Elastic strips

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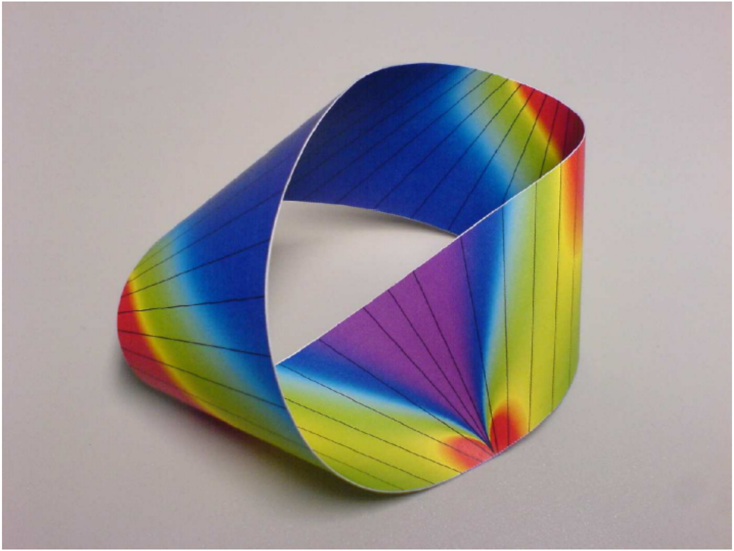
Elastic Moebius strips



Developpable strips

- Let $\gamma : [0, L] \to \mathbb{R}^3$ be an arclength parametrized curve with Frenet frame T, N, B. Assume that there is a smooth function $\lambda : [0, L] \to \mathbb{R}$ such that the curvature and torsion of γ satisfy

$$\tau = \lambda\kappa.$$

- Then $f : [0, L] \times [-\epsilon, \epsilon] \to \mathbb{R}^3$ defined by

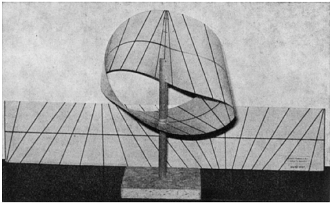
$$f(s, t) = \gamma(s) + t(B(s) + \lambda T(s))$$

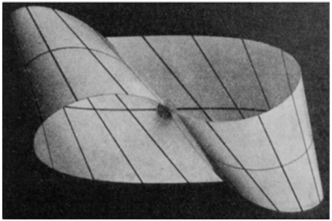
parametrizes a developpable strip of width 2ϵ.

- α the angle between the curve and the normal to the rulings $\rightsquigarrow$

$$\lambda = \tan \alpha$$

- Photographs 1961 by W. Wunderlich





- The bending energy

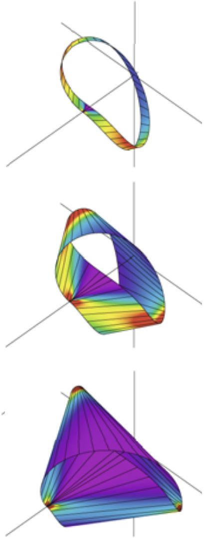
$$E = \int H^2 dA$$

can be expressed as

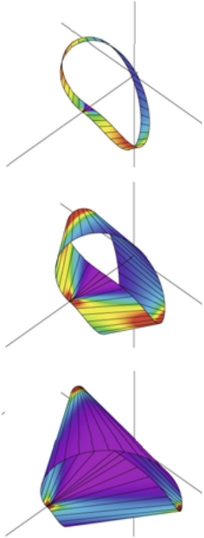
$$E = \int_0^L \kappa^2 (1 + \lambda^2)^2 \frac{\log(1 + \epsilon\lambda') - \log(1 - \epsilon\lambda')}{\lambda'} ds.$$

