Panizzut - Roch
Technical University Berlin
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Discrete Geometry I

Exercise Sheet 1

Exercise 1 - Convexity

a) A set $K \subseteq \mathbb{R}^n$ is convex if for any two points $\mathbf{x}, \mathbf{y} \in K$ the segment $[\mathbf{x}, \mathbf{y}]$ is contained in K.
Show that K is convex if and only if $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i \in K$ for all $k \geq 1$, $\mathbf{x}_1, \ldots, \mathbf{x}_k \in K$, $\lambda_1, \ldots \lambda_k \in \mathbb{R}$ such that $\lambda_1, \ldots \lambda_k \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.

b) Let $K \subseteq \mathbb{R}^n$. Prove that
$\operatorname{conv} K = \{\lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k \,:\, \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset K, \lambda_i \geq 0 \ \forall i \in \{1, \ldots, k\} \text{ and } \sum_{i=1}^{k} \lambda_i = 1\}$.

Exercise 2 - Examples of polytopes

a) Let $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$ be the standard basis vectors in $\mathbb{R}^{n+1}$. Show that

$$\operatorname{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\} = \Big\{\mathbf{x} \in \mathbb{R}^{n+1} \,:\, \sum_{i=1}^{n+1} x_i = 1,\ x_i \geq 0 \ \forall i \in \{1, \ldots, n+1\}\Big\}.$$

This polytope is known as the *standard n-dimensional simplex* Δ_n.

b) Let $\{\{1, -1\}^d\}$ be the set of vectors $\mathbf{x} \in \mathbb{R}^n$ such that $x_i = 1$ or $x_i = -1$ for every $i \in \{1, \ldots, n\}$.
Show that

$$\operatorname{conv}\{\{1, -1\}^d\} = \{\mathbf{x} \in \mathbb{R}^n \,:\, -1 \leq x_i \leq 1 \ \forall i \in \{1, \ldots, n\}\}.$$

This polytope is known as the *n-dimensional cube C_d* (or hypercube).

c) Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors in $\mathbb{R}^n$
Show that

$$\operatorname{conv}\{\mathbf{e}_1, -\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_n\} = \Big\{\mathbf{x} \in \mathbb{R}^n \,:\, \sum_{i=1}^{n} x_i \leq 1\Big\}.$$

This polytope is known as the *n-dimensional crosspolytope C_n^{Δ}*.

Exercise 3 - Affine dependence

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$. Show the following assertions:

a) $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are affinely dependent if and only if $\begin{pmatrix} \mathbf{x}_1 \\ 1 \end{pmatrix}, \ldots \begin{pmatrix} \mathbf{x}_k \\ 1 \end{pmatrix}$ are linearly dependent in $\mathbb{R}^{n+1}$.

Here we are using the notation $\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = (x_1, \ldots, x_n, 1)^T$.

b) If $k \geq n+2$, then $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are affinely dependent.