Correction to: Couplings via comparison principle and exponential ergodicity of SPDEs in the hypoelliptic setting

Oleg Butkovsky[1, 2] Michael Scheutzow[3]

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Abstract

This is a correction to the article [BS20]. The proof of the existence of the invariant measure π in [BS20, Theorem 2.4] had an error. We provide a correct proof here.

In our recent paper [BS20], the existence of the invariant measure π in Theorem 2.4 was proved using an incorrect argument. In this note, we correct this error and provide the correct proof.

Let E be a Polish space equipped with the complete metric ρ, and let $(P_t)_{t\geq 0}$ be a Markov transition function over E. We use the same notation also for the semigroup corresponding to this transition function. Let $W_{\rho\wedge 1}$ denote the corresponding Wasserstein (Kantorovich-Rubinstein) metric, see [BS20, Section 2]. We showed in [BS20, p. 1020, lines 1-8] that under the conditions of [BS20, Theorem 2.4] for any $x \in E$

$$W_{\rho\wedge 1}(P_t(x,\cdot), P_s(x,\cdot)) \to 0, \quad \text{as } s,t \to \infty,$$

and hence there exists a measure π such that

$$W_{\rho\wedge 1}(P_t(x,\cdot), \pi) \to 0, \quad \text{as } t \to \infty.$$

However, since we do not assume that the semigroup $(P_t)_{t\geq 0}$ is Feller, this does **not** necessarily imply that the measure π is invariant for $(P_t)_{t\geq 0}$, as was claimed in our paper [BS20, p. 1020, lines 12-13]. Indeed, consider the following simple counterexample.

Example 1. Consider a Polish space $E := \{0, 1, \frac{1}{2}, \frac{1}{4}, ...\}$ equipped with the Euclidean metric ρ. Let $(P_t)_{t\geq 0}$ correspond to a Markov process that jumps to the next state (in the given order) at rate 1. Then the sequence $(P_t(x,\cdot))_{t\geq 0}$ is Cauchy with respect to $W_\rho = W_{\rho\wedge 1}$ for any $x \in E$. Furthermore, the transition probabilities $P_t(x,\cdot)$ converge weakly to δ_0 as $t \to \infty$ for any $x \in E$. On the other hand, the measure δ_0 is not invariant for this Markov process.

[1]Weierstrass Institute, Mohrenstrasse 39, 10117, Berlin. Email: oleg.butkovskiy@gmail.com

[2]Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489, Berlin.

[3]Technische Universität Berlin, MA 7-5, Strasse des 17 Juni 136, 10623 Berlin, Germany. Email: ms@math.tu-berlin.de

Nevertheless, we still claim that under the assumptions of [BS20, Theorem 2.4], the semigroup $(P_t)_{t\geq 0}$ has an invariant measure, and thus the statement of [BS20, Theorem 2.4] holds true. The main idea is to show that the sequence of measures $(P_t(x,\cdot))_{t\geq 0}$ is Cauchy with respect to a Kolmogorov metric introduced below.

Let $\preceq$ be a partial order on E and suppose that the set

$$\Gamma := \{(x,y) \in E \times E \colon x \preceq y\} \tag{1}$$

is closed (condition (2.1) of [BS20]). A subset A of E is called increasing if $x \in A$ and $x \preceq y$ implies $y \in A$. We denote by $\mathscr{J}$ the set of measurable and increasing subsets of E and by $\mathscr{G}$ the set of measurable and increasing functions $E \to [0,1]$. We assume that the transition function $(P_t)_{t\geq 0}$ is order-preserving, that is, it maps $\mathscr{G}$ to $\mathscr{G}$. Let $\mathcal{P}(E)$ be the set of all probability measures on $(E, \mathcal{B}(E))$.

