Discrete Geometry I
Summer term 2022
Panizzut, Roch

Exercise Sheet 10

Exercise 1 (Graphs of simple polytopes)
Remember the proof of the lecture where we showed that the combinatorics of a simple d-polytope P is determined by its graph $G(P)$. Let H be a connected k-regular subgraph of $G(P)$ with $k \leq d$ and let $\mathcal{O}$ be a good acyclic orientation of $G(P)$ with respect to which the vertex set $\operatorname{vert}(H)$ is initial.

a) How can the corresponding k-face $F < P$ of P be extracted from H?

b) Why is $G(F)$ a subgraph of H?

c) Why does $G(F) = H$ follow?

Exercise 2 (h - Polynomials)
Let P be a simple d-dimensional polytope. Let $\mathcal{O}$ be an acyclic orientation of $G(P)$ and let $h_k^{\mathcal{O}}$ be the number of vertices of $G(P)$ with in-degree k. We define

$$f^{\mathcal{O}} := h_0^{\mathcal{O}} + 2h_1^{\mathcal{O}} + 4h_2^{\mathcal{O}} + \ldots + 2^k h_k^{\mathcal{O}} + \ldots + 2^d h_d^{\mathcal{O}} \ .$$

a) Give two good and two bad acyclic orientations of the 3-cube and compute $f^{\mathcal{O}}$.

b) Show that $f^{\mathcal{O}} \geq f$ holds, where f denotes the number of non-empty faces of P.

c) Show that $\mathcal{O}$ is good if and only if $f = f^{\mathcal{O}}$ holds.

Exercise 3 (Steinitz' Theorem)
Steinitz' Theorem for 3-polytopes states that a graph G is the graph of a 3-dimensional polytope if and only if G is simple, planar and 3-connected. Prove the easy direction.

Exercise 4 (Nested cones theorem)
Let P be a d-polytope. Pick a vertex v of P. For every $k \geq 1$ define the cone

$$C_k := \operatorname{cone}\{u - v \colon u \in \operatorname{vert}(P), \delta(v,u) = k\} \ .$$

Prove that this yields a descending sequence of cones.

Exercise 5 (Cyclic polytopes)
Let $n > d \geq 4$. Show that the graph of the cyclic d-polytope $C_d(n)$ is complete: Give for any two vertices u and v of P a linear functional supporting P, which is maximized precisely on $\operatorname{conv}\{u, v\}$.

Exercise 6 (Dimensionally ambiguous polytopes)
A d-polytope P is called *dimensionally ambiguous* if there is a polytope Q of differenct dimension $\dim(Q) \neq \dim(p)$ with isomorphic graph $G(Q) \cong G(P)$.

a) Show that d-simplex is dimensionally ambiguous for $d \geq 5$ but not for $d \leq 4$.

b) Show that 3-polytopes and simple 4-polytopes cannot be dimensionally ambiguous.

c) Is there a chance that the 5-cube C_5 is dimensionally ambiguous?

d) Show that if P is a 0/1-polytope whose graph is isomorphic to $G(C_d)$, then P is affinely isomorphic to C_d.