}} [\Phi(r\,\xi)] \quad \text{for all} \quad r \in \mathbb{R}. \tag{6}$$

Note that as soon as Φ is continuous on $\mathbb{R}^d \setminus \{0\}$ or Φ is a radial, the function $\mathcal{A}_d \Phi$ is well-defined. By definition $\mathcal{A}_d \Phi$ is an even function, i.e., $\mathcal{A}_d \Phi(r) = \mathcal{A}_d \Phi(-r)$, $r \in \mathbb{R}$ and we have for $d = 1$ that

$$\mathcal{A}_1 \Phi(r) = \tfrac{1}{2} \left(\Phi(r) + \Phi(-r) \right) \quad \text{for all} \quad r \in \mathbb{R}.$$

Moreover, we obtain by definition that

$$\mathcal{A}_d(\Phi \circ Q) = \mathcal{A}_d \Phi \quad \text{for all} \quad Q \in \mathrm{O}(d).$$

The operator $\mathcal{A}_d$ is the inverse of $\mathcal{R}_d$, meaning that for every even function $F \colon \mathbb{R} \to \mathbb{R}$ and $r \geqslant 0$ it holds

$$(\mathcal{A}_d \circ \mathcal{R}_d) F(r) = \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}} [F(\|r\xi\|)] = F(|r|) = F(r), \tag{7}$$

and conversely for every radial function $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$ we have

$$(\mathcal{R}_d \circ \mathcal{A}_d)\Phi(x) = (\mathcal{A}_d\Phi)(\|x\|) = \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}\left[\Phi(\|x\|\xi)\right] = \Phi(x). \tag{8}$$

The following theorem considers special radial functions of the form (3).

Theorem 2.1. *Let $d \in \mathbb{N}$, $d \geq 3$ and let $f \in L^1_{\mathrm{loc}}([0,\infty))$. Then the function $F\colon [0,\infty) \to \mathbb{R}$ fulfilling the slicing relation*

$$\mathcal{R}_d F = \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}\left[f\left(|\langle \cdot, \xi\rangle|\right)\right] \tag{9}$$

is determined by the Abel-type integral

$$F(s) = c_d \int_0^1 f(ts)(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t = c_d\,\frac{1}{s}\int_0^s f(t)\Big(1 - \frac{t^2}{s^2}\Big)^{\frac{d-3}{2}}\,\mathrm{d}t, \tag{10}$$

where $c_d \coloneqq \frac{2\omega_{d-2}}{\omega_{d-1}}$ and ω_{d-1} denotes the surface measure of $\mathbb{S}^{d-1}$.

The theorem was proved in a more general form for projections on subspaces in [38, Lem. 2.1] and for the above special case of a projection onto a line also in [21]. For convenience, we add the short proof in Appendix A.

The following theorem, whose proof is given in Appendix B, clarifies smoothness properties of the function F in the Abel-like integral.

Theorem 2.2. *For $d \in \mathbb{N}$ with $d \geq 3$, let $f \in L^1_{\mathrm{loc}}([0,\infty))$ for odd d and $f \in L^p_{\mathrm{loc}}([0,\infty))$ with $p > 2$ for even d. Then the function F defined by (10) is $\lfloor (d-2)/2 \rfloor$-times continuously differentiable on $(0,\infty)$. Moreover, if d is odd, then the $\lfloor (d-2)/2 \rfloor$-th derivative of F is absolutely continuous.*

In the rest of this paper, we are interested in characterizing f from given F, i.e., the inversion of the Abel-like integral transform (10), where we want to use the Fourier analytic tools.

3. Slicing of L^1 Functions

We start with functions F that are Fourier transforms of absolutely integrable functions. Let $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$, be the Banach space of p-integrable functions. The *Fourier transform* $\mathcal{F}_d\colon L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$ is an injective, linear operator defined for $\Phi \in L^1(\mathbb{R}^d)$ by

$$\hat{\Phi} = \mathcal{F}_d[\Phi] \coloneqq \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}\langle x,\cdot\rangle}\,\Phi(x)\,\mathrm{d}x. \tag{11}$$

If $\hat{\Phi} \in L^1(\mathbb{R}^d)$, then the inverse Fourier transform reads as

$$\Phi = \mathcal{F}_d^{-1}[\hat{\Phi}] \coloneqq \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i}\langle \cdot, v\rangle} \hat{\Phi}(v) \, \mathrm{d}v.$$

On even functions, and in particular radial functions, the Fourier transform coincides with its inverse. Moreover, the Fourier transform of a real-valued, radial function is real-valued again, and we have $\widehat{\Phi \circ Q} = \hat{\Phi} \circ Q$ for all $Q \in \mathrm{O}(d)$. For $\rho \colon \mathbb{R} \to \mathbb{R}$, we define the *multiplication operator* by

$$\mathcal{M}_d \rho(r) \coloneqq \rho(|r|)|r|^{d-1} \quad \text{for all} \quad r \in \mathbb{R}.$$

By definition, $\mathcal{M}_d \rho$ is an even function. We obtain the following inversion result.

Proposition 3.1. *Let $d \geq 3$. Assume that $F \colon [0, \infty) \to \mathbb{R}$ fulfills $\mathcal{R}_d F = \mathcal{F}_d[\mathcal{R}_d \rho]$ for some function $\rho \colon [0, \infty) \to \mathbb{R}$ with $\mathcal{R}_d \rho \in L^1(\mathbb{R}^d)$. Then the function $f \colon [0, \infty) \to \mathbb{R}$ given by the corresponding even function*

$$f = \tfrac{\omega_{d-1}}{2} (\mathcal{F}_1 \circ \mathcal{M}_d)[\rho] \in \mathcal{C}_0(\mathbb{R}), \tag{12}$$

fulfills (9), where ρ is also considered evenly extended here. If in addition $\mathcal{R}_d F \in L^1(\mathbb{R}^d)$, then

$$f = \tfrac{\omega_{d-1}}{2} (\mathcal{F}_1 \circ \mathcal{M}_d \circ \mathcal{A}_d \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)[F]. \tag{13}$$

Proof. Using that $v = \|v\|\xi$, where $\xi \in \mathbb{S}^{d-1}$, we obtain by assumption

$$\begin{aligned}
\mathcal{R}_d F(x) &= \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\langle x, v\rangle} \rho(\|v\|) \, \mathrm{d}v \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi\rangle r} \rho(r) r^{d-1} \, \mathrm{d}r \, \mathrm{d}\xi \\
&= \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi\rangle r} \rho(r)|r|^{d-1} \, \mathrm{d}r \, \mathrm{d}\xi \\
&= \frac{1}{2} \int_{\mathbb{S}^{d-1}} \mathcal{F}_1[\mathcal{M}_d \rho](|\langle x, \xi\rangle|) \, \mathrm{d}\xi \\
&= \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}} \left[\tfrac{\omega_{d-1}}{2} (\mathcal{F}_1 \circ \mathcal{M}_d)[\rho](|\langle x, \xi\rangle|) \right].
\end{aligned}$$

On the other hand, we have by Theorem 2.1 that f with (10) fulfills

$$\mathcal{R}_d F(x) = \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}} \left[f\left(|\langle x, \xi\rangle|\right)\right].$$

This implies that $f = \frac{\omega_{d-1}}{2}(\mathcal{F}_1 \circ \mathcal{M}_d)[\rho]$ fulfills (9). Since $\mathcal{R}_d \rho \in L^1(\mathbb{R}^d)$, we know that $\mathcal{M}_d \rho \in L^1(\mathbb{R})$ and hence its Fourier transform is continuous, so that $f \in \mathcal{C}_0(\mathbb{R})$ is even.

If in addition $\mathcal{R}_d F \in L^1(\mathbb{R}^d)$, then $\mathcal{R}_d \rho = \mathcal{F}_d^{-1}[\mathcal{R}_d F]$ and by (7) further $\rho = (\mathcal{A}_d \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)F$. Plugging this into (12), we obtain the second assertion. □

Let us note that another characterization of Fourier transforms of radial L^1 functions is given by the following remark, see, e.g. [16].

Remark 3.2. *The Fourier transform of a radial function $\mathcal{R}_d\rho \in L^1(\mathbb{R}^d)$, $d \geq 2$ is also a radial function and can be written as*

$$\mathcal{F}_d\big[\mathcal{R}_d\rho\big](x) = \|x\|^{1-d/2} \int_0^\infty \rho(r) r^{d/2} J_{d/2-1}(2\pi r\|x\|)\,\mathrm{d}r,$$

where $J_{d/2-1}$ denotes the Bessel function of first kind of order $d/2 - 1$.

Combining Theorem 2.2 and Proposition 3.1 gives the following corollary.

Corollary 3.3. *The Fourier transform of any radial function from $L^1(\mathbb{R}^d)$ is $\lfloor (d-2)/2 \rfloor$ times continuously differentiable on $\mathbb{R}^d \setminus \{0\}$.*

Proof. Let $\Phi = \mathcal{R}_d\rho \in L^1(\mathbb{R}^d)$ for some $\rho\colon [0,\infty) \to \mathbb{R}$. Then the Fourier transform $\mathcal{F}_d[\Phi]$ exists and is radial. Therefore, a function $F\colon [0,\infty) \to \mathbb{R}$ exists with $\mathcal{R}_dF = \mathcal{F}_d[\mathcal{R}_d\rho] = \mathcal{F}_d[\Phi]$. By Proposition 3.1, it follows that f and F satisfy (10) as well as $f \in \mathcal{C}(\mathbb{R})$. By Theorem 2.2 the function F is $\lfloor (d-2)/2 \rfloor$ times continuously differentiable on $(0,\infty)$. The smoothness of the Euclidean norm on $\mathbb{R}^d \setminus \{0\}$ yields the assertion. □

The "dimension walk" between Fourier transforms of radial functions in different dimensions was discussed, e.g. in [11, 17]. Various examples of sliced transform pairs (F, f) were given in [21, 38]. Here are two interesting ones.

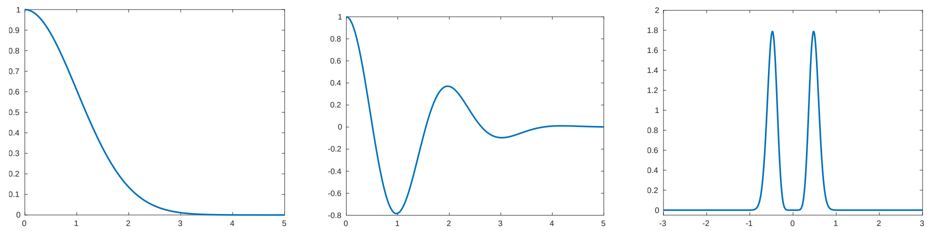
Example 3.4. *i) If $F(x) = \exp\big(-\frac{x^2}{2}\big)$ is a Gaussian, then $\rho(x) = (2\pi)^{d/2} \exp(-2\pi^2 x^2)$ in Theorem 3.1 is a Gaussian as well and we obtain $f = \mathcal{F}_1[\mathcal{M}_d\rho] = \mathcal{F}_1[\rho(|\cdot|)|\cdot|^{d-1}]$ on $\mathbb{R}$ or conversely $\mathcal{F}_1 f = \mathcal{F}_1^{-1} f = \mathcal{M}_d\rho$. The function f is the confluent hypergeometric distribution function, i.e.,*

$$F(x) = \exp\left(-\frac{x^2}{2}\right) \quad\Longleftrightarrow\quad f(x) = {}_1F_1\left(\frac{d}{2}, \frac{1}{2}, -\frac{x^2}{2}\right).$$

The graphs of these functions are depicted for dimension $d = 10$ in Fig. 2. The function $\mathcal{R}_dF$ is positive definite in every dimension $d \in \mathbb{N}$, and the function f is positive definite in one dimension by Bochner's theorem 4.10. However, f shows oscillations, which increase with the dimension. Its Fourier transform $\mathcal{F}_1 f$ has only two modes, which get separated more far from each other with an increasing dimension, but keep their shapes.

ii) For Riesz kernels, both F and f have the same structure, more precisely, for $r > -1$, we have

$$F(x) = x^r \quad\Longleftrightarrow\quad f(x) = \frac{\sqrt{\pi}\,\Gamma(\frac{d+r}{2})}{\Gamma(\frac{d}{2})\,\Gamma(\frac{r+1}{2})}\, x^r. \tag{14}$$



F $\qquad$ f $\qquad$ $\mathcal{F}_1 f$

Figure 2: Abel-type transform of the confluent hypergeometric distribution f results in the Gaussian F $(d = 10)$.