Definition 2. The *Kolmogorov metric* on the space of probability measures on $\mathcal{P}(E)$ is defined as

$$\kappa(\mu,\nu) := \sup_{A\in\mathscr{J}} |\mu(A) - \nu(A)|, \quad \mu,\nu \in \mathcal{P}(E).$$

Proposition 3. *We have*

$$\kappa(\mu,\nu) = \sup_{g\in\mathscr{G}} \Big| \int_E g(x)\mu(dx) - \int_E g(x)\nu(dx) \Big|. \tag{2}$$

Proof. Since the function $g := \mathbb{1}_A$ is increasing for any set $A \in \mathscr{J}$, we see that the left-hand side of (2) is smaller than the right-hand side. To derive the converse inequality, we note that for any $g \in \mathscr{G}$ we have

$$\begin{aligned} \Big|\int_E g(x)\mu(dx) - \int_E g(x)\nu(dx)\Big| &= \Big|\int_E \int_0^1 \mathbb{1}(g(x) \geq y)\, dy\mu(dx) - \int_E\int_0^1 \mathbb{1}(g(x)\geq y)\, dy\nu(dx)\Big| \\ &= \Big|\int_0^1 \big(\mu(\{x : g(x) \geq y\}) - \nu(\{x : g(x)\geq y\})\big)\, dy\Big| \\ &\leq \int_0^1 \kappa(\mu,\nu)\, dy = \kappa(\mu,\nu), \end{aligned} \tag{3}$$

where in (3) we used Theorem 2 and the fact that the set $\{x : g(x) \geq y\}$ is increasing for any $y \in [0,1]$. □

It is known that the Kolmogorov metric κ is complete in the case $E = \mathbb{R}^d$, equipped with the following partial order: $x \preceq y$ if each coordinate $x_i \leq y_i$ [CR98]. However, we were unable to find any results that establish completeness of the metric for a general Polish space. The closest result we are aware of is [KS19, Theorem 4.1], which proves completeness under additional assumptions on E. Nevertheless, the following holds.

Lemma 4. *Let $(\mu_t)_{t\geq 0}$ be a Cauchy sequence of probability measures on E with respect to κ. Let $\pi \in \mathcal{P}(E)$. Suppose further that*

$$\mu_t \to \pi, \quad \textit{weakly, as } t \to \infty.$$

Then

$$\kappa(\mu_t, \pi) \to 0, \quad \textit{as } t \to \infty. \tag{4}$$

Proof. Fix $\varepsilon > 0$. Let $t_\varepsilon \in \mathbb{N}$ be such that $\kappa(\mu_t, \mu_s) < \varepsilon$ whenever $s, t \geq t_\varepsilon$. Fix any $s > t_\varepsilon$. Then for any $t > t_\varepsilon$ by [KS19, Theorem 3.1], there exists a pair of random variables $(X_{t,s}, Y_{t,s})$ taking values in E such that

$$\mathsf{P}(X_{t,s} \preceq Y_{t,s}) > 1 - \varepsilon; \quad \text{Law}(X_{t,s}) = \mu_t,\ \text{Law}(Y_{t,s}) = \mu_s.$$

Note that for fixed s the sequence of pairs $(X_{t,s}, Y_{t,s})_{t > t_\varepsilon}$ is tight in $E \times E$ because the sequence $(\mu_t)_{t \geq 0}$ is tight. Using Prokhorov's theorem and passing to a converging subsequence, we see that there exists a pair of random variables (X_s, Y_s) such that

$$(X_{t,s}, Y_{t,s}) \to (X_s, Y_s), \quad \text{weakly, as } t \to \infty; \quad \text{Law}(X_s) = \pi,\ \text{Law}(Y_s) = \mu_s.$$

Furthermore since the set Γ defined in (1) is closed, the Portmanteau theorem implies

$$\mathsf{P}(X_s \preceq Y_s) = \mathsf{P}((X_s, Y_s) \in \Gamma) \geq \limsup_{t \to \infty} \mathsf{P}((X_{t,s}, Y_{t,s}) \in \Gamma) > 1 - \varepsilon.$$

Thus, using again [KS19, Theorem 3.1], we see

$$\sup_{A \in \mathcal{J}} (\pi(A) - \mu_s(A)) < \varepsilon.$$

Similarly, we get $\sup_{A \in \mathcal{J}} (\mu_s(A) - \pi(A)) < \varepsilon$, which yields $\kappa(\mu_s, \pi) < \varepsilon$. Since s was an arbitrary number in (t_ε, ∞), this implies (4). □

Now we have all the ingredients to prove the key step towards establishing the existence of the invariant measure.

Lemma 5. *Suppose that all the assumptions of [BS20, Theorem 2.4] hold. Then, for every $x \in E$, the sequence of measures $(P_t(x, \cdot))_{t \geq 0}$ is Cauchy with respect to κ.*

Proof. The proof is similar to the proof of the Cauchy property of $(P_t(x, \cdot))_{t \geq 0}$ with respect to the Wasserstein metric in [BS20, Section 5.1].

