Constraint Willmore Surfaces

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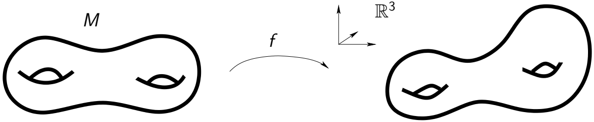
Joint work with C. Bohle, P. Peters

Conformal immersions

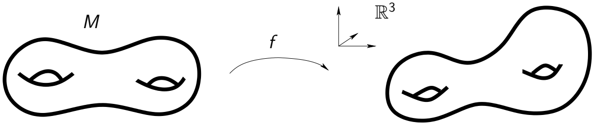
- Conformal structure on oriented $M^2 \leftrightarrow$ complex structure $J : TM \rightarrow TM, J^2 = -I$



- (Garcia, Ruedy 1961/71) Every Riemann surface can be conformally immersed into R^3.

Constraint Willmore surfaces

- Compact constraint Willmore surfaces:
  critical points of Willmore functional for surfaces of a fixed conformal type

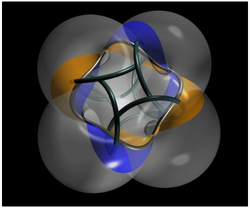


- For spheres: Only one conformal type $\rightsquigarrow$
  Constraint Willmore $\Rightarrow$ Willmore
- For tori: Willmore conjecture proven for some conformal types (Li & Yau, Montiel & Ross)

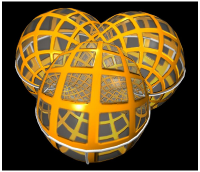
CMC Surfaces in Spaceforms

(Thomsen, 1923)
Minimal in some spaceform
$\Longleftrightarrow$ Willmore + isothermic.

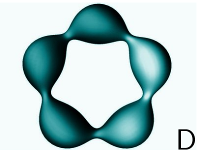
- Clifford torus in S^3
- Willmore spheres in R^3
- this torus in H^3:



(Burstall, Pedit, - 1997)
CMC in some spaceform
$\Longrightarrow$ constrained Willmore + isothermic.
For tori the converse holds as well.

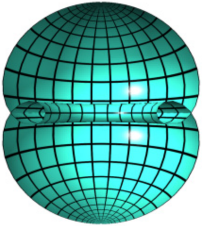


Wente torus in R^3



Delaunay torus in S^3

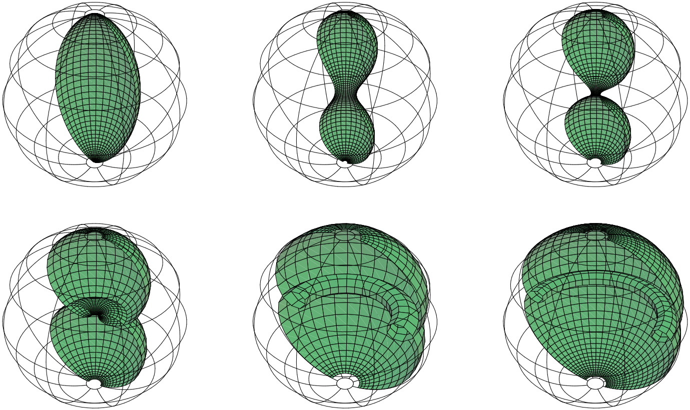
A remarkable immersed sphere



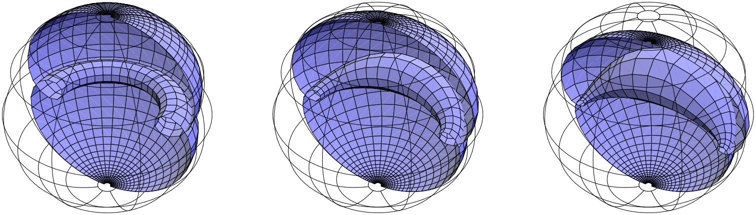
- $H^2 g$ has constant curvature
- 1-soliton sphere (Taimanov, Peters)
- cmc 1 in H^3

- $f : S^2 \longrightarrow \mathbb{R}^3$ is C^∞
- $f|_{S^2 - \{p_1, p_2\}}$ is constraint Willmore