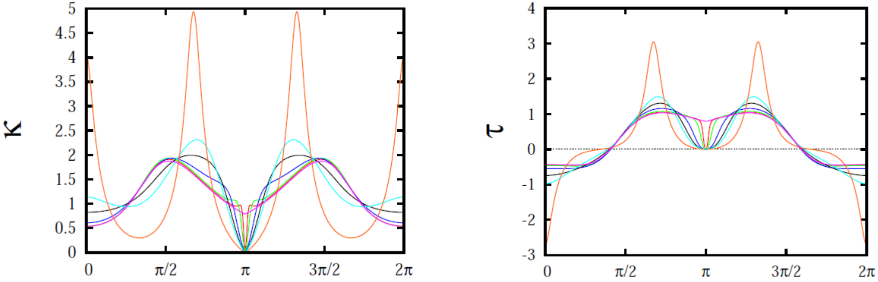
- Wunderlich 1961

- Starostin and van der Heijden computed in 2007 the Euler-Lagrange equations
- Variational complex $\rightsquigarrow$ Computer algebra
- Numerical study of energy-minimizing Moebius bands



- In the limit $\epsilon \to 0$ the curvature κ jumps from $+1$ to -1 at the inflection point.
- Torsion τ is continuous with value 1.
- Angle between curve and rulings jumps from 45° to -45°.
- Energy minimizing Moebius bands are only C^1.




κ
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
π/2
π
3π/2
2π
τ
4
3
2
1
0
-1
-2
-3
0
π/2
π
3π/2
2π

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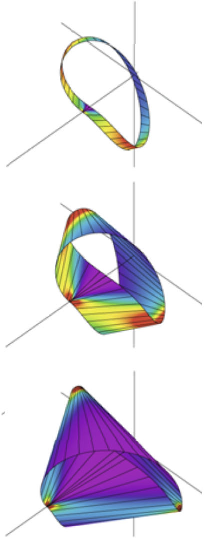


- In the limit $\epsilon \to 0$ the energy becomes

$$E = \int_0^L \kappa^2 (1 + \lambda^2)^2 ds.$$

- Introduced by Sadowski in 1930 as

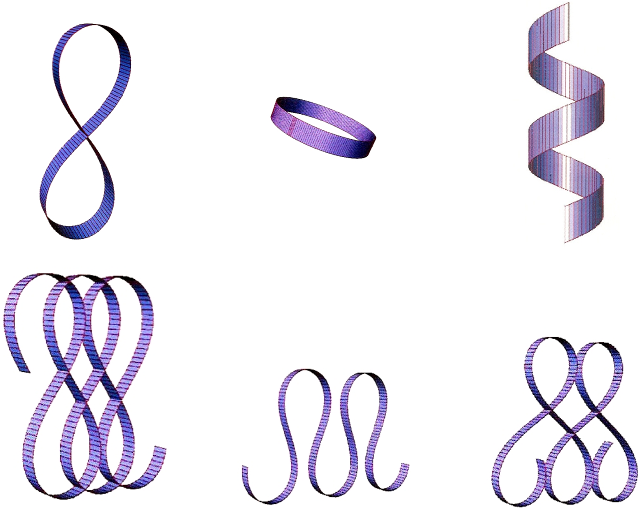
$$E = \int_0^L \frac{(\kappa^2 + \tau^2)^2}{\kappa^2} ds.$$

- E has to be minimized among strips with fixed length $\rightsquigarrow$ Langrange multiplier μ in Euler-Lagrange equations.

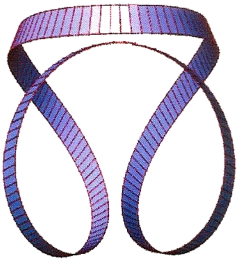
Computed in 2005 by Hagan, subsequently corrected by Rominger and Chubelaschwili