Fix $x, y \in E$. Let $\{X^x(s), s \geq 0\}$ and $\{X^y(s), s \geq 0\}$ be independent Markov processes with the transition function $(P_t)_{t \geq 0}$ and the initial conditions $X^x(0) = x$ and $X^y(0) = y$. Introduce stopping times

$$\tau_{x \preceq y} := \inf\{n \in \mathbb{Z}_+ : X^x(n) \preceq X^y(n)\},$$
$$\tau_{y \preceq x} := \inf\{n \in \mathbb{Z}_+ : X^y(n) \preceq X^x(n)\}.$$

Then, using consecutively [FS24, Theorem 3.5(i)] and [BS20, p. 1019, lines 14–16], we get

$$\sup_{g \in \mathscr{G}} \big| \mathsf{E} g(X_t^x) - \mathsf{E} g(X_t^y) \big| \leq \mathsf{P}\big(\tau_{x \preceq y} > t\big) \vee \mathsf{P}\big(\tau_{y \preceq x} > t\big) \leq C\big(1 + V(x) + V(y)\big) e^{-\lambda t}, \quad (5)$$

for a constant $C > 0$. Then for any $s, t \geq 0$, $x \in E$ we derive

$$\begin{aligned}
\kappa(P_t(x, \cdot), P_{t+s}(x, \cdot) &= \sup_{g \in \mathscr{G}} |P_t g(x) - P_{t+s} g(x)| = \sup_{g \in \mathscr{G}} \Big| P_t g(x) - \int_E P_t g(y) P_s(x, dy) \Big| \\
&\leq \sup_{g \in \mathscr{G}} \int_E |P_t g(x) - P_t g(y)| P_s(x, dy) \\
&\leq \int_E C\big(1 + V(x) + V(y)\big) e^{-\lambda t} P_s(x, dy) \\
&\leq C(1 + 2V(x) + \frac{K}{\gamma}) e^{-\lambda t} \to 0 \quad \text{as } t \to \infty,
\end{aligned}$$

where in the penultimate line we used (5), and the last inequality follows from [BS20, Formula (2.2)]. Thus the sequence $(P_t(x, \cdot))_{t \geq 0}$ is Cauchy with respect to κ. □

Now we can complete the proof of the existence of the invariant measure.

Corrected proof of existence of invariant measure for (P_t) in [BS20, Theorem 2.4]. Fix arbitrary $x \in E$. By [BS20, p. 1020, lines 1-8] there exists a measure $\pi \in \mathcal{P}(E)$ such that

$$W_{\rho\wedge 1}(P_t(x,\cdot),\pi) \to 0, \quad \text{as } t \to \infty.$$

By Theorem 5, the sequence of measures $(P_t(x,\cdot))_{t\geq 0}$ is Cauchy with respect to κ. Therefore, Theorem 4 yields

$$\kappa(P_t(x,\cdot),\pi) \to 0, \qquad \text{as } t \to \infty. \tag{6}$$

Take arbitrary $f \in \mathscr{G}$ and $s \geq 0$. We derive

$$\begin{aligned}
\int_E f(z) P_s\pi(dz) &= \int_E P_s f(z)\pi(dz) \\
&= \lim_{n\to\infty} \int_E P_s f(z) P_t(x,dz) \qquad (7)\\
&= \lim_{n\to\infty} \int_E f(z) P_{t+s}(x,dz) \\
&= \int_E f(z)\pi(dz). \qquad (8)
\end{aligned}$$

Here in (7), we used that the semigroup P_t maps bounded increasing measurable functions to bounded increasing measurable functions ([BS20, Assumption 1, Theorem 2.3]). Therefore, $P_s f \in \mathscr{G}$, and thus (7) follows from Theorem 3 and (6). Identity (8) follows from (6) and the fact that $f \in \mathscr{G}$. Thus, $P_s\pi(A) = \pi(A)$ for each $A \in \mathcal{J}$. Since two probability measures which agree on all measurable and increasing sets are equal (see, e.g., [FS24, Lemma 2.8] or [KK78, Lemma 1]) and since $s > 0$ is arbitrary it follows that π is invariant for $(P_t)_{t\geq 0}$. □

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