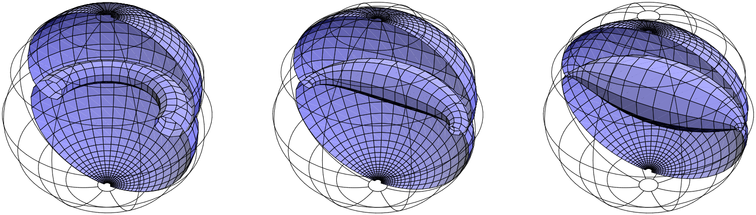
CMC-1 Surfaces of revolution in H^3



CMC-1 Surfaces in H^3 with 3 smooth ends, $W = 16\pi$



CMC-1 Surfaces in H^3 with 4 smooth ends, $W = 16\pi$



To Do

- Define conformal constraint carefully
- Euler Lagrange Equation?
- Non-compact surfaces?
- Other functionals (Area, Volume, . . .)?

Conformal Variations

- Conformal variation with compact support of $f : M \to \mathbb{R}^3$:
  - $f_t(x) = f(x)$ for all $x \in M - K$, $K \subset M$ some compact set.
  - all f_t conformal

- Infinitesimal conformal variation of f :
  vector field Y with compact support along f such that $\dot{J} = 0$ for all infinitesimal variations $\dot{f} = Y$.

Infinitesimal Conformal Variations

- Normal variation $\dot{f} = uN, \quad u \in C_0^\infty(M) \quad \rightsquigarrow$

  $$\dot{J} = 2u\mathring{A}J =: \delta(u) \in \Gamma_0(End_-(TM))$$

- Tangential variation $\dot{f} = df(X), \quad X \in \Gamma_0(TM) \quad \rightsquigarrow$

  $$\dot{J} = \mathcal{L}_X J \text{ (Lie derivative)}$$

- $u \in C_0^\infty(M)$ decribes the normal part uN of a conformal variation $\Leftrightarrow$ there exists $X \in \Gamma_0(TM)$ such that

  $$\delta(u) = \mathcal{L}_X J$$

The adjoint δ^* of δ

The adjoint of

$$\delta : C_0^\infty(M) \to \Gamma_0(End_-(TM))$$

is given by

$$\delta^* : \Gamma(K^2) \to \Omega^2(M)$$

$$\delta^*(q)(X, Y) = 4Re(q(\mathring{A}JX, Y) - q(\mathring{A}JX, Y))$$

Constrained $\mathcal{F}$-Critical Immersions

- Let $f \mapsto \mathcal{F}(f)$ be a reparametrization-invariant functional for immersions $f : M \to \mathbb{R}^3$. f is called constrained $\mathcal{F}$-critical if

  $$\frac{d}{dt}\Big|_{t=0}\mathcal{F}(f_t) = 0$$

  for all compactly supported infinitesimal conformal deformations $\dot{f} = Y$.

$\rightsquigarrow$ constrained Willmore, constrained minimal, volume critical ...

Gradients of Functionals $\mathcal{F}$

- There is a 2-form $\operatorname{grad}\mathcal{F}$ on M such that for every compactly supported variation f_t of f with

$$\dot{f} = uN + df(X)$$

one has

$$\tfrac{d}{dt}\big|_{t=0}\mathcal{F}(f_t) = \textstyle\int_M u \operatorname{grad}\mathcal{F}$$

- $\mathcal{F} =$ surface area $\rightsquigarrow$ $\operatorname{grad}\mathcal{F} = -2HdA$
- $\mathcal{F} =$ enclosed volume $\rightsquigarrow$ $\operatorname{grad}\mathcal{F} = dA$
- $\mathcal{F} =$ Willmore $\rightsquigarrow$ $\operatorname{grad}\mathcal{F} = d * dH - 2H(H^2 - K)dA$

Euler-Lagrange Equation

Theorem 1 : Let $f : M \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface M. If there is a holomorphic quadratic differential $q \in H^0(K^2)$ such that

$$\operatorname{grad}(\mathcal{F}) = \delta^*(q)$$

then f is $\mathcal{F}$-critical.