4. Slicing of Tempered Distributions

In this section, we extend our considerations to radial tempered distributions, where we partially build up on results in [17]. This allows to show a generalization of Theorem 3.1.

4.1. Radial Schwartz Functions

For a smooth function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and an integer $m \in \mathbb{N}$, define

$$\|\varphi\|_m \coloneqq \max_{\beta \in \mathbb{N}^d, |\beta| \leqslant m} \|(1 + \|\cdot\|)^m D^\beta \varphi\|_\infty. \tag{15}$$

The space of *Schwartz functions* $\mathcal{S}(\mathbb{R}^d)$ consists of all smooth functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_m < \infty$ for all $m \in \mathbb{N}$. A sequence $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ converges to $\varphi \in \mathcal{S}(\mathbb{R}^d)$ if

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_m = 0 \quad \text{for all} \quad m \in \mathbb{N}.$$

The Fourier transform (11) is a linear, bijective and continuous map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. A Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is called *radial* if $\varphi = \varphi \circ Q$ for all $Q \in \mathrm{O}(d)$. The *space of radial Schwartz functions* is denoted by

$$\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \coloneqq \{\varphi \in \mathcal{S}(\mathbb{R}^d) : \varphi \circ Q = \varphi \text{ for all } Q \in \mathrm{O}(d)\}.$$

In particular, for $d = 1$, we obtain the even Schwartz functions $\mathcal{S}_{\mathrm{rad}}(\mathbb{R})$.

The following theorem shows that the rotation and the averaging operator are well-defined on radial Schwartz functions. We will use ψ to denote one-dimensional Schwartz functions and φ to address d-dimensional Schwartz functions.

Theorem 4.1. *i) The rotation operator $\mathcal{R}_d \colon \mathcal{S}_{\mathrm{rad}}(\mathbb{R}) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ given by (5) is linear and continuous. In particular, there exist constants $b_m > 0$ such that $\|\mathcal{R}_d \psi\|_m \leqslant b_m \|\psi\|_{4m}$ for all $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ and all $m \in \mathbb{N}$.*

ii) The averaging operator $\mathcal{A}_d\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ *given by* (6) *is well-defined and continuous with* $\|\mathcal{A}_d\varphi\|_m \leqslant d^m \|\varphi\|_m$ *for all* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and all* $m \in \mathbb{N}$.

A rough sketch of the proof can be found in [17]. We give a rigorous proof in Appendix C. The following corollary is a direct consequence of the above theorem and the relations (7) and (8). It shows that $\mathcal{A}_d$ restricted to $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ is bijective.

Corollary 4.2. *i) The concatenated operator* $\mathcal{R}_d \circ \mathcal{A}_d\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ *is a continuous projection onto* $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, *i.e., it is surjective and* $(\mathcal{R}_d \circ \mathcal{A}_d)^2 = \mathcal{R}_d \circ \mathcal{A}_d$.

ii) The operator $\mathcal{A}_d\colon \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ *is a homeomorphism, i.e., it is bijective and continuous with continuous inverse,*

$$\mathcal{A}_d \circ \mathcal{R}_d = \mathrm{Id}_{\mathcal{S}_{\mathrm{rad}}(\mathbb{R})} \quad \textit{and} \quad \mathcal{R}_d \circ \mathcal{A}_d = \mathrm{Id}_{\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)}.$$

4.2. Radial Tempered Distributions

The space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^d)$ consists of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. A sequence $T_k \in \mathcal{S}'(\mathbb{R}^d)$ converges to $T \in \mathcal{S}'(\mathbb{R}^d)$ if $\lim_{k\to\infty}\langle T_k, \varphi\rangle = \langle T, \varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The Fourier transform $\mathcal{F}_d\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is the linear, continuous operator defined for a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \mathcal{F}_d T, \varphi\rangle \coloneqq \langle T, \mathcal{F}_d \varphi\rangle \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

A distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ is called *radial* if

$$\langle T, \varphi \circ Q\rangle = \langle T, \varphi\rangle \quad \text{for all} \quad Q \in \mathrm{O}(d),\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The *space of all radial tempered distributions* is denoted by

$$\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d) \coloneqq \{T \in \mathcal{S}'(\mathbb{R}^d) : \langle T, \varphi \circ Q\rangle = \langle T, \varphi\rangle \text{ for all } Q \in \mathrm{O}(d) \text{ and all } \varphi \in \mathcal{S}(\mathbb{R}^d)\}.$$

Since we obtain for $T \in \mathcal{S}'(\mathbb{R}^d)$ and every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\langle \mathcal{F}_d T, \varphi \circ Q\rangle = \langle T, \mathcal{F}_d(\varphi \circ Q)\rangle = \langle T, (\mathcal{F}_d\varphi) \circ Q\rangle = \langle T, \mathcal{F}_d\varphi\rangle = \langle \mathcal{F}_d T, \varphi\rangle,$$

we conclude that the Fourier transform of a radial tempered distribution is again radial.

The proof of the following lemma can be found in [17, Prop. 3.2]. Based on Corollary 4.2 i), the operator $\mathcal{R}_d \circ \mathcal{A}_d$ maps $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, so that the lemma finally states that radial distributions can be determined by testing just with radial Schwartz functions.

Lemma 4.3. *For* $T \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)$ *and every* $\varphi \in \mathcal{S}(\mathbb{R}^d)$, *it holds* $\langle T, (\mathcal{R}_d \circ \mathcal{A}_d)\varphi\rangle = \langle T, \varphi\rangle$. *In particular, a radial distribution is uniquely determined by its application to radial Schwartz functions.*

Next, we define the operator $\mathcal{R}_d^\star\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R})$ by

$$\langle \mathcal{R}_d^\star T, \psi\rangle \coloneqq \langle T, (\mathcal{R}_d \circ \mathcal{A}_1)\psi\rangle \quad \text{for all} \quad \psi \in \mathcal{S}(\mathbb{R}).$$

Indeed, $\mathcal{R}_d^\star T \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R})$ for every $T \in \mathcal{S}'(\mathbb{R}^d)$, since we have for every $\psi \in \mathcal{S}(\mathbb{R})$ that

$$\langle \mathcal{R}_d^\star T, \psi(-\cdot)\rangle = \langle T, \mathcal{R}_d\left(\mathcal{A}_1\psi(-\cdot)\right)\rangle = \langle T, \mathcal{R}_d\left(\mathcal{A}_1\psi\right)\rangle = \langle \mathcal{R}_d^\star T, \psi\rangle.$$

Further, we introduce the operator $\mathcal{A}_d^*\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)$ by

$$\langle \mathcal{A}_d^*\tau, \varphi\rangle \coloneqq \langle \tau, \mathcal{A}_d\varphi\rangle \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

Clearly, $\mathcal{A}_d^*\tau \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)$ for every $\tau \in \mathcal{S}'(\mathbb{R})$, since we get for every $Q \in \mathrm{O}(d)$ and every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\langle \mathcal{A}_d^*\tau, \varphi \circ Q\rangle = \langle \tau, \mathcal{A}_d(\varphi \circ Q)\rangle = \langle \tau, \mathcal{A}_d\varphi\rangle = \langle \mathcal{A}_d^*\tau, \varphi\rangle.$$

Note that $\mathcal{A}_d^*$ is the adjoint operator of $\mathcal{A}_d\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$, while $\mathcal{R}_d^\star$ is the radial extension of the adjoint of $\mathcal{R}_d\colon \mathcal{S}_{\mathrm{rad}}(\mathbb{R}) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R})$. In the following, we will use τ to denote one-dimensional distributions and T for d-dimensional distributions. The next proposition shows that $\mathcal{R}_d^\star$ and $\mathcal{A}_d^*$ become bijective when restricted to radial distributions.

Lemma 4.4. *The restrictions $\mathcal{R}_d^\star\colon \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R})$ and $\mathcal{A}_d^*\colon \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}) \to \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)$ are bijective and inverse to each other, i.e.,*

$$\mathcal{R}_d^\star \circ \mathcal{A}_d^* = \mathrm{Id}_{\mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)} \quad \textit{and} \quad \mathcal{A}_d^* \circ \mathcal{R}_d^\star = \mathrm{Id}_{\mathcal{S}'_{\mathrm{rad}}(\mathbb{R})}.$$

Proof. Let $T \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)$. For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we see that $\mathcal{A}_d\varphi$ is even and obtain by Lemma 4.3 that

$$\langle (\mathcal{A}_d^* \circ \mathcal{R}_d^\star)T, \varphi\rangle = \langle T, (\mathcal{R}_d \circ \mathcal{A}_1 \circ \mathcal{A}_d)\varphi\rangle = \langle T, (\mathcal{R}_d \circ \mathcal{A}_d)\varphi\rangle = \langle T, \varphi\rangle.$$

Let $\tau \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R})$. Then, it follows for all $\psi \in \mathcal{S}(\mathbb{R})$ by Corollary 4.2 ii) that

$$\langle (\mathcal{R}_d^\star \circ \mathcal{A}_d^*)\tau, \psi\rangle = \langle \tau, (\mathcal{A}_d \circ \mathcal{R}_d \circ \mathcal{A}_1)\psi\rangle = \langle \tau, (\mathcal{A}_d \circ \mathcal{R}_d)\left(\mathcal{A}_1\psi\right)\rangle = \langle \tau, \psi\rangle. \qquad \square$$

4.3. Slicing of Radial Regular Tempered Distributions

In the following, we are mainly interested in tempered distributions of function type, where we skip the word "tempered" in the following. A distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ is called *regular* if it is generated by a function $T \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ via

$$\langle T, \varphi\rangle \coloneqq \int_{\mathbb{R}^d} T(x)\varphi(x)\,\mathrm{d}x \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Regular distributions were characterized in [48]. Clearly, radial regular distributions arise from radial functions. A function $T \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is *slowly increasing* if there exist $c, k, R > 1$ such that

$$|T(x)| \le c\|x\|^k \quad \text{for all} \quad \|x\| > R, \tag{16}$$

see, e.g., [50]. The notion of slow increase is sometimes defined slightly different in the literature, requiring (16) to hold on $\mathbb{R}^d$. Every slowly increasing function generates a regular distribution, but the converse does not hold true. We are interested in even functions F such that $\mathcal{R}_d F$ is a regular distribution on $\mathbb{R}^d$. Note that this is a weaker assumption than saying that F itself is an even regular distribution on $\mathbb{R}$. Then, we can associate to F a distribution

$$f \coloneqq (\mathcal{F}_1 \circ \mathcal{R}_d^\star \circ \mathcal{F}_d^{-1})[\mathcal{R}_d F] = (\mathcal{F}_1 \circ \mathcal{R}_d^\star \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)F. \tag{17}$$

Note that $\mathcal{F}_d^{-1}[\mathcal{R}_d F]$ is in general not a function again. However, by the following proposition, we will see that (17) coincides with (13) if this is the case.

Proposition 4.5. *i) Let $T \in \mathcal{S}'_{\mathrm{rad}}(\mathbb{R}^d)$ be a radial regular tempered distribution. Then*

$$\langle \mathcal{R}_d^\star T, \psi\rangle = \tfrac{\omega_{d-1}}{2}\langle (\mathcal{M}_d \circ \mathcal{A}_d)T, \psi\rangle \quad \text{for all} \quad \psi \in \mathcal{S}(\mathbb{R}).$$

ii) For a function $F\colon [0,\infty) \to \mathbb{R}$, let $\mathcal{R}_d F$ as well as $\mathcal{F}_d^{-1}[\mathcal{R}_d F]$ be regular distributions. Then f in (17) has the form

$$f = \tfrac{\omega_{d-1}}{2}(\mathcal{F}_1 \circ \mathcal{M}_d \circ \mathcal{A}_d \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)F.$$

Note that the factor $\omega_{d-1}/2$ is hidden in $\mathcal{R}_d^\star$.