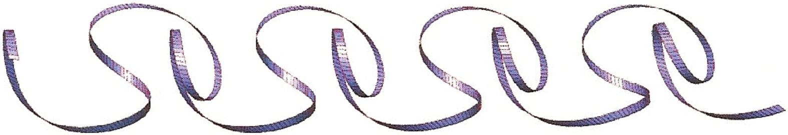
$$
\begin{aligned}
0 &= (\kappa'(1+\lambda^2)^2 + 2\kappa(1+\lambda^2)\lambda\lambda')' \\
&+ \frac{\kappa}{2}(\kappa^2(1+\lambda^2)^2 - \mu) \\
&+ \lambda\kappa(\kappa^2(1+\lambda^2)^2\lambda + (\frac{\kappa'}{\kappa}(1+\lambda^2)2\lambda))' + (1+\lambda^2)2\lambda)'') \\
0 &= (\kappa^2(1+\lambda^2)^2\lambda + (\frac{\kappa'}{\kappa}(1+\lambda^2)2\lambda)' + ((1+\lambda^2)2\lambda)'')' \\
&+ \kappa\lambda(\kappa'(1+\lambda^2)^2 + 2\kappa(1+\lambda^2)\lambda\lambda')
\end{aligned}
$$

Planar elastic curves and helices yield elastic strips with $\lambda = const$.



Other periodic examples can be found by numerical search (Rominger 2007).





Theorem (Chubelaschwili 2009): A strip is elastic $\Leftrightarrow$

the *force vector*

$$
\begin{aligned}
\mathbf{b} &= \frac{1}{2}(\kappa^2(1+\lambda^2)^2+\mu)\,\mathbf{T} \\
&+ (\kappa'(1+\lambda^2)^2+2\kappa(1+\lambda^2)\lambda\lambda')\,\mathbf{N} \\
&- (\kappa^2(1+\lambda^2)^2\lambda+(\frac{\kappa'}{\kappa}(1+\lambda^2)2\lambda)'+((1+\lambda^2)2\lambda)'')\,\mathbf{B}
\end{aligned}
$$

is constant.

Theorem (Chubelaschwili 2009): For an elastic strip

the *torque vector*

$$\begin{aligned} \mathbf{a} &= 2\,\kappa\,\lambda\,(1+\lambda^2)\,\mathbf{T} \\ &+ \frac{1}{\kappa}\,(2\,\kappa\,\lambda\,(1+\lambda^2))'\,\mathbf{N} \\ &+ \kappa\,(1+\lambda^2)\,(1-\lambda^2)\,\mathbf{B} \\ &- \mathbf{b}\times\gamma \end{aligned}$$

is constant.

Force-free elastic strips

- The force **b** and the torque **a** have to be applied to the end point of the strip to keep it in equilibrium.
- They come from the boundary terms of the first variation formula used to derive the Euler-Lagrange equations.

Definition: An elastic strip is called *force-free* if the force vector **b** vanishes.

- For force-free elastic strips the bending energy is critical even if the end point of γ is allowed to move, only the frame at the end point is held fixed.

Force-free elastic strips

- No condition on end point of γ
  $\rightsquigarrow$ variational problem for the tangent image **T**

- For a force-free elastic strip the Lagrange multiplier μ is positive
  $\rightsquigarrow$ normalize to 1 by scaling.

- $\rightsquigarrow$ We are looking for critical points of

$$\tilde{E} = 1/2 \int_0^L (\kappa^2(1+\lambda^2)^2 + 1)ds.$$

- For a force-free elastic strip κ does not vanish $\rightsquigarrow$ tangent image $\mathbf{T}$ is a regular curve in S^2 of curvature λ:

$$\begin{aligned}\mathbf{T}' &= \quad\quad +\kappa\,\mathbf{N} \\ \mathbf{N}' &= -\kappa\mathbf{T} \quad\quad + \kappa\,\lambda\,\mathbf{B} \\ \mathbf{T}' &= \quad\quad -\kappa\,\lambda\,\mathbf{N}\end{aligned}$$

- $d\tilde{s} = \kappa\, ds \rightsquigarrow \gamma$ can be reconstructed from an arclength parametrization of $\mathbf{T}$ as

$$\gamma(\tilde{s}) = \int_0^{\tilde{s}} \frac{1}{\kappa}\,\mathbf{T}\, d\tilde{s}.$$

