Panizzut - Roch
Technical University Berlin
Summer semester 2022

Discrete Geometry I

Exercise Sheet 2

Exercise 1 ⋆ - Minkowski sums

Let K_1, $K_2 \subseteq \mathbb{R}^n$. We define the *Minkowski sum* of K_1 and K_2 as

$$K_1 + K_2 = \{\mathbf{k}_1 + \mathbf{k}_2 \,:\, \mathbf{k}_1 \in K_1 \text{ and } \mathbf{k}_2 \in K_2\}.$$

a) Show that $\operatorname{conv}(K_1 + K_2) = \operatorname{conv} K_1 + \operatorname{conv} K_2$.

b) Show that the Minkowski sum of two polytopes is a polytope.

Exercise 2 - Affine independence and dimension

a) Show that $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ are affinely independent if and only for any $m \in \{1, \ldots, k\}$ the vectors $\mathbf{x}_i - \mathbf{x}_m$, $1 \leq i \neq m \leq k$ are linearly independent.

b) Let A be an affine subspace of $\mathbb{R}^n$. Show that $\dim A$ is the maximum number of affinely independent points in A minus one.

Exercise 3 ⋆ - Simplices and polytopes

Show that every polytope $P \subset \mathbb{R}^n$ with m vertices is the linear image of the standard $(m-1)$-dimensional simplex $\Delta_{m-1} \subset \mathbb{R}^m$. In other words, show that there is a matrix $M \in \mathbb{R}^{n \times m}$ such that $P = \{M\mathbf{x} \,:\, \mathbf{x} \in \Delta_{m-1}\}$.

Exercise 4 ⋆ - Vertices

Prove that a point $\mathbf{v}$ of a polytope P cannot be written as convex combination of two distinct points of P if and only if $\mathbf{v}$ is a vertex of P.