Proof. i) Firstly, if T is a radial regular distribution, then the same holds true for $(\mathcal{M}_d \circ \mathcal{A}_d)T$. We set $\rho \coloneqq \mathcal{A}_d T$ which is by definition an even function. Since T is radial, we have by (8) that $T = \mathcal{R}_d \rho$. Further, since $\mathcal{R}_d^\star T$ as well as $\mathcal{M}_d \rho$ is even, it suffices by Lemma 4.3 to reduce ourselves to $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$. Then $\mathcal{A}_1 \psi = \psi$ and we obtain

$$\begin{aligned}
\langle \mathcal{R}_d^\star T, \psi\rangle = \langle T, (\mathcal{R}_d \circ \mathcal{A}_1)\psi\rangle &= \int_{\mathbb{R}^d} \rho(\|x\|)\psi(\|x\|)\,\mathrm{d}x = \omega_{d-1}\int_0^\infty \rho(r) r^{d-1}\psi(r)\,\mathrm{d}r \\
&= \frac{\omega_{d-1}}{2}\int_{\mathbb{R}} \rho(r)|r|^{d-1}\psi(r)\,\mathrm{d}r = \frac{\omega_{d-1}}{2}\int_{\mathbb{R}} \mathcal{M}_d\rho(r)\psi(r)\,\mathrm{d}r \\
&= \frac{\omega_{d-1}}{2}\langle \mathcal{M}_d \rho, \psi\rangle = \frac{\omega_{d-1}}{2}\langle (\mathcal{M}_d \circ \mathcal{A}_d)T, \psi\rangle.
\end{aligned}$$

ii) This part follows directly from (17) and part i) applied to $T = \mathcal{F}_d^{-1}[\mathcal{R}_d F]$. □

Under the assumptions of Proposition 3.1, we can apply part ii) of Proposition (4.5), therefore we can consider (17) as a generalization of (13) to regular distributions. The following theorem establishes conditions such that the pairs of functions (F, f) fulfill the slicing property (9). Indeed, it includes functions F for which the Fourier transform of $\mathcal{R}_d F$ is not regular, as the already mentioned function $F(x) = x^r$, $r > -1$, from Example 3.4 or positive definite functions $\mathcal{R}_d F$ having just a positive measure as Fourier transform. To prove the theorem, we need the following lemma, which can be shown following the lines of [50, Thm. 5.20].

Lemma 4.6. *Let $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be slowly increasing. If Φ is continuous in $z \in \mathbb{R}^d$, then*

$$\Phi(z) = \lim_{m\to\infty} \langle \Phi, \varphi_{d,m,z} \rangle, \tag{18}$$

where $\varphi_{d,m,z}(x) \coloneqq (m/\pi)^{d/2} \mathrm{e}^{-m\|x-z\|^2}$.

Theorem 4.7. *Let $F\colon [0,\infty) \to \mathbb{R}$ such that both $\mathcal{R}_d F$ and*

$$f \coloneqq (\mathcal{F}_1 \circ \mathcal{R}_d^\star \circ \mathcal{F}_d^{-1})[\mathcal{R}_d F]$$

are regular, slowly increasing functions. If $f \in \mathcal{C}(\mathbb{R})$ and F is continuous in $\|z\|$, then

$$F(\|z\|) = \mathbb{E}_{\xi\sim\mathcal{U}_{\mathbb{S}^{d-1}}} f(|\langle z, \xi\rangle|). \tag{19}$$

Proof. For $m \in \mathbb{N}$, we use the above Schwartz function $\varphi_{d,m,z}$, which has the Fourier transform $\hat\varphi_{d,m,z}(v) = \mathrm{e}^{-2\pi\mathrm{i}\langle z,v\rangle}\mathrm{e}^{-\pi^2\|v\|^2/m}$. Since $\mathcal{R}_d F$ is continuous in $\|z\|$ and slowly increasing, we obtain by Lemma 4.6 that

$$F(\|z\|) = \lim_{m\to\infty} \langle \mathcal{R}_d F, \varphi_{d,m,z}\rangle = \lim_{m\to\infty} \langle (\mathcal{F}_d^{-1}[\mathcal{R}_d F], \hat\varphi_{d,m,z}\rangle.$$

By Lemma 4.4, we realize that

$$(\mathcal{A}_d^* \circ \mathcal{F}_1^{-1}) f = (\mathcal{A}_d^* \circ \mathcal{F}_1^{-1} \circ (\mathcal{F}_1 \circ \mathcal{R}_d^\star \circ \mathcal{F}_d^{-1})[\mathcal{R}_d F] = (\mathcal{A}_d^* \circ \mathcal{R}_d^\star) \circ \mathcal{F}_d^{-1}[\mathcal{R}_d F] = \mathcal{F}_d^{-1}[\mathcal{R}_d F],$$

so that

$$F(\|z\|) = \lim_{m\to\infty} \langle (\mathcal{A}_d^* \circ \mathcal{F}_1^{-1}) f, \hat\varphi_{d,m,z}\rangle = \lim_{m\to\infty} \mathbb{E}_{\xi\sim\mathcal{U}_{\mathbb{S}^{d-1}}} \left[\langle f, \mathcal{F}_1^{-1}[(\hat\varphi_{d,m,z})_\xi]\rangle\right],$$

where for any $\xi \in \mathbb{S}^{d-1}$ and $r \in \mathbb{R}$,

$$(\hat\varphi_{d,m,z})_\xi(r) \coloneqq \hat\varphi_{d,m,z}(r\xi) = \mathrm{e}^{-2\pi\mathrm{i}\langle r\xi, z\rangle}\,\mathrm{e}^{-\pi^2|r|^2/m} = \hat\varphi_{1,m,\langle\xi,z\rangle}(r).$$

It follows that

$$F(\|z\|) = \lim_{m\to\infty} \left[\mathbb{E}_{\xi\sim\mathcal{U}_{\mathbb{S}^{d-1}}} \langle f, \varphi_{1,m,\langle z,\xi\rangle}\rangle\right]. \tag{20}$$

Since f is continuous slowly increasing and even, we obtain again by (18) that

$$\lim_{m\to\infty} \langle f, \varphi_{1,m,\langle z,\xi\rangle}\rangle = f(\langle z,\xi\rangle) = f(|\langle z,\xi\rangle|)$$

It remains to show that we can interchange the limit and integration in (20). Since f is slowly increasing, there exist $k \in \mathbb{N}$, $c > 0$ and $R > 0$ such that $|f(r)| \leqslant c|r|^k$ for all $r \geqslant R$. We choose R large enough so that $r \leqslant 2r^2 - k$ for all $r \geqslant R$ and $|f(r)| \leqslant c|r|^k$ for all $r \geqslant R - \|z\|$.
For any $r, s \in \mathbb{R}$, the convexity of $|\cdot|^k$ implies that

$$|r+s|^k = |\tfrac{1}{2}(2r) + \tfrac{1}{2}(2s)|^k \leqslant \tfrac{1}{2}|2r|^k + \tfrac{1}{2}|2s|^k = 2^{k-1}(|r|^k + |s|^k). \tag{21}$$

Setting $\varphi_{d,m} \coloneqq \varphi_{d,m,0}$, we split up the integral

$$\begin{aligned} |\langle f, \varphi_{1,m,s}\rangle| = |\langle f(\cdot + s), \varphi_{1,m}\rangle| &\leqslant \int_{\mathbb{R}} |f(r+s)\varphi_{1,m}(r)| \,\mathrm{d}r \\ &= \int_{|r|\leqslant R} |f(r+s)|\varphi_{1,m}(r)\,\mathrm{d}r + \int_{|r|>R} |f(r+s)|\varphi_{1,m}(r)\,\mathrm{d}r. \end{aligned}$$

Using that $\varphi_{1,m}$ is positive and its integral is one, we estimate the first part

$$\int_{|r|\leqslant R} |f(r+s)|\varphi_{1,m}(r)\,\mathrm{d}r \leqslant \max_{|r|\leqslant R} |f(r+s)| \int_{|r|\leqslant R} \varphi_{1,m}(r)\,\mathrm{d}r \leqslant \max_{|r|\leqslant R} |f(r+s)|.$$

For the second part, we have by (21) for all $|s| \leqslant \|z\|$ that

$$\begin{aligned} \int_{|r|>R} |f(r+s)|\varphi_{1,m}(r)\,\mathrm{d}r &\leqslant c2^{k-1} \int_{|r|>R} (|r|^k + |s|^k)\varphi_{1,m}(r)\,\mathrm{d}r \\ &\leqslant c2^{k-1}\left(|s|^k + 2\int_R^\infty r^k \varphi_{1,m}(r)\,\mathrm{d}r\right) \\ &\leqslant c2^{k-1}\left(|s|^k + 2(m/\pi)^{1/2}\int_R^\infty (2mr^{k+1} - kr^{k-1})\mathrm{e}^{-mr^2}\,\mathrm{d}r\right) \\ &= c2^{k-1}\left(|s|^k + 2(m/\pi)^{1/2}\left[-r^k\mathrm{e}^{-mr^2}\right]_R^\infty\right) \\ &= c2^{k-1}\left(|s|^k + 2R^k(m/\pi)^{1/2}\mathrm{e}^{-mR^2}\right). \end{aligned}$$

Due to the growth of the exponential, we can find m_0 such that $2R^k(m/\pi)^{1/2}\mathrm{e}^{-mR^2} \leqslant 1$ for all $m \geqslant m_0$. Now let $s = \langle z, \xi\rangle$. Then we have $|s| \leqslant \|z\|$ and for $m \geqslant m_0$ it follows

$$\begin{aligned} |\langle f, \varphi_{1,m,s}\rangle| &\leqslant \max_{|r|\leqslant R} |f(r+s)| + c2^{k-1}|s|^m + 2R^k\varphi_{1,m}(R) \\ &\leqslant \max_{|t|\leqslant R+\|z\|} |f(t)| + c2^{k-1}(\|z\|^m + 1). \end{aligned}$$

This bound is independent of ξ and m, therefore we can apply Lebesgue's dominated convergence theorem to (20) and finally obtain (19). □

The conditions in Theorem 4.7 on $\mathcal{R}_d F$ are fulfilled if F is continuous on $(0, \infty)$, slowly increasing, and $F(r)r^{d-1}$ is bounded for $r \searrow 0$.

The following two examples show applications of Theorem 4.7.