Theorem: Let $\mathbf{T}\colon [0, \tilde{L}] \to S^2$ be an arclength parametrized curve with curvature λ. Then among all strips $\gamma : [0, \tilde{L}] \to \mathbb{R}^3$ with tangent image $\mathbf{T}$ the one given by

$$\gamma(\tilde{s}) = \int_0^{\tilde{s}} (1 + \lambda^2)\, \mathbf{T}\, d\tilde{s}$$

has minimal Sadowski functional $\tilde{E}$ and

$$\tilde{E} = \int_0^{\tilde{L}} (1 + \lambda^2)\, d\tilde{s}.$$

γ has curvature

$$\kappa = 1/(1 + \lambda)^2.$$

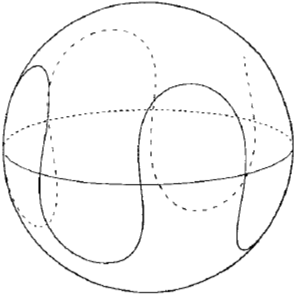
Corollary: The tangent images of force-free elastic strips are elastic curves in S^2, in fact critical points of

$$\int_0^{\tilde{L}} (1 + \lambda^2) \, d\tilde{s}.$$

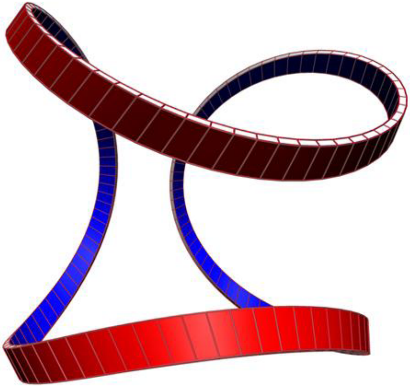
Conversely, for any such spherical curve $\mathbf{T}$ the space curve

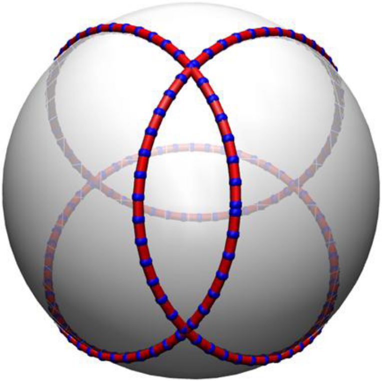
$$\gamma = \int (1 + \lambda^2) \, \mathbf{T} \, d\tilde{s}$$

defines a force-free elastic strip.



Closed force-free strips

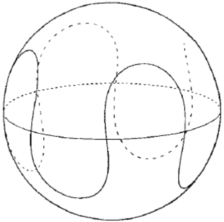




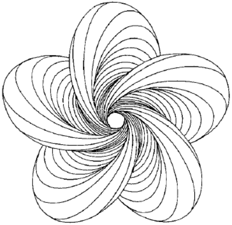
- All possible adapted frames (lifted to S^3) define the *frame cylinder*

$$F : [0, L] \times S^1 \to S^3$$

of a space curve γ.



- F is the preimage of the tangent image $\mathbf{T}$ under the Hopf map $S^3 \to S^2$.



Corollary: The frame cylinder of a force-free elastic strip is Willmore in S^3.

Given

- any arclength-parametrized spherical elastic curve $B : [0, L] \to S^2$ without inflection points
- its unit normal $T = B \times B'$
- its curvature λ.

Then

$$\gamma : [0, L] \to \mathbb{R}^3$$

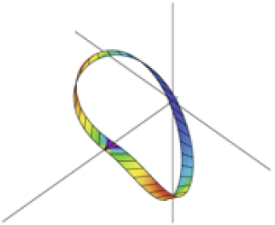
$$\gamma = \int (1 + \frac{1}{\lambda^2}) T$$

defines an elastic strip.

Momentum strips are never closed.

At any point $\gamma(s)$ of a force-free elastic strip the following are equivalent:

- The rulings make an angle of 45° with γ.
- The curvature of the tangent image satisfies $\lambda(s) = 1$.
- A glueing construction is possible where the curvature κ is discontinuous but nevertheless we still have an elastic strip in the sense of balanced force $\mathbf{b}$ and torque $\mathbf{a}$.



Open questions

- Closed solutions by cut and paste?
- Stable ones?
- Moebius band?
- Other integrable classes of elastic strips?
- Maybe all elastic strips come from an integrable system?

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