Theorem 2 : If M is compact, then also the converse is true: For every $\mathcal{F}$-critical conformal immersion $f : M \to \mathbb{R}^3$ there is a holomorphic quadratic differential $q \in H^0(K^2)$ such that

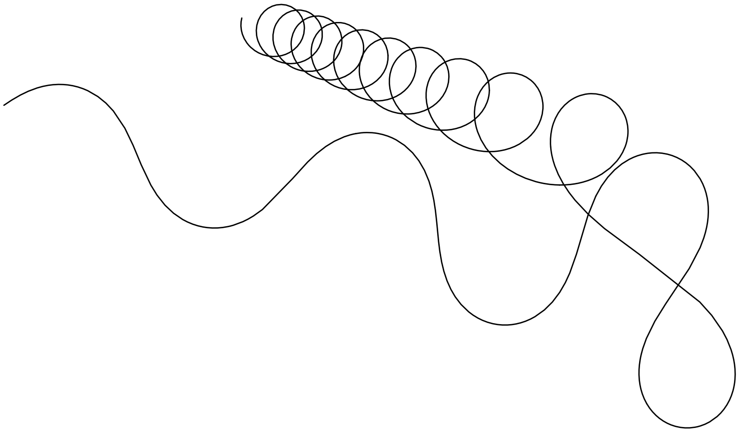
$$\operatorname{grad}(\mathcal{F}) = \delta^*(q).$$

Proof

$$
\begin{tikzcd}
u & & X \\
C_0^\infty(M) \arrow[r, "\delta"] \arrow[d, dotted, leftrightarrow] & \Gamma_0(\mathrm{End}_-(TM)) \arrow[d, dotted, leftrightarrow] & \Gamma_0(TM) \arrow[l, "\mathcal{L}J"'] \arrow[d, dotted, leftrightarrow] \\
\Omega^2(M) & \Gamma(K^2) \arrow[l, "\delta^*"] \arrow[r, "\bar\partial"'] & \Omega^2(T^*M) \\
\operatorname{grad} \mathcal{F} & q & 0
\end{tikzcd}
$$

Burstall Cylinder

There is a 1-parameter family of plane curves γ such that the cylinder over γ is constraint Willmore.

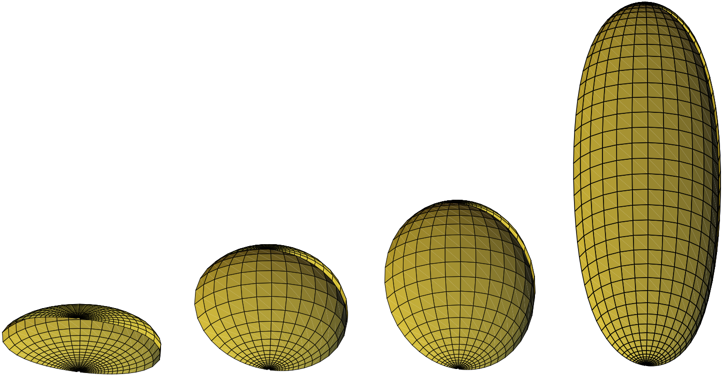


Constrained Minimal Surfaces in $\mathbb{R}^3$

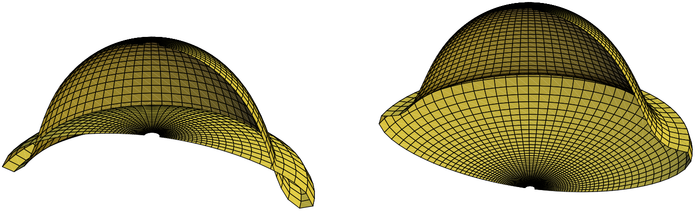
Diplom thesis F. Sziegoleit 2004

- Cylinders over arbitrary plane curves are constraint minimal
- Only round cylinders are in addition constraint volume critical
- There is a one-parameter family of embedded smooth spheres of revolution that are constraint minimal when two points are deleted

Constrained Minimal Spheres with Smooth Ends



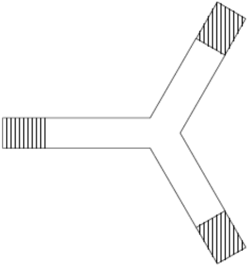
Constrained Minimal Spheres with Non-Smooth Ends



Counterexample in non-compact case

A constraint minimal surface in $\mathbb{R}^3$ with *no* holomorphic quadratic differential q satisfying

$$\text{grad}(\mathcal{F}) = \delta^*(q)$$



Hopf Tori in S^3

- $h : S^3 \to S^2$ Hopf fibration

- $T^2 = h^{-1}(\gamma)$, γ a closed curve in S^2

  $\rightsquigarrow$ T^2 isometric to $\mathbb{R}^2/\Gamma$, Γ generated by

  $$(0,1), \qquad (A/2, L/2)$$

  $L =$ length of γ, $A =$ area enclosed by γ

- All Hopf tori are constraint minimal as well as volume critical in S^3