Example 4.8. *For the Riesz kernel $F(x) = |x|$ from Example 3.4, the Fourier transform of $\mathcal{R}_d F$ does not exist in the classical sense, but as a tempered distribution. We utilize $\varphi_{d,m} \coloneqq \varphi_{d,m,0}$ from Lemma 4.6. Set $f \coloneqq (\mathcal{F}_1 \circ \mathcal{R}_d^\star \circ \mathcal{F}_d^{-1})[\mathcal{R}_d F]$, see (17). Let $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$. Since $\mathcal{R}_d\hat{\psi} - \hat{\psi}(0)\hat{\varphi}_{d,m} \in \mathcal{S}(\mathbb{R}^d)$ as well as all its first order derivatives vanish at 0, Wendland [50, Thm. 8.16] yields*

$$\begin{aligned}\langle f, \psi\rangle &= \langle \mathcal{F}_d^{-1}[\mathcal{R}_d F], \mathcal{R}_d\hat{\psi}\rangle = \langle \mathcal{F}_d^{-1}[\mathcal{R}_d F], \mathcal{R}_d\hat{\psi} - \hat{\psi}(0)\hat{\varphi}_{d,m}\rangle + \hat{\psi}(0)\langle \mathcal{F}_d^{-1}[\mathcal{R}_d F], \hat{\varphi}_{d,m}\rangle \\ &= \frac{-2}{\pi\omega_d}\int_{\mathbb{R}^d} \frac{\hat{\psi}(\|x\|) - \hat{\psi}(0)\hat{\varphi}_{d,m}(x)}{\|x\|^{d+1}}\,\mathrm{d}x + \hat{\psi}(0)\int_{\mathbb{R}^d}\|x\|\varphi_{d,m}(x)\,\mathrm{d}x. \qquad (22)\end{aligned}$$

The limit for $m \to \infty$ of the last term vanishes when we apply Lemma 4.6 to $\mathcal{R}_d F = \|\cdot\|$ and $z = 0$, i.e.,

$$\lim_{m\to\infty}\int_{\mathbb{R}^d}\|x\|\varphi_{d,m}(x)\,\mathrm{d}x = 0.$$

Since $\hat{\varphi}_{d,m}(x) = \mathrm{e}^{-\pi^2\|x\|^2/m} = \hat{\varphi}_{1,m}(\|x\|)$, we obtain for the first term

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\mathcal{R}_d\hat{\psi}(x) - \hat{\psi}(0)\hat{\varphi}_{d,m}(x)}{\|x\|^{d+1}}\,\mathrm{d}x &= \int_{\mathbb{R}^d} \frac{\hat{\psi}(\|x\|) - \hat{\psi}(0)\hat{\varphi}_{1,m}(\|x\|)}{\|x\|^{d+1}}\,\mathrm{d}x \\ = \omega_{d-1}\int_0^\infty \frac{\hat{\psi}(r) - \hat{\psi}(0)\hat{\varphi}_{1,m}(r)}{r^{d+1}} r^{d-1}\,\mathrm{d}r &= \frac{\omega_{d-1}}{2}\int_{\mathbb{R}} \frac{\hat{\psi}(r) - \hat{\psi}(0)\hat{\varphi}_{1,m}(r)}{r^2}\,\mathrm{d}r.\end{aligned}$$

Then we have

$$\langle f, \psi\rangle = \frac{-\omega_{d-1}}{\pi\omega_d}\lim_{m\to\infty}\int_{\mathbb{R}} \frac{\hat{\psi}(r) - \hat{\psi}(0)\hat{\varphi}_{1,m}(r)}{r^2}\,\mathrm{d}r.$$

Employing (22) backwards for dimension 1, we finally obtain

$$\langle f, \psi\rangle = \frac{\omega_{d-1}\omega_1}{2\omega_d}\langle \mathcal{F}_1\mathcal{R}_1^\star\mathcal{F}_1^{-1}[\mathcal{R}_1 F], \psi\rangle = \frac{\pi\omega_{d-1}}{\omega_d}\int_{\mathbb{R}}|x|\psi(x)\,\mathrm{d}x.$$

Therefore, we have $f(x) = \frac{\pi\omega_{d-1}}{\omega_d}|x|$ for $x \in \mathbb{R}$, which gives an alternative proof of (14). Both f and F are slowly increasing and continuous functions, so the assumptions of Theorem 4.7 are satisfied.

Example 4.9. *We consider the fundamental solution, also known as Green's function, of the Helmholtz operator, namely for some $k_0 > 0$*

$$F(r) = \frac{\mathrm{i}}{4}\left(\frac{k_0}{2\pi r}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(k_0 r), \quad \text{for all} \quad r \ge 0,$$

where $H_a^{(1)}$ is the Hankel function of the first kind and order a. We have

$$\langle \mathcal{F}_d[\mathcal{R}_d F], \varphi\rangle = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d} \frac{\varphi(x)}{4\pi^2\|x\|^2 - k_0^2 - \mathrm{i}\varepsilon}\,\mathrm{d}x,$$

see [25]. The significance of this example is that the forward simulation of a scattering problem can be done via the convolution with $\mathcal{R}_d F$, cf. [12]. By [25], $\mathcal{R}_d F$ is indeed a regular tempered distribution on $\mathbb{R}^d$. However, the asymptotic form $|H_a(r)| \sim \sqrt{2/(\pi r)}$ for $r \to \infty$, see [29, 10.2.5], shows that $\mathcal{R}_d F \notin L^1(\mathbb{R}^d)$, so we are not in the setting of Theorem 3.1. Let $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$. We obtain with transformation to polar coordinates that

$$\left\langle \mathcal{R}_d^\star \mathcal{F}_d^{-1}[\mathcal{R}_d F], \psi \right\rangle = \langle \mathcal{F}_d[\mathcal{R}_d F], \mathcal{R}_d \psi \rangle = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \frac{\psi(\|x\|)}{4\pi^2 \|x\|^2 - k_0^2 - \mathrm{i}\varepsilon} \,\mathrm{d}x$$
$$= \frac{\omega_{d-1}}{2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \frac{\psi(r)|r|^{d-1}}{4\pi^2 r^2 - k_0^2 - \mathrm{i}\varepsilon} \,\mathrm{d}r.$$

By Theorem 4.7, we have

$$\langle f, \psi \rangle = \left\langle \mathcal{F}_1 \mathcal{R}_d^\star \mathcal{F}_d^{-1}[\mathcal{R}_d F], \psi \right\rangle = \left\langle \mathcal{R}_d^\star \mathcal{F}_d^{-1} \mathcal{A}_d \mathcal{P}_d[F], \mathcal{F}_1[\psi] \right\rangle$$
$$= \frac{\omega_{d-1}}{2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|r|^{d-1}}{4\pi^2 r^2 - k_0^2 - \mathrm{i}\varepsilon} \psi(s) \mathrm{e}^{-2\pi \mathrm{i} r s} \,\mathrm{d}s \,\mathrm{d}r$$
$$= \omega_{d-1} \lim_{\varepsilon \searrow 0} \int_0^\infty \int_{\mathbb{R}} \frac{r^{d-1}}{4\pi^2 r^2 - k_0^2 - \mathrm{i}\varepsilon} \psi(s) \cos(2\pi r s) \,\mathrm{d}s \,\mathrm{d}r.$$

In particular, for $d = 2$ we have

$$\langle f, \psi \rangle = 2\pi \lim_{\varepsilon \searrow 0} \int_0^\infty \int_{\mathbb{R}} \frac{r}{4\pi^2 r^2 - k_0^2 - \mathrm{i}\varepsilon} \psi(s) \cos(2\pi r s) \,\mathrm{d}s \,\mathrm{d}r.$$

Noting that

$$\frac{r}{4\pi^2 r^2 - k_0^2 - \mathrm{i}\varepsilon} = \frac{-r}{k_0^2 + \mathrm{i}\varepsilon} {}_1F_0\big(1; -; -\tfrac{4\pi^2}{k_0^2 + \mathrm{i}\varepsilon} r^2\big),$$

where F is the hypergeometric function, we have by [10, 8.19(19)] that

$$\langle f, \psi \rangle = \frac{1}{4\sqrt{\pi}} \lim_{\varepsilon \searrow 0} \int_0^\infty \psi(s) G_{1,3}^{2,1}\left(-\tfrac{1}{4}(k_0^2 + \mathrm{i}\varepsilon) s^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix} \right) \mathrm{d}s,$$

where G denotes the Meijer-G function defined by

$$G_{p,q}^{m,n}\left(z \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) := \frac{1}{2\pi \mathrm{i}} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \,\mathrm{d}s,$$

where L is a certain loop in the complex plane,

see [14, Sect. 9.3]. Aside from its poles, the G function has a jump discontinuity along the positive real axis due to taking the main branch of z^s. We obtain

$$\langle f, \psi \rangle = \frac{1}{4\sqrt{\pi}} \lim_{\varepsilon \searrow 0} \int_0^\infty \psi(s) \overline{G_{1,3}^{2,1}\left(\tfrac{1}{4}(-k_0^2 + \mathrm{i}\varepsilon) s^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix} \right)} \mathrm{d}s,$$

and the integrand depends continuously on $\varepsilon \geq 0$. Assuming the limit and the integral can be interchanged, we obtain

$$f(s) = \frac{1}{4\sqrt{\pi}} \overline{G^{2,1}_{1,3}\left(-\tfrac{1}{4}k_0^2 s^2 \middle| \begin{matrix} 0 \\ 0, 0, \tfrac{1}{2} \end{matrix}\right)}.$$

Conversely, we can verify that f indeed fulfills (10) using from [34] the integral formula 2.24.2 and the relation 8.4.23.1, in particular for $s > 0$,

$$\begin{aligned}
F(s) &= \frac{2}{\pi} \int_0^1 f(ts)(1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t \\
&= \frac{1}{4\pi^{3/2}} \int_0^1 \overline{G^{2,1}_{1,3}\left(-\tfrac{1}{4}k_0^2 u s^2 \middle| \begin{matrix} 0 \\ 0, 0, \tfrac{1}{2} \end{matrix}\right)} (1-u)^{-\frac{1}{2}} u^{-\frac{1}{2}} \,\mathrm{d}u \\
&= \frac{1}{4\pi} \overline{G^{2,2}_{2,4}\left(-\tfrac{1}{4}k_0^2 s^2 \middle| \begin{matrix} \tfrac{1}{2}, 0 \\ 0, 0, \tfrac{1}{2}, 0 \end{matrix}\right)} = \frac{1}{4\pi} \overline{G^{2,0}_{0,2}\left(-\tfrac{1}{4}k_0^2 s^2 \middle| \begin{matrix} - \\ 0, 0 \end{matrix}\right)} \\
&= \frac{1}{2\pi} \overline{K_0(\mathrm{i}k_0 s)} = \frac{\mathrm{i}}{4} H_0^{(1)}(k_0 s),
\end{aligned}$$

where K_0 is the modified Bessel function of the second kind and the relation of the G functions with different orders follows directly from its definition.

4.4. Slicing of positive definite functions

By $\mathcal{M}(\mathbb{R}^d)$, we denote the set of finite Borel measures on $\mathbb{R}^d$ and by $\mathcal{M}_+(\mathbb{R}^d)$ the subset of positive measures. The space $\mathcal{M}(\mathbb{R}^d)$ with the total variation norm $\|\cdot\|_{\mathrm{TV}}$ is a Banach space. Actually, it can be seen as a subspace of $\mathcal{S}'(\mathbb{R}^d)$ in the following sense, see, e.g. [31, Sect. 4.4]: by Riesz' representation theorem, it can be identified with the dual space $\mathcal{C}_0'(\mathbb{R}^d) \cong \mathcal{M}(\mathbb{R}^d)$ via the isometric isomorphism $\mu \mapsto T_\mu$ given by

$$\langle T_\mu, \varphi \rangle \coloneqq \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu \quad \text{for all} \quad \varphi \in \mathcal{C}_0(\mathbb{R}^d).$$

Since $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of $(\mathcal{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$, every $\mu \in \mathcal{M}(\mathbb{R}^d)$ can be actually identified with a linear functional T_μ on the Schwartz space, which is also continuous with respect to the convergence in $\mathcal{S}(\mathbb{R}^d)$ by

$$|\langle T_\mu, \varphi \rangle| \leq \|\mu\|_{\mathrm{TV}} \|\varphi\|_\infty = \|\mu\|_{\mathrm{TV}} \|\varphi\|_0 \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

with $\|\cdot\|_0$ from (15). Thus, $T_\mu \in \mathcal{S}'(\mathbb{R}^d)$, i.e., every measure from $\mathcal{M}(\mathbb{R}^d)$ corresponds to a tempered distribution, but not conversely. The Fourier transform $\mathcal{F}_d \colon \mathcal{M}(\mathbb{R}^d) \to \mathcal{C}_b(\mathbb{R}^d)$ of measures is an injective, linear transform defined by

$$\mathcal{F}_d[\mu] \coloneqq \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}\langle \cdot, v \rangle} \,\mathrm{d}\mu(v). \tag{23}$$

Note that $\mathcal{F}_d[\mu]$ is also known as the characteristic function of μ. If μ is absolutely continuous with respect to the Lebesgue measure with density $\Phi \in L^1(\mathbb{R}^d)$, then (23) becomes (11). If T_μ is the tempered distribution associated to the measure μ, then it holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\langle \mathcal{F}_d T_\mu, \varphi \rangle = \langle T_\mu, \mathcal{F}_d \varphi \rangle = \int_{\mathbb{R}^d} \mathcal{F}_d \varphi \, \mathrm{d}\mu = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) \mathrm{e}^{-2\pi \mathrm{i} \langle x, v \rangle} \, \mathrm{d}x \, \mathrm{d}\mu(v)$$
$$= \int_{\mathbb{R}^d} \varphi \, \mathcal{F}_d[\mu] \, \mathrm{d}x = \langle T_{\mathcal{F}_d[\mu]}, \varphi \rangle,$$

so that $\mathcal{F}_d T_\mu = T_{\mathcal{F}_d[\mu]}$.

If we want to sample from a measure, we are only interested in positive, bounded measures, therefore we consider probability measures. The Fourier transform of positive measures is related with so-called positive definite functions. A continuous (not necessary radial) function $\Phi \colon \mathbb{R}^d \to \mathbb{C}$ is called *positive definite* if for all $N \in \mathbb{N}$, all pairwise distinct $x_j \in \mathbb{R}^d$, and all $\alpha_j \in \mathbb{C}$, $j = 1, \ldots, N$, it holds

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \overline{\alpha_k} \Phi(x_j - x_k) \geq 0.$$

Positive definite functions are bounded, more precisely $\|\Phi\|_\infty = \Phi(0)$. Functions F such that the radial functions $\mathcal{R}_d F$ are positive definite *in every dimension* $d \in \mathbb{N}$ were characterized by Schoenberg via completely monotone functions [41]. A well-known example of such a function is the Gaussian function. Bochner's theorem [4] relates the Fourier transform of positive measures with positive definite functions.

Theorem 4.10 (Bochner)**.** *Any positive definite function $\Phi \colon \mathbb{R}^d \to \mathbb{R}$ is the Fourier transform of a positive measure and conversely. If $\Phi(0) = 1$, then it is the Fourier transform of a probability measure, i.e., there exists $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ such that $\Phi = \mathbb{E}_{v \sim \mu}\left[\mathrm{e}^{-2\pi \mathrm{i} \langle \cdot, v \rangle}\right]$.*

Using the above relations of measures and tempered distributions, we obtain the following one-to-one correspondence between positive definite radial functions and their sliced versions.

Corollary 4.11. *Let $F \colon [0, \infty) \to \mathbb{R}$ such that $\mathcal{R}_d F$ is positive definite. Then $f = \mathcal{F}_1 \mathcal{R}_d^\star \mathcal{F}_d^{-1}[\mathcal{R}_d F]$ is positive definite on $\mathbb{R}$, fulfills the slicing formula (3), and is $\lfloor \frac{d-2}{2} \rfloor$ times continuously differentiable on $(0, \infty)$. Conversely, for every even, positive definite function f on $\mathbb{R}$, the radial function $\mathcal{R}_d F$ given by (3) is positive definite on $\mathbb{R}^d$.*

Proof. Bochner's Theorem 4.10 implies that $\mathcal{F}_d^{-1}[\mathcal{R}_d F] \in \mathcal{M}_+(\mathbb{R}^d)$. Since a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is positive if and only if $\langle \mu, \varphi \rangle \geq 0$ for every non-negative function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, see [31, Sect. 4.4], also $\mathcal{R}_d^\star \mathcal{F}_d^{-1}[\mathcal{R}_d F]$ is a positive measure and again by Bochner's theorem f is a positive definite function on $\mathbb{R}$. The slicing identity follows from Theorem 4.7

by identifying measures with distributions, since both $\mathcal{R}_d F$ and f are continuous and bounded and are therefore slowly increasing. By Theorem 2.2 the function F is $\lfloor \frac{d-2}{2} \rfloor$ times continuously differentiable on $(0, \infty)$.

The converse follows analogously. □

For odd dimension d, [49, Thm. 7] shows that any radial positive definite function F is $\lfloor \frac{d}{2} \rfloor$ times differentiable if $\mathcal{R}_d F \in L^1(\mathbb{R}^d)$. In comparison, Corollary 4.11 gives one derivative less, but only requires $\mathcal{R}_d F$ being positive definite without further assumptions on F.

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A. Proof of Theorem 2.1

Let $x \in \mathbb{R}^d$ with $\|x\| = r$. Denote by U_x an orthogonal matrix such that $U_x x = \|x\| e_1$, where e_1 is the first unit vector. Then, it holds

$$\langle x, \xi \rangle = \langle r U_x^{\mathrm{T}} e_1, \xi \rangle = r \langle U_x^{\mathrm{T}} e_1, \xi \rangle = r \langle e_1, U_x \xi \rangle$$

and consequently

$$\begin{aligned}\mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}[f(|\langle x, \xi \rangle|)] &= \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}[f(r|\langle e_1, U_x \xi \rangle|)] \\ &= \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}[f(r|\langle e_1, \xi \rangle|)] = \mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}[f(r|\xi_1|)].\end{aligned}$$

We write $\xi = \xi_1 e_1 + \sqrt{1 - \xi_1^2}(0, \xi_{2:d})$ with $\xi_1 \in [-1, 1]$ and $\xi_{2:d} \coloneqq (\xi_2, \ldots, \xi_d) \in \mathbb{S}^{d-2}$. Applying [2, (1.16)], which holds for $d \geq 3$, we obtain

$$\begin{aligned}\mathbb{E}_{\xi \sim \mathcal{U}_{\mathbb{S}^{d-1}}}[f(|\langle x, \xi \rangle|)] &= \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(r|\xi_1|) \,\mathrm{d}\mathbb{S}^{d-1}(\xi) \\ &= \frac{1}{\omega_{d-1}} \int_{-1}^{1} \int_{\mathbb{S}^{d-2}} f(r|\xi_1|) \,\mathrm{d}\mathbb{S}^{d-2}(\xi_{2:d}) (1 - \xi_1^2)^{\frac{d-3}{2}} \,\mathrm{d}\xi_1 \\ &= \frac{\omega_{d-2}}{\omega_{d-1}} \int_{-1}^{1} f(r|t|)(1 - t^2)^{\frac{d-3}{2}} \,\mathrm{d}t = c_d \int_0^1 f(rt)(1 - t^2)^{\frac{d-3}{2}} \,\mathrm{d}t. \quad \square\end{aligned}$$

B. Proof of Theorem 2.2

To prove Theorem 2.2 we need some technical lemmas.

Lemma B.1. *For all $0 < h < s$ and $0 < t < s - h$, it holds*

$$\frac{1}{h}\left(\left(1 - \frac{t^2}{s^2}\right)^{\frac{1}{2}} - \left(1 - \frac{t^2}{(s-h)^2}\right)^{\frac{1}{2}}\right) < \frac{1}{s-t}\left(1 - \frac{t^2}{s^2}\right)^{\frac{1}{2}}.$$

Proof. Since $h < s$ and $s > 0$, we have $-3s + h = -2s - (s - h) < 0$ and therefore

$$t < s - h = \frac{s-h}{2s-h}(2s - h) = \frac{2s^2 - 3sh + h^2}{2s - h} < \frac{2s^2}{2s - h}.$$

We can multiply the inequality with $t(2s - h)$ and obtain

$$\begin{aligned}0 < 2s^2 t - t^2(2s - h) &= t\left((s-h)^2 + s^2\right) + th(2s - h) - t^2(2s - h) \\ &< t\left((s-h)^2 + s^2\right) + (s - h)s(2s - h) - t^2(2s - h).\end{aligned}$$

Multiplying the inequality with $h > 0$, a straightforward calculation yields

$$\begin{aligned} 0 &< ht\left((s-h)^2+s^2\right)+(s-h)s(2s-h)h-h(2s-h)t^2 \\ &= (s-h+t)(s-t)s^2-(s-h-t)(s+t)(s-h)^2. \end{aligned}$$

Further, since $s-t$, s^2, and $(s-h)^2$ are positive, we obtain

$$\begin{aligned} 0 &< \frac{(s-h)+t)}{(s-h)^2}-\frac{((s-h)-t)}{s-t}\frac{s+t}{s^2} \\ &= \frac{1}{s-h-t}\Big(\frac{(s-h+t)(s-h-t)}{(s-h)^2}-\frac{(s-h-t)^2}{(s-t)^2}\frac{(s^2-t^2)}{s^2}\Big) \\ &= \frac{1}{s-h-t}\left(\left(1-\frac{t^2}{(s-h)^2}\right)-\left(1-\frac{h}{s-t}\right)^2\left(1-\frac{t^2}{s^2}\right)\right). \end{aligned}$$

Multiplying with $s-h-t>0$, reordering and taking the square root yields

$$\left(1-\frac{h}{s-t}\right)\left(1-\frac{t^2}{s^2}\right)^{\frac{1}{2}} < \left(1-\frac{t^2}{(s-h)^2}\right)^{\frac{1}{2}}.$$

Finally, we rearrange the equation and divide by $h > 0$ to get the assertion

$$\frac{1}{h}\left(\left(1-\frac{t^2}{s^2}\right)^{\frac{1}{2}}-\left(1-\frac{t^2}{(s-h)^2}\right)^{\frac{1}{2}}\right) < \frac{1}{s-t}\left(1-\frac{t^2}{s^2}\right)^{\frac{1}{2}}. \qquad \square$$

Lemma B.2. *Let $f \in L^1_{\mathrm{loc}}([0,\infty))$ if $\nu \geqslant 1$ and $f \in L^p_{\mathrm{loc}}([0,\infty))$ with $p > 2$ if $\nu = 1/2$. For $s > 0$, we define*

$$I_\nu f(s) := \int_0^s f(t)\left(1-\frac{t^2}{s^2}\right)^\nu \mathrm{d}t, \tag{24}$$

then it holds that

$$\frac{\mathrm{d}}{\mathrm{d}s}I_\nu f(s) = \frac{2\nu}{s^3}I_{\nu-1}g(s), \qquad g(t) := f(t)t^2.$$

Proof. We show that

$$\lim_{h\to 0}\frac{I_\nu f(s+h)-I_\nu f(s)}{h} = \frac{2\nu}{s^3}I_{\nu-1}g(s).$$

1. First, we consider $h > 0$, i.e., the right-sided limit

$$\begin{aligned} \lim_{h\searrow 0}\frac{I_\nu f(s+h)-I_\nu f(s)}{h} &= \lim_{h\searrow 0}\frac{1}{h}\int_s^{s+h} f(t)\left(1-\frac{t^2}{(s+h)^2}\right)^\nu \mathrm{d}t \qquad (25) \\ &+ \lim_{h\searrow 0}\int_0^s f(t)\frac{1}{h}\left(\left(1-\frac{t^2}{(s+h)^2}\right)^\nu-\left(1-\frac{t^2}{s^2}\right)^\nu\right)\mathrm{d}t. \end{aligned}$$

We show that the first summand is zero, while the second one equals $2\nu s^{-3}I_{\nu-1}g(s)$.

1.1. Concerning the first limit, we have for $h > 0$ and $s < t < s+h$ that

$$\left|1 - \frac{t^2}{(s+h)^2}\right| \leqslant 1 - \frac{s^2}{(s+h)^2} = h\frac{2s+h}{(s+h)^2},$$

and further by the monotony of the power function, for $0 < h < 1$, that

$$\begin{aligned}\left|\frac{1}{h}\int_s^{s+h} f(t)\left(1 - \tfrac{t^2}{(s+h)^2}\right)^{\nu} \mathrm{d}t\right| &\leqslant \frac{1}{h}\int_s^{s+h} |f(t)| \left(h\frac{2s+h}{(s+h)^2}\right)^{\nu} \mathrm{d}t \\ &\leqslant \left(\frac{2s+1}{s^2}\right)^{\nu} h^{\nu-1}\int_s^{s+h} |f(t)|\,\mathrm{d}t.\end{aligned}$$

If $\nu \geqslant 1$ and $f \in L^1_{\mathrm{loc}}([0,\infty))$, then $\nu - 1 \geqslant 0$ and we can estimate

$$\left(\frac{2s+1}{s^2}\right)^{\nu} h^{\nu-1}\int_s^{s+h} |f(t)|\,\mathrm{d}t \leqslant \left(\frac{2s+1}{s^2}\right)^{\nu}\int_s^{s+h} |f(t)|\,\mathrm{d}t \xrightarrow{h\searrow 0} 0.$$

If $\nu = 1/2$, we assumed that $f \in L^p_{\mathrm{loc}}([0,\infty))$ with $p > 2$ and thus $f \in L^2_{\mathrm{loc}}([0,\infty)) \subset L^p_{\mathrm{loc}}([0,\infty))$. Then we get by Hölder's inequality with

$$\begin{aligned}\left(\frac{2s+1}{s^2}\right)^{\nu} h^{\nu-1}\int_s^{s+h} |f(t)|\,\mathrm{d}t &\leqslant \left(\frac{2s+1}{s^2}\right)^{\frac{1}{2}} h^{-\frac{1}{2}}\left(\int_s^{s+h} |f(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_s^{s+h}\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\frac{2s+1}{s^2}\right)^{\frac{1}{2}}\left(\int_s^{s+h} |f(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \xrightarrow{h\searrow 0} 0.\end{aligned}$$

1.2. Concerning the second limit in (25), we use that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(1 - \frac{t^2}{s^2}\Big)^{\nu} = \frac{2t^2\nu}{s^3}\Big(1 - \frac{t^2}{s^2}\Big)^{\nu-1}.$$

If $\nu \geqslant 1$ and $f \in L^1_{\mathrm{loc}}([0,\infty))$, then the mean value theorem implies

$$\begin{aligned}h^{-1}\left|\left(1 - \tfrac{t^2}{(s+h)^2}\right)^{\nu} - \left(1 - \tfrac{t^2}{s^2}\right)^{\nu}\right| &\leqslant \sup_{\xi\in(0,h)}\left|2t^2\nu\Big(1 - \frac{t^2}{(s+\xi)^2}\Big)^{\nu-1}\frac{1}{(s+\xi)^3}\right| \\ \leqslant \sup_{\xi\in(0,h)} 2t^2\nu\Big(1 + \frac{t^2}{(s+\xi)^2}\Big)^{\nu-1}\frac{1}{(s+\xi)^3} &\leqslant 2t^2\nu\Big(1 + \frac{s^2}{s^2}\Big)^{\nu-1}\frac{1}{s^3} = \nu 2^{\nu}\frac{t^2}{s^3} \leqslant \nu 2^{\nu} s^{-1}.\end{aligned}$$

Hence $t \mapsto f(t)\nu 2^{\nu} s^{-1}$ is an h independent integrable majorant and Lebesgue's dominated convergence theorem gives

$$\lim_{h\searrow 0}\int_0^s f(t)\frac{1}{h}\left(\left(1 - \tfrac{t^2}{(s+h)^2}\right)^{\nu} - \left(1 - \tfrac{t^2}{s^2}\right)^{\nu}\right)\mathrm{d}t = \frac{2\nu}{s^3}\int_0^s f(t)t^2\left(1 - \tfrac{t^2}{s^2}\right)^{\nu-1}\mathrm{d}t. \tag{26}$$

If $\nu = 1/2$ and $f \in L^p_{\mathrm{loc}}([0,\infty))$ with $p > 2$, then

$$\frac{1}{h}\left|\left(1 - \tfrac{t^2}{(s+h)^2}\right)^{\frac{1}{2}} - \left(1 - \tfrac{t^2}{s^2}\right)^{\frac{1}{2}}\right| \leqslant \sup_{\xi\in(0,h)}\left|\frac{t^2}{(s+\xi)^3}\Big(1 - \frac{t^2}{(s+\xi)^2}\Big)^{-\frac{1}{2}}\right| \leqslant \frac{t^2}{s^3}\Big(1 - \frac{t^2}{s^2}\Big)^{-\frac{1}{2}}.$$

Denote by $q = (1-p^{-1})^{-1}$ the Hölder conjugate to p. Since $p > 2$ it holds $q < 2$ and also

$$\left(1 - \frac{t^2}{s^2}\right)^{-\frac{1}{2}q} = \left(\frac{s+t}{s^2}\right)^{-\frac{q}{2}} (s-t)^{-\frac{q}{2}}.$$

For $t \in (0, s)$, both $t^2 s^{-3}$ and $((s+t)s^{-2})^{-q/2}$ are bounded, and $(s-t)^{-q/2}$ is integrable on $[0, s]$. Therefore, $t \mapsto t^2 s^{-3}((s+t)s^{-2})^{-q/2}(s-t)^{-q/2} f(t)$ is an integrable majorant. Hence, Lebesgue's dominated convergence theorem yields (26).

2. Next, we deal with the left-sided limit

$$\begin{aligned}\lim_{h \nearrow 0} \frac{I_\nu f(s+h) - I_\nu f(s)}{h} &= \lim_{h \searrow 0} \frac{I_\nu f(s) - I_\nu f(s-h)}{h} \\ &= \lim_{h \searrow 0} \int_0^s \frac{f(t)}{h} \left(\left(1 - \tfrac{t^2}{s^2}\right)^\nu - \left(1 - \tfrac{t^2}{(s-h)^2}\right)^\nu \mathbb{1}_{[0,s-h]}(t)\right) \mathrm{d}t.\end{aligned}$$

Define the functions $m_h, m \colon (0, s) \to \mathbb{R}$ by

$$m_h(t) := \frac{1}{h}\left(\left(1 - \tfrac{t^2}{s^2}\right)^\nu - \left(1 - \tfrac{t^2}{(s-h)^2}\right)^\nu \mathbb{1}_{[0,s-h]}(t)\right) \quad \text{and} \quad m(t) := \frac{\mathrm{d}}{\mathrm{d}s}\left(1 - \frac{t^2}{s^2}\right)^\nu.$$

Note that for every $t \in (0, s)$ we have $\lim_{h \to 0} m_h(t) = m(t)$.

2.1. Let $0 < t < s - h$. We choose $0 < h < s/2$. If $\nu \geqslant 1$ and $f \in L^1_{\mathrm{loc}}([0, \infty))$, then the mean value theorem implies

$$\begin{aligned}|m_h(t)| &= \frac{1}{h}\left|\left(1 - \tfrac{t^2}{(s-h)^2}\right)^\nu - \left(1 - \tfrac{t^2}{s^2}\right)^\nu\right| \leqslant \sup_{\xi \in (0,h)} \left|2\nu\left(1 - \frac{t^2}{(s-\xi)^2}\right)^{\nu-1} \frac{t^2}{(s-\xi)^3}\right| \\ &\leqslant \sup_{\xi \in (0,h)} 2\nu\left(1 + \frac{t^2}{(s-\xi)^2}\right)^{\nu-1} \frac{t^2}{(s-s/2)^3} \leqslant 2\nu\left(1 + \frac{s^2}{(s-s/2)^2}\right)^{\nu-1} \frac{t^2}{(s/2)^3} \\ &\leqslant 16\nu \frac{t^2}{s^3} 5^{\nu-1} \leqslant 16 \cdot 5^{\nu-1} \nu s^{-1},\end{aligned}$$

which is bounded independently of t and s. Hence, Lebesgue's dominated convergence theorem implies

$$\lim_{h \nearrow 0} \frac{I_\nu(f)(s+h) - I_\nu(s)}{h} = \frac{2\nu}{s^3} \int_0^s f(t) t^2 \left(1 - \tfrac{t^2}{s^2}\right)^{\nu-1} \mathrm{d}t.$$

For $\nu = 1/2$ and $f \in L^p_{\mathrm{loc}}([0, \infty))$ with $p > 2$ and Hölder conjugate $q < 2$ to p, we apply Lemma B.1 to obtain

$$|m_h(t)| \leqslant w(t) := \frac{1}{s-t}\left(1 - \frac{t^2}{s^2}\right)^{\frac{1}{2}} = \frac{1}{s}\left(\frac{s+t}{s-t}\right)^{\frac{1}{2}}, \tag{27}$$

which is q integrable. The claim follows again by using the Hölder inequality and the dominated convergence. 2.2. Let $s - h \leqslant t < s$ and $0 < h < s$. For $\nu \geqslant 1$, we see that m_h is integrable. For $\nu = 1/2$, we have $s - t \leqslant h$ and consequently

$$|m_h(t)| = \frac{1}{h}\left(1 - \frac{t^2}{s^2}\right)^{\frac{1}{2}} \leqslant \frac{1}{s-t}\left(1 - \frac{t^2}{s^2}\right)^{\frac{1}{2}} = w(t),$$

see (27). Finally, we use Lebesgue's dominated convergence theorem again. $\square$

Proof of Theorem 2.2. Since $s \mapsto s^{-1}$ is smooth on $(0, \infty)$, the differentiability of F in (10) depends only on the integral $I_{(d-3)/2} f$ defined in (24). By Lemma B.2, we have for $f \in L^1_{\text{loc}}([0, \infty))$ if $d \geq 5$ as well as for $f \in L^p_{\text{loc}}([0, \infty))$ with $p > 2$ if $\nu = 1/2$ and $g_k(t) := f(t)t^{2k}$ that

$$\frac{\mathrm{d}}{\mathrm{d}s} I_\nu f(s) = \frac{2\nu}{s^3} I_{\nu-1} g_1(s). \tag{28}$$

Set $\nu := (d-3)/2$. We show that for each $n = 0, \ldots, \lfloor \nu \rfloor$, there exist smooth functions $r_0, \ldots, r_n$ on $(0, \infty)$ such that

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n} \left(\frac{c_d}{s} I_\nu(s) \right) = \sum_{k=0}^{n} r_k(s) I_{\nu-k} g_k(s).$$

For $n = 0$, we obtain that $r_0 = c_d/s$. Assume the assertion holds for $n < \lfloor \nu \rfloor$. For $k = 0, \ldots, n$, we have $g_k \in L^p_{\text{loc}}([0, \infty))$ with $p > 2$ if $\nu = 1/2$ and $p = 1$ if $\nu \geqslant 1$. Also $\nu - k \geqslant \lfloor \nu \rfloor - n \geqslant 1$, so that (28) yields

$$\begin{aligned} \frac{\mathrm{d}^{n+1}}{\mathrm{d}s^{n+1}} F(s) &= \sum_{k=0}^{n} \left(r'_k(s) I_{\nu-k} g_k(s) + r_k(s) \frac{2\nu}{s^3} I_{\nu-k-1} g_{k+1}(s) \right) \\ &= \sum_{k=0}^{n+1} \left(r'_k(s) + r_{k-1}(s) \frac{2\nu}{s^3} \right) I_{\nu-k} g_k(s), \quad r'_{n+1} = r_{-1} := 0. \end{aligned} \tag{29}$$

Hence F is $\lfloor \nu \rfloor$ times differentiable. Moreover, the parameter integrals I_k, $k = 0, \ldots, \lfloor \nu \rfloor$ are absolutely continuous, which follows from Lemma B.2 for $k \geq 1$ and from the definition of I_0. Hence, also the $\lfloor \nu \rfloor$-th derivative of F is absolutely continuous.

If d is odd, it holds $\lfloor \nu \rfloor = \lfloor (d-2)/2 \rfloor$ and we are done. If d is even, then $\lfloor \nu \rfloor = (d-4)/2$ and $\nu - \lfloor \nu \rfloor = 1/2$. For $f \in L^p_{\text{loc}}([0, \infty))$ and $p > 2$, we obtain by (29) and Lemma B.2 that the $\lfloor \nu \rfloor + 1$ th derivative of F also exits and is absolutely continuous. This finishes the proof. $\square$

C. Proof of Theorem 4.1

To prove the theorem, we need several auxiliary lemmata.

Lemma C.1. *For $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(0) = 0$, the function*

$$V[\psi](x) := \begin{cases} \psi(x)/x & \text{if } x \neq 0, \\ \psi'(0) & \text{if } x = 0. \end{cases} \tag{30}$$

is in $\mathcal{S}(\mathbb{R})$ *and the respective map* $V\colon \{\psi \in \mathcal{S}(\mathbb{R}) \mid \psi(0)=0\} \to \mathcal{S}(\mathbb{R})$ *is continuous with* $\|V[\psi]\|_n \leqslant a_n\|\psi\|_{n+1}$*, where*

$$a_n := \max_{m=0,\dots,n}\left\{\max\left\{\sum_{l=0}^{m}\frac{2^n}{m-l+1}\binom{m}{l}, \sum_{l=0}^{m}\frac{m!}{l!}\right\}\right\} \geqslant 1.$$

Proof. Since $\psi(0)=0$, we see that $V[\psi]$ is continuous. We note that if a function $\phi \in \mathcal{C}(\mathbb{R})$ is differentiable on $\mathbb{R}\setminus\{0\}$ and $\lim_{x\to 0}\phi'(x)$ exists, then it is differentiable in 0.
Part 1: We show by induction that for $m \geqslant 1$ the m-th derivative of $V[\psi]$ is continuous and can be represented as

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}V[\psi](x) = x^{-m-1}\sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{m!}{l!} \quad \text{for all} \quad x \in \mathbb{R}\setminus\{0\}. \tag{31}$$

For $m=1$ we have for $x \neq 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}V[\psi](x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi(x)}{x} = \frac{\psi'(x)x-\psi(x)}{x^2} = x^{-m-1}\sum_{l=0}^{1}\psi^{(l)}(x)x^l(-1)^{l+1}\frac{1!}{l!}.$$

By L'Hospital it holds

$$\lim_{x\to 0}\frac{\mathrm{d}}{\mathrm{d}x}V[\psi](x) = \lim_{x\to 0}\frac{\psi'(x)x-\psi(x)}{x^2} = \lim_{x\to 0}\frac{\psi''(x)x}{2x} = \frac{\psi^{(2)}(0)}{2}.$$

Since $V[\psi]$ is continuous on $\mathbb{R}$ and $\lim_{x\to 0}V[\psi]'(x)$ is finite, we have $V[\psi]'(0) = \psi^{(2)}(0)/2$. In particular, the derivative is continuous.

Now assume, that (31) holds for all derivatives less or equal m and all derivatives less or equal m are continuous. We can compute the derivative

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}V[\psi](x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-m-1}\sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{m!}{l!}\right)$$
$$= x^{-m-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{m!}{l!}\right) - (m+1)x^{-m-2}\sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{m!}{l!}.$$

With the calculation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{m!}{l!}\\
&= \sum_{l=0}^{m}\psi^{(l+1)}(x)x^l(-1)^{l+m}\frac{m!}{l!} + \sum_{l=1}^{m}\psi^{(l)}(x)x^{l-1}(-1)^{l+m}\frac{m!}{(l-1)!}\\
&= \sum_{l=0}^{m}\psi^{(l+1)}(x)x^l(-1)^{l+m}\frac{m!}{l!} + \sum_{l=0}^{m-1}\psi^{(l+1)}(x)x^l(-1)^{l+1+m}\frac{m!}{l!}\\
&= \psi^{(m+1)}(x)x^m(-1)^{2m}\frac{m!}{m!} = \psi^{(m+1)}(x)x^m,
\end{aligned} \tag{32}$$

we obtain that

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}V[\psi](x) = x^{-m-1}\,\psi^{(m+1)}(x)x^m - x^{-m-2}\sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{(m+1)!}{l!}$$
$$= x^{-m-2}\sum_{l=0}^{m+1}\psi^{(l)}(x)x^l(-1)^{l+(m+1)}\frac{(m+1)!}{l!},$$

which shows (31). Since ψ is a Schwartz function, it is clear that $\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}V[\psi](x)$ is continuous on $\mathbb{R}\setminus\{0\}$. Now we use again L'Hospital and (32) to obtain

$$\lim_{x\to 0}\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}V[\psi](x) = \lim_{x\to 0}\frac{\sum_{l=0}^{m+1}\psi^{(l)}(x)x^l(-1)^{l+(m+1)}\frac{(m+1)!}{l!}}{x^{m+2}}$$
$$= \lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\sum_{l=0}^{m+1}\psi^{(l)}(x)x^l(-1)^{l+(m+1)}\frac{(m+1)!}{l!}}{\frac{\mathrm{d}}{\mathrm{d}x}x^{(m+1)+1}} = \lim_{x\to 0}\frac{\psi^{(m+2)}(x)x^{m+1}}{(m+1)x^{m+1}} = \frac{\psi^{(m+2)}(0)}{m+1}.$$

This yields that $V[\psi]\in\mathcal{C}^{m+1}(\mathbb{R})$ and this induction is finished. *Part 2:* Next we show that $V[\psi]$ is a Schwartz function and that V is continuous. Let $x\in\mathbb{R}$ and $m\in\mathbb{N}$ be fixed. We use the Taylor expansion with the Lagrange reminder,

$$\psi^{(l)}(x) = \sum_{k=l}^{m}\frac{\psi^{(k)}(0)}{(k-l)!}x^{k-l} + \frac{\psi^{(m+1)}(\xi_l(x))}{(m-l+1)!}x^{m-l+1} \quad \text{for some} \quad |\xi_l(x)| < |x|.$$

With the representation (31), we see that

$$x^{m+1}\frac{\mathrm{d}^m}{\mathrm{d}x^m}V[\psi](x) = \sum_{l=0}^{m}\psi^{(l)}(x)x^l(-1)^{l+m}\frac{m!}{l!}$$
$$= x^{m+1}\sum_{l=0}^{m}\frac{\psi^{(m+1)}(\xi_l(x))}{(m-l+1)!}(-1)^{l+m}\frac{m!}{l!} + \sum_{l=0}^{m}\sum_{k=l}^{m}\frac{\psi^{(k)}(0)}{(k-l)!}x^k(-1)^{l+m}\frac{m!}{l!}$$
$$= x^{m+1}\sum_{l=0}^{m}\frac{\psi^{(m+1)}(\xi_l(x))}{m-l+1}(-1)^{l+m}\binom{m}{l} + \sum_{k=0}^{m}x^k\psi^{(k)}(0)\sum_{l=0}^{k}\frac{(-1)^{l+m}}{(k-l)!}\frac{m!}{l!}.$$

If $k\geqslant 1$, we have by the binomial theorem

$$\sum_{l=0}^{k}\frac{(-1)^{l+m}}{(k-l)!}\frac{m!}{l!} = (-1)^m\frac{m!}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l} = 0.$$

Since $\psi(0)=0$, only the first sum remains and we have

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}V[\psi](x) = \sum_{l=0}^{m}\psi^{(m+1)}(\xi_l(x))\frac{(-1)^{l+m}}{m-l+1}\binom{m}{l} \quad \text{with} \quad |\xi_l(x)|\leqslant|x|. \tag{33}$$

Fixing some natural number $m \leqslant n$, we have for $x \in [-1,1] \setminus \{0\}$ using (33), that

$$\left|(1+|x|)^n \frac{\mathrm{d}^m}{\mathrm{d}x^m} V[\psi](x)\right| = \left|(1+|x|)^n \sum_{l=0}^{m} \psi^{(m+1)}(\xi_l(x)) \frac{(-1)^{l+m}}{m-l+1} \binom{m}{l}\right|$$

$$\leqslant (1+|x|)^n \sup_{y\in[-1,1]} \psi^{(m+1)}(y) \sum_{l=0}^{m} \frac{1}{m-l+1} \binom{m}{l}$$

$$\leqslant 2^n \|\psi\|_{n+1} \sum_{l=0}^{m} \frac{1}{m-l+1} \binom{m}{l} \leqslant a_n.$$

On the other hand, if $|x| > 1$, we have by (31)

$$\left|(1+|x|)^n \frac{\mathrm{d}^m}{\mathrm{d}x^m} V[\psi](x)\right| \leqslant (1+|x|)^n \sum_{l=0}^{m} \left|\psi^{(l)}(x)\right| \frac{m!}{l!} \leqslant a_n \|\psi\|_{n+1}.$$

Since $V[\psi]$ is smooth all its derives are continuous, we have

$$\|V[\psi]\|_n = \sup_{m=0,\ldots,n} \sup_{x\in\mathbb{R}\setminus\{0\}} \left|(1+|x|)^n \frac{\mathrm{d}^m}{\mathrm{d}x^m} V[\psi](x)\right| \leqslant a_n \|\psi\|_{n+1} < \infty.$$

Thus $V[\psi]$ is a Schwartz function and the operator V is continuous. □

Lemma C.2. *The operator*

$$W\colon \mathcal{S}_{\mathrm{rad}}(\mathbb{R}) \to \mathcal{S}_{\mathrm{rad}}(\mathbb{R}), \quad W := V \circ \tfrac{\mathrm{d}}{\mathrm{d}x}, \tag{34}$$

or more precisely $W[\psi](x) = V[\psi'](x) = \psi'(x)/x$ *if* $x \neq 0$ *and* $W[\psi](0) = \psi''(0)$ *is linear and continuous with* $\|W[\psi]\|_m \leqslant a_m \|\psi\|_{m+2}$ *for all* $m \in \mathbb{N}$ *and all* $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$.

Proof. If $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$, then ψ is even and ψ' is odd. Therefore, $\psi'(0) = 0$ and $V[\psi']$ is well-defined by Lemma C.1. The function $W[\psi](x) = \psi'(x)/x$ is again even, so that W is well-defined and we can estimate the m norm of $W[\psi]$ by Lemma C.1 as

$$\|W[\psi]\|_m = \|V(\psi')\|_m \leqslant a_m \|\psi'\|_{m+1} \leqslant a_m \|\psi\|_{m+2}.$$ □

Proof of Theorem 4.1 i). The directional derivative of $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ in direction $\xi \in \mathbb{S}^{d-1}$ is

$$\partial_\xi(\psi \circ \|\cdot\|_2)(0) = \lim_{t\to 0} \frac{\psi(\|t\xi\|) - \psi(\|0\|)}{t} = \lim_{t\to 0} \frac{\psi(|t|) - \psi(0)}{t} = \psi'(0) = 0,$$

and, for $x \neq 0$, by the chain rule,

$$\partial_\xi(\psi \circ \|\cdot\|_2)(x) = \psi'(\|x\|) \left\langle \frac{x}{\|x\|}, \xi \right\rangle.$$

Recalling the definition of V in (30), we have the representation

$$\partial_\xi(\psi \circ \|\cdot\|_2)(x) = \langle x, \xi\rangle V[\psi](\|x\|) \quad \text{for all} \quad x \in \mathbb{R}^d. \tag{35}$$

In the following, we show inductively that for $\alpha \in \mathbb{N}^d$ there are polynomials $p_{\alpha,1}, \dots, p_{\alpha,|\alpha|}$ of degree $\leqslant k$ such that for every even, smooth function $\psi\colon \mathbb{R} \to \mathbb{R}$ it holds

$$D^\alpha(\psi \circ \|\cdot\|_2)(x) = \sum_{k=1}^{|\alpha|} p_{\alpha,k}(x) \cdot W^k[\psi](\|x\|) \quad \text{for all} \quad x \in \mathbb{R}^d, \tag{36}$$

where $W^k = W \circ W \circ \ldots \circ W$ exactly k times, see (34). For $|\alpha| = 1$, we write $\alpha = e_l$ and choose $p_{\alpha,1}(x) = x_l$. By (35), it holds for all $x \in \mathbb{R}^d$ that

$$D^\alpha(\psi \circ \|\cdot\|_2)(x) = x_k \frac{\psi'(\|x\|)}{\|x\|} = p_{\alpha,1}(x) W[\psi](\|x\|).$$

Now assume (36) holds for $|\alpha| < n$ and choose $\beta \in \mathbb{N}^d$ with $|\beta| = n$. Find some nonzero entry l of β and define $\alpha := \beta - e_l$. Since $|\alpha| < n$, we can apply the induction hypothesis and obtain with Lemma C.2

$$\begin{aligned}
D^\beta(\psi \circ \|\cdot\|_2)(x) &= \partial_{e_l}(D^\alpha(\psi \circ \|\cdot\|_2))(x) = \partial_{e_l} \sum_{k=1}^{|\alpha|} p_{\alpha,k}(x) W^k[\psi](\|x\|) \\
&= \sum_{k=1}^{|\alpha|} \left(\partial_{e_l} p_{\alpha,k}(x) W^k[\psi](\|x\|) + p_{\alpha,k}(x) x_l \frac{W^k[\psi]'(\|x\|)}{\|x\|} \right) \\
&= \sum_{k=1}^{|\alpha|} \partial_{e_l} p_{\alpha,k}(x) W^k[\psi](\|x\|) + \sum_{k=2}^{|\alpha|+1} p_{\alpha,k-1}(x) x_l W^k[\psi](\|x\|) \\
&= \sum_{k=1}^{|\beta|} p_{\beta,k}(x) W^k[\psi](\|x\|),
\end{aligned}$$

where we set $p_{\beta,1}(x) := \partial_{e_l} p_{\alpha,1}(x)$, $p_{\beta,|\beta|}(x) := p_{\alpha,|\alpha|}(x) x_l$, and $p_{\beta,k}(x) := \partial_{e_l} p_{\alpha,k}(x) + p_{\alpha,k-1}(x) x_l$ for $k = 2, \dots, |\beta| - 1$. This shows (36).

Let $n \in \mathbb{N}$. Since $\deg p_{\alpha,k} \leqslant k$ for all $\alpha \in \mathbb{N}^d$ we can find $c_n > 0$ such that

$$|p_{\alpha,k}(x)| \leqslant c_n (1 + \|x\|)^n \quad \text{for all} \quad |\alpha| \leqslant n, \ k = 1, \dots, |\alpha|.$$

For any $x \in \mathbb{R}^d$, we have by the continuity of W in Corollary C.2

$$\begin{aligned}
\big\|\psi \circ \|\cdot\|_2\big\|_n &= \sup_{|\alpha|\leqslant n} \|(1+\|\cdot\|)^n D^\alpha(\psi \circ \|\cdot\|_2)\|_\infty \\
&\leqslant \sup_{|\alpha|\leqslant n} \left\|(1+\|\cdot\|)^n \sum_{k=1}^{|\alpha|} p_{\alpha,k}(x) W^k[\psi] \circ \|\cdot\|\right\|_\infty \\
&\leqslant c_n \sup_{m=0,\dots,n} \sum_{k=1}^{m} \left\|(1+|\cdot|)^{2n} W^k[\psi]\right\|_\infty \\
&\leqslant c_n \sum_{k=1}^{n} \|W^k[\psi]\|_{2n} \leqslant c_n \sum_{k=1}^{n} \prod_{j=1}^{k} a_j \|\psi\|_{2n+2k} \leqslant n c_n \prod_{j=1}^{n} a_j \|\psi\|_{4n}.
\end{aligned}$$

Setting $b_n \coloneqq n c_n \prod_{j=1}^{n} a_j$ finishes the proof. □

Lemma C.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function and $Q \in \mathrm{O}(d)$, then $\varphi \circ Q \in \mathcal{S}(\mathbb{R}^d)$ and $\|\varphi \circ Q\|_m \leqslant d^m \|\varphi\|_m$ for all $m \in \mathbb{N}$.*

Proof. For $k \in [d]$ and $z \in \mathbb{R}^d$, the chain rule implies

$$\begin{aligned}
|\partial_{e_k}(\varphi \circ Q)(z)| &= |\langle D(\varphi \circ Q)(z), e_k\rangle| = |\langle \nabla\varphi(Q(z)) \cdot Q, e_k\rangle| = |\langle (\nabla\varphi \circ Q)(z), Q_k\rangle| \\
&\leqslant \sum_{j=1}^{d} |\partial_{e_j}\varphi(Qz)| \cdot |Q_{k,j}| \leqslant \sum_{j=1}^{d} |\partial_{e_j}\varphi(Qz)| \leqslant d \cdot \max_{j=1,\dots,d} |\partial_{e_j}\varphi(Qz)|.
\end{aligned}$$

For $\beta \in \mathbb{N}^d$ we obtain inductively

$$|D^\beta(\varphi \circ Q)(z)| \leqslant d^{|\beta|} \max_{|\beta'|=|\beta|} |D^{\beta'}\varphi(Qz)|.$$

Finally, we have for $m \in \mathbb{N}$ that

$$\begin{aligned}
\|\varphi \circ Q\|_m &= \max_{|\beta|\leqslant m} \sup_{x\in\mathbb{R}^d} |(1+\|c\|)^m D^\beta[\varphi \circ Q](x)| \\
&\leqslant \max_{|\beta|\leqslant m} d^{|\beta|} \sup_{x\in\mathbb{R}^d} \|(1+\|x\|)^m (D^\beta\varphi)(Qx)| = d^m \|\varphi\|_m.
\end{aligned}$$

Therefore $\varphi \circ Q \in \mathcal{S}(\mathbb{R}^d)$. □

Lemma C.4. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \mathbb{S}^{d-1}$. Then*

$$\varphi_\xi \colon \mathbb{R} \to \mathbb{R}, \quad r \mapsto \varphi(r\xi)$$

is a one-dimensional Schwartz function and $\|\varphi_\xi\|_m \leqslant d^m \|\varphi\|_m$ for all $m \in \mathbb{N}_0$.

Proof. First, let $\xi = e_1$ and $n \in \mathbb{N}_0$. we see that φ_{e_1} is smooth with

$$D^n \varphi_{e_1}(r) = \left(\tfrac{\mathrm{d}}{\mathrm{d}r}\right)^n \varphi(re_1) = D^\beta \varphi(re_1),$$

where $\beta = ne_1 \in \mathbb{N}_0^d$. Therefore, we obtain for $m \in \mathbb{N}_0$ that

$$\|\varphi_{e_1}\|_m = \sup_{x_1 \in \mathbb{R}} \left|(1 + |x_1|)^m D^\beta \varphi(x_1 e_1)\right| \leqslant \sup_{x \in \mathbb{R}^d} \left|(1 + \|x\|)^m D^\beta \varphi(x)\right| \leqslant \|\varphi\|_m.$$

For arbitrary $\xi \in \mathbb{S}^{d-1}$, we take some $Q \in \mathrm{O}(d)$ such that $Qe_1 = \xi$, then $\varphi_\xi = (\varphi \circ Q)_{e_1}$. By Lemma C.3, we obtain $\varphi_\xi \in \mathcal{S}(\mathbb{R})$ and

$$\|\varphi_\xi\|_m = \|(\varphi \circ Q)_{e_1}\|_m \leqslant \|\varphi \circ Q\|_m \leqslant d^m \|\varphi\|_m.$$

□

Proof of Theorem 4.1 ii). Since $\mathbb{S}^{d-1}$ is compact and φ is continuous, $\mathcal{A}_d\varphi(r)$ is well-defined. For distinct $r, s \in \mathbb{R}$, we can find $t \in \mathbb{R}$ between r and s by the mean value theorem, such that

$$\left|\frac{\varphi_\xi(r) - \varphi_\xi(s)}{r - s}\right| = |\varphi_\xi'(t)| \leqslant \|\varphi_\xi\|_1 \leqslant d\|\varphi\|_1.$$

Therefore we get by Lebesgue's dominated convergence theorem that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{A}_d\varphi(r) &= \lim_{s \to r} \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \frac{\varphi_\xi(s) - \varphi_\xi(r)}{r - s} \mathrm{d}\xi = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \lim_{s \to r} \frac{\varphi_\xi(s) - \varphi_\xi(r)}{r - s} \,\mathrm{d}\xi \\ &= \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \varphi_\xi'(r) \,\mathrm{d}\xi. \end{aligned}$$

Inductively, we obtain with Lemma C.4 for any $r \in \mathbb{R}$ and $n \leqslant m$, that

$$\begin{aligned} \left|(1 + |r|)^m \mathcal{A}_d\varphi^{(n)}(r)\right| &\leqslant \frac{(1 + |r|)^m}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \left|\varphi_\xi^{(n)}(r)\right| \,\mathrm{d}\xi \\ &\leqslant \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \|\varphi_\xi\|_m \,\mathrm{d}\xi \leqslant d^m \|\varphi\|_m. \end{aligned}$$

Clearly $\mathcal{A}_d\varphi$ is an even function and therefore in $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. □

D. Riemann-Liouville Fractional Integrals and Derivatives

In this section, we briefly describe the relation between Abel-type integrals (4) and Riemann-Liouville fractional integrals together with their inversion formula via fractional derivatives. For more information, we refer e.g. to [39, 40].

Let $\alpha > 0$. Then, for $f \in L^1(0, \infty)$, the *Riemann–Liouville fractional integral* is given by

$$(I_+^\alpha f)(s) \coloneqq \frac{1}{\Gamma(\alpha)} \int_0^s f(t)\,(s - t)^{\alpha - 1} \,\mathrm{d}t, \quad \text{for all} \quad s \in (0, \infty),$$

and the *fractional derivative* by

$$(D^{\alpha}_{+} f)(s) \coloneqq \frac{1}{\Gamma(n-\alpha)} \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^{n} \int_0^s f(t)\,(s-t)^{n-\alpha-1}\,\mathrm{d}t,$$

where $n = \lfloor \alpha \rfloor + 1$, see [40, Thm. 2.4]. For $f \in L^1(0,b)$ and $F \in I^{\alpha}_{+}(L^1(0,\infty))$, it holds

$$D^{\alpha}_{+} I^{\alpha}_{+} f = f \quad \text{and} \quad I^{\alpha}_{+} D^{\alpha}_{+} F = F.$$

In order to invert the transform (10) by these formulas, we set $\alpha \coloneqq {}^{(d-1)}/_{2}$, where $d \geq 2$. Reparameterizing F and using the substitution $t \to \sqrt{t}$, we obtain

$$\begin{aligned}
\frac{1}{c_d} F(\sqrt{s}) &= s^{-\frac{d-2}{2}} \int_0^{\sqrt{s}} f(t)\,(s-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t \\
&= \frac{1}{2}\, s^{-\frac{d-2}{2}} \int_0^s \frac{f(\sqrt{t})}{\sqrt{t}}\,(s-t)^{\frac{d-3}{2}}\,\mathrm{d}t \\
&= \frac{1}{2}\, s^{-\frac{d-2}{2}}\, \Gamma\Big(\frac{d-1}{2}\Big) \Big(I_{+}^{\frac{d-1}{2}} \frac{f(\sqrt{\cdot})}{\sqrt{\cdot}}\Big)(s),
\end{aligned}$$

and consequently

$$\frac{f(\sqrt{s})}{\sqrt{s}} = \frac{2}{c_d \Gamma(\frac{d-1}{2})} \Big(D_{+}^{\frac{d-1}{2}} F(\sqrt{\cdot})\, \cdot^{\frac{d-2}{2}}\Big)(s).$$

Under the assumption $f \in L^1(\mathbb{R})$, which is equivalent to $\frac{f(\sqrt{\cdot})}{\sqrt{\cdot}} \in L^1(\mathbb{R})$, the transformation $f \mapsto F$ is injective and this gives with $\nu \coloneqq 0$ if d is odd and $\nu \coloneqq \frac{1}{2}$ if d is even,

$$\begin{aligned}
f(\sqrt{s}) &= \frac{2\sqrt{s}}{c_d \Gamma(\frac{d-1}{2})} \Big(D_{+}^{\frac{d-1}{2}} F(\sqrt{\cdot})\, \cdot^{\frac{d-2}{2}}\Big)(s) \\
&= \frac{2s}{c_d \Gamma(\frac{d-1}{2})\, \Gamma(\lfloor \frac{d+1}{2} \rfloor - \frac{d-1}{2})} \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^{\lfloor \frac{d+1}{2} \rfloor} \int_0^s F(\sqrt{t})\, t^{\frac{d-2}{2}}\, (s-t)^{-\nu}\,\mathrm{d}t \\
&= \frac{4\sqrt{s}}{c_d \Gamma(\frac{d-1}{2})\, \Gamma(\lfloor \frac{d+1}{2} \rfloor - \frac{d-1}{2})} \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^{\lfloor \frac{d+1}{2} \rfloor} \int_0^{\sqrt{s}} F(t)\, t^{d-1}\, (s-t^2)^{-\nu}\,\mathrm{d}t,
\end{aligned}$$

so that

$$f(s) = \frac{4s}{c_d \Gamma(\frac{d-1}{2})\, \Gamma(1-\nu)} \Big(\frac{\mathrm{d}}{\mathrm{d}(s^2)}\Big)^{\lfloor \frac{d+1}{2} \rfloor} \int_0^s F(t)\, t^{d-1}\, (s^2-t^2)^{-\nu}\,\mathrm{d}